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The fundamental operator tuples associated with the symmetrized polydisc

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Abstract. A commuting tuple of operators $(S_1, \dots, S_{n-1}, P)$, defined on a Hilbert space $\mathcal{H}$, for which the closed symmetrized polydisc is a spectral set, is called a Γ_n-*contraction*. To every Γ_n-contraction, there is a unique operator tuple $(A_1, \dots, A_{n-1})$, defined on $\overline{Ran}(I - P^*P)$, such that

$$S_i - S_{n-i}^* P = D_P A_i D_P, \quad D_P = (I - P^*P)^{\frac{1}{2}}, \qquad i = 1, \dots, n-1.$$

This is called the *fundamental operator tuple* or $\mathcal{F}_O$-*tuple* associated with the Γ_n-contraction. The $\mathcal{F}_O$-tuple of a Γ_n-contraction completely determines the structure of a Γ_n-contraction and provides operator model and complete unitary invariant for them. In this note, we analyze the $\mathcal{F}_O$-tuples and find some intrinsic properties of them. Given a Γ_n-contraction $(S_1, \dots, S_{n-1}, P)$ and $n-1$ operators $A_1, \dots, A_{n-1}$ defined on $\overline{Ran} D_P$, we provide a necessary and sufficient condition under which $(A_1, \dots, A_{n-1})$ becomes the $\mathcal{F}_O$-tuple of $(S_1, \dots, S_{n-1}, P)$. Also for given tuples of operators $(A_1, \dots, A_{n-1})$ and $(B_1, \dots, B_{n-1})$, defined on a Hilbert space E, we find a necessary condition and a sufficient condition under which there exist a Hilbert space $\mathcal{H}$ and a Γ_n-contraction $(S_1, \dots, S_{n-1}, P)$ on $\mathcal{H}$ such that $(A_1, \dots, A_{n-1})$ becomes the $\mathcal{F}_O$-tuple of $(S_1, \dots, S_{n-1}, P)$ and $(B_1, \dots, B_{n-1})$ becomes the $\mathcal{F}_O$-tuple of the adjoint $(S_1^*, \dots, S_{n-1}^*, P^*)$.

Contents

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1. Introduction and preliminaries

The *symmetrized n-disk* or simply the *symmetrized polydisc*

$$\Gamma_n^\circ = \mathbb{G}_n = \left\{ \left(\sum_{1\leq i\leq n} z_i, \sum_{1\leq i<j\leq n} z_i z_j, \dots, \prod_{i=1}^{n} z_i \right) : |z_i| < 1, i = 1, \dots, n \right\}$$

is a polynomially convex but non-convex domain which has attracted considerable attentions in past two decades because of its rich function theory, complex geometry, operator theory and its connection with the appealing and difficult problem of μ-synthesis. We mention here only a few of the numerous references that are relevant to the operator theory on the symmetrized polydisc of several dimensions, [1, 2, 8, 5, 6, 7, 9, 10, 11, 12, 14, 15]. An interested reader can also see the references there. A tuple of commuting operators $(S_1, \dots, S_{n-1}, P)$ defined on a complex Hilbert space $\mathcal{H}$ is said to be a Γ_n-*contraction* if Γ_n is a spectral set for $(S_1, \dots, S_{n-1}, P)$, i.e., if the Taylor joint spectrum $\sigma_T(S_1, \dots, S_{n-1}, P) \subseteq \Gamma_n$ and the von Neumann inequality

$$\|f(S_1, \dots, S_{n-1}, P)\| \leq \|f\|_{\infty, \Gamma_n} = \sup_{(z_1, \dots, z_n) \in \Gamma_n} |f(z_1, \dots, z_n)|$$

holds for all rational functions f with singularities off Γ_n. It was shown by the second named author of this article ([12], Theorem 3.3) that to every Γ_n-contraction $(S_1, \dots, S_{n-1}, P)$, there is a unique operator tuple $(A_1, \dots, A_{n-1})$ such that

$$S_i - S_{n-i}^* P = D_P A_i D_P, \quad \text{for each } i = 1, \dots, n-1.$$

For its pivotal role in deciphering the structure of a Γ_n-contraction [11], producing an operator model and constituting a complete unitary invariant for Γ_n-contractions [5, 6, 12, 13, 14] $(A_1, \dots, A_{n-1})$ is called the *fundamental operator tuple* or the $\mathcal{F}_O$*-tuple* of $(S_1, \dots, S_{n-1}, P)$.

There are three main results in this paper. First, Theorem 2.1 provides a necessary and sufficient condition under which an operator tuple $(A_1, \dots, A_{n-1})$ becomes the $\mathcal{F}_O$-tuple of a given Γ_n-contraction. A natural question arises; given two tuples of operators $(A_1, \dots, A_{n-1})$ and $(B_1, \dots, B_{n-1})$ defined on some certain Hilbert spaces, does there exist a Γ_n-contraction $(S_1, \dots, S_{n-1}, P)$ such that $(A_1, \dots, A_{n-1})$ becomes the $\mathcal{F}_O$-tuple of $(S_1, \dots, S_{n-1}, P)$ and $(B_1, \dots, B_{n-1})$ becomes the $\mathcal{F}_O$-tuple of its adjoint $(S_1^*, \dots, S_{n-1}^*, P^*)$? We answer this question in Lemma 3.1 and Theorem 3.4 by finding separately a necessary condition and a sufficient condition. This is considered to be the second main result of this paper. In [14], it was shown that the commutators $[A_i^*, A_j]$, where $[P, Q] = PQ - QP$, are the key ingredients in representing the distinguished varieties in Γ_n. Also in [12] and [13], we have seen that the same commutators determined the conditional dilation on Γ_n and produced a complete unitary invariant for

