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Geometry of semi-invariant lightlike product manifolds

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ABSTRACT. We study the geometry of semi-invariant lightlike submanifolds of an indefinite Kaehler manifold $\tilde{M}$ admitting a quarter-symmetric non-metric connection $\tilde{\mathcal{D}}$. We establish conditions for a proper totally umbilical semi-invariant lightlike submanifold of $\tilde{M}$ admitting $\tilde{\mathcal{D}}$ to be a totally geodesic semi-invariant lightlike submanifold. We further derive some characterization theorems for a semi-invariant lightlike submanifold of $\tilde{M}$ admitting $\tilde{\mathcal{D}}$ to be a semi-invariant lightlike product manifold. Finally, we obtain some necessary and sufficient conditions for a semi-invariant lightlike submanifold of $\tilde{M}$ admitting $\tilde{\mathcal{D}}$ to be a minimal lightlike submanifold.

CONTENTS

1. Introduction

It is known that there are many similarities between the geometry of non-degenerate submanifolds and Riemannian submanifolds of a semi-Riemannian manifold. When the induced metric on the submanifold is degenerate, then the geometry of submanifolds becomes remarkably different from the geometry of non-degenerate submanifolds. Moreover, with respect to the casual character of a curve, there are spacelike curves, timelike curves, and

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lightlike curves. Analogous to the similarity between Riemannian and non-degenerate submanifolds, spacelike and timelike curves also have several similarities with Riemannian curves. But the geometry of lightlike curves is remarkably different from the geometry of non-degenerate curves. As a generalization of the geometry of lightlike curves, the geometry of lightlike submanifolds of semi-Riemannian manifolds was established by Duggal and Bejancu in [7]. Since in the geometry of lightlike submanifolds, the tangent and normal bundles have a non-empty intersection, therefore the geometry of lightlike submanifolds becomes more complicated than that of non-degenerate submanifolds of semi-Riemannian manifolds. Duggal and Bejancu [7] used this geometry to fill an important missing part in the general theory of submanifolds. Since the screen distribution of a lightlike submanifold is not unique, therefore, it is not possible to generalize all the concepts of the classical theory of submanifolds to lightlike submanifolds. Hence, to establish unique screen distribution, we recently explored the concept of screen conformal lightlike submanifolds of a semi-Riemannian manifold in [12].

It is well known that most of the branches of physics and mathematics use the geometry of manifolds and their submanifolds endowed with indefinite metric (metric with non-zero index), so the classical theory of submanifolds endowed with definite metric may not be useful there. Moreover, the geometry of lightlike submanifolds is important in the general theory of relativity, as lightlike submanifolds act as models of various types of horizons, namely Killing horizons, dynamical and conformal horizons etc. Lightlike geodesics are also interpreted as the worldline of photons in the general theory of relativity, and hence, a lightlike submanifold is taken as a photon surface [6]. Furthermore, the concept of lightlike submanifolds has potential applications in the study of asymptotically flat spacetimes, radiation and electromagnetic fields, event horizons of the Kerr and Kruskal black holes etc., see [7], [10]. Hence, the notion of lightlike submanifolds has become of remarkable importance in the present scenario.

Golab [11] introduced a quarter-symmetric linear connection $\bar{\nabla}$ on a differentiable manifold as a linear connection whose torsion tensor $\tilde{T}$ is of the form

$$\tilde{T}(X,Y) = \pi(Y)\phi(X) - \pi(X)\phi(Y), \tag{1}$$

for any vector fields X, Y on a manifold, where π is a 1-form and ϕ is a tensor of the type $(1,1)$. If the quarter-symmetric linear connection $\bar{\nabla}$ is not a metric connection, then $\bar{\nabla}$ is called a quarter-symmetric non-metric connection. Kilic and Bahadir [14] studied screen semi-invariant lightlike hypersurfaces of a semi-Riemannian product manifold admitting a quarter-symmetric non-metric connection. Recently in [13], we also studied radical screen transversal lightlike submanifolds of an indefinite Kaehler manifold admitting a quarter-symmetric non-metric connection. In the process of development of geometry of lightlike submanifolds, we contribute to establish the geometry of semi-invariant lightlike submanifolds of an indefinite Kaehler manifold admitting a quarter-symmetric non-metric connection.

2. Lightlike submanifolds

Let $(\tilde{M}, \tilde{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$. Let (M, g) be an $m-$dimensional submanifold of $\tilde{M}$ and g be the induced metric of $\tilde{g}$ on M. If g is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\tilde{M}$. For a degenerate metric g on M, $T_xM^{\perp}$ is also a degenerate $n-$dimensional subspace of $T_x\tilde{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces, but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$, which is known as the radical (null) subspace. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\tilde{M}$ is called an $r-$lightlike submanifold [7], and $Rad(TM)$ is called the radical distribution on M. The screen distribution, denoted by $S(TM)$, is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is, $TM = Rad(TM) \perp S(TM)$. Let $S(TM^{\perp})$ be a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$, which is non-degenerate with respect to $\tilde{g}$. Let $tr(TM)$ be a complementary (but not orthogonal) vector bundle to TM in $T\tilde{M} \mid_M$, then $tr(TM) = ltr(TM) \perp S(TM^{\perp})$, where $ltr(TM)$ is complementary to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and is an arbitrary lightlike transversal vector bundle of M. Hence

$$T\tilde{M} \mid_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}).$$

Let $\mathcal{U}$ be a local coordinate neighbourhood of M, then the local quasi-orthonormal field of frames on $\tilde{M}$ along M is

$$\{\xi_1, ..., \xi_r, X_{r+1}, ..., X_m, N_1, ..., N_r, W_{r+1}, ..., W_n\},$$

where $\{\xi_i\}_{i=1}^{r}$, $\{N_i\}_{i=1}^{r}$ are lightlike basis of $\Gamma(Rad(TM)|_{\mathcal{U}})$, $\Gamma(ltr(TM)|_{\mathcal{U}})$, respectively and $\{X_a\}_{a=r+1}^{m}$, $\{W_\alpha\}_{\alpha=r+1}^{n}$ are orthonormal basis of $\Gamma(S(TM)|_{\mathcal{U}})$, $\Gamma(S(TM^{\perp})|_{\mathcal{U}})$, respectively. These local quasi-orthonormal field of frames on $\tilde{M}$ satisfy

$$\tilde{g}(N_i, \xi_j) = \delta^i_j, \quad \tilde{g}(N_i, N_j) = \tilde{g}(N_i, X_a) = \tilde{g}(N_i, W_\alpha) = 0. \tag{2}$$

Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$, then the Gauss and Weingarten formulas are $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and $\tilde{\nabla}_X U = -A_U X + \nabla^t_X U$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here, ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on TM, which is known as the second fundamental form, A_U is a linear operator on M and known as the shape operator. Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad \tilde{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{3}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))-$valued and $\Gamma(S(TM^{\perp}))-$valued respectively, therefore, they are respectively known as the lightlike second fundamental form and the screen second fundamental form on M. In particular, we have

$$\tilde{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \quad \tilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{4}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Then, using (3) and (4), we obtain

$$\tilde{g}(h^s(X,Y),W) + \tilde{g}(Y, D^l(X,W)) = \tilde{g}(A_W X, Y). \tag{5}$$

We can induce some new geometric objects on the screen distribution $S(TM)$ of M. Let P be the projection morphism of TM on $S(TM)$, then

$$\nabla_X PY = \nabla^*_X PY + h^*(X,PY), \quad \nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{6}$$

$\{\nabla^*_X PY, A^*_{\xi} X\}$ and $\{h^*(X,Y), \nabla^{*t}_X \xi\}$ belongs to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$, respectively. h^* and A^* are $Rad(TM)-$valued and $S(TM)-$valued bilinear forms and are known as the screen second fundamental form and the screen shape operator of $S(TM)$, respectively. Using (3) and (6), we obtain

$$\tilde{g}(h^l(X,PY),\xi) = g(A^*_{\xi} X, PY), \quad \tilde{g}(h^*(X,PY),N) = g(A_N X, PY), \tag{7}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

From the geometry of non-degenerate submanifolds, it is known that the induced connection ∇ on a non-degenerate submanifold is a metric connection. Unfortunately, this is not true for lightlike submanifolds, as

$$(\nabla_X g)(Y,Z) = \tilde{g}(h^l(X,Y),Z) + \tilde{g}(h^l(X,Z),Y), \tag{8}$$

for any $X, Y, Z \in \Gamma(TM)$.

Next, Barros and Romero [3] defined indefinite Kaehler manifolds as below:

Definition 2.1. *Let $(\tilde{M}, J, \tilde{g})$ be an indefinite almost Hermitian manifold and $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$ with respect to $\tilde{g}$. Then, $\tilde{M}$ is called an indefinite Kaehler manifold if J is parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla} J = 0$.*

Recently, Atcen and Kilic [1] defined semi-invariant lightlike submanifolds of semi-Riemannian product manifolds. Later, Bahadir and Kilic [2] defined semi-invariant lightlike submanifolds of indefinite Kaehler manifolds as below:

Definition 2.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite Kaehler manifold $(\tilde{M}, J, \tilde{g})$. If*

$$J(Rad(TM)) \subset S(TM), \quad J(ltr(TM)) \subset S(TM), \quad J(S(TM^{\perp})) \subset S(TM),$$

then M is called a semi-invariant lightlike submanifold of an indefinite Kaehler manifold $(\tilde{M}, J, \tilde{g})$.

If we set $L_1 = J(Rad(TM))$, $L_2 = J(ltr(TM))$ and $L_3 = J(S(TM^{\perp}))$ then

$$S(TM) = L_0 \perp (L_1 \oplus L_2) \perp L_3,$$

where L_0 is a $(m-n-2r)-$dimensional distribution of M and hence

$$TM = L_0 \perp (L_1 \oplus L_2) \perp L_3 \perp Rad(TM).$$

Thus, we have the following decomposition

$$T\tilde{M} = L_0 \perp (L_1 \oplus L_2) \perp L_3 \perp S(TM^{\perp}) \perp (Rad(TM) \oplus ltr(TM)).$$

Denote $L = L_0 \perp L_1 \perp Rad(TM)$ and $L' = L_2 \perp L_3$, then $TM = L \oplus L'$, where L and L' are invariant and anti-invariant distributions with respect to J, respectively.

Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of an indefinite Kaehler manifold $(\tilde{M}, J, \tilde{g})$, then for each X tangent to M, JX can be written as

$$JX = \tau X + \omega_l X + \omega_s X = \tau X + \omega X, \tag{9}$$

where ω_l and ω_s are projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively. Therefore, τX and ωX are the tangential and the transversal parts of JX, respectively. In addition, for any $V \in \Gamma(tr(TM))$, JV can be written as

$$JV = \nu V, \tag{10}$$

where νV is the tangential part of JV.

Example 1. *Let $\tilde{M} = \boldsymbol{R}_4^8$ be a $8-$dimensional manifold with signature $(-, -, +, +, -, -, +, +)$ and $\{x_i\}_{i=1}^{8}$ be the standard coordinate system of $\boldsymbol{R}_4^8$. If we take $J(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7)$, then $J^2 = -I$ and J is an almost complex structure on $\boldsymbol{R}_4^8$. Assume that the submanifold M of $\tilde{M}$ is defined by $x_1 = -\sqrt{2}t_2 + \sqrt{2}t_4 + \sqrt{2}t_5 - \sqrt{2}t_6$, $x_2 = \sqrt{2}t_1 - \sqrt{2}t_3 - \sqrt{2}t_4 + \sqrt{2}t_5 + 3\sqrt{2}t_6$, $x_3 = t_1 - t_2 + t_3 - t_5 + t_6$, $x_4 = t_1 + t_2 - t_3 + t_4 + t_6$, $x_5 = -\sqrt{2}t_2 + \sqrt{2}t_4 - \sqrt{2}t_5 - 3\sqrt{2}t_6$, $x_6 = \sqrt{2}t_1 + \sqrt{2}t_3 + \sqrt{2}t_4 + \sqrt{2}t_5 - 3\sqrt{2}t_6$, $x_7 = -t_1 - t_2 + t_3 + 3t_4 - 2t_5 - 5t_6$, $x_8 = t_1 - t_2 + t_3 + 2t_4 + 3t_5 - t_6$. Then, M becomes a semi-invariant lightlike submanifold of $\boldsymbol{R}_4^8$, for complete proof see [2].*

