New York Journal of Mathematics

New York J. Math. **26** (2020) 1184–1212.

Composition operators on distinct Bergman spaces over planar domains

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Abstract. In this paper, we will consider composition operators defined between distinct Bergman spaces over planar domains. The smoothness on boundary of the domains plays an important role in our study. On one hand, an essential extension of Littlewood's Subordination Principle is obtained. Precisely, for each holomorphic map that is defined between bounded domains of smooth boundaries, the associated composition operator is always bounded. This essentially depends on a "standard decomposition" of holomorphic functions over a classical domain, bounded by finitely many disjoint circles. On the other hand, the situation becomes complex if domains with cusp boundary points are concerned, and there exists a link between the boundary behavior of the function symbol and the boundedness of the associated composition operator, where a detailed discussion is presented. Finally, we give estimates of norms for some classes of such composition operators. A deep interplay of function theory, geometry, and operator theory is revealed.

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Received August 22, 2020.

2010 *Mathematics Subject Classification.* Primary: 47B33; Secondary: 30H05.

Key words and phrases. Bergman spaces, composition operators, domains of C^2-boundary.

This work is partially supported by NSFC (12071134, 11471113). We thank the referee for his/her suggestions to improve the paper.

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ISSN 1076-9803/2020

1. Introduction

Let Ω_1 and Ω_2 be bounded domains in the complex plane $\mathbb{C}$, and let $\mathcal{H}(\Omega_1)$ and $\mathcal{H}(\Omega_2)$ be the class of all holomorphic maps over Ω_1 and Ω_2, respectively. Each holomorphic map $\varphi : \Omega_1 \to \Omega_2$ induces a composition operator $C_\varphi : f \mapsto f \circ \varphi$ from $\mathcal{H}(\Omega_2)$ to $\mathcal{H}(\Omega_1)$. The study of composition operators on various spaces of holomorphic functions has attracted a lot of attention in the past four decades. The reader can consult, for example, the references [CM] and [Sh2] for an overview of many aspects on the theory of composition operators, and also refer to [CHZ, CKS, CZ2, CZ1, KSZ, GM, Ma, ManPZ, MaS, OSZ, Sm, Sh1].

In this paper, we focus on the Bergman space $L_a^2(\Omega)$ where Ω is a bounded planar domain. Precisely, let dA denote the area measure over Ω, and the Bergman space $L_a^2(\Omega)$ consists of all holomorphic functions f such that

$$\|f\|^2 = \frac{1}{m(\Omega)} \int_\Omega |f(z)|^2 dA(z) < \infty,$$

where $m(\Omega)$ denotes the area measure of Ω.

One main motivation of this study arises from Littlewood's Subordination Principle, which says that a holomorphic selfmap φ of the unit disk $\mathbb{D}$ with $\varphi(0) = 0$ gives a bounded composition operator on the Bergman space $L_a^2(\mathbb{D})$ (or the Hardy space $H^2(\mathbb{D})$). As well known, it has a direct generalization.

Theorem 1.1 (Littlewood's Subordination Principle)**.** *If $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic map. Then $C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L_a^2(\mathbb{D})$ to $L_a^2(\mathbb{D})$.*

One soon sees that a plain generalization of Littlewood's Subordination Principle fails on high dimensional domain, even if $\mathbb{D}$ is replaced by a unit ball in $\mathbb{C}^d (d \geq 2)$ (see Examples 1 and 2 in [Wo2]). Then it is natural for us to elaborate on holomorphic maps between planar domains and the associated composition operators between distinct Bergman spaces. The first question is posed as follows.

If φ is a holomorphic map from the unit disk to the annulus Ω, does $C_\varphi : f \to f \circ \varphi$ define a bounded composition operator from $L_a^2(\Omega)$ to $L_a^2(\mathbb{D})$?

The answer turns out to be yes (see Theorem 3.2). Moreover, if ψ is a holomorphic map from the annulus to the unit disk, then the analogue is also true (see Theorem 3.1); the map $f \to f \circ \psi$ defines a bounded composition operator from $L_a^2(\mathbb{D})$ to $L_a^2(\Omega)$. But the ideas for the proofs of these two facts are quite different. Furthermore, we have the following generalized theorem, whose proof depends heavily on a decomposition of holomorphic functions on special domains (see Lemma 3.5).

Theorem 1.2. *Suppose both Ω_0 and Ω are bounded domains of C^2-boundary. Let $\varphi : \Omega_0 \to \Omega$ be a holomorphic map. Then the map $C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L_a^2(\Omega)$ to $L_a^2(\Omega_0)$.*

The definition of C^2-boundary is presented in Section 2, and it simply guarantees some regularity of the boundary of a domain. We caution the reader that a bounded domain of C^2-boundary is always finite connected.

In Theorem 1.2 the smoothness on the boundaries of domains Ω_0 and Ω does play an important role in our study. If either $\partial\Omega_0$ or $\partial\Omega$ has at least a cusp point, then Theorem 1.2 would fail. For example, a conformal map from a triangle $\triangle$ onto the unit disk never defines a bounded composition operator from $L_a^2(\mathbb{D})$ to $L_a^2(\triangle)$ (see Example 5.5).

The above examples indicate that cusp boundary points of a domain may destroy the boundedness of a composition operator. However, for bounded planar domains Ω_1 and Ω_2, if a holomorphic map φ from Ω_1 to Ω_2 behaves well at "bad" points on $\partial\Omega_1$, then φ still induces a bounded composition operator (a point $p \in \partial\Omega$ is called a bad point if it is not a regular boundary point of class C^1). This is formulated as below.

Theorem 1.3. *Suppose that Ω_1 is a bounded domain of piecewise C^2-boundary, and Ω_2 is a bounded domain of C^2-boundary. If $\varphi : \Omega_1 \to \Omega_2$ is a holomorphic map such that for each bad point p on $\partial\Omega_1$, there is a neighborhood U_p such that $\varphi(U_p \cap \Omega_1)$ is contained in a compact subset of Ω_2. Then the map $C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L_a^2(\Omega_2)$ to $L_a^2(\Omega_1)$.*

In general, the situation for domains of cusp boundary points is complex. It is known that if φ is a selfmap of the unit disk and its image is contained in a polygon inscribed in $\mathbb{D}$, then $C_\varphi : f \mapsto f \circ \varphi$ defines a compact operator on $L_a^2(\mathbb{D})$ [ShT]. Inspired by this we obtain the following result, but the idea is essentially different from that in [ShT].

Theorem 1.4. *Let Ω be a domain of C^2-boundary and Σ be a convex polygon. If φ is a holomorphic map from Ω to Σ, then $C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L_a^2(\Sigma)$ to $L_a^2(\Omega)$.*

The convexity of Σ can not be dropped in Theorem 1.4. To see this, let Σ be a concave polygon and φ is a conformal map from $\mathbb{D}$ onto Σ. Later, by Theorem 2.8 it follows that there exists a point ζ on $\partial\mathbb{D}$ such that $\varphi'(z)$ tends to zero as $z \to \zeta$. Since $f \circ \varphi = (f \circ \varphi\, \varphi')\dfrac{1}{\varphi'}$ and $\dfrac{1}{\varphi'}$ is unbounded, it is easy to see that the map $C_\varphi : f \to f \circ \varphi$ fails to give a bounded composition operator from $L_a^2(\Sigma)$ to $L_a^2(\mathbb{D})$.

Another motivation is to find a link between the boundedness of composition operators C_φ and the behavior of function symbols φ. This line has been attacked under various settings. For example, in [Wo1] Wogen presented a sufficient and necessary condition for boundedness of C_φ on the Hardy space $H^2(\mathbb{B}_n)$, where φ is a C^3-map on the closed unit ball $\overline{\mathbb{B}_n}$, and he gave some geometric characterizations for bounded composition operators in [Wo2]. The Bergman-space version of Wogen's result are fully discussed in [KS], and later Koo and Wang [KW] generalized Wogen's work under a wider setting, where the symbol φ is not necessarily holomorphic, the domain is the unit ball or the polydisk, and the underling spaces are (harmonic) weighted Bergman spaces.

All characterizations mentioned above require certain local analytical treatise to deal with smoothness of the symbol φ, and we give a try to investigate how some properties of φ globally act on the boundedness of the map C_φ. For this, let us turn to holomorphic maps between two polygons and the Bergman spaces over the polygons. The following present a sufficient condition on a map φ such that C_φ is bounded. For a point $\xi \in \partial\Omega$, if $\mathcal{N}$ is a neighborhood of ξ, then $\mathcal{N} \cap \Omega$ is called an inside-neighborhood of ξ.

Theorem 1.5. *Let Σ_0 be a polygon, and Σ_1 a convex polygon. Suppose φ is a holomorphic map from Σ_0 to Σ_1. If for each vertex p of Σ_0, one of the following hold:*

(i) *φ maps an inside-neighborhood of p into a compact subset of Σ_1;*

(ii) *φ is continuous at p, $\varphi(p)$ is a vertex of Σ_1, and $\frac{|\varphi(z)-\varphi(p)|^{\pi/\beta-1}}{|z-p|^{\pi/\alpha-1}}$ is bounded on an inside-neighborhood of p, where α and β denotes the interior angles at p and at $\varphi(p)$, respectively.*

Then $C_\varphi : f \to f\circ\varphi$ defines a bounded composition map from $L^2_a(\Sigma_1)$ to $L^2_a(\Sigma_0)$.

If both Σ_0 and Σ_1 are convex, Condition (ii) in fact says that φ satisfies a Lipschtz condition on an inside-neighborhood of the vertex p; that is, there is a constant M such that

$$|\varphi(z) - \varphi(p)| \leq M|z-p|^s,$$

where $s = \frac{\pi/\alpha-1}{\pi/\beta-1} > 0$.