$C_{.0}$ Γ_n-contractions. In this article, we choose a couple of Γ_n-contractions $(S_1, \dots, S_{n-1}, P), (S'_1, \dots, S'_{n-1}, P)$ instead of taking only one Γ_n-contraction and study the commutators $[A_i, A'_j], [A_i^*, A'_j]$ and also $[A_i, A_j], [A_i^*, A_j]$, where $(A'_1, \dots, A'_{n-1})$ is the $\mathcal{F}_O$-tuple of the Γ_n-contraction $(S'_1, \dots, S'_{n-1}, P)$. As a consequence, we obtain a few new interrelations between a pair of Γ_n-contractions and their $\mathcal{F}_O$-tuples which are presented in the third main result Theorem 2.9 and its corollaries. En route we find few more interesting properties of the $\mathcal{F}_O$-tuple of a Γ_n-contraction. Our results on one hand generalize the existing similar results for Γ_2-contractions [4], and on the other hand reflect new light on the possibility of extending operator theory from Γ_2 to Γ_n for $n > 2$. Indeed, there are notable major differences in operator theory when we move from 2 to higher dimensions and the main underlying reason is that rational dilation succeeds on the symmetrized n-disk for $n = 2$, ([5]) but fails for $n \geq 3$ ([13]).

Note. The arxiv (https://arxiv.org/archive/math) reference [14] has been split into several parts for publications and the present article is one of them.

2. Properties of the fundamental operator tuples ($\mathcal{F}_O$-tuples)

We begin this section with a necessary and sufficient condition under which a tuple of operator becomes the $\mathcal{F}_O$-tuple of a Γ_n-contraction. This is one of the main results of this paper.

Theorem 2.1. *A tuple of operators $(A_1, \dots, A_{n-1})$ defined on $\mathcal{D}_P$ is the $\mathcal{F}_O$-tuple of a Γ_n-contraction $(S_1, \dots, S_{n-1}, P)$ if and only if $(A_1, \dots, A_{n-1})$ satisfy the following operator equations in $X_1, \dots, X_{n-1}$:*

$$D_P S_i = X_i D_P + X_{n-i}^* D_P P \ , \quad i = 1, \dots, n-1.$$

Proof. First let $(A_1, \dots, A_{n-1})$ be the $\mathcal{F}_O$-tuple of $(S_1, \dots, S_{n-1}, P)$. Then

$$S_i - S_{n-i}^* P = D_P A_i D_P \text{ for } i = 1, \dots, n-1.$$

Now

$$\begin{aligned} D_P(A_i D_P + A_{n-i}^* D_P P) &= (S_i - S_{n-i}^* P) + (S_{n-i}^* - P^* S_i)P \\ &= (I - P^* P) S_i \\ &= D_P^2 S_i. \end{aligned}$$

Therefore, if $J = D_P S_i - A_i D_P - A_{n-i}^* D_P P$ then $J : \mathcal{H} \to \mathcal{D}_P$ and also $D_P J = 0$. Now

$$\langle Jh, D_P h' \rangle = \langle D_P Jh, h' \rangle = 0 \quad \text{ for all } h, h' \in \mathcal{H}.$$

This shows that $J = 0$ and hence $A_i D_P + A_{n-i}^* D_P P = D_P S_i$.

Conversely, let $(X_1, \dots, X_{n-1})$ be a tuple of operators on $\mathcal{D}_P$ such that

$$D_P S_i = X_i D_P + X_{n-i}^* D_P P \text{ for } i = 1, \dots, n-1 \, .$$

Also suppose that $(F_1, \dots, F_{n-1})$ is the $\mathcal{F}_O$-tuple of $(S_1, \dots, S_{n-1}, P)$. We need to show that $(X_1, \dots, X_{n-1}) = (F_1, \dots, F_{n-1})$. Since we just proved that $(F_1, \dots, F_{n-1})$ satisfies the above mentioned operator equations, we have

$$F_i D_P + F_{n-i}^* D_P P = D_P S_i = X_i D_P + X_{n-i}^* D_P P.$$

and consequently

$$(X_i - F_i) D_P + (X_{n-i} - F_{n-i})^* D_P P = (X_{n-i} - F_{n-i}) D_P + (X_i - F_i)^* D_P P = 0.$$

Let for each i

$$Y_i = X_i - F_i \,,\, Y_{n-i} = X_{n-i} - F_{n-i}.$$

Then for each i

$$Y_i D_P + Y_{n-i}^* D_P P = Y_{n-i} D_P + Y_i^* D_P P = 0. \tag{2.1}$$

To complete the proof, we need to show that $Y_1 = \cdots = Y_{n-1} = 0$. We have

$$\begin{aligned}
& Y_i D_P + Y_{n-i}^* D_P P = 0 \\
\text{or} \quad & Y_i D_P = -Y_{n-i}^* D_P P \\
\text{or} \quad & D_P Y_i D_P = -D_P Y_{n-i}^* D_P P \\
\text{or} \quad & D_P Y_i^* D_P = P^* D_P Y_i^* D_P P = P^{*2} D_P Y_i^* D_P P^2 = \cdots
\end{aligned}$$

We obtained the equalities in the last line by applying (2.1). Thus we have

$$D_P Y_i^* D_P = P^{*n} D_P Y_i^* D_P P^n \tag{2.2}$$

for all $n = 1, 2, \dots$. Now consider the series

$$\begin{aligned}
\sum_{n=0}^{\infty} \|D_P P^n h\|^2 &= \sum_{n=0}^{\infty} \langle D_P P^n h, D_P P^n h \rangle \\
&= \sum_{n=0}^{\infty} \langle P^{*n} D_P^2 P^n h, h \rangle \\
&= \sum_{n=0}^{\infty} \langle P^{*n} (I - P^* P) P^n h, h \rangle \\
&= \sum_{n=0}^{\infty} \langle (P^{*n} P^n - P^{*n+1} P^{n+1} h, h \rangle \\
&= \sum_{n=0}^{\infty} (\|P^n h\|^2 - \|P^{n+1} h\|^2) \\
&= \|h\|^2 - \lim_{n \to \infty} \|P^n h\|^2.
\end{aligned}$$

$$\|h\| \geq \|Ph\| \geq \|P^2 h\| \geq \cdots \geq \|P^n h\| \geq \cdots \geq 0.$$