3. Quarter symmetric non-metric connection

Let $\tilde{\nabla}$ be a Levi-Civita connection on an indefinite Kaehler manifold $(\tilde{M}, J, \tilde{g})$. If we set

$$\tilde{\mathcal{D}}_X Y = \tilde{\nabla}_X Y + \pi(Y)JX, \tag{11}$$

for any $X, Y \in \Gamma(T\tilde{M})$, then using the fact that $\tilde{\nabla}$ is a torsion-free metric connection, we obtain

$$(\tilde{\mathcal{D}}_X \tilde{g})(Y, Z) = -\pi(Y)\tilde{g}(JX, Z) - \pi(Z)\tilde{g}(JX, Y), \tag{12}$$

and

$$\tilde{T}^{\tilde{\mathcal{D}}}(X, Y) = \pi(Y)JX - \pi(X)JY, \tag{13}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$, where $\tilde{T}^{\tilde{\mathcal{D}}}$ is a torsion tensor of the connection $\tilde{\mathcal{D}}$ and π is a 1-form associated with the vector field U on $\tilde{M}$, that is, $\pi(X) = \tilde{g}(X, U)$. From (12) and (13), $\tilde{D}$ becomes a quarter symmetric non-metric connection on $\tilde{M}$. Moreover, $\tilde{M}$ admits a tensor field J of the type (1,1), therefore for any $X, Y \in \Gamma(T\tilde{M})$, we have

$$\tilde{\mathcal{D}}_X JY = J\tilde{\mathcal{D}}_X Y + \pi(JY)JX + \pi(Y)X. \tag{14}$$

Remark 1. *Throughout in this paper, we suppose $\tilde{M}$ as an indefinite Kaehler manifold admitting a quarter-symmetric non-metric connection $\tilde{\mathcal{D}}$, unless otherwise stated.*

Let M be a semi-invariant lightlike submanifold of $\tilde{M}$ and $\mathcal{D}$ be the induced linear connection on M from $\tilde{\mathcal{D}}$. Then, the Gauss formula is

$$\tilde{\mathcal{D}}_X Y = \mathcal{D}_X Y + \tilde{h}^l(X, Y) + \tilde{h}^s(X, Y), \tag{15}$$

for any $X, Y \in \Gamma(TM)$, where $\mathcal{D}_X Y \in \Gamma(TM)$ and $\tilde{h}^l$, $\tilde{h}^s$ are the lightlike second fundamental form, the screen second fundamental form of M, respectively. On substituting (3), (15) in (11) and on comparing the tangential and transversal components both sides, we obtain

$$\mathcal{D}_X Y = \nabla_X Y + \pi(Y)\tau X, \tag{16}$$

$$\tilde{h}^l(X, Y) = h^l(X, Y) + \pi(Y)\omega_l X, \tag{17}$$

$$\tilde{h}^s(X, Y) = h^s(X, Y) + \pi(Y)\omega_s X. \tag{18}$$

Further, using (8), (12), (13) and (15)-(18), we get

$$(\mathcal{D}_X g)(Y, Z) = (\nabla_X g)(Y, Z) - \pi(Y)g(\tau X, Z) - \pi(Z)g(\tau X, Y), \tag{19}$$

and

$$T^{\mathcal{D}}(X, Y) = \pi(Y)\tau X - \pi(X)\tau Y, \tag{20}$$

for any $X, Y, Z \in \Gamma(TM)$, where $T^{\mathcal{D}}$ is the induced torsion tensor of the induced connection $\mathcal{D}$ on M. Hence, from (19) and (20), we have the following observation immediately.

Theorem 3.1. *Let M be a semi-invariant lightlike submanifold of $\tilde{M}$. Then, the induced connection $\mathcal{D}$ is also a quarter symmetric non-metric connection on M.*

Next, the Weingarten formulas with respect to the quarter symmetric non-metric connection $\tilde{\mathcal{D}}$ are given as

$$\tilde{\mathcal{D}}_X N = -\tilde{A}_N X + \tilde{\nabla}^l_X N + \tilde{\mathcal{D}}^s(X, N), \tag{21}$$

$$\tilde{\mathcal{D}}_X W = -\tilde{A}_W X + \tilde{\nabla}^s_X W + \tilde{\mathcal{D}}^l(X, W), \tag{22}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$, where $\tilde{\nabla}^l$ and $\tilde{\nabla}^s$ are the linear connections on $ltr(TM)$ and $S(TM^\perp)$, respectively, and both $\tilde{A}_N$, $\tilde{A}_W$ are linear operators on $\Gamma(TM)$. Using (4), (9), (11), (21), (22) and on comparing the tangential components, we obtain

$$\tilde{A}_N X = A_N X - \pi(N)\tau X, \quad \tilde{A}_W X = A_W X - \pi(W)\tau X. \tag{23}$$

Now, define differential $1-$forms η_i locally on TM as $\eta_i(X) = g(X, N_i)$, for any $X \in \Gamma(TM)$. Let P be the projection of TM on $S(TM)$, then any $X \in \Gamma(TM)$, can be written as $X = PX + \sum_{i=1}^r \eta_i(X)\xi_i$, where $\{\xi_i\}_{i=1}^r$ is a basis for $Rad(TM)$. Therefore

$$\mathcal{D}_X PY = \mathcal{D}^*_X PY + \tilde{h}^*(X, PY), \quad \mathcal{D}_X \xi = -\tilde{A}^*_\xi X + \tilde{\nabla}^{*t}_X \xi, \tag{24}$$

where $\{\mathcal{D}^*_X PY, \tilde{A}^*_\xi X\}$ belongs to $\Gamma(S(TM))$ and $\{\tilde{h}^*(X, PY), \tilde{\nabla}^{*t}_X \xi\}$ belongs to $\Gamma(Rad(TM))$. Using (16) and (24), we also obtain

$$\mathcal{D}^*_X PY = \nabla^*_X PY + \pi(PY)P\tau X, \tag{25}$$

$$\tilde{h}^*(X, PY) = h^*(X, PY) + \pi(PY)\sum_{i=1}^r \eta_i(\tau X)\xi_i, \tag{26}$$

$$\tilde{A}^*_\xi X = A^*_\xi X - \pi(\xi)P\tau X, \quad \tilde{\nabla}^{*t}_X \xi = \nabla^{*t}_X \xi + \pi(\xi)\sum_{i=1}^r \eta_i(\tau X)\xi_i. \tag{27}$$

Further, using (7), (17), (23), (26), and (27), we have

$$\begin{aligned} g(\tilde{h}^*(X, PY), N_j) &= g(\tilde{A}_{N_j} X, PY) + \pi(N_j)g(P\tau X, PY) \\ &\quad + \pi(PY)\eta_j(\tau X). \end{aligned} \tag{28}$$

Since the induced connection ∇^* of ∇ is a metric connection on the screen distribution, then using (25), the induced connection $\mathcal{D}^*$ of $\mathcal{D}$ on the screen distribution $S(TM)$ satisfies

$$(\mathcal{D}^*_X g)(PY, PZ) = -\pi(PY)g(P\tau X, PZ) - \pi(PZ)g(PY, P\tau X), \tag{29}$$

and the induced torsion tensor T^* of $\mathcal{D}^*$ is given by

$$T^*(PX, PY) = \pi(PY)P\tau X - \pi(PX)P\tau Y + h^*(Y, PX) - h^*(X, PY). \tag{30}$$