So far, the above context has presented a bird's eye view of boundedness of such composition operators defined by general holomorphic maps between bounded planar domains, either of smooth boundary or of boundary with cusp points. If one takes a look on composition operators defined by conformal maps, then one will find an interplay between function theory and geometry. For example, it is clear that every conformal selfmap of the unit disk simply gives a bounded composition operator on the Bergman space $L^2_a(\mathbb{D})$. However, if φ is a holomorphic automorphism of a polygon Σ, and if $C_\varphi : f \mapsto f\circ\varphi$ defines a bounded composition operator on $L^2_a(\Sigma)$, then φ maps vertices of Σ to vertices of Σ (see Theorem 6.1). Since three boundary values fix a holomorphic automorphism φ, only finite members of φ give a bounded composition operator C_φ on $L^2_a(\Sigma)$. In particular, if Σ is a triangle or a quadrilateral, the only candidate for φ is the identity map unless Σ is an equilateral triangle or a parallelogram (see Examples 6.3 and 6.4). We also give norm estimates for some composition operators.

This paper is arranged as follows. Section 2 contains some notations and preparatary lemmas. In Section 3, we will present the proof of Theorem 1.2.

Section 4 establishes Theorem 1.3 and also give some examples of composition operators defined by holomorphic maps on domains of cusp boundary points. The proofs of Theorems 1.4 and 1.5 are contained in Section 5, and Section 6 deals with composition operators defined by conformal maps. Section 7 presents norm estimates for some classes of composition operators.

2. Some preparations

In this section, we will present basic notions and preparatary lemmas, including C^k-boundary ($1 \le k \le \infty$), Carathéodory's theorem, and Schwarz-Christoffel transformation.

In the sequel, two kinds of domains will be involved: domains of C^∞-boundary and polygons. The definition of C^∞-boundary is needed [BerG, p. 22, Definition 1.4.1].

Definition 2.1. An open subset Ω of $\mathbb{R}^2$ is said to have a regular boundary of class $C^k (1 \le k \le \infty)$ if for each point p on $\partial\Omega$, there exist a neighborhood U_p of p, a neighborhood V_p of 0 in $\mathbb{R}^2$, and a diffeomorphism $\varphi_p : U_p \to V_p$ of class C^k such that $\varphi_p(p) = 0$,

$$\varphi_p(U_p \cap \overline{\Omega}) = V_p \cap \{(x, y) \in \mathbb{R}^2 : x \le 0\},$$

and the Jacobian determinant $J(\varphi_p)$ of φ_p is positive in U_p.

Throughout this paper, a domain Ω is said to be *of C^k-boundary* if it has a regular boundary of class C^k.

Example 2.2. *The unit ball and the ellipse are domains of C^∞-boundary.*

More generally, given $1 \le k \le \infty$, if for each point $p \in \partial\Omega$ there is a neighborhood U_p of p such that by omitting a translation and a rotation transformation, $\partial\Omega \cap U_p$ is the image of the function

$$y = f(x), \ -\delta < x < \delta,$$

where $f \in C^k[-\delta, \delta]$, with $p = (0, f(0))$, and

$$\Omega \cap U_p = V_p \cap \{(x, y) \in \mathbb{R}^2 : y > f(x), -\delta < x < \delta\} \tag{2.1}$$

for some neighborhood of V_p, then Ω is of C^k-boundary.

To see this, let $p \in \partial\Omega$, and define the map $\varphi_p : \Omega \cap U_p \to \mathbb{R}^2$ by

$$(x, y) \mapsto (f(x) - y, x).$$

Since each point (x, y) in $\Omega \cap U_p$ satisfies $y > f(x)$ by (2.1), and $J(\varphi_p) = 1$, one can shrink U_p such that φ_p becomes a diffeomorphism from $\Omega \cap U_p$ onto $V_p \cap \{(x, y) \in \mathbb{R}^2 : x \le 0\}$, where V_p is a neighborhood of 0 in $\mathbb{R}^2$. Thus Ω is of C^k-boundary provided that f is of class C^k.

In view of Example 2.2, one can construct more domains of C^k-boundaries.

Example 2.3. *For $\varepsilon > 0$, let Ω_ε be the domain bounded by*

$$\{(x,y) \in \mathbb{R}^2 : x^4 + y^2 = \varepsilon^2\}.$$

Then Ω_ε is of C^∞-boundary.

Furthermore, one can construct a band for Ω_ε. Precisely, one can cut Ω_ε into two parts through the y-axis, and then glue a rectangle of height 2ε in the middle. The new domain is denoted by Ω'_ε, and we claim that Ω'_ε is of C^2-boundary. To this end, observe that at a neighborhood of $(0,\varepsilon) \in \partial\Omega_\varepsilon$

$$y = \sqrt{\varepsilon^2 - x^4}.$$

By taking derivative, one obtains that $y'(x) = -2x^3(\varepsilon^2 - x^4)^{-\frac{1}{2}}$, and

$$y''(x) = -6x^2(\varepsilon^2 - x^4)^{-\frac{1}{2}} - 4x^6(\varepsilon^2 - x^4)^{-\frac{3}{2}}.$$

Then it is straightforward to check that the function

$$h(x) = \begin{cases} y(x), & x \geq 0; \\ 0, & x < 0. \end{cases}$$

is in class C^2, and it follows that Ω'_ε is of C^2-boundary.

By using the idea in Example 2.3, one can construct a band of C^∞-boundary by replacing Ω_ε with the domain

$$\{(x,y) \in \mathbb{R}^2 : x^2 \exp(-\frac{\varepsilon^2}{x^2}) + y^2 = \varepsilon^2 e^{-1}\}.$$

For each $r \in (0,1)$, write

$$A_r = \{z \in \mathbb{C} : r < |z| < \frac{1}{r}\}.$$

The construction in Example 2.3 can be used to prove the following lemma.

Lemma 2.4. *For $0 < r < 1$, each annulus A_r is contained in a union of finitely many simply connected sub-domains of A_r bounded by C^∞-boundary.*

Proof. Let ρ be such that $r < \rho < 1$. It is obvious that the compact set $\{z \in \mathbb{C} : \rho \leq |z| \leq \frac{1}{\rho}\}$ is contained in a union of finitely many disks in the annulus A_r. Thus it is enough to show that the domains $\{z \in \mathbb{C} : r < |z| < \rho\}$ and $\{z \in \mathbb{C} : \frac{1}{\rho} < |z| < \frac{1}{r}\}$ are contained in a union of finitely many simply connected sub-domains $\Omega_j (1 \leq j \leq N)$ of A_r bounded by C^∞-boundary. Observe that the function $z \mapsto \frac{1}{z}$ maps $\{z \in \mathbb{C} : r < |z| < \rho\}$ conformally to $\{z \in \mathbb{C} : \frac{1}{\rho} < |z| < \frac{1}{r}\}$, and it suffices to consider the domain $\{z \in \mathbb{C} : r < |z| < \rho\}$.

Since $\{z \in \mathbb{C} : |z| = r\}$ can be divided into finitely many arc segments, which are biholomorphic to a same line segment, to prove the lemma boils down to showing that $\{z \in \mathbb{C} : 0 < Rez < 1, 0 < Imz < \varepsilon\}$ can be covered by finitely many simply connected sub-domains of C^∞-boundary, lying in the upper half plane. This follows directly from the constructions in Example 2.3 and the comments below it, because $\{z \in \mathbb{C} : 0 < Re\, z < 1, 0 < Im\, z < \varepsilon\}$

can be covered by finitely many copies of translations of Ω'_ε. The proof is complete. □

A slight generalization of Lemma 2.4 is presented as follows.

Corollary 2.5. *A bounded open set Ω in $\mathbb{C}$ of C^∞-boundary is contained in a union of finitely many simply connected sub-domains $\Omega_j (1 \leq j \leq N)$ of Ω bounded by C^∞-boundary.*

We emphasize here that a bounded open set in $\mathbb{C}$ of C^∞-boundary is necessarily a simply connected domain or a multiply connected domain with finitely many holes [BerG, p. 24, Proposition 1.4.7].

Corollary 2.5 follows directly from a combination of Lemma 2.4 and Theorem 2.6 below.

Theorem 2.6. *Every bounded open set Ω_1 in $\mathbb{C}$ of $C^k(1 \leq k \leq \infty)$-boundary is biholomorphic to a bounded open set Ω_2, whose components are analytic curves. Let $h : \Omega_1 \to \Omega_2$ be such a map. Then both h and h^{-1} have C^{k-1}-extension up to the boundaries.*

The case $k \geq 2$ of Theorem 2.6 is of special interest, where both h and h^{-1} have bounded derivatives. The case of $k = \infty$ is due to Bell and Krantz [BeK], and an alternative reference is [BerG, pp. 431-432, Propositions 4.8.22, 4.8.23]. The proofs for $1 \leq k < \infty$ can be founded in Warschawski's work in the book [Pom].

In the study of boundedness of composition operators, our first question is whether a conformal map between distinct domains induces bounded composition operators between Bergman spaces on these domains. We have the following lemma, which will be used frequently later.

Lemma 2.7. *Suppose that $\varphi : \Omega_1 \to \Omega_2$ is a conformal map between two planar domains Ω_1 and Ω_2. Then $C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L^2_a(\Omega_2)$ to $L^2_a(\Omega_1)$ if and only if φ' is bounded away from zero.*

Proof. Let Ω_1 and Ω_2 be planar domains, and let $\varphi : \Omega_1 \to \Omega_2$ be a conformal map. Then the map U_φ defined by

$$U_\varphi f = f \circ \varphi\, \varphi',\ f \in L^2_a(\Omega_2)$$

proves to be a unitary operator from $L^2_a(\Omega_2)$ onto $L^2_a(\Omega_1)$. By the observation that

$$f \circ \varphi = \frac{1}{\varphi'} (f \circ \varphi\, \varphi') = \frac{1}{\varphi'} U_\varphi f, f \in L^2_a(\Omega_2),$$

$C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L^2_a(\Omega_2)$ to $L^2_a(\Omega_1)$ if and only if $h \mapsto h\frac{1}{\varphi'}, h \in L^2_a(\Omega_1)$ defines a bounded multiplication operator, which holds if and only if $\frac{1}{\varphi'}$ is bounded. Then it is direct to get the desired conclusion. □

By Theorem 2.6, if both Ω_1 and Ω_2 are of smooth boundary, then the conformal map $\varphi : \Omega_1 \to \Omega_2$ induces a bounded composition map from $L_a^2(\Omega_2)$ to $L_a^2(\Omega_1)$. However, things change if one of Ω_1 and Ω_2 has cusp boundary points, and more techniques are needed (see Section 5).