So $\lim\limits_{n\to\infty} \|P^n h\|^2$ exists. Therefore, the series is convergent and thus we have $\lim\limits_{n\to\infty} \|D_P P^n h\|^2 = 0$. So

$$\begin{aligned} \|D_P Y_i^* D_P h\| &= \|P^{*n} D_P Y_i^* D_P P^n h\| \text{ by (2.2)} \\ &\leq \|P^{*n}\| \|D_p Y_i^*\| \|D_P P^n h\| \\ &\leq \|D_p Y_i^*\| \|D_P P^n h\| \to 0. \end{aligned}$$

So $D_P Y_i^* D_P = 0$ and hence $Y_i = 0$ for each $i = 1, \dots, n-1$.

□

The next few results will provide some useful algebraic relations between Γ_n-contractions and their $\mathcal{F}_O$-tuples.

Theorem 2.2. *Suppose $(S_1, \dots, S_{n-1}, P)$ and $(S_1', \dots, S_{n-1}', P)$ are two commuting Γ_n-contractions on a Hilbert space $\mathcal{H}$. Let $(A_1, \dots, A_{n-1})$ and $(A_1', \dots, A_{n-1}')$ be the commuting $\mathcal{F}_O$-tuples of $(S_1, \dots, S_{n-1}, P)$ and $(S_1', \dots, S_{n-1}', P)$, respectively and suppose $A_i A_j' = A_j' A_i$ for any $i, j = 1, \dots, n-1$. Then for each $i, j = 1, \dots, n-1$ we have*

$$S_i^* S_j' - S_{n-j}'^* S_{n-i} = D_P (A_i^* A_j' - A_{n-j}'^* A_{n-i}) D_P.$$

Proof. We have that $(A_1, \dots, A_{n-1})$ is a commuting tuple satisfying

$$S_i - S_{n-i}^* P = D_P A_i D_P\,, \text{ for } i = 1, \dots, n-1.$$

and $(A_1', \dots, A_{n-1}')$ is a commuting tuple satisfying

$$S_j' - S_{n-j}'^* P = D_P A_j' D_P\,, \text{ for } j = 1, \dots, n-1.$$

Then

$$\begin{aligned} S_i^* S_j' &= S_i^* \left(S_{n-j}'^* P + D_P A_j' D_P \right) \\ &= S_i^* S_{n-j}'^* P + S_i^* D_P A_j' D_P \\ &= S_{n-j}'^* S_i^* P + S_i^* D_P A_j' D_P \\ &= S_{n-j}'^* \left(S_{n-i} - D_P A_{n-i} D_P \right) + S_i^* D_P A_j' D_P \\ &= S_{n-j}'^* S_{n-i} - S_{n-j}'^* D_P A_{n-i} D_P + S_i^* D_P A_j' D_P. \end{aligned}$$

Now from Theorem 2.1 we have

$$S_i^* D_P = D_P A_i^* + P^* D_P A_{n-i}$$

and

$$S_{n-j}'^* D_P = D_P A_{n-j}'^* + P^* D_P A_j'.$$

Then

$$\begin{aligned} & S_i^* D_P A_j' - S_{n-j}'^* D_P A_{n-i} \\ =& \left(D_P A_i^* A_j' + P^* D_P A_{n-i} A_j' \right) - \left(D_P A_{n-j}'^* A_{n-i} + P^* D_P A_j' A_{n-i} \right) \\ =& D_P A_i^* A_j' - D_P A_{n-j}'^* A_{n-i}. \end{aligned}$$

Therefore, $S_i^* S_j' - S_{n-j}'^* S_{n-i} = D_P \left(A_i^* A_j' - A_{n-j}'^* A_{n-i} \right) D_P$. □

A direct consequence of the previous theorem is the following.

Corollary 2.3. *Let $(S_1, \dots, S_{n-1}, P)$ be a Γ_n-contraction with commuting $\mathcal{F}_O$-tuple $(A_1, \dots, A_{n-1})$. Then for each $i, j = 1, \dots, n-1$ we have*

$$S_i^* S_j - S_{n-j}^* S_{n-i} = D_P(A_i^* A_j - A_{n-j}^* A_{n-i}) D_P.$$

Lemma 2.4. *Let $(S_1, \dots, S_{n-1}, P)$ be a Γ_n-contraction on a Hilbert space $\mathcal{H}$ and let $(A_1, \dots, A_{n-1})$ and $(B_1, \dots, B_{n-1})$ be the $\mathcal{F}_O$-tuples of $(S_1, \dots, S_{n-1}, P)$ and $(S_1^*, \dots, S_{n-1}^*, P^*)$, respectively. Then*

$$D_P A_i = (S_i D_P - D_{P^*} B_{n-i} P)|_{\mathcal{D}_P} \text{ for } i = 1, \dots, n-1.$$

Proof. For $h \in \mathcal{H}$, we have

$$\begin{aligned} (S_i D_P - D_{P^*} B_{n-i} P) D_P h &= S_i (I - P^* P) h - (D_{P^*} B_{n-i} D_{P^*}) P h \\ &= S_i h - S_i P^* P h - (S_{n-i}^* - S_i P^*) P h \\ &= S_i h - S_i P^* P h - S_{n-i}^* P h + S_i P^* P h \\ &= (S_i - S_{n-i}^* P) h = (D_P A_i) D_P h. \end{aligned}$$