Let $\{\xi_i\}_{i=1}^r$ be a lightlike basis of $\Gamma(Rad(TM)|_{\mathcal{U}})$ on a coordinate neighbourhood $\mathcal{U}$ of M, then locally, (24) can be written as

$$\mathcal{D}_X PY = \mathcal{D}^*_X PY + \sum_{i=1}^{r} \tilde{h}^*_i(X, PY)\xi_i, \tag{31}$$

$$\mathcal{D}_X \xi_i = -\tilde{A}^*_{\xi_i} X + \sum_{j=1}^{r} \mu_{ij}(X)\xi_j, \tag{32}$$

where $\tilde{h}^*_i(X, PY) = \tilde{g}(\tilde{h}^*(X, PY), N_i)$ and $\mu_{ij}(X) = \tilde{g}(\tilde{\nabla}^{*t}_X \xi_i, N_j) = -\rho_{ji}(X)$. Next, we recall the following theorem from [7] for later uses.

Theorem 3.2. *Let M be an $r-$lightlike submanifold with $r < min\{m, n\}$ of a semi-Riemannian manifold $\bar{M}$. Then, the following assertions are equivalent*

(i) *$S(TM)$ is integrable.*
(ii) *h^* is symmetric on $\Gamma(S(TM))$.*
(iii) *A_N is self-adjoint on $\Gamma(S(TM))$ with respect to g.*

Using the Theorem 3.2, we obtain a necessary and sufficient condition for the integrability of screen distribution of a semi-invariant lightlike submanifold of $\tilde{M}$ as below:

Theorem 3.3. *Let M be a semi-invariant lightlike submanifold of $\tilde{M}$. Then, the screen distribution $S(TM)$ is integrable, if and only if, $1-$forms $\eta_i, 1 \leq i \leq r$, are closed forms on $S(TM)$.*

Proof. Since the torsion tensor T of $\mathcal{D}$ does not vanish, then using (20), (31), and (32), we get

$$\begin{aligned}
[X, Y] &= \mathcal{D}^*_X PY - \mathcal{D}^*_Y PX + \sum_{i=1}^{r}\{\eta_i(X)\tilde{A}^*_{\xi_i} Y - \eta_i(Y)\tilde{A}^*_{\xi_i} X\} \\
&+ \sum_{i=1}^{r}\{\tilde{h}^*_i(X, PY) - \tilde{h}^*_i(Y, PX) + X(\eta_i(Y)) - Y(\eta_i(X))\}\xi_i \\
&+ \sum_{i,j=1}^{r}\{\eta_i(Y)\mu_{ij}(X) - \eta_i(X)\mu_{ij}(Y)\}\xi_j \\
&+ \sum_{i=1}^{r}\{\eta_i(X)P\tau Y - \eta_i(Y)P\tau X\}\pi(\xi_i) \\
&+ \sum_{i,k=1}^{r}\{\eta_k(X)\eta_i(\tau Y) - \eta_k(Y)\eta_i(\tau X)\}\pi(\xi_k)\xi_i \\
&+ \sum_{i=1}^{r}\{\pi(PX)\eta_i(\tau Y) - \pi(PY)\eta_i(\tau X)\}\xi_i \\
&+ \pi(PX)P\tau Y - \pi(PY)P\tau X.
\end{aligned}$$

On taking scalar product both sides with N_l, we obtain

$$\begin{aligned} g([X,Y],N_l) &= \tilde{h}^*_l(X,PY) - \tilde{h}^*_l(Y,PX) + X(\eta_l(Y)) - Y(\eta_l(X)) \\ &+ \sum_{i=1}^{r}\{\eta_i(Y)\mu_{il}(X) - \eta_i(X)\mu_{il}(Y)\} \\ &+ \sum_{k=1}^{r}\{\eta_k(X)\eta_l(\tau Y) - \eta_k(Y)\eta_l(\tau X)\}\pi(\xi_k) \\ &+ \pi(PX)\eta_l(\tau Y) - \pi(PY)\eta_l(\tau X). \end{aligned} \tag{33}$$

Let $X,Y \in \Gamma(S(TM))$ then using the definition of η_i in (33), we obtain

$$2d\eta_i(X,Y) = \tilde{h}^*_i(Y,PX) - \tilde{h}^*_i(X,PY), \tag{34}$$

where $1 \leq i \leq r$. By using (26), for any $X,Y \in \Gamma(S(TM))$, we have

$$\tilde{h}^*_i(Y,PX) - \tilde{h}^*_i(X,PY) = h^*_i(Y,PX) - h^*_i(X,PY). \tag{35}$$

Hence, using the Theorem 3.2 with (34) and (35), the result follows directly. □

Thus, using the Theorem 3.2 and the Theorem 3.3 with (29) and (30), we have the following result immediately.

Theorem 3.4. *Let M be a semi-invariant lightlike submanifold of $\tilde{M}$. Then, the induced connection $\mathcal{D}^*$ is also a quarter symmetric non-metric connection on the screen distribution $S(TM)$ if 1−forms $\eta_i, 1 \leq i \leq r$, are closed forms on $S(TM)$.*

Now, after differentiating (9) and (10), we have the following lemmas for further uses.

Lemma 3.5. *Let M be a semi-invariant lightlike submanifold of $\tilde{M}$. Then*

$$(\mathcal{D}_X\tau)Y = \tilde{A}_{\omega_Y}X + \nu\tilde{h}(X,Y) + \pi(JY)\tau X + \pi(Y)X, \tag{36}$$

$$(\mathcal{D}_X\omega)Y = -\tilde{h}(X,\tau Y) - \tilde{\mathcal{D}}^t(X,\omega Y) + \pi(JY)\omega X, \tag{37}$$

for any $X,Y \in \Gamma(TM)$, where

$$(\mathcal{D}_X\tau)Y = \mathcal{D}_X\tau Y - \tau\mathcal{D}_XY, \quad (\mathcal{D}_X\omega)Y = \tilde{\nabla}^t_X\omega Y - \omega(\mathcal{D}_XY). \tag{38}$$

4. Totally umbilical semi-invariant lightlike submanifolds

Definition 4.1. *A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\bar{M},\bar{g})$ is called a totally umbilical lightlike submanifold [8], if there exists a smooth transversal curvature vector field $H \in \Gamma(tr(TM))$ on M, such that $h(X,Y) = H\bar{g}(X,Y)$, for $X,Y \in \Gamma(TM)$. Further using (3), M is a totally umbilical lightlike submanifold, if and only if, on each coordinate neighbourhood $\mathcal{U}$, there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ such that*

$$h^l(X,Y) = H^l\bar{g}(X,Y), \quad h^s(X,Y) = H^s\bar{g}(X,Y), \tag{39}$$

for any $X, Y \in \Gamma(TM)$. A lightlike submanifold is said to be totally geodesic if $h(X, Y) = 0$, or equivalently if $H^l = 0$ and $H^s = 0$, for any $X, Y \in \Gamma(TM)$.