For this, we introduce a notion called (a, θ, ε)-*circular section.* Given $a \in \mathbb{C}, \theta \in (0, 2\pi)$, and $\varepsilon > 0$, an (a, θ, ε)-circular section is a domain Ω satisfying

$$\{z : 0 < \arg(z - a)e^{-it} < \theta, |z - a| < \varepsilon\} \subseteq \Omega,$$

and

$$\{z : 0 < \arg(z - a)e^{-it} < \theta, |z - a| < \varepsilon'\} \supseteq \Omega$$

for some $t \in \mathbb{R}$ and $\varepsilon' > 0$, and a is called the vertex of (a, θ, ε)-circular section Ω. The following is of considerable importance in our study.

Theorem 2.8. *Suppose φ is a biholomorphic map from an (a, α, ε)-circular section Ω_0 onto a (b, β, δ)-circular section Ω_1, and φ is continuous at a such that $b = \varphi(a)$. Then the following hold:*

- *(i) If $\alpha > \beta$, then $\lim_{z \to a} \varphi'(z) = \infty$.*
- *(ii) If $\alpha < \beta$, then $\lim_{z \to a} \varphi'(z) = 0$.*
- *(iii) If $\alpha = \beta$, then $\varphi'(z)$ is bounded and bounded away form zero if $|z - a|$ is small enough.*

In particular, suppose that Σ is a polygon and φ is a holomorphic automorphism of Σ. Let a and b be points on $\partial\Sigma$, $b = \varphi(a)$, α and β be the interior angles at a and b, respectively. Then φ is a biholomorphic map from an (a, α, ε)-circular section onto a (b, β, δ)-circular section.

Proof of Theorem 2.8. Without loss of generality, assume $a = b = 0$ and

$$\{z : 0 < \arg z < \alpha, |z| < \varepsilon\} \subseteq \Omega_0 \quad \text{and} \quad \{z : 0 < \arg z < \beta, |z| < \delta\} \subseteq \Omega_1.$$

Also, we assume that both Ω_0 and Ω_1 are of C^2-boundary except at the vertex 0. Define

$$f_0(z) = z^{\frac{\pi}{\alpha}}, 0 < \arg z < \alpha,$$

and

$$f_1(z) = z^{\frac{\pi}{\beta}}, 0 < \arg z < \beta,$$

with $f_0(1) = f_1(1) = 1$. Let Σ_0 and Σ_1 be the images of Ω_0 and Ω_1 under f_0 and f_1, respectively; that is,

$$\Sigma_0 = f_0(\Omega_0), \Sigma_1 = f_1(\Omega_1).$$

Then $H(z) = f_1 \circ \varphi \circ f_0^{-1}(z), z \in \Sigma_0$ defines a biholomorphic map from Σ_0 onto Σ_1. Since both Σ_0 and Σ_1 are of C^2 boundary, by Theorem 2.6 both H' and $(H^{-1})'$ are bounded by a positive constant M.

Rewrite $w_1 = f_0(z)$, $w_2 = H(w_1)$, and $w = f_1^{-1}(w_2)$. Then

$$w = f_1^{-1} \circ H \circ f_0(z) = \varphi(z),$$

and by taking derivatives

$$\varphi'(z) = \frac{dw}{dz} = \frac{dw}{dw_2}\frac{dw_2}{dw_1}\frac{dw_1}{dz} = \frac{\beta}{\alpha} z^{\frac{\pi}{\alpha}-1} w_2^{\frac{\beta}{\pi}-1} \frac{dw_2}{dw_1}.$$

Since both H' and $(H^{-1})'$ are bounded by M, there is a positive constant C such that

$$|w_2| \le C|w_1|, |w_1| \le C|w_2|,$$

which gives

$$|\varphi'(z)| \le M\frac{\beta}{\alpha} C^{\frac{\beta}{\pi}-1} z^{\frac{\pi}{\alpha}-1} w_1^{\frac{\beta}{\pi}-1} = M\frac{\beta}{\alpha} C^{\frac{\beta}{\pi}-1} z^{\frac{\beta}{\alpha}-1}.$$

If $\alpha > \beta$, it immediately follows that $\lim_{z\to a} \varphi'(z) = \infty$.

For $\alpha < \beta$, note that φ^{-1} is a biholomorphic map from the (b, β, δ)-circular section Ω_1 to (a, α, ε)-circular section Ω_0. Similarly, one obtains that $\lim_{w\to b} (\varphi^{-1}(w))' = \infty$, which leads to $\lim_{z\to a} \varphi'(z) = 0$.

If $\beta = \alpha$, by similar reasoning $\varphi'(z)$ is bounded as z tends to a and $(\varphi^{-1}(w))'$ is bounded as w tends to b. Therefore, $\varphi'(z)$ is bounded and bounded away form zero as z tends to a. □

Remark 2.9. The definition of (a, θ, ε)-circular section Ω can be generalized. Recall that the domain Ω is contained in the angular domain Σ

$$\{z : 0 < \arg(z - a)e^{-it} < \theta\}$$

for some $t \in \mathbb{R}$, where $\partial\Sigma$ consists of two half lines $\{z : \arg(z - a)e^{-it} = 0\}$ and $\{z : \arg(z - a)e^{-it} = \theta\}$. In fact, these half lines can be replaced by circular arcs. In this case, Theorem 2.8 still remains true, with a bit modification of the proof.

We record two theorems on complex analysis that will be needed later.

One is Carathéodory's theorem [Ah], which states that if Ω is a Jordan domain, then the conformal map f from $\mathbb{D}$ onto Ω extends to a continuous bijective map F from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$ and F maps $\partial\mathbb{D}$ bijectively onto $\partial\Omega$. If three boundary values of f are assigned, then f is uniquely determined [Ru, pp. 290-291]. These immediately lead to the following theorem.

Theorem 2.10 (Carathéodory's theorem). *If Ω_1 and Ω_2 are Jordan domains, then each conformal map f from Ω_1 to Ω_2 extends to a continuous bijective map from $\overline{\Omega_1}$ to $\overline{\Omega_2}$. If three boundary values of f are assigned, then f is necessarily unique.*

Specifically, if in Theorem 2.10 Ω_1 is allowed to be the half plane, and Ω_2 be a polygon, the conformal map is given by Schwarz-Christoffel transformation [Neh, Pom].

Theorem 2.11 (Schwarz-Christoffel). *Let Σ be a polygon in the complex plane with vertices $a_1, \cdots, a_n$ in counterclockwise order and interior angles*

$\alpha_1, \cdots, \alpha_n$. *Then for each conformal map f from the upper half plane onto Σ satisfying $f(\infty) = a_n$, f can be expressed as*

$$f(z) = A \int_{z_0}^{z} \prod_{i=1}^{n-1} (\zeta - z_i)^{\frac{\alpha_i}{\pi} - 1} d\zeta + C,$$

where A and C are suitable constants, and $z_0 < z_1 < z_2 < \cdots < z_{n-1}$ are real numbers such that $f(z_k) = a_k$, $1 \leq k \leq n-1$.

3. Composition operators induced by holomorphic maps between bounded domains of smooth boundary

This section is devoted to the proof of Theorem 1.2.

To begin with, we establish two stepping-stone theorems. Recall that for each $r \in (0,1)$,

$$A_r = \{z \in \mathbb{C} : r < |z| < \frac{1}{r}\}.$$

The following tells us that a holomorphic map from an annulus to the unit disk induces a bounded composition operator between the associated Bergman spaces.

Theorem 3.1. *If $\varphi : A_r \to \mathbb{D}$ is a holomorphic map, then the map $C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L_a^2(\mathbb{D})$ to $L_a^2(A_r)$.*

Proof. Let f be a function in $L_a^2(\mathbb{D})$. For each compact subset K of A_r, $f \circ \varphi$ is bounded on K, and thus $\int_K |f \circ \varphi(z)|^2 dA(z) < \infty$. Then it suffices to show that for a positive number $\rho \, (r < \rho < 1)$,

$$\int_{\{r < |z| < \rho\}} |f \circ \varphi(z)|^2 dA(z) < \infty,$$

and

$$\int_{\{\frac{1}{\rho} < |z| < \frac{1}{r}\}} |f \circ \varphi(z)|^2 dA(z) < \infty.$$

By Lemma 2.4, $\{r < |z| < \rho\}$ is contained in a union of finitely many simply connected sub-domains $\Omega_j (1 \leq j \leq N)$ of A_r, each bounded by a C^∞-boundary Jordan curve. For each j $(1 \leq j \leq N)$, denote by ψ_j the Riemann mapping from Ω_j onto $\mathbb{D}$, and its inverse by ϕ_j.

By Theorem 2.6, ϕ_j' is bounded on $\mathbb{D}$, and then there is a constant C such that $|\phi_j'|^2 \leq C$ for all j. Thus

$$\begin{aligned} \int_{\Omega_j} |f \circ \varphi(z)|^2 dA(z) &= \int_{\mathbb{D}} |f \circ \varphi \circ \phi_j(z)|^2 |\phi_j'(z)|^2 dA(z) \\ &\leq C \int_{\mathbb{D}} |f \circ \varphi \circ \phi_j(z)|^2 dA(z) \\ &\leq C' \int_{\mathbb{D}} |f(z)|^2 dA(z), \end{aligned}$$

where C' is a constant independent on f, and the last inequality follows from Littlewood's Subordination Principle. Hence,

$$\int_{\{r<|z|<\rho\}} |f\circ\varphi(z)|^2 dA(z) \le \sum_{j=1}^{N} \int_{\Omega_j} |f\circ\varphi(z)|^2 dA(z) < \infty.$$

By the same reasoning as above, one obtains

$$\int_{\{\frac{1}{\rho}<|z|<\frac{1}{r}\}} |f\circ\varphi(z)|^2 dA(z) < \infty$$

to finish the proof. □

By exchanging the positions of the disk $\mathbb{D}$ and the annulus A_r, one has an analogue of Theorem 3.1, but with a different proof.