Hence,

$$D_P A_i = (S_i D_P - D_{P^*} B_{n-i} P)|_{\mathcal{D}_P}.$$

□

Lemma 2.5. *Let $(S_1, \dots, S_{n-1}, P)$ be a Γ_n-contraction on a Hilbert space $\mathcal{H}$ and let $(A_1, \dots, A_{n-1})$ and $(B_1, \dots, B_{n-1})$ be the $\mathcal{F}_O$-tuples of $(S_1, \dots, S_{n-1}, P)$ and $(S_1^*, \dots, S_{n-1}^*, P^*)$, respectively. Then*

$$P A_i = B_i^* P|_{\mathcal{D}_P}, \text{ for } i = 1, \dots, n-1.$$

Proof. We observe here that the operators on both sides are defined from $\mathcal{D}_P$ to $\mathcal{D}_{P^*}$. Let $h, h' \in \mathcal{H}$ be any two elements. Then

$$\begin{aligned} & \langle (P A_i - B_i^* P) D_P h, D_{P^*} h' \rangle \\ = \ & \langle D_{P^*} P A_i D_P h, h' \rangle - \langle D_{P^*} B_i^* P D_P h, h' \rangle \\ = \ & \langle P (D_P A_i D_P) h, h' \rangle - \langle (D_{P^*} B_i^* D_{P^*}) P h, h' \rangle \\ = \ & \langle P (S_i - S_{n-i}^* P) h, h' \rangle - \langle (S_i - P S_{n-i}^*) P h, h' \rangle \\ = \ & \langle (P S_i - P S_{n-i}^* P - S_i P + P S_{n-i}^* P) h, h' \rangle = 0. \end{aligned}$$

Hence $P A_i = B_i^* P|_{\mathcal{D}_P}$ for $i = 1, \dots, n-1$ and the proof is complete. □

Lemma 2.6. *Let $(S_1, \dots, S_{n-1}, P)$ be a Γ_n-contraction on a Hilbert space $\mathcal{H}$ and let $(A_1, \dots, A_{n-1})$ and $(B_1, \dots, B_{n-1})$ be the $\mathcal{F}_O$-tuples of $(S_1, \dots, S_{n-1}, P)$ and $(S_1^*, \dots, S_{n-1}^*, P^*)$ respectively. Then for $i = 1, \dots, n-1$,*

$$(A_i^* D_P D_{P^*} - A_{n-i} P^*)|_{\mathcal{D}_{P^*}} = D_P D_{P^*} B_i - P^* B_{n-i}^*.$$

Proof. For $h \in \mathcal{H}$, we have

$$\begin{aligned}
& (A_i^* D_P D_{P^*} - A_{n-i} P^*) D_{P^*} h \\
= \;& A_i^* D_P (I - PP^*) h - A_{n-i} P^* D_{P^*} h \\
= \;& A_i^* D_P h - A_i^* D_P P P^* h - A_{n-i} D_P P^* h \\
= \;& A_i^* D_P h - (A_i^* D_P P + A_{n-i} D_P) P^* h \\
= \;& A_i^* D_P h - D_P S_{n-i} P^* h \qquad [\text{ by Lemma } (2.1)] \\
= \;& (S_i D_P - D_{P^*} B_{n-i} P)^* h - D_P S_{n-i} P^* h \qquad [\text{by Lemma } 2.4] \\
= \;& D_P S_i^* h - P^* B_{n-i}^* D_{P^*} h - D_P S_{n-i} P^* h \\
= \;& D_P (S_i^* - S_{n-i} P^*) h - P^* B_{n-i}^* D_{P^*} h \\
= \;& D_P D_{P^*} B_i D_{P^*} h - P^* B_{n-i}^* D_{P^*} h \\
= \;& (D_P D_{P^*} B_i - P^* B_{n-i}^*) D_{P^*} h.
\end{aligned}$$

□

The following theorem is another main result of this section.

Theorem 2.7. *Suppose $(S_1, \dots, S_{n-1}, P)$ and $(S'_1, \dots, S'_{n-1}, P)$ are two commuting Γ_n-contractions on a Hilbert space $\mathcal{H}$. Let $(A_1, \dots, A_{n-1})$ and $(A'_1, \dots, A'_{n-1})$ be the commuting $\mathcal{F}_O$-tuples of $(S_1, \dots, S_{n-1}, P)$ and $(S'_1, \dots, S'_{n-1}, P)$, respectively and suppose $A_i A'_j = A'_j A_i$ for any $i, j = 1, \dots, n-1$. Suppose $(B_1, \dots, B_{n-1})$ and $(B'_1, \dots, B'_{n-1})$ are the $\mathcal{F}_O$-tuples of $(S_1^*, \dots, S_{n-1}^*, P^*)$ and $(S_1'^*, \dots, S_{n-1}'^*, P^*)$ respectively. If P has dense range, then the following identities hold for $i, j = 1, \dots, n-1$:*

(i) $[A_i, A_j'^*] = [A'_{n-j}, A_{n-i}^*]$
(ii) $[B_i, B_{n-j}] = [B'_i, B'_{n-j}] = 0$
(iii) $[B_i^*, B'_j] = [B_{n-j}'^*, B_{n-i}]$.