Theorem 4.2. *Let M be a proper totally umbilical semi-invariant lightlike submanifold of $\tilde{M}$ such that the distribution L_0 is integrable. Then, M is a totally geodesic semi-invariant lightlike submanifold of $\tilde{M}$ with respect to $\tilde{\mathcal{D}}$.*

Proof. Using (37) for any $X, Y \in \Gamma(L_0)$, we derive $\omega\mathcal{D}_X Y = \tilde{h}(X, \tau Y)$. Since M is a totally umbilical semi-invariant lightlike submanifold, then

$$\omega(\mathcal{D}_X Y - \mathcal{D}_Y X) = \tilde{g}(X, JY)\tilde{H} - \tilde{g}(Y, JX)\tilde{H}.$$

As $X, Y \in \Gamma(L_0)$, then $\omega(\tau X) = \omega(\tau Y) = 0$. Hence using (20), we obtain $\omega[X, Y] = \tilde{g}(X, JY)\tilde{H} - \tilde{g}(Y, JX)\tilde{H}$. Further for $X = JY$, the integrability of the distribution L_0 implies $2\tilde{g}(X, X)\tilde{H} = 0$. Hence, the non-degeneracy of the distribution L_0 gives $\tilde{H} = 0$. Thus, the assertion is complete. □

Theorem 4.3. *Let M be a proper totally umbilical semi-invariant lightlike submanifold of $\tilde{M}$, then $\tilde{H}^l = 0$.*

Proof. Using the Kaehlerian property of $\tilde{M}$ for $Z \in \Gamma(L_3)$, we have $\tilde{\nabla}_Z(JZ) = J(\tilde{\nabla}_Z Z)$. On using (3), (4), (9), (10), and equating tangential components of the resulting equation on both sides, we obtain

$$-A_{JZ}Z = \tau\nabla_Z Z + \nu(h^l(Z, Z)) + \nu(h^s(Z, Z)).$$

Further, on taking inner product both sides with $J\xi$, for any $\xi \in \Gamma(Rad(TM))$, we get

$$g(A_{JZ}Z, J\xi) + \tilde{g}(h^l(Z, Z), \xi) = 0. \tag{40}$$

Take $Y = J\xi$, $X \in \Gamma(L_3)$ in (5) and on using the totally umbilical property of M, we get

$$g(A_W X, J\xi) = g(h^s(X, J\xi), W) = g(X, J\xi)\tilde{g}(H^s, W) = 0,$$

for $W \in \Gamma(S(TM^\perp))$. Using this result in (40), it follows that $\tilde{g}(h^l(Z, Z), \xi) = 0$, further using (17), we obtain $\tilde{g}(\tilde{h}^l(Z, Z), \xi) = 0$. Since M is a totally umbilical semi-invariant lightlike submanifold, therefore $\tilde{g}(Z, Z)\tilde{g}(\tilde{H}^l, \xi) = 0$. Hence, the non-degeneracy of L_3 and (2) give $\tilde{H}^l = 0$. □

Theorem 4.4. *Let M be a proper totally umbilical semi-invariant lightlike submanifold of $\tilde{M}$. Let the screen distribution $S(TM)$ be a parallel distribution with respect to ∇, then $\tilde{H}^s = 0$.*

Proof. Assume that the screen distribution $S(TM)$ be a parallel distribution with respect to ∇, that is, $\nabla_X Y \in \Gamma(S(TM))$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(S(TM))$. Using (6), it implies $h^*(X, PY) = 0$, further from (7), A_N is a $Rad(TM)-$valued operator. Let $W \in \Gamma(S(TM^\perp))$, then $g(\nabla_X Y, W) = 0$ implies

$$\tilde{g}(\tilde{\nabla}_X Y, W) - \tilde{g}(h^s(X, Y), W) = 0. \tag{41}$$

Choose $Y = JN \in \Gamma(Jltr(TM))$, then $\tilde{g}(\tilde{\nabla}_X Y, W) = \tilde{g}(A_N X, JW)$. Since A_N is a $Rad(TM)-$valued operator, we get $\tilde{g}(\tilde{\nabla}_X Y, W) = 0$. Using this result in (41), it follows that $\tilde{g}(h^s(X,Y), W) = 0$, further (18) implies $\tilde{g}(\tilde{h}^s(X,Y), W) = 0$. Since M is a totally umbilical lightlike submanifold of $\tilde{M}$, therefore $\tilde{g}(X,Y)\tilde{g}(\tilde{H}^s, W) = 0$. As $X \in \Gamma(TM)$, $Y = JN \in \Gamma(Jltr(TM))$, then $\tilde{g}(X,Y)$ does not vanish and therefore $\tilde{g}(\tilde{H}^s, W) = 0$. Hence, the non-degeneracy of $S(TM^{\perp})$ gives $\tilde{H}^s = 0$. □

Thus, from the Theorem 4.3 and the Theorem 4.4, we have the following important observation.

Theorem 4.5. *Let M be a proper totally umbilical semi-invariant lightlike submanifold of $\tilde{M}$. Let the screen distribution $S(TM)$ be a parallel distribution with respect to ∇. Then M is a totally geodesic semi-invariant lightlike submanifold of $\tilde{M}$.*

Theorem 4.6. *Let M be a proper totally umbilical semi-invariant lightlike submanifold of $\tilde{M}$. Then either $\pi(X) = 0$, for any $X \notin \Gamma(L_3)$ or the dim $L_3 =$ dim $S(TM^{\perp}) = 1$.*

Proof. Let $X, Y \in \Gamma(L_3) = \Gamma(J(S(TM^{\perp})))$ then from (36), it follows that

$$-\tau \mathcal{D}_X Y = \tilde{A}_{\omega Y} X + J\tilde{h}(X,Y) + \pi(Y)X.$$

On taking inner product both sides with respect to X, we get

$$\tilde{g}(\tilde{A}_{\omega Y} X, X) = -\tilde{g}(J\tilde{h}^s(X,Y), X) - \pi(Y)\tilde{g}(X,X).$$