Theorem 3.2. *If ψ is a holomorphic map from $\mathbb{D}$ to A_r. Then the map $C_\psi : f \to f\circ\psi$ defines a bounded composition operator from $L_a^2(A_r)$ to $L_a^2(\mathbb{D})$.*

Proof. Let f be a function in $L_a^2(A_r)$ and expand the Laurent series of f as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,\ r < |z| < \frac{1}{r}.$$

Write $f_0(z) = \sum_{n=0}^{\infty} c_n z^n$ and $f_1(z) = \sum_{n=-\infty}^{-1} c_n z^n$. Since

$$\int_{A_r} |f(z)|^2 dA(z) < \infty,$$

both f_0 and f_1 lie in $L_a^2(A_r)$. By direct computations, there is a constant $c > 0$ satisfying

$$\int_{A_r} |z^n|^2 dA(z) \ge c \int_{\mathbb{D}_{\frac{1}{r}}} |z^n|^2 dA(z),\ n = 0, 1, \cdots,$$

which implies that $f_0 \in L_a^2(\mathbb{D}_{\frac{1}{r}})$, with $\mathbb{D}_{\frac{1}{r}} = \{z \in \mathbb{C} : |z| < \frac{1}{r}\}$. Then by Littlewood's Subordination Principle, $f_0\circ\psi \in L_a^2(\mathbb{D})$.

Since $f_1 \in L_a^2(A_r)$, a change of variable yields that $f_1(\frac{1}{z}) \in L_a^2(A_r)$, and by similar reasoning as above one obtains that $f_1(\frac{1}{z}) \in L_a^2(\mathbb{D}_{\frac{1}{r}})$. Note that $\frac{1}{\psi} : \mathbb{D} \to A_r$, by applying Littlewood's Subordination Principle

$$f_1\circ\psi = f_1(\frac{1}{z})\circ\frac{1}{\psi} \in L_a^2(\mathbb{D}).$$

Therefore, by $f\circ\psi = f_0\circ\psi + f_1\circ\psi$ we get $f\circ\psi \in L_a^2(\mathbb{D})$ as desired. □

To extend Theorems 3.1 and 3.2 to multiply connected domains of C^∞-boundary, we need some preparations. It is well known that each doubly connected domain is biholomorphic to the annulus A_r for some $r \in (0,1)$ [Go, Chapter 5]. This, along with Theorem 2.6, yields the following lemma.

Lemma 3.3. *Suppose Ω is a doubly connected domain bounded by two disjoint circles. Then there is a conformal map ψ from Ω onto A_r for some $r < 1$. Furthermore, both ψ' and $\frac{1}{\psi'}$ are bounded on Ω.*

Using Lemma 3.3 we get an immediate consequence of Theorem 3.2.

Corollary 3.4. *Suppose Ω is a doubly-connected domain of C^∞-boundary and ψ is a holomorphic map from $\mathbb{D}$ to Ω. Then the map $C_\psi : f \to f \circ \psi$ defines a bounded composition operator from $L^2_a(\Omega)$ to $L^2_a(\mathbb{D})$.*

We also need the following lemma, which gives the decomposition of a function in the Bergman space over a multiply connected domain. It is of independent interest.

Lemma 3.5. *Let D_0 be an open disk and C_0 is the boundary of D_0. Let $C_1, \cdots, C_k$ be k disjoint circles contained in the disk D_0, and Ω be the domain bounded by $C_0, \cdots, C_k$. Then for each $f \in L^2_a(\Omega)$, there is a decomposition*

$$f = f_0 - \sum_{j=1}^{k} f_j,$$

where $f_0 \in L^2_a(D_0)$, and $f_j \in L^2_a(D \backslash \overline{D_j})$ for $1 \le j \le k$, where D_j stands for the domain bounded by C_j, and D is any bounded domain containing D_0.

Proof. For $1 \le j \le k$, write $C_j = \{z \in \mathbb{C} : |z - a_j| = r_j\}$ and define $C_0 = \{z \in \mathbb{C} : |z - a_0| = r_0\}$. Rewrite

$$\int_{C_0} \frac{f(w)}{w-z} dw = \lim_{t \to r_0^+} \int_{|z-a_0|=t} \frac{f(w)}{w-z} dw,$$

and

$$\int_{C_j} \frac{f(w)}{w-z} dw = \lim_{t \to r_j^-} \int_{|z-a_j|=t} \frac{f(w)}{w-z} dw, \; 1 \le j \le k. \tag{3.1}$$

Note that for each $z \in \Omega$, as t tends closely enough to r_j, an application of Cauchy's formula [Ru, p.218, Theorem 10.35] gives that the integral $\int_{|z-a_j|=t} \frac{f(w)}{w-z} dw$ does not depend on the choice of t. Therefore, the right hand side of (3.1) makes sense.

Again by Cauchy's formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(w)}{w-z} dw - \sum_{j=1}^{k} \frac{1}{2\pi i} \int_{C_j} \frac{f(w)}{w-z} dw, z \in \Omega$$

Set

$$f_j(z) = \frac{1}{2\pi i}\int_{C_j}\frac{f(w)}{w-z}dw, z\in\Omega, 1\le j\le k. \tag{3.2}$$

and we will prove that f_j enjoys the desired property in Lemma 3.5.

For this, let $1 \le j \le k$, $z \in \Omega$, and fix an s_j close to r_j ($s_j > r_j$). With $|w - a_j| = s_j$,

$$\frac{1}{w-z} = \frac{1}{(w-a_j)-(z-a_j)} = -\frac{1}{z-a_j}(1-\frac{w-a_j}{z-a_j})^{-1} = -\sum_{n=0}^{\infty}\frac{(w-a_j)^n}{(z-a_j)^{n+1}}. \tag{3.3}$$

Since $f \in L^2_a(\{r_j < |z - a_j| < s_j\})$ and f extends analytically to a neighborhood of $\{z : |z - a_j| = s_j\}$, expanding the Laurent series of f yields that

$$f(w) = \sum_{m=-\infty}^{\infty} c_m(w-a_j)^m, r_j < |w-a_j| \le s_j \tag{3.4}$$

With (3.3) and (3.4) substituted in (3.2), one gets

$$\begin{aligned}
f_j(z) &= \frac{1}{2\pi i}\int_{|w-a_j|=s_j}\frac{f(w)}{w-z}dw \\
&= -\frac{1}{2\pi i}\int_{|w-a_j|=s_j}\sum_{n=0}^{\infty}\frac{f(w)(w-a_j)^n}{(z-a_j)^{n+1}}dw \\
&= -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{|w-a_j|=s_j}\sum_{m=-\infty}^{\infty}c_m(w-a_j)^m(w-a_j)^n dw\frac{1}{(z-a_j)^{n+1}} \\
&= -\sum_{m=-\infty}^{-1}c_m(z-a_j)^m.
\end{aligned}$$

Also noting that $f \in L^2_a(\{r_j < |z - a_j| < s_j\})$, by straightforward computations one gets that $f_j \in L^2_a(D\backslash\overline{D_j})$ for $1 \le j \le k$, where D can be any bounded domain containing D_0. By similar discussions as above, we have $f_0 \in L^2_a(D_0)$ to finish the proof. □

Now we are ready to prove that a holomorphic map from a simply connected domain to a multiply connected domain (both of smooth boundaries) induces a bounded composition operator.

Proposition 3.6. *Suppose that D is a simply connected domain of C^2-boundary, and that Ω is a multiply connected domain bounded by $k + 1$ disjoint closed curves of class C^2. Let $\varphi : D \to \Omega$ be a holomorphic map. Then the map $C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L^2_a(\Omega)$ to $L^2_a(D)$.*

Proof. First assume $D = \mathbb{D}$, and Ω is of the form in Lemma 3.5. Then for each $f \in L_a^2(\Omega)$,

$$f = f_0 - \sum_{j=1}^{k} f_j, \tag{3.5}$$

where $f_0 \in L_a^2(D)$, and $f_j \in L_a^2(D\backslash\overline{D_j})$ for $1 \leq j \leq k$. Since $\varphi : \mathbb{D} \to \Omega$ can also be regarded as a map from $\mathbb{D}$ to $D\backslash\overline{D_1}, \cdots$, or $D\backslash\overline{D_k}$, by Corollary 3.4 we have $f_j \circ \varphi \in L_a^2(\mathbb{D})$ for $1 \leq j \leq k$. With $\varphi : \mathbb{D} \to \Omega$ considered as a map from $\mathbb{D}$ to D, by applying Littlewood's Subordination Principle one gets $f_0 \circ \varphi \in L_a^2(\mathbb{D})$. Therefore, by (3.5) it follows that $f \circ \varphi \in L_a^2(\mathbb{D})$.

Next let $D = \mathbb{D}$, and Ω is a general multiply connected domain bounded by $k + 1$ disjoint closed curves of class C^2. Then there is a biholomorphic map ϕ from a domain Ω' onto Ω, where Ω' is of the form in Lemma 3.5. Moreover, by Theorem 2.6 ϕ' is away from zero, and by Theorem 2.7 C_ϕ is bounded. Since each map $\varphi : \mathbb{D} \to \Omega$ can be written as the composition of ϕ with a map $h : \mathbb{D} \to \Omega'$, $\varphi = \phi \circ h$,

$$C_\varphi = C_h C_\phi.$$

By the discussions in the above paragraph, C_h is bounded as well as C_φ, and then C_φ is bounded as desired.

Finally, by another application of Theorem 2.6, it is easy to see that the unit disk $\mathbb{D}$ can be replaced with a simply connected domain bounded by a C^2-boundary. The proof is finished. □

Remark 3.7. Proposition 3.6 is also true if Ω is replaced by a simply connected domain bounded by a C^2-boundary. This follows from a combination of Theorem 2.6 and Littlewood's Subordination Principle.

Now it comes to the proof of our main result (=Theorem 1.2) in this section.