Proof. (i) By Theorem 2.1, we have for each $i = 1, \dots, n-1$ that $D_P S_i = A_i D_P + A_{n-i}^* D_P P$ and $D_P S'_i = A'_i D_P + A_{n-i}'^* D_P P$. Multiplying $D_P S_i = A_i D_P + A_{n-i}^* D_P P$ by $D_P A'_{n-j}$ from the left we get,

$$\begin{aligned}
& D_P A'_{n-j} D_P S_i = D_P A'_{n-j} A_i D_P + D_P A'_{n-j} A_{n-i}^* D_P P \\
\Rightarrow & (S'_{n-j} - S_j'^* P) S_i = D_P A'_{n-j} A_i D_P + D_P A'_{n-j} A_{n-i}^* D_P P \\
\Rightarrow & (S'_{n-j} S_i - S_j'^* S_i P) = D_P A'_{n-j} A_i D_P + D_P A'_{n-j} A_{n-i}^* D_P P.
\end{aligned}$$

Similarly, multiplying $D_P S'_{n-j} = A'_{n-j} D_P + A_j'^* D_P P$ by $D_P A_i$ from the left we get

$$\begin{aligned}
& D_P A_i D_P S'_{n-j} = D_P A_i A'_{n-j} D_P + D_P A_i A_j'^* D_P P \\
\Rightarrow & (S_i - S_{n-i}^* P) S'_{n-j} = D_P A'_{n-j} A_i D_P + D_P A_i A_j'^* D_P P \\
\Rightarrow & (S'_{n-j} S_i - S_{n-i}^* S'_{n-j} P) = D_P A'_{n-j} A_i D_P + D_P A_i A_j'^* D_P P.
\end{aligned}$$

On subtraction we get

$$\begin{aligned}
&(S_j'^* S_i - S_{n-i}^* S_{n-j}')P = D_P(A_i A_{n-j}' - A_{n-j}' A_i)D_P \\
&\qquad\qquad\qquad\qquad\qquad + D_P(A_i A_j'^* - A_{n-j}' A_{n-i}^*)D_P P \\
\Rightarrow & D_P(A_j'^* A_i - A_{n-i}^* A_{n-j}')D_P P = D_P(A_i A_j'^* - A_{n-j}' A_{n-i}^*)D_P P \\
\Rightarrow & D_P(A_i A_j'^* - A_j'^* A_i + A_{n-i}^* A_{n-j}' - A_{n-j}' A_{n-i}^*)D_P P \\
\Rightarrow & D_P\left([A_i, A_j'^*] + [A_{n-i}^*, A_{n-j}']\right) D_P P = 0 \\
\Rightarrow & D_P\left([A_i, A_j'^*] + [A_{n-i}^*, A_{n-j}']\right) D_P = 0 \quad \text{[since Ran}P \text{ is dense in } \mathcal{H}] \\
\Rightarrow & [A_i, A_j'^*] = [A_{n-j}', A_{n-i}^*].
\end{aligned}$$

(ii) From Lemma 2.5, we have that $PA_i = B_i^* P|_{\mathcal{D}_P}$, for $i, j = 1, \dots, n-1$. Therefore,

$$\begin{aligned}
& PA_i A_{n-j} D_P = B_i^* P A_{n-j} D_P \\
\Rightarrow & PA_{n-j} A_i D_P = B_i^* P A_{n-j} D_P \\
\Rightarrow & B_{n-j}^* B_i^* P D_P = B_i^* B_{n-j}^* P D_P \\
\Rightarrow & [B_i^*, B_{n-j}^*] D_{P^*} P = 0 \\
\Rightarrow & [B_i^*, B_{n-j}^*] = 0 \\
\Rightarrow & [B_i, B_{n-j}] = 0.
\end{aligned}$$

Similarly, for each $i, j = 1, \dots, n-1$ we have

$$[B_i', B_{n-j}'] = 0.$$

(iii) Applying Theorem 2.2 for Γ_n-contractions $(S_1^*, \dots, S_{n-1}^*, P^*)$ and $(S_1'^*, \dots, S_{n-1}'^*, P^*)$ we get $S_i S_j'^* - S_{n-j}' S_{n-i}^* = D_{P^*}(B_i^* B_j' - B_{n-j}'^* B_{n-i})D_{P^*}$. From Lemma 2.4, $D_P A_i = (S_i D_P - D_{P^*} B_{n-i} P)|_{\mathcal{D}_P}$. Multiplying by $A_{n-j}' D_P$ from right we get

$$\begin{aligned}
& D_P A_i A_{n-j}' D_P = (S_i D_P - D_{P^*} B_{n-i} P) A_{n-j}' D_P \\
\Rightarrow & D_P A_i A_{n-j}' D_P = S_i D_P A_{n-j}' D_P - D_{P^*} B_{n-i} P A_{n-j}' D_P \\
\Rightarrow & D_P A_{n-j}' A_i D_P = S_i (S_{n-j}' - S_j'^* P) - D_{P^*} B_{n-i} P A_{n-j}' D_P \\
\Rightarrow & D_P A_i A_{n-j}' D_P = S_i S_{n-j}' - S_i S_j'^* P - D_{P^*} B_{n-i} B_{n-j}'^* P D_P.
\end{aligned}$$

Similarly, multiplying $D_P A_{n-j}' = (S_{n-j}' D_P - D_{P^*} B_j' P)|_{\mathcal{D}_P}$ by $A_i D_P$ from right we get

$$\begin{aligned}
& D_P A_{n-j}' A_i D_P = (S_{n-j}' D_P - D_{P^*} B_j' P) A_i D_P \\
\Rightarrow & D_P A_{n-j}' A_i D_P = S_{n-j}' D_P A_i D_P - D_{P^*} B_j' P A_i D_P \\
\Rightarrow & D_P A_{n-j}' A_i D_P = S_{n-j}' (S_i - S_{n-i}^* P) - D_{P^*} B_j' P A_i D_P \\
\Rightarrow & D_P A_{n-j}' A_i D_P = S_{n-j}' S_i - S_{n-j}' S_{n-i}^* P - D_{P^*} B_j' B_i^* P D_P.
\end{aligned}$$