Using (18) and (23), we have $\tilde{g}(A_{JY}X, X) = \tilde{g}(h^s(X,Y), JX)$, further using (5), it implies $\tilde{g}(h^s(X,X), JY) = \tilde{g}(h^s(X,Y), JX)$. Then using (18), we obtain

$$\tilde{g}(\tilde{h}^s(X,X), JY) - \pi(X)\tilde{g}(X,Y) = \tilde{g}(\tilde{h}^s(X,Y), JX) - \pi(Y)\tilde{g}(X,X).$$

Since M is a totally umbilical lightlike submanifold, therefore

$$\begin{aligned} \tilde{g}(X,X)\tilde{g}(\tilde{H}^s, JY) - \pi(X)\tilde{g}(X,Y) &= \tilde{g}(X,Y)\tilde{g}(\tilde{H}^s, JX) \\ &\quad -\pi(Y)\tilde{g}(X,X). \end{aligned} \tag{42}$$

On interchanging the role of X and Y in the last equation, we have

$$\tilde{g}(Y,Y)\tilde{g}(\tilde{H}^s, JX) - \pi(Y)\tilde{g}(X,Y) = \tilde{g}(X,Y)\tilde{g}(\tilde{H}^s, JY) - \pi(X)\tilde{g}(Y,Y). \tag{43}$$

Hence from (42) and (43), we obtain

$$\Big\{\tilde{g}(\tilde{H}^s, JX) + \pi(X)\Big\}\Big\{\frac{\tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}(X,Y)^2}{\tilde{g}(X,Y)}\Big\} = 0. \tag{44}$$

Since $1-$form $\pi(X)$, associated with the vector field U on $\tilde{M}$, is given by $\pi(X) = \tilde{g}(X,U)$, therefore (44) becomes

$$\Big\{\tilde{g}(J\tilde{H}^s - U, X)\Big\}\Big\{\frac{\tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}(X,Y)^2}{\tilde{g}(X,Y)}\Big\} = 0. \tag{45}$$

As L_3 is a non-degenerate distribution, therefore for non-null vector fields X and Y from L_3, (45) implies either $U = J\tilde{H}^s \in \Gamma(J(S(TM^\perp)))$, that is, $\pi(Z) = \tilde{g}(J\tilde{H}^s, Z) = 0$, for any $Z \notin \Gamma(L_3)$ or X and Y are linearly dependent. Thus, the proof is complete. $\square$

Corollary 4.7. *Let M be a proper totally umbilical semi-invariant lightlike submanifold of $\tilde{M}$ such that dim $L_3 \neq 1$, then $H^s = 2\tilde{H}^s$.*

Proof. Let $dim\ L_3 \neq 1$ then from (45), we have

$$\tilde{g}(J\tilde{H}^s - U, X) = 0, \tag{46}$$

for any $X \in \Gamma(L_3)$. Since M is a proper totally umbilical semi-invariant lightlike submanifold, then from (18), we derive

$$\tilde{H}^s = H^s + \frac{\pi(X)}{\tilde{g}(X,X)} JX.$$

On using above expression in (46), it follows that $0 = \tilde{g}(JH^s, X) - 2\pi(X) = \tilde{g}(JH^s - 2U, X)$, then the non-degeneracy of L_3 gives $JH^s = 2U$. Moreover, from the Theorem 4.6, we have $J\tilde{H}^s = U$. Hence, the proof is complete. $\square$

5. Semi-invariant lightlike product manifolds

Definition 5.1. *A semi-invariant lightlike submanifold M of an indefinite Kaehler manifold $\tilde{M}$ is called a semi-invariant lightlike product manifold if the distributions L and L' define totally geodesic foliations in M.*

Theorem 5.2. *Let M be a semi-invariant lightlike submanifold of $\tilde{M}$. Then the distribution L defines a totally geodesic foliation in M, if and only if, $\tilde{h}(X, JY) = 0$, or equivalently $(\mathcal{D}_X\omega)Y = 0$, for any $X, Y \in \Gamma(L)$.*

Proof. Using the definition of a semi-invariant lightlike submanifold, the distribution L defines a totally geodesic foliation in M, if and only if, $\mathcal{D}_XY \in \Gamma(L)$, for any $X, Y \in \Gamma(L)$, or equivalently $\tilde{g}(\mathcal{D}_XY, J\xi) = 0$ and $\tilde{g}(\mathcal{D}_XY, JW) = 0$, for any $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$. Suppose $X, Y \in \Gamma(L)$, then from (14), it follows that

$$g(\mathcal{D}_XY, J\xi) = -\tilde{g}(J\tilde{\mathcal{D}}_XY, \xi) = -\tilde{g}(\tilde{\mathcal{D}}_XJY, \xi) = -\tilde{g}(\tilde{h}^l(X, JY), \xi),$$

for any $\xi \in \Gamma(Rad(TM))$ and similarly

$$g(\mathcal{D}_XY, JW) = -g(\tilde{h}^s(X, JY), W),$$

for any $W \in \Gamma(S(TM^\perp))$. Hence, the distribution L defines a totally geodesic foliation in M, if and only if, $\tilde{h}(X, JY) = 0$, or using (37), $(\mathcal{D}_X\omega)Y = 0$, for any $X, Y \in \Gamma(L)$. $\square$

Theorem 5.3. *Let M be a semi-invariant lightlike submanifold of $\tilde{M}$. Then the distribution L' defines a totally geodesic foliation in M, if and only if, $\tilde{h}^*(X, Y) = 0$ and $\tilde{A}_{JY}X$ has no component in $\Gamma(L_0 \perp JRad(TM))$, for any $X, Y \in \Gamma(L')$.*

Proof. Using the definition of a semi-invariant lightlike submanifold, the distribution L' defines a totally geodesic foliation in M, if and only if, $\mathcal{D}_X Y \in \Gamma(L')$, for any $X, Y \in \Gamma(L')$, or equivalently $\tilde{g}(\mathcal{D}_X Y, N) = 0$, $\tilde{g}(\mathcal{D}_X Y, Z) = 0$, and $\tilde{g}(\mathcal{D}_X Y, JN) = 0$, for any $N \in \Gamma(ltr(TM))$ and $Z \in \Gamma(L_0)$. Using (11), (15), (21), (22), and (28), we derive

$$\tilde{g}(\mathcal{D}_X Y, N) = \tilde{g}(\tilde{\mathcal{D}}_X Y, N) = -\tilde{g}(Y, \tilde{\nabla}_X N) = g(Y, \tilde{A}_N X) = g(\tilde{h}^*(X, Y), N),$$

$$g(\mathcal{D}_X Y, Z) = \tilde{g}(\tilde{\mathcal{D}}_X Y, Z) = \tilde{g}(J\tilde{\nabla}_X Y, JZ) = \tilde{g}(\tilde{\mathcal{D}}_X JY, JZ) = -\tilde{g}(\tilde{A}_{JY} X, JZ),$$

$$g(\mathcal{D}_X Y, JN) = -\tilde{g}(\tilde{\nabla}_X JY, N) = -\tilde{g}(\tilde{\mathcal{D}}_X JY, N) = \tilde{g}(\tilde{A}_{JY} X, N).$$

Hence, the proof is complete. □

We know that for a CR-submanifold of a Kaehler manifold, Chen [5] proved the following important characterization theorem.