Theorem 3.8. *Suppose both Ω_0 and Ω are bounded domains of C^2-boundary. Let $\varphi : \Omega_0 \to \Omega$ be a holomorphic map. Then the map $C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L_a^2(\Omega)$ to $L_a^2(\Omega_0)$.*

Proof. Each bounded domains of C^∞-boundary is biholomorphic to a bounded domain bounded by finitely many analytic Jordan curves, and the latter domain is biholomorphic to either the unit disk $\mathbb{D}$ or a domain bounded by a big circle and finitely many disjoint circles inside it (as in Lemma 3.5). In view of Theorem 2.6, to prove Theorem 3.8 it is enough to consider the special case of Ω_0 being $\mathbb{D}$ or the domain in Lemma 3.5.

As done in the proof of Theorem 3.1, write Ω_0 as a union of finitely many simply connected sub-domains $\Omega_j (1 \leq j \leq N)$ of Ω_0 bounded by C^∞-boundary Jordan curves, and a compact subset K. Then for each $f \in L_a^2(\Omega_0)$,

$$\int_{\Omega_0} |f \circ \varphi(z)|^2 dA(z) \leq \int_K |f \circ \varphi(z)|^2 dA(z) + \sum_{j=1}^{N} \int_{\Omega_j} |f \circ \varphi(z)|^2 dA(z).$$

It is clear that $\int_K |f \circ \varphi(z)|^2 dA(z) < \infty$ follows from the boundedness of $f \circ \varphi$ on compact sets, and by Proposition 3.6 and Remark 3.7, for each j

$$\int_{\Omega_j} |f \circ \varphi(z)|^2 dA(z) < \infty.$$

Hence $\int_{\Omega_0} |f \circ \varphi(z)|^2 dA(z) < \infty$ for each $f \in L^2_a(\Omega_0)$. Then by an application of the closed graph theorem, the map $f \to f \circ \varphi$ defines a bounded composition operator from $L^2_a(\Omega)$ to $L^2_a(\Omega_0)$. $\square$

In conclusion, a holomorphic map between two bounded domains of smooth boundaries gives arise to a bounded composition operator between Bergman spaces over these domains. Next section will elaborate on the case that at least one domain has some cusp boundary points.

4. Bounded domains of piecewise smooth boundary

In this section, we study boundedness of composition operators induced by holomorphic maps between two bounded domains, where one is of piecewise smooth boundary. As one will see, the behavior of those maps at "bad" points do have impact on boundedness of composition operators.

For an open set Ω in $\mathbb{C}$, let us call a point p on $\partial\Omega$ a regular point of class $C^k (1 \leq k \leq \infty)$ if it satisfies the condition in Definition 2.1. If $p \in \partial\Omega$ and p is not a regular boundary point of class C^1, let us call p a bad point. For example, each vertex of a polygon is a bad point. A simply connected domain can have infinitely many bad points.

Example 4.1. *Let L_r denote the segment connecting 0 and $\frac{1}{q}e^{ir}$, where $r = \frac{p}{q}$ is a rational number, p and q are positive integers such that*

$$\gcd(p, q) = 1 \, (q > 0).$$

Set

$$\Omega = (\mathbb{D} \backslash [0, 1]) \backslash \bigcup_{r \in (0,1) \cap \mathbb{Q}} L_r.$$

Then 0 and all points $\frac{1}{q}e^{ir}$ are bad points on $\partial\Omega$.

The following result (=Theorem 4.2) can be regarded as a slight generalization of Theorem 1.2. It indicates that if both Ω_1 and Ω_2 have reasonably smooth boundary, and if a holomorphic map $\varphi : \Omega_1 \to \Omega_2$ behaves well at bad points on $\partial\Omega_1$, then φ induces a bounded composition operator.

Theorem 4.2. *Suppose that Ω_1 is a bounded domain of piecewise C^2-boundary, and Ω_2 is a bounded domain of C^2-boundary. If $\varphi : \Omega_1 \to \Omega_2$ is a holomorphic map such that for each bad point p on $\partial\Omega_1$, there is a neighborhood U_p such that $\varphi(U_p \cap \Omega_1)$ is contained in a compact subset of Ω_2. Then the map $f \to f \circ \varphi$ defines a bounded composition operator from $L^2_a(\Omega_2)$ to $L^2_a(\Omega_1)$.*

Proof. If there is no bad point on $\partial\Omega_1$, then the conclusion follows from Theorem 3.8. Otherwise, by assumption one can find finitely many open disks $U_1, \cdots, U_k$, whose union covers these bad points, and the union

$$\cup_{j=1}^{k}\varphi(U_j \cap \Omega_1)$$

is contained in a compact subset of Ω_2. Therefore, for each $f \in L_a^2(\Omega_2)$

$$\int_{\cup_{j=1}^{k}U_j\cap\Omega_1} |f\circ\varphi(z)|^2 dA(z) < \infty.$$

Besides, since Ω_1 is of piecewise C^2-boundary, there exists a perturbation Ω_1' of Ω_1 with C^2-boundary, $\Omega_1' \subseteq \Omega_1$, and $\Omega_1 \backslash \Omega_1' \subseteq \cup_{j=1}^{k}U_j \cap \Omega_1$. Hence by Theorem 3.8

$$\int_{\Omega_1'} |f\circ\varphi(z)|^2 dA(z) < \infty,$$

where f is considered as a function in $L_a^2(\Omega_1')$. Since

$$\Omega_1 \subseteq \Omega_1' \cup (\cup_{j=1}^{k}U_j \cap \Omega_1),$$

$$\int_{\Omega_1} |f\circ\varphi(z)|^2 dA(z) \le \int_{\Omega_1'} |f\circ\varphi(z)|^2 dA(z) + \int_{\cup_{j=1}^{k}U_j\cap\Omega_1} |f\circ\varphi(z)|^2 dA(z) < \infty.$$

Then an application of the closed graph theorem leads to the desired conclusion. $\square$

In general, if either Ω_1 or Ω_2 has bad points, and if $\varphi : \Omega_1 \to \Omega_2$ is a holomorphic map, then it can happen that C_φ is not bounded.

Example 4.3. *Put $U = \{z \in \mathbb{C} : |z-1| < 1, Imz > 0\}$. One can construct a conformal map φ from U onto $\mathbb{D}$ by defining*

$$\varphi(z) = \frac{(\frac{1}{z}-\frac{1}{2})^2 + i}{(\frac{1}{z}-\frac{1}{2})^2 - i}, z \in U.$$

By direct computations,

$$\varphi'(z) = \frac{4i}{[(\frac{1}{z}-\frac{1}{2})^2 - i]^2}(\frac{1}{z}-\frac{1}{2})\frac{1}{z^2}.$$

Hence $\lim_{z\to 0}\varphi'(z) = \lim_{z\to 2}\varphi'(z) = 0$, where the limits are taken for $z \in U$. Since $\frac{1}{\varphi'}$ is unbounded, by Lemma 2.7 $f \mapsto f\circ\varphi$ is a (densely-defined) unbounded operator from $L_a^2(\mathbb{D})$ to $L_a^2(U)$.

On the other hand, φ' is bounded on U and therefore $\frac{1}{(\varphi^{-1})'}$ is bounded on $\mathbb{D}$. This, combined with Lemma 2.7, implies that

$$g \mapsto g\circ\varphi^{-1}, g \in L_a^2(U)$$

defines a bounded composition from $L_a^2(U)$ to $L_a^2(\mathbb{D})$.

To conclude this section, we give one more example.

Example 4.4. *Let $\Omega_0 = \mathbb{D}\backslash[-\frac{1}{2}, \frac{1}{2}]$, and there is an $r\,(0 < r < 1)$ such that there is a conformal map φ from Ω_0 onto A_r, where*

$$A_r = \{z \in \mathbb{C} : r < |z| < \frac{1}{r}\}.$$

By applying Theorem 2.8, φ' is away from zero. Then by Lemma 2.7 we conclude that C_φ induces a bounded composition operator from $L^2_a(A_r)$ to $L^2_a(\Omega_0)$. By similar reasoning, $C_{\varphi^{-1}}$ defines an unbounded composition operator from $L^2_a(\Omega_0)$ to $L^2_a(A_r)$.

There exist abundant domains that have finitely many cusp points, such as the polygons. In the following section, the focus will be on the case where one of the domain is a polygon.

5. Composition operators induced by holomorphic maps from a domain to a polygon

This section will study boundedness of composition operators induced by holomorphic maps from a domain to a polygon.

Our first result (=Theorem 1.4) shows that a holomorphic map from a smooth domain to a convex polygon always induces a bounded composition operator.

Theorem 5.1. *Let Ω be a domain bounded by a C^2-boundary and Σ be a convex polygon. If φ is a holomorphic map from Ω to Σ, then $C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L^2_a(\Sigma)$ to $L^2_a(\Omega)$.*

If Σ is a polygon but is not convex, then C_φ is not necessarily bounded even if φ is a biholomorphic map from a simply connected domain Ω to Σ.

Proof. In view of Theorem 2.6, to deal with a domain bounded by a C^2-boundary is equivalent to treating with a domain bounded by finitely many disjoint circles. By the strategy in Lemma 3.1, such a domain can be covered by finitely many simply connected domains of C^2-boundary. Again by Theorem 2.6, this reduces to the case of $\Omega = \mathbb{D}$.

First, assume that φ is a conformal map from $\mathbb{D}$ to Σ. By Lemma 2.7, the boundedness of C_φ is equivalent to the boundedness of $\frac{1}{\varphi'}$. Assume conversely $\frac{1}{\varphi'}$ is not bounded on $\mathbb{D}$. Then there is a point ξ on the unit circle $\partial\mathbb{D}$ such that $\frac{1}{\varphi'(z)}$ is unbounded as z tends to ξ. However, since Σ is convex, all its interior angles at vertices or at points on edges are less than π. Since either $\varphi(\xi)$ is a vertex of Σ or $\varphi(\xi)$ lies on an edge of Σ, by Theorem 2.8 either $\lim_{z\to\xi} \varphi'(z) = \infty$ or $\varphi'(z)$ is bounded away form zero as z tends to ξ. In each case, $\frac{1}{\varphi'(z)}$ is bounded as z tends to ξ, which is a contradiction. Thus C_φ is bounded.

In general, φ is a holomorphic map from $\mathbb{D}$ to Σ. In this case, let h be a conformal map from $\mathbb{D}$ to Σ and put $\phi = h^{-1} \circ \varphi$, a self-map of $\mathbb{D}$.