Subtracting those two equations we get

$$\begin{aligned}
&D_P(A_iA'_{n-j} - A'_{n-j}A_i)D_P = (S'_{n-j}S^*_{n-i} - S_iS'^*_j)P \\
&\qquad\qquad\qquad\qquad + D_{P^*}(B'_jB^*_i - B_{n-i}B'^*_{n-j})PD_P \\
&\Rightarrow (S_iS'^*_j - S'_{n-j}S^*_{n-i})P + D_{P^*}(B_{n-i}B'^*_{n-j} - B'_jB^*_i)D_{P^*}P = 0 \\
&\Rightarrow D_{P^*}(B^*_iB'_j - B'^*_{n-j}B_{n-i})D_{P^*}P + D_{P^*}(B_{n-i}B'^*_{n-j} - B'_jB^*_i)D_{P^*}P = 0 \\
&\Rightarrow D_{P^*}([B^*_i, B'_j] + [B_{n-i}, B'^*_{n-j}])D_{P^*}P = 0 \\
&\Rightarrow [B^*_i, B'_j] = [B'^*_{n-j}, B_{n-i}].
\end{aligned}$$

□

A direct consequence of the previous theorem is the following.

Corollary 2.8. *Let $(S_1, \ldots, S_{n-1}, P)$ be a Γ_n-contraction acting on a Hilbert space $\mathcal{H}$ and let $(A_1, \ldots, A_{n-1})$ and $(B_1, \ldots, B_{n-1})$ be the $\mathcal{F}_O$-tuples of $(S_1, \ldots, S_{n-1}, P)$ and $(S^*_1, \ldots, S^*_{n-1}, P^*)$, respectively. If $[A_i, A_{n-j}] = 0$ for each $i, j = 1, \ldots, n-1$ and if P has dense range, then the following identities hold for $i, j = 1, \ldots, n-1$:*

(i) $[A^*_j, A_i] = [A^*_{n-i}, A_{n-j}]$
(ii) $[B_i, B_{n-j}] = 0$
(iii) $[B^*_i, B_j] = [B^*_{n-j}, B_{n-i}]$.

Lemma 2.9. *Let $(S_1, \ldots, S_{n-1}, P)$ and $(S'_1, \ldots, S'_{n-1}, P)$ be two Γ_n-contractions on a Hilbert space $\mathcal{H}$ such that P is invertible. Let $(A_1, \ldots, A_{n-1})$, $(A'_1, \ldots, A'_{n-1})$, $(B_1, \ldots, B_{n-1})$ and $(B'_1, \ldots, B'_{n-1})$ be as in previous theorem. Then $[A_i, A_j] = 0 = [A'_i, A'_j]$, for $i, j = 1, \ldots, n-1$ if and only if $[B_i, B_j] = 0 = [B'_i, B'_j]$, for $i, j = 1, \ldots, n-1$.*

Proof. Suppose that $[A_i, A_j] = 0 = [A'_i, A'_j]$ for $i, j = 1, \ldots, n-1$. Since P has dense range, by part (ii) of above theorem, we get $[B_i, B_j] = 0 = [B'_i, B'_j]$ for $i, j = 1, \ldots, n-1$.

Conversely, let $[B_i, B_j] = 0 = [B'_i, B'_j]$ for $i, j = 1, \ldots, n-1$. Since P is invertible, P^* has dense range too. So applying previous theorem for Γ_n-contractions $(S^*_1, \ldots, S^*_{n-1}, P^*)$ and $(S'^*_1, \ldots, S'^*_{n-1}, P^*)$, we get $[A_i, A_j] = 0 = [A'_i, A'_j]$ for $i, j = 1, \ldots, n-1$. □

Corollary 2.10. *Let $(S_1, \ldots, S_{n-1}, P)$ be a Γ_n-contraction on a Hilbert space $\mathcal{H}$ such that P is invertible. Let $(A_1, \ldots, A_{n-1})$ and $(B_1, \ldots, B_{n-1})$ be as in previous theorem. Then $[A_i, A_{n-j}] = 0$, for $i, j = 1, \ldots, n-1$ if and only if $[B_i, B_{n-j}] = 0$, for $i, j = 1, \ldots, n-1$.*

3. Admissible fundamental operator tuples

We recall from [16], the notion of characteristic function of a contraction introduced by Sz.-Nagy and Foias. For a contraction P defined on a Hilbert space $\mathcal{H}$, let Λ_P be the set of all complex numbers for which the operator

$I - zP^*$ is invertible. For $z \in \Lambda_P$, the characteristic function of P is defined as

$$\Theta_P(z) = [-P + zD_{P^*}(I - zP^*)^{-1}D_P]|_{\mathcal{D}_P}. \tag{3.1}$$

By virtue of the relation $PD_P = D_{P^*}P$ (Section I.3 of [16]), $\Theta_P(z)$ maps $\mathcal{D}_P = \overline{\text{Ran}}D_P$ into $\mathcal{D}_{P^*} = \overline{\text{Ran}}D_{P^*}$ for every z in Λ_P. Since for each $z \in \mathbb{D}$, $\Theta_P(z)$ maps $\mathcal{D}_P$ into $\mathcal{D}_{P^*}$, Θ_P induces a multiplication operator M_{Θ_P} from $H^2(\mathbb{D}) \otimes \mathcal{D}_P$ into $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$, defined by

$$M_{\Theta_P} f(z) = \Theta_P(z) f(z), \text{ for all } f \in H^2(\mathbb{D}) \otimes \mathcal{D}_P \text{ and } z \in \mathbb{D}.$$

Note that $M_{\Theta_P}(M_z \otimes I_{\mathcal{D}_P}) = (M_z \otimes I_{\mathcal{D}_{P^*}})M_{\Theta_P}$.