Theorem 5.4. *A CR-submanifold of a Kaehler manifold is a CR-product, if and only if, τ is parallel, that is, $\nabla\tau = 0$.*

We generalize similar result for a semi-invariant lightlike submanifold of $\tilde{M}$ as below:

Theorem 5.5. *Let M be a semi-invariant lightlike submanifold of $\tilde{M}$. Then M is a semi-invariant lightlike product manifold, if τ is parallel with respect to connection $\mathcal{D}$, that is, $(\mathcal{D}_X\tau)Y = 0$, for any $X, Y \in \Gamma(TM)$.*

Proof. Let M be a semi-invariant lightlike submanifold of $\tilde{M}$ such that $(\mathcal{D}_X\tau)Y = 0$, for any $X, Y \in \Gamma(TM)$. Suppose $X, Y \in \Gamma(L')$, then $\tau Y = 0$, further from $(\mathcal{D}_X\tau)Y = 0$, we get $\tau\mathcal{D}_X Y = 0$. Hence, L' defines a totally geodesic foliation in M. Next, assume that $X, Y \in \Gamma(L)$, then $\omega Y = 0$, and further from (36), we obtain

$$\nu\tilde{h}(X, Y) = -\pi(\tau Y)\tau X - \pi(Y)X. \tag{47}$$

From the Theorem 5.2, the distribution L defines a totally geodesic foliation in M, if and only if, $\tilde{h}(X, \tau Y) = 0$, or equivalently $\tilde{g}(\tilde{h}^l(X, \tau Y), \xi) = 0$ and $\tilde{g}(\tilde{h}^s(X, \tau Y), W) = 0$, for any $X, Y \in \Gamma(L)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$. Using (47), we have

$$\begin{aligned}
\tilde{g}(\tilde{h}^l(X, \tau Y), \xi) &= \tilde{g}(J\tilde{h}^l(X, \tau Y), J\xi) \\
&= -\pi(\tau^2 Y)\tilde{g}(\tau X, J\xi) - \pi(\tau Y)\tilde{g}(X, J\xi) \\
&= 0,
\end{aligned}$$

similarly, $\tilde{g}(\tilde{h}^s(X, \tau Y), W) = 0$. Hence, the distribution L defines a totally geodesic foliation in M, consequently, M is a semi-invariant lightlike product manifold. □

Remark 2. The converse of the above theorem does not hold.

Proof. Let the distribution L' defines a totally geodesic foliation in M, then $\tau\mathcal{D}_X Y = 0$, for any $X, Y \in \Gamma(L')$. Also for $Y \in \Gamma(L')$, we have $\tau Y = 0$, this implies $\mathcal{D}_X \tau Y = 0$. Hence using (38), we obtain $(\mathcal{D}_X \tau)Y = 0$, for any $X, Y \in \Gamma(L')$.

Next, let the distribution L defines a totally geodesic foliation in M, then from the Theorem 5.2, $\tilde{h}(X, \tau Y) = 0$, for any $X, Y \in \Gamma(L)$. Since $\tilde{M}$ is a Kaehler manifold, therefore from (3), it follows that $h(X, JY) = Jh(X, Y)$, and further using (17) and (18), we have $\tilde{h}(X, JY) = J\tilde{h}(X, Y)$, for any $X, Y \in \Gamma(L)$. Hence from (14), we obtain $(\mathcal{D}_X \tau)Y = \mathcal{D}_X \tau Y - \tau\mathcal{D}_X Y \neq 0$. Thus, the proof is complete. □

Theorem 5.6. *Let M be a semi-invariant lightlike submanifold of $\tilde{M}$ such that $(\mathcal{D}_X \omega)Y = 0$, for any $X, Y \in \Gamma(TM)$. Suppose there exists a transversal vector bundle of M, which is parallel along L' with respect to quarter symmetric non-metric connection on M, that is, $\tilde{\mathcal{D}}_X V \in \Gamma(tr(TM))$, for any $V \in \Gamma(tr(TM))$ and $X \in \Gamma(L')$. Then M is a semi-invariant lightlike product manifold.*

Proof. From the hypothesis of the theorem and the Theorem 5.2, it is obvious that the distribution L defines a totally geodesic foliation in M. Let $\tilde{\mathcal{D}}_X V \in \Gamma(tr(TM))$, for any $V \in \Gamma(tr(TM))$ and $X \in \Gamma(L')$, then from (21) and (22), we have $\tilde{A}_V X = 0$. Further, using the Theorem 5.3, we derive

$$g(\mathcal{D}_X Y, N) = g(Y, \tilde{A}_N X) = 0, \quad g(\mathcal{D}_X Y, Z) = g(JZ, \tilde{A}_{JY} X) = 0$$

and

$$g(\mathcal{D}_X Y, JN) = g(N, \tilde{A}_{JY} X) = 0.$$

Hence, L' defines a totally geodesic foliation in M, consequently, M is a semi-invariant lightlike product manifold. □

Theorem 5.7. *Let M be a totally umbilical semi-invariant lightlike submanifold of $\tilde{M}$. Then M is a semi-invariant lightlike product manifold, if and only if, $\tilde{h}(X, JY) = 0$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(L)$.*

Proof. Let M be a semi-invariant lightlike product manifold, then the distributions L and L' define totally geodesic foliation in M. Hence, from the Theorem 5.2, it follows that $\tilde{h}(X, JY) = 0$, for any $X, Y \in \Gamma(L)$. Moreover, M is totally umbilical, therefore $\tilde{h}(X, JY) = g(X, JY)\tilde{H} = 0$, for any $X \in \Gamma(L')$ and $Y \in \Gamma(L)$. Thus, $\tilde{h}(X, JY) = 0$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(L)$.