Then the map $f \to f \circ \varphi = f \circ h \circ \phi$ is the multiplication of two bounded composition operators, C_h and C_ϕ, and hence $C_\varphi : f \to f \circ \varphi$ defines a bounded composition operator from $L_a^2(\Sigma)$ to $L_a^2(\Omega)$. $\square$

Theorem 5.1 assumes that Σ is a convex polygon. In general, this assumption can be replaced by the condition that Σ is a domain bounded by piece-wise C^2-boundaries, and all interior angles of Σ are less than π. This observation just follows from the proof of Theorem 5.1. For example,

$$\Sigma = \triangle \setminus \overline{D_1 \cup D_2},$$

where $D_1 = \{z : |z + 0.9| < 1\}$, $D_2 = \{z : |z - 0.9| < 1\}$, and $\triangle$ is a triangle containing $\overline{D_1 \cup D_2}$.

The following example illustrates another side of Theorem 5.1.

Example 5.2. *Let Ω be a simply connected domain of C^2-boundary and Σ be a polygon. Then there is a holomorphic map φ from Σ to Ω such that $C_\varphi : f \to f \circ \varphi$ defines a (densely-defined) unbounded composition operator from $L_a^2(\Omega)$ to $L_a^2(\Sigma)$.*

To see this, it is enough to consider the case of $\Omega = \mathbb{D}$ by Theorem 2.6. Let φ be a conformal map from $\mathbb{D}$ onto Σ. Since Σ has an interior angle that is less than π, by an application of Theorem 2.8 $\frac{1}{\varphi'}$ is not bounded. Then Lemma 2.7 leads to that C_φ is not bounded.

If φ is a holomorphic map between two polygons, things are different from Theorem 5.1. We present as below a sufficient condition on φ to define a bounded composition operator C_φ. Geometrically, it says that the boundedness of C_φ depends heavily on the behavior of φ near the vertices of a polygon.

Theorem 5.3. *Let Σ_0 be a polygon, and Σ_1 a convex polygon. Suppose φ is a holomorphic map from Σ_0 to Σ_1. If for each vertex p of Σ_0, one of the following hold:*

(i) *φ maps an inside-neighborhood of p into a compact subset of Σ_1;*
(ii) *φ is continuous at p, $\varphi(p)$ is a vertex of Σ_1, and $\frac{|\varphi(z)-\varphi(p)|^{\pi/\beta-1}}{|z-p|^{\pi/\alpha-1}}$ is bounded on an inside-neighborhood of p, where α and β denotes the interior angles at p and at $\varphi(p)$, respectively.*

Then C_φ defines a bounded composition map from $L_a^2(\Sigma_1)$ to $L_a^2(\Sigma_0)$.

In Condition (ii), if $\alpha = \beta$, then the boundedness of $\frac{|\varphi(z)-\varphi(p)|^{\pi/\beta-1}}{|z-p|^{\pi/\alpha-1}}$ reduces to that of $\frac{|\varphi(z)-\varphi(p)|}{|z-p|}$.

Proof. To prove that C_φ is a bounded composition map from $L_a^2(\Sigma_1)$ to $L_a^2(\Sigma_0)$, we need show that for each f in $L_a^2(\Sigma_1)$,

$$\int_{\Sigma_0} |f \circ \varphi(z)|^2 dA(z) < \infty.$$

The idea is to cut the n-gon Σ_0 into $n+1$ blocks: n small corners Ω_p at vertices p (these corners are (p,α,ε)-circular sections) and one big block Σ'. Precisely, one can choose a sufficiently small $\varepsilon>0$, and for each vertex p the corner Ω_p is nothing but a (p,α,ε)-circular section, where $\alpha=\alpha(p)$ is the interior angle at p. Such a corner Ω_p is required of C^2-boundary except at the vertex p. There exists a domain Σ' of C^2-boundary such that

$$\Sigma_0-\bigcup_p \Omega_p\subseteq \Sigma'\subseteq \Sigma_0-\bigcup_p O(p,\varepsilon/2),$$

where p runs over all vertices of the polygon Σ_0. Since $\Sigma_0=\bigcup_p \Omega_p\cup\Sigma'$, it is enough to prove that for each f in $L^2_a(\Sigma_1)$,

$$\int_{\Sigma'}|f\circ\varphi(z)|^2dA(z)<\infty,$$

and $\int_{\Omega_p}|f\circ\varphi(z)|^2dA(z)<\infty$ for each vertex p.

Since $\varphi:\Sigma'\to\Sigma_1$ is a holomorphic map, by Theorem 5.1 $C_{\varphi|_{\Sigma'}}$ defines a bounded composition operator from $L^2_a(\Sigma')$ to $L^2_a(\Sigma_0)$, and therefore

$$\int_{\Sigma'}|f\circ\varphi(z)|^2dA(z)<\infty.$$

To finish the proof, it remains to prove that for each vertex p,

$$\int_{\Omega_p}|f\circ\varphi(z)|^2dA(z)<\infty.$$

For this, there are two case to distinguish: either φ maps an inside-neighborhood of p into a compact subset of Σ_1, or φ satisfies Condition (ii).

If there is an inside-neighborhood U of p such that $\varphi(U)$ is contained in a compact subset of Σ_1, then the prescribed number ε can be chosen enough small such that $\Omega_p\subseteq\Sigma_0\cap O(p,\varepsilon)\subseteq U$, which gives that $f\circ\varphi$ is bounded on Ω_p, forcing $\int_{\Omega_p}|f\circ\varphi(z)|^2dA(z)<\infty$.

Now assume φ satisfies Condition (ii), and this will be the main focus of our discussion. For convenience, assume $p=\varphi(p)=0$, let Ω_p be contained in the angular domain $\{z:0<\arg z<\alpha\}$, and its image $\varphi(\Omega_p)$ is contained in $\Sigma_1\cap\{z:0<\arg z<\beta\}$.

First we construct holomorphic maps ϕ, V and T such that

$$\varphi|_{\Omega_p}=V\circ\phi\circ T.$$

To be precise, set

$$T(z)=z^{\frac{\pi}{\alpha}},0<\arg z<\alpha,$$

and

$$V(z)=z^{\frac{\beta}{\pi}},0<\arg z<\pi.$$

Let Ω_p^* be a $(\varphi(p),\beta,\delta)$-circular section of C^2-boundary (except one possible cusp point $\varphi(p)$) such that

$$\varphi(\Omega_p)\subseteq\Omega_p^*.$$

Let $\Omega_p' = T(\Omega_p)$, $\Omega_p'' = V^{-1}(\Omega_p^*)$, and $\phi = V^{-1} \circ \varphi \circ T^{-1} : \Omega_p' \to \Omega_p''$. Then

$$\varphi|_{\Omega_p} = V \circ \phi \circ T.$$

Since

$$\frac{1}{T'(z) \cdot V'(\phi \circ T(z))} = \frac{1}{z^{\pi/\alpha - 1}(\phi \circ T(z))^{\beta/\pi - 1}} = \frac{\varphi(z)^{\pi/\beta - 1}}{z^{\pi/\alpha - 1}} \tag{5.1}$$

by assumption (ii) $\dfrac{1}{T'(z) \cdot V'(\phi \circ T(z))}$ is bounded on Ω_p.

Since $f \in L_a^2(\Sigma_1)$,

$$\int_{\Omega_p''} |f \circ V(z)V'(z)|^2 dA(z) = \int_{\Omega_p^*} |f(z)|^2 dA(z) < \infty.$$

Since both Ω_p' and Ω_p'' are of C^2-boundary and $\phi : \Omega_p' \to \Omega_p''$ is a holomorphic map, applying Theorem 3.8 leads to that

$$\int_{\Omega_p'} |(f \circ V \cdot V') \circ \phi(z)|^2 dA(z) < \infty,$$

and noting that $T : \Omega_p \to \Omega_p'$ is a biholomorphic map, one obtains

$$\int_{\Omega_p} |T'(z)(f \circ V \cdot V') \circ (\phi \circ T)(z)|^2 dA(z) < \infty.$$

Since

$$f \circ \varphi = f \circ V \circ \phi \circ T = T' \cdot (f \circ V \cdot V') \circ (\phi \circ T) \frac{1}{T'(z) \cdot V'(\phi \circ T(z))}$$

then by boundedness of $\dfrac{1}{T'(z) \cdot V'(\phi \circ T(z))}$ (see (5.1))

$$\int_{\Omega_p} |f \circ \varphi|^2 dA(z) < \infty.$$

The proof of Theorem 5.3 is complete. □

Also, Theorems 5.1 and 5.3 are true for curvilinear polygons (a domain bounded by finitely many circular arcs is called a curvilinear polygon [Neh]), and the proofs are just the same. For example,

$$\{z : Imz > 1, |z + \sqrt{3}i| > 2, |z| < 1\}$$

is a curvilinear polygon with vertices -1 and 1.

Theorem 5.3 fails if the condition that Σ_1 is convex is dropped. This is illustrated by the following.

Example 5.4. *Let $\triangle$ be the equilateral triangle with vertices $-1, 1$ and $\sqrt{3}i$, and let Σ be the polygon $\triangle$ minus the square with vertices $-\frac{1}{4}, \frac{1}{4}, \frac{1+i}{4}$, and $\frac{-1+i}{4}$. Let φ be the conformal map from $\triangle$ onto Σ, preserving the vertices $-1, 1$ and $\sqrt{3}i$. Then by Theorem 2.8 Condition (ii) in Theorem 5.3 holds.*

However, by Theorem 2.10 there is a unique point p on the edge of $\triangle$ such that $\varphi(p) = \frac{-1+i}{4}$, at which the interior angle equals $\frac{7\pi}{4} > \pi$. Again by an application of Theorem 2.8, $\lim_{z\to p}\varphi'(z) = 0$. Then by Lemma 2.7 C_φ is not bounded.

The following construction can be found in many textbooks of complex variable.