Lemma 3.1. *Let $(A_1, \dots, A_{n-1})$ and $(B_1, \dots, B_{n-1})$ be the $\mathcal{F}_O$-tuples of a Γ_n-contraction $(S_1, \dots, S_{n-1}, P)$ and its adjoint $(S_1^*, \dots, S_{n-1}^*, P^*)$, respectively. Then for each $i = 1, \dots, n-1$,*

$$(A_i^* + A_{n-i}z)\Theta_{P^*}(z) = \Theta_{P^*}(z)(B_i + B_{n-i}^* z) \text{ for all } z \in \mathbb{D}. \tag{3.2}$$

Proof. We have that

$$\begin{aligned}
& (A_i^* + A_{n-i}z)\Theta_{P^*}(z) \\
= \; & (A_i^* + A_{n-i}z)(-P^* + \sum_{n=0}^{\infty} z^{n+1} D_P P^n D_{P^*}) \\
= \; & (-A_i^* P^* + \sum_{n=1}^{\infty} z^n A_i^* D_P P^{n-1} D_{P^*}) \\
& \quad + (-zA_{n-i}P^* + \sum_{n=2}^{\infty} z^n A_{n-i} D_P P^{n-2} D_{P^*}) \\
= \; & -A_i^* P^* + z(A_i^* D_P D_{P^*} - A_{n-i}P^*) \\
& \quad + \sum_{n=2}^{\infty} z^n (A_i^* D_P P^{n-1} D_{P^*} + A_{n-i} D_P P^{n-2} D_{P^*}) \\
= \; & -A_i^* P^* + z(A_i^* D_P D_{P^*} - A_{n-i}P^*) \\
& \quad + \sum_{n=2}^{\infty} z^n (A_i^* D_P P + A_{n-i} D_P) P^{n-2} D_{P^*} \\
= \; & -P^* B_i + z(D_P D_{P^*} B_i - P^* B_{n-i}^*) + \sum_{n=2}^{\infty} z^n D_P S_2 P^{n-2} D_{P^*}.
\end{aligned}$$

The last equality follows by using Theorem 2.1, Lemma 2.5 and Lemma 2.6. Also we have

$$\begin{aligned}
& \Theta_{P^*}(z)(B_i + B_{n-i}^* z) \\
= \quad & (-P^* + \sum_{n=0}^{\infty} z^{n+1} D_P P^n D_{P^*})(B_i + B_{n-i}^* z) \\
= \quad & (-P^* B_i + \sum_{n=1}^{\infty} z^n D_P P^{n-1} D_{P^*} B_i) \\
& \qquad + (-zP^* B_{n-i}^* + \sum_{n=2}^{\infty} z^n D_P P^{n-2} D_{P^*} B_{n-i}^*) \\
= \quad & -P^* B_i + z(D_P D_{P^*} B_i - P^* B_{n-i}^*) \\
& \qquad + \sum_{n=2}^{\infty} z^n (D_P P^{n-1} D_{P^*} B_i + D_P P^{n-2} D_{P^*} B_{n-i}^*) \\
= \quad & -P^* B_i + z(D_P D_{P^*} B_i - P^* B_{n-i}^*) \\
& \qquad + \sum_{n=2}^{\infty} z^n D_P P^{n-2} (P D_{P^*} B_i + D_P B_{n-i}^*) \\
= \quad & -P^* B_i + z(D_P D_{P^*} B_i - P^* B_{n-i}^*) + \sum_{n=2}^{\infty} z^n D_P P^{n-2} S_{n-i} D_{P^*} \\
= \quad & -P^* B_i + z(D_P D_{P^*} B_i - P^* B_{n-i}^*) + \sum_{n=2}^{\infty} z^n D_P S_{n-i} P^{n-2} D_{P^*}.
\end{aligned}$$

Hence for $i = 1, \dots, n-1$ we have $(A_i^* + A_{n-i} z)\Theta_{P^*}(z) = \Theta_{P^*}(z)(B_i + B_{n-i}^* z)$ for all $z \in \mathbb{D}$ and the proof is complete. $\square$

Note 3.2. Under the hypotheses of Theorem 3.1, the following equations hold:

$$(B_i^* + B_{n-i} z)\Theta_P(z) = \Theta_P(z)(A_i + A_{n-i}^* z), \qquad \text{for all } z \in \mathbb{D}. \tag{3.3}$$

We are now in a position to present one of the main results of this paper. We first state a result from the literature which provides a characterization of Γ_n-unitaries. We shall use this result in the proof of the main theorem.

Theorem 3.3 ([7], Theorem 4.2)**.** *Let $(S_1, \dots, S_{n-1}, P)$ be a commuting tuple of bounded operators. Then the following are equivalent.*

(1) *$(S_1, \dots, S_{n-1}, P)$ is a Γ_n-unitary,*
(2) *P is a unitary, $(\frac{n-1}{n} S_1, \frac{n-2}{n} S_2, \dots, \frac{1}{n} S_{n-1})$ is a Γ_{n-1}-contraction and $S_i = S_{n-i}^* P$ for $i = 1, \dots, n-1$.*

Theorem 3.4. *Let P be a $C_{\cdot 0}$ contraction on a Hilbert space $\mathcal{H}$. Let $A_1, \dots, A_{n-1} \in \mathcal{B}(\mathcal{D}_P)$ and $B_1, \dots, B_{n-1} \in \mathcal{B}(\mathcal{D}_{P^*})$ be such that they satisfy* *equations (3.3) and*

$$\left(\frac{n-1}{n}(B_1^*+B_{n-1}z), \frac{n-2}{n}(B_2^*+B_{n-2}z), \dots, \frac{1}{n}(B_{n-1}^*+B_1z)\right)$$

are Γ_{n-1}-contractions for all $z \in \mathbb{T}$. Then there exists a Γ_n-contraction $(S_1, \dots, S_{n-1}, P)$ such that $(A_1, \dots, A_{n-1})$ is the $\mathcal{F}_O$-tuple of $(S_1, \dots, S_{n-1}, P)$ and $(B_1, \dots, B_{n-1})$ is the $\mathcal{F}_O$-tuple of $(S_1^*, \dots, S_{n-1}^*, P^*)$.