Conversely, let $\tilde{h}(X, JY) = 0$, for any $X, Y \in \Gamma(L)$, then from the Theorem 5.2, the distribution L defines a totally geodesic foliation in M. Next, let $X, Y \in \Gamma(L')$, then the distribution L' defines a totally geodesic foliation in M, if and only if, $\tau\mathcal{D}_X Y = 0$. For $Z \in \Gamma(L_0)$, using (11), (14), and (36), we obtain

$$g(\tau\mathcal{D}_X Y, Z) = -\tilde{g}(\tilde{A}_{\omega Y} X, Z) = \tilde{g}(J\tilde{\mathcal{D}}_X Y, Z) = \tilde{g}(Y, J\tilde{h}(X, Z)) = 0.$$

Then, the non-degeneracy of the distribution L_0 implies $\tau\mathcal{D}_X Y = 0$. Hence, the proof is complete. □

When M is a hypersurface of a $4-$dimensional Minkowski space, then as a particular case, Duggal and Bejancu [7] defined a minimal lightlike submanifold. A general notion of a minimal lightlike submanifold of a semi-Riemannian manifold has been given by Bejan and Duggal[4] as follows:

Definition 5.8. *Let M be a lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then M is said to be minimal if*

(i) $h^s = 0$ *on* $Rad(TM)$ *and*
(ii) *trace* $h = 0$, *where trace is written with respect to g restricted to* $S(TM)$.

Let M is a proper totally umbilical semi-invariant lightlike submanifold of $\tilde{M}$ such that L_0 is integrable or $S(TM)$ is parallel with respect to ∇, then from the Theorem 4.2 and the Theorem 4.5, it follows that M is minimal.

Lemma 5.9. *Let M be a lightlike submanifold of $\tilde{M}$. Then $\tilde{h}^l = 0$ on $Rad(TM)$.*

Proof. Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$, then from the Koszul Formula, it follows that

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X\tilde{g}(Y,Z) + Y\tilde{g}(X,Z) - Z\tilde{g}(X,Y) - \tilde{g}(X,[Y,Z]) \\ &\quad + \tilde{g}(Y,[Z,X]) + \tilde{g}(Z,[X,Y]), \end{aligned} \tag{48}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$. Particularly, on taking $X = \xi$, $Y = \xi'$, and $Z = \xi''$ in (48), we obtain $\tilde{g}(\tilde{\nabla}_\xi \xi', \xi'') = 0$. Then from (11), we have

$$\tilde{g}(\tilde{\mathcal{D}}_\xi \xi', \xi'') - \pi(\xi')\tilde{g}(J\xi, \xi'') = 0. \tag{49}$$

Next, if ξ is a local section of $Rad(TM)$, then $\tilde{g}(J\xi, \xi) = 0$, this further implies, either $J\xi \in \Gamma(S(TM))$ or $J\xi \in \Gamma(S(TM^\perp))$. Therefore, (49) becomes $\tilde{g}(\tilde{\mathcal{D}}_\xi \xi', \xi'') = 0$, and on using (15), we get $\tilde{h}^l(\xi, \xi') = 0$, for any $\xi, \xi' \in \Gamma(Rad(TM))$. Hence, the proof is complete. □

Theorem 5.10. *Let M be a proper totally umbilical lightlike submanifold of $\tilde{M}$. Then M is minimal, if and only if, M is totally geodesic.*

Proof. Let M be a minimal lightlike submanifold of $\tilde{M}$, then $\tilde{h}^s(X,Y) = 0$, for any $X, Y \in \Gamma(Rad(TM))$, and from the Lemma 5.9, $\tilde{h}^l = 0$, on $Rad(TM)$. Let $\{e_1, ..., e_{m-r}\}$ be an orthonormal basis of $S(TM)$, then using (39), we obtain

$$trace\ \tilde{h}(e_i, e_i) = \sum_{i=1}^{m-r} \{\epsilon_i \tilde{g}(e_i, e_i)\tilde{H}^l + \epsilon_i \tilde{g}(e_i, e_i)\tilde{H}^s\},$$

this further implies $trace\ \tilde{h}(e_i, e_i) = (m-r)\tilde{H}^l + (m-r)\tilde{H}^s$. Since M is minimal and $ltr(TM) \cap S(TM^\perp) = \{0\}$, therefore $\tilde{H}^l = 0$ and $\tilde{H}^s = 0$. Hence, M is totally geodesic and the converse is immediate. □

Theorem 5.11. *Let M be a proper totally umbilical semi-invariant lightlike submanifold of $\tilde{M}$. Then M is minimal, if and only if, trace $\tilde{h}^s = 0$ on $L_0 \perp L_3$.*

Proof. Let M be a totally umbilical lightlike submanifold, then from (39), we have $\tilde{h}^s(X,Y) = \tilde{g}(X,Y)\tilde{H}^s = 0$, for any $X, Y \in \Gamma(Rad(TM))$. Therefore, M is minimal, if and only if, *trace* $\tilde{h}|_{S(TM)} = 0$ or equivalently

$$\sum_{i=1}^{a} \tilde{h}(X_i, X_i) + \sum_{j=1}^{r} \tilde{h}(J\xi_j, J\xi_j) + \sum_{j=1}^{r} \tilde{h}(JN_j, JN_j) + \sum_{k=1}^{b} \tilde{h}(JW_k, JW_k) = 0,$$

where $a = dim(L_0)$ and $b = dim(L_3)$. Since M is totally umbilical, using (39), we have $\tilde{h}(J\xi_j, J\xi_j) = \tilde{h}(JN_j, JN_j) = 0$. Therefore, using the Theorem 4.3, M is minimal, if and only if

$$\sum_{i=1}^{a} \tilde{h}^s(X_i, X_i) + \sum_{k=1}^{b} \tilde{h}^s(JW_k, JW_k) = 0,$$

or equivalently, *trace* $\tilde{h}^s = 0$ on $L_0 \perp L_3$. $\square$

Remark 3. *Let $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$, then from (3), (4), and (12), we obtain*

$$\begin{aligned} \tilde{g}(\tilde{A}_W X, Y) &= \tilde{g}(\tilde{h}^s(X,Y), W) + \tilde{g}(Y, \tilde{\mathcal{D}}^l(X,W)) \\ &\quad -\pi(Y)\tilde{g}(JX, W) - \pi(W)\tilde{g}(JX, Y). \end{aligned} \tag{50}$$

Particularly, put $X = Y \in \Gamma(L_0)$, then (50) implies

$$\tilde{g}(\tilde{h}^s(X,X), W) = \tilde{g}(\tilde{A}_W X, X). \tag{51}$$

If we choose $X = Y \in \Gamma(L_3)$, then (50) implies

$$\tilde{g}(\tilde{h}^s(X,X), W) = \tilde{g}(\tilde{A}_W X, X) - \pi(X)\tilde{g}(JX, W). \tag{52}$$

Thus, from (51) and (52), it does not follow that M is minimal, if and only if, trace $\tilde{A}_W = 0$ on $L_0 \perp L_3$.

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