Example 5.5. *Let $\triangle$ be the equilateral triangle with vertices $-1, 1$ and $\sqrt{3}i$. Then one can construct a conformal map φ from the unit disk onto the triangle $\triangle$. For this, let f be a conformal map of upper half-plane onto $\triangle$. Precisely, by Theorem 2.11 there are two constants C_0 and C_1 such that*

$$f(z) = C_0 \int_0^z w^{-\frac{2}{3}}(w-1)^{-\frac{2}{3}}dw + C_1.$$

If we require that $f(0) = -1$ and $f(1) = 1$, we get

$$f(z) = (-1+\sqrt{3}i)\frac{\Gamma(\frac{2}{3})}{\Gamma^2(\frac{1}{3})}\int_0^z w^{-\frac{2}{3}}(w-1)^{-\frac{2}{3}}dw - 1,$$

where Γ denotes the Gamma function.

Let h be a conformal map from the unit disk to the upper half-plane, and put $\varphi = h\circ f$. By Theorem 5.1 C_φ induces a bounded composition operator from $L_a^2(\mathbb{D})$ to $L_a^2(\triangle)$.

On the other hand, by a combination of Lemma 2.7 and Theorem 2.8 one can prove that $C_{\varphi^{-1}}$ is an unbounded composition operator from a dense subspace of $L_a^2(\triangle)$ to $L_a^2(\mathbb{D})$.

So far, we have given characterizations for boundedness of a composition operator arising from a holomorphic map from a domain to a polygon under mild setting. The next content will focus on conformal holomorphic maps between polygons, and present a close link between the rigidity of polygons' geometry and the boundedness of the composition operators defined by such maps.

6. Composition operators induced by conformal maps between polygons

This section will treat with composition operators induced by conformal maps between polygons. One will see how the rigidity of polygons' geometry and the behavior of conformal maps together work on the boundedness of the associated composition operators.

Every conformal selfmap of the unit disk simply gives a bounded composition operator on the Bergman space $L_a^2(\mathbb{D})$. However, if the unit disk is replaced by polygons, rarely does a conformal map define a bounded composition operator.

Theorem 6.1. *Suppose Σ is a polygon and φ is a holomorphic automorphism of Σ. If $C_\varphi : f \mapsto f \circ \varphi$ defines a bounded composition operator from $L_a^2(\Sigma)$ to $L_a^2(\Sigma)$, then φ maps vertices of Σ to vertices of Σ. In this case, for each vertex a of Σ, the interior at a equals the interior at $\varphi(a)$.*

Proof. Suppose Σ is a polygon, and φ is a holomorphic automorphism of Σ such that $C_\varphi : f \mapsto f \circ \varphi$ defines a bounded composition operator $L_a^2(\Sigma)$ to $L_a^2(\Sigma)$.

To derive a contradiction, assume that φ does not map all vertices of Σ to vertices. For this, there are two cases to distinguish: Σ is a convex polygon or Σ is not a convex polygon.

Case I: Σ is a convex polygon.

Since φ does not map all vertices of Σ to vertices, there is a vertex a such that $\varphi(a)$ lie on an edge of the polygon Σ, and the interior angle at $\varphi(a)$ equals π. But the interior angle at a is less than π since Σ is convex. Then by Theorem 2.8 $\lim\limits_{z\to a} \varphi'(z) = 0$. It follows from Lemma 2.7 that C_φ is not bounded, which contradicts with the assumption in Theorem 6.1.

Case II: Σ is not a convex polygon.

In this case, let $a_1, \cdots, a_j$; $b_1, \cdots, b_k$ be all vertices of Σ, and $\alpha_1, \cdots, \alpha_j$; $\beta_1, \cdots, \beta_k$ be the interior angles at these vertices, respectively, and

$$\alpha_i > \pi, \beta_l < \pi, \quad 1 \le i \le j, 1 \le l \le k.$$

By Theorem 2.10, φ extends to a continuous bijection from $\partial\Sigma$ onto $\partial\Sigma$. By the same reasoning as in Case I, for each i the interior angle at $\varphi^{-1}(a_i)$ is not less than $\alpha_i (\alpha_i > \pi)$, and thus $\varphi^{-1}(a_i)$ lies in $\{a_1, \cdots, a_j\}$. Therefore

$$\{\varphi^{-1}(a_1), \cdots, \varphi^{-1}(a_j)\} = \{a_1, \cdots, a_j\}.$$

This immediately gives that

$$\{a_1, \cdots, a_j\} = \{\varphi(a_1), \cdots, \varphi(a_j)\}. \tag{6.1}$$

Similarly, for each l the interior angle at $\varphi(b_l)$ is not larger than $\beta_l (\beta_1 < \pi)$, and thus $\varphi(b_l)$ lies in $\{b_1, \cdots, b_k\}$. This yields that

$$\{\varphi(b_1), \cdots, \varphi(b_k)\} = \{b_1, \cdots, b_k\},$$

which, along with (6.1), shows that φ maps all vertices of Σ to vertices. This is a contradiction.

Now it remains to prove that for each vertex a of Σ, the interior at a equals the interior at $\varphi(a)$. Let q be the number of vertices of Σ, which are

$$v_1, \cdots, v_q$$

in anti-clockwise direction. Let $[n]$ denote the element $n + q\mathbb{Z}$ in the group $\mathbb{Z}_q \equiv \mathbb{Z}/q\mathbb{Z}$. such that

$$\varphi(v_i) = v_{[i+d]}, \, i = 1, \cdots, q.$$

If $d = 0$, then φ preserves all vertices of Σ, and the proof is finished. If $1 \le d < q$, there is a least nonnegative integer m such that

$$v_{[i+(m+1)d]} = v_i.$$

Let $\gamma_1, \cdots, \gamma_m, \gamma_{m+1}$ denote the interior angles at

$$v_i, v_{i+d}, \cdots, v_{[i+md]}, v_{[i+(m+1)d]} = v_i,$$

respectively. Since φ maps one vertex to the next one in the list above, by Theorem 2.8

$$\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_m \ge \gamma_{m+1} = \gamma_1.$$

Hence $\gamma_1 = \gamma_2$; that is, the interior angle at v_i equals the interior angle at $v_{[i+d]} = \varphi(v_i)$. By arbitrariness of i, for each vertex a of Σ the interior at a equals the interior at $\varphi(a)$. The proof is complete. $\square$

Some special cases of Theorem 6.1 are of independent interest.

Example 6.2. *Suppose Σ is a regular n-gon centered at zero, and φ is a holomorphic automorphism of Σ. If $C_\varphi : f \mapsto f \circ \varphi$ defines a bounded composition operator from $L_a^2(\Sigma)$ to $L_a^2(\Sigma)$, we conclude that there is a positive integer $k < n$ such that $\varphi(z) = \omega^k z$, where $\omega = \exp(\frac{2\pi i}{n})$.*

To see this, note that by Theorem 6.1 φ maps vertices of Σ to vertices of Σ. By Theorem 2.10, three assigned boundary values of a conformal map between Jordan domains determine uniquely the map φ, and hence φ is of the form as above.

Also, by Theorem 6.1 it is almost routine to show the following examples.

Example 6.3. *Suppose Σ is a triangle. Then there is a holomorphic automorphism map φ that induces a bounded composition operator on $L_a^2(\Sigma)$ with $\varphi(z) \ne z$, $z \in \Sigma$ if and only if Σ is an equilateral triangle.*

Example 6.4. *Suppose Σ is a quadrilateral. Then there is a holomorphic automorphism map φ that induces a bounded composition operator on $L_a^2(\Sigma)$ with $\varphi(z) \ne z$, $z \in \Sigma$ if and only if Σ is a parallelogram. In this case, φ is a rotation preserving vertices of Σ.*

7. Estimates for norms of composition operators

This section will give norm estimates for some classes of composition operators.

To begin with, we recall a fact [Zhu, Chapter 11]. In the Bergman space $L_a^2(\mathbb{D})$, if φ is a holomorphic selfmap of $\mathbb{D}$, then C_φ is bounded and

$$\|C_\varphi\| \le \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}.$$

For a holomorphic selfmap φ of $\mathbb{D}$, we denote

$$\mathfrak{v}(\varphi) = \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}.$$

Proposition 7.1. *If φ is a holomorphic map from $\mathbb{D}$ to A_r, then the composition operator $C_\varphi : L_a^2(A_r) \to L_a^2(\mathbb{D})$ is bounded and $\|C_\varphi\| \leq \mathfrak{t}(\varphi)$, where*

$$\mathfrak{t}(\varphi) = \mathfrak{v}(r\varphi) + \mathfrak{v}(\frac{r}{\varphi})\sqrt{\max\{\frac{1}{r^4}, (\frac{1}{r^4} - 1)\frac{1}{-8\ln r}\}}.$$

Proof. First note that if $h : \mathbb{D} \to \mathbb{D}_{\frac{1}{r}}$ is a holomorphic map, then $rh(z)$ is a holomorphic selfmap of $\mathbb{D}$, and by direct computations,

$$\|C_h\| = \|C_{rh}\| \leq \mathfrak{v}(rh),$$

where $C_h : L_a^2(\mathbb{D}_{\frac{1}{r}}) \to L_a^2(\mathbb{D})$ and $C_{rh} : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ are composition operators defined on distinct Bergman spaces.