Proof. Let us define $W : \mathcal{H} \to H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ by

$$W(h) = \sum_{n=0}^{\infty} z^n \otimes D_{P^*}P^{*n}h \text{ for all } h \in \mathcal{H}.$$

It is evident that

$$\begin{aligned}||Wh||^2 = \sum_{n=0}^{\infty} ||D_{P^*}P^{*n}h||^2 &= \sum_{n=0}^{\infty}\left(||P^{*n}h||^2 - ||P^{*n+1}h||^2\right)\\ &= ||h||^2 - \lim_{n\to\infty} ||P^{*n}h||^2.\end{aligned}$$

Therefore W is an isometry if P is a pure contraction. It is obvious that

$$W^*(z^n \otimes \xi) = P^n D_{P^*}\xi \text{ for all } \xi \in \mathcal{D}_{P^*} \text{ and } n \geq 0.$$

Also if M_z is the multiplication operator on $H^2(\mathbb{D})$ and if $M = M_z \otimes I$ on $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$, then we have

$$M^*Wh = T_z^*\left(\sum_{n=0}^{\infty} z^n D_{P^*}P^{*n}h\right) = \sum_{n=0}^{\infty} z^n D_{P^*}P^{*n+1}h = WP^*h.$$

Therefore $M^*W = WP^*$. Since

$$\left(\frac{n-1}{n}(B_1^*+B_{n-1}z), \frac{n-2}{n}(B_2^*+B_{n-2}z), \dots, \frac{1}{n}(B_{n-1}^*+B_1z)\right)$$

is a Γ_{n-1}-contraction for all $z \in \mathbb{T}$, it follows from Theorem 3.3 that the multiplication operator tuple $(M_{B_1^*+B_{n-1}z}, \dots, M_{B_{n-1}^*+B_1z}, M_z)$ on $L^2(\mathcal{D}_{P^*})$ is a Γ_n-unitary. Obviously the Toeplitz operator tuple

$$(T_{B_1^*+B_{n-1}z}, \dots, T_{B_{n-1}^*+B_1z}, T_z) \text{ on } H^2(\mathcal{D}_{P^*}),$$

by being the restriction of $(M_{B_1^*+B_{n-1}z}, \dots, M_{B_{n-1}^*+B_1z}, M_z)$ to the joint invariant subspace $H^2(\mathcal{D}_{P^*})$, is a Γ_n-isometry. Again since $H^2(\mathcal{D}_{P^*})$ and $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ are isomorphic, the Γ_n-isometry on the space $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ that corresponds to $(T_{B_1^*+B_{n-1}z}, \dots, T_{B_{n-1}^*+B_1z}, T_z)$ is

$$(I \otimes B_1^* + M_z \otimes B_{n-1}, \dots, I \otimes B_{n-1}^* + M_z \otimes B_1, M_z \otimes I).$$

Let us define

$$S_i = W^*M_iW \text{ for } i = 1, \dots, n-1\,,$$

where

$$M_i = I \otimes B_i^* + M_z \otimes B_{n-i} \text{ for } i = 1, \dots, n-1.$$

Equations (3.3) tell us that $RanM_{\Theta_P}$ is invariant under M_i for $i = 1, \dots, n-1$ which is same as saying that $RanW = (RanM_{\Theta_P})^\perp$ is invariant under M_i^* for $i = 1, \dots, n-1$.

Since $(M_1, \dots, M_{n-1}, M)$ is a Γ_n-isometry, $(S_1, \dots, S_{n-1}, P)$ is a Γ_n-contraction by being the compression of $(M_1, \dots, M_{n-1}, M)$. We now show that $(B_1, \dots, B_{n-1})$ is the $\mathcal{F}_O$-tuple of $(S_1^*, \dots, S_{n-1}^*, P^*)$. For each $i = 1, \dots, n-1$ we have that

$$\begin{aligned} S_i^* - S_{n-i}P^* &= W^*M_i^*W - W^*M_{n-i}WW^*M^*W \\ &= W^*M_i^*W - W^*M_{n-i}M^*W \\ &= W^*[(I \otimes B_i) + (M_z^* \otimes B_{n-i}^*) - (M_z^* \otimes B_{n-i}^*) \\ &\quad - (M_zM_z^* \otimes B_i)]W \\ &= D_{P^*}B_iD_{P^*}. \end{aligned}$$

To obtain the equalities above, we used the fact that $RanW$ is invariant under M_z^* and that $I - M_zM_z^*$ is a rank one projection. By the uniqueness of $\mathcal{F}_O$-tuple of a Γ_n-contraction, we conclude that $(B_1, \dots, B_{n-1})$ is the fundamental operator tuple of $(S_1^*, \dots, S_{n-1}^*, P^*)$. Let $(Y_1, \dots, Y_{n-1})$ be the $\mathcal{F}_O$-tuple of $(S_1, \dots, S_{n-1}, P)$. Then by the first part of this theorem, we have for each $i = 1, \dots, n-1$ that

$$(B_i^* + B_{n-i}z)\Theta_P(z) = \Theta_P(z)(Y_i + Y_{n-i}^*z) \text{ for all } z \in \mathbb{D}.$$

By this and the fact that all B_i satisfy equations (3.3), for some operators $A_1, \dots, A_{n-1} \in \mathcal{B}(\mathcal{D}_P)$, we have that

$$A_i + A_{n-i}^*z = Y_i + Y_{n-i}^*z \text{ for all } i = 1, \dots, n-1$$

and for all $z \in \mathbb{D}$. Therefore, $Y_i = A_i$ for each i and consequently $(A_1, \dots, A_{n-1})$ is the $\mathcal{F}_O$-tuple of $(S_1, \dots, S_{n-1}, P)$ and the proof is complete. $\square$

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