For each function f in $L_a^2(A_r)$, let $f = f_0 + f_1$ be the decomposition in the proof of Theorem 3.2. That is,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,\ r < |z| < \frac{1}{r},$$

$f_0(z) = \sum_{n=0}^{\infty} c_n z^n$ and $f_1(z) = \sum_{n=-\infty}^{-1} c_n z^n$. By the above paragraph,

$$\begin{aligned}
\frac{\|f_0 \circ \varphi\|^2}{\|f\|^2} &= \frac{\frac{1}{m(\mathbb{D})}\int_{\mathbb{D}} |f_0 \circ \varphi(z)|^2 dA(z)}{\frac{1}{m(A_r)}\int_{A_r} |f(z)|^2 dA(z)} \\
&\leq \mathfrak{v}^2(r\varphi)\frac{\frac{1}{m(\mathbb{D}_{\frac{1}{r}})}\int_{\mathbb{D}_{\frac{1}{r}}} |f_0(z)|^2 dA(z)}{\frac{1}{m(A_r)}\int_{A_r} |f(z)|^2 dA(z)} \leq \mathfrak{v}^2(r\varphi).
\end{aligned}$$

By the proof of Theorem 3.2, since $f_1 \in L_a^2(A_r)$, $f_1(\frac{1}{z}) \in L_a^2(A_r)$, and then $f_1(\frac{1}{z})$ can be considered as a function on $\mathbb{D}_{\frac{1}{r}}$. Since $f_1 \circ \varphi = f_1(\frac{1}{z}) \circ \frac{1}{\varphi}$,

$$\frac{1}{m(\mathbb{D})}\int_{\mathbb{D}} |f_1 \circ \varphi(z)|^2 dA(z) \leq \mathfrak{v}^2(\frac{r}{\varphi})\frac{1}{m(\mathbb{D}_{\frac{1}{r}})}\int_{\mathbb{D}_{\frac{1}{r}}} |f_1(\frac{1}{z})|^2 dA(z),$$

and thus

$$\begin{aligned}
\frac{\|f_1\circ\varphi\|^2}{\|f\|^2} &= \frac{\frac{1}{m(\mathbb{D})}\int_{\mathbb{D}}|f_1\circ\varphi(z)|^2dA(z)}{\frac{1}{m(A_r)}\int_{A_r}|f(z)|^2dA(z)}\\
&\leq \mathfrak{v}^2(\frac{r}{\varphi})\frac{\frac{1}{m(\mathbb{D}_{\frac{1}{r}})}\int_{\mathbb{D}_{\frac{1}{r}}}|f_1(\frac{1}{z})|^2dA(z)}{\frac{1}{m(A_r)}\int_{A_r}|f(z)|^2dA(z)}\\
&\leq \mathfrak{v}^2(\frac{r}{\varphi})\frac{m(A_r)}{m(\mathbb{D}_{\frac{1}{r}})}\sup_{n\geq 1}\frac{\int_{\mathbb{D}_{\frac{1}{r}}}|z|^{2n}dA(z)}{\int_{A_r}|z|^{-2n}dA(z)}\\
&\leq \mathfrak{v}^2(\frac{r}{\varphi})(1-r^4)\max\{\frac{1}{r^4(1-r^4)},\frac{1}{r^4}\frac{1}{-8\ln r}\}\\
&\leq \mathfrak{v}^2(\frac{r}{\varphi})\max\{\frac{1}{r^4},(\frac{1}{r^4}-1)\frac{1}{-8\ln r}\}.
\end{aligned}$$

This inequality, combined with (7.1), immediately gives that

$$\frac{\|f\circ\varphi\|}{\|f\|}\leq\frac{\|f_0\circ\varphi\|}{\|f\|}+\frac{\|f_1\circ\varphi\|}{\|f\|}\leq\mathfrak{v}(r\varphi)+\mathfrak{v}(\frac{r}{\varphi})\sqrt{\max\{\frac{1}{r^4},(\frac{1}{r^4}-1)\frac{1}{-8\ln r}\}}.$$

□

It is worthy to point out that if U is an annulus conformally isomorphic to A_r, then there is a map of the form $z\mapsto az+b(a\neq 0)$ which maps A_r to U. For each holomorphic map $\varphi:\mathbb{D}\to A_r$, $a\varphi(z)+b$ maps $\mathbb{D}$ to U. It is direct to check that

$$\|C_\varphi\|=\|C_{a\varphi+b}\|.$$

In this situation, we define

$$\mathfrak{t}(a\varphi+b)=\mathfrak{t}(\varphi).$$

Remind that the modulus of an annulus $\{z:r<|z-a|<R\}$ is the ratio $\frac{R}{r}$, and each planar doubly connected domains Ω are conformally isomorphic to an annular A_r where r is uniquely determined [Go]. In this case, the modulus of Ω, $\mathfrak{m}(\Omega)$, is defined to be the modulus of A_r.

The following context will provide an estimate of the norm of a composition operator defined by a map from $\mathbb{D}$ to the domain Ω in Lemma 3.5.

For this, let D_0 be an open disk and C_0 is the boundary of D_0. Let $C_1,\cdots,C_k$ be k disjoint circles contained in the disk D_0, and Ω be the domain bounded by $C_0,\cdots,C_k$. For $1\leq i\leq k$, let Ω_i be the minimal annulus containing Ω such that $\partial\Omega_i\supseteq C_i$, let Ω_i' be the maximal annulus contained in Ω such that $\partial\Omega_i\supseteq C_i$, and put

$$\varphi_i=\varphi:\mathbb{D}\to\Omega_i.$$

Then C_{φ_i} denotes the composition operator from $L_a^2(\Omega_i)$ to $L_a^2(\mathbb{D})$, and $\mathfrak{t}(\varphi_i)$ is well defined. Denote by Ω_0 the largest annulus contained in Ω such that

$\partial\Omega_0 \supseteq C_0$, and define

$$\varphi_0 = \varphi : \mathbb{D} \to D_0.$$

Proposition 7.2. *Let Ω be the domain in Lemma 3.5, and let φ_i, Ω_i, Ω_i' be defined as above. If φ is a holomorphic map from $\mathbb{D}$ to Ω, then for the composition operator $C_\varphi : L_a^2(\Omega) \to L_a^2(\mathbb{D})$,*

$$\begin{aligned} \|C_\varphi\| &\leq \mathfrak{t}(\varphi_0)\sqrt{\frac{m(\Omega)}{m(D_0)}\frac{1}{1-1/\mathfrak{m}^2(\Omega_0)}} \\ &+ \sum_{1\leq i\leq k} \mathfrak{t}(\varphi_i)\sqrt{\frac{m(\Omega)}{m(\Omega_i)}\max\{\frac{1-1/\mathfrak{m}^2(\Omega_i)}{1-1/\mathfrak{m}^2(\Omega_i')}, \frac{\ln\mathfrak{m}(\Omega_i)}{\ln\mathfrak{m}(\Omega_i')}\}}. \end{aligned}$$

Proof. For each $f \in L_a^2(\Omega)$, let $f = f_0 - \sum_{i=1}^k f_i$ be the decomposition in the proof of Lemma 3.5, where $f_0 \in L_a^2(D_0)$, and $f_i \in L_a^2(D\backslash\overline{D_i})$ for $1 \leq i \leq k$, where D is any bounded domain containing Ω. Since D can be chosen sufficiently large, we have $f_i \in L_a^2(\Omega_i)$ for $1 \leq i \leq k$.

For $1 \leq i \leq k$,

$$\begin{aligned} \frac{\|f_i\circ\varphi\|^2}{\|f\|^2} &= \frac{\frac{1}{m(\mathbb{D})}\int_{\mathbb{D}}|f_i\circ\varphi(z)|^2dA(z)}{\frac{1}{m(\Omega)}\int_\Omega|f(z)|^2dA(z)} \\ &\leq \mathfrak{t}^2(\varphi_i)\frac{\frac{1}{m(\Omega_i)}\int_{\Omega_i}|f_i(z)|^2dA(z)}{\frac{1}{m(\Omega)}\int_\Omega|f(z)|^2dA(z)} \\ &\leq \mathfrak{t}^2(\varphi_i)\frac{m(\Omega)}{m(\Omega_i)}\frac{\int_{\Omega_i}|f_i(z)|^2dA(z)}{\int_{\Omega_i'}|f(z)|^2dA(z)} \\ &\leq \mathfrak{t}^2(\varphi_i)\frac{m(\Omega)}{m(\Omega_i)}\sup_{n\geq1}\frac{\int_{\Omega_i}|z-a_i|^{-2n}dA(z)}{\int_{\Omega_i'}|z-a_i|^{-2n}dA(z)} \\ &\leq \mathfrak{t}^2(\varphi_i)\frac{m(\Omega)}{m(\Omega_i)}\max\{\frac{1-1/\mathfrak{m}^2(\Omega_i)}{1-1/\mathfrak{m}^2(\Omega_i')}, \frac{\ln\mathfrak{m}(\Omega_i)}{\ln\mathfrak{m}(\Omega_i')}\}. \end{aligned}$$

In a similar way, one gets

$$\begin{aligned} \frac{\|f_0\circ\varphi\|^2}{\|f\|^2} &= \frac{\frac{1}{m(\mathbb{D})}\int_{\mathbb{D}}|f_0\circ\varphi(z)|^2dA(z)}{\frac{1}{m(\Omega)}\int_\Omega|f(z)|^2dA(z)} \\ &\leq \mathfrak{t}^2(\varphi_0)\frac{\frac{1}{m(D_0)}\int_{D_0}|f_0(z)|^2dA(z)}{\frac{1}{m(\Omega)}\int_\Omega|f(z)|^2dA(z)} \\ &\leq \mathfrak{t}^2(\varphi_0)\frac{m(\Omega)}{m(D_0)}\frac{\int_{D_0}|f_0(z)|^2dA(z)}{\int_{\Omega_0}|f(z)|^2dA(z)} \\ &\leq \mathfrak{t}^2(\varphi_0)\frac{m(\Omega)}{m(D_0)}\frac{1}{1-1/\mathfrak{m}^2(\Omega_0)}. \end{aligned}$$

Therefore, by $f = f_0 - \sum_{1\le i\le k} f_i$, it follows that

$$\begin{aligned}\frac{\|f\circ\varphi\|}{\|f\|} &\le \frac{\|f_0\circ\varphi\|}{\|f\|} + \sum_{1\le i\le k}\frac{\|f_i\circ\varphi\|}{\|f\|}\\ &\le \mathfrak{t}(\varphi_0)\sqrt{\frac{m(\Omega)}{m(D_0)}\frac{1}{1-1/\mathfrak{m}^2(\Omega_0)}}\\ &+ \sum_{1\le i\le k}\mathfrak{t}(\varphi_i)\sqrt{\frac{m(\Omega)}{m(\Omega_i)}\max\{\frac{1-1/\mathfrak{m}^2(\Omega_i)}{1-1/\mathfrak{m}^2(\Omega_i')},\frac{\ln\mathfrak{m}(\Omega_i)}{\ln\mathfrak{m}(\Omega_i')}\}}.\end{aligned}$$

The proof is completed. □

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This paper is available via http://nyjm.albany.edu/j/2020/26-49.html.