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A lower bound for the doubly slice genus from signatures

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Abstract. The doubly slice genus of a knot in the 3-sphere is the minimal genus among unknotted orientable surfaces in the 4-sphere for which the knot arises as a cross-section. We use the classical signature function of the knot to give a new lower bound for the doubly slice genus. We combine this with an upper bound due to C. McDonald to prove that for every nonnegative integer N there is a knot where the difference between the slice and doubly slice genus is exactly N, refining a result of W. Chen which says this difference can be arbitrarily large.

Contents

1. Introduction

In what follows all manifolds are topological, compact, and oriented, and embeddings are locally flat, although our results also hold in the smooth category. A basic 4-dimensional measurement for the complexity of a knot $K \subset S^3$ is the *slice genus* $g_4(K)$, defined as the minimal genus among connected properly embedded surfaces in D^4 that have the knot as boundary. Doubling such a surface along its boundary produces a closed connected surface in S^4 for which the knot appears as a cross section. This doubled surface will be genus minimising among surfaces in S^4 for which the knot appears as a cross section, but will in general be a knotted surface embedding.

A connected surface in S^4 is *unknotted* if it bounds an embedded 3-dimensional handlebody in S^4. Unknotted surfaces with the knot K as cross section are easily produced by doubling a Seifert surface for K that has been pushed in to D^4. The *doubly slice genus* $g_{ds}(K)$, which was first

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defined in [8, §5], is the minimal genus among unknotted surfaces in S^4 for which the knot arises as a cross-section. Writing $g_3(K)$ for the minimal genus among Seifert surfaces for K, it is immediate from the above discussion that

$$2g_4(K) \leq g_{ds}(K) \leq 2g_3(K).$$

Further comparison of these quantities is fairly subtle, but we will show in this article that classical abelian knot invariants can be employed for this purpose.

A choice of Seifert surface for a knot $K \subset S^3$ and a choice of basis for the first homology gives rise to a Seifert matrix V. Then given $\omega \in S^1 \subset \mathbb{C}$ the *ω-signature* of K is defined as the signature of the complex hermitian matrix

$$\sigma_\omega(K) := \operatorname{sgn}\left((1-\omega)V + (1-\omega^{-1})V^T\right).$$

Theorem 1.1. *Let K be a knot in S^3. The doubly slice genus of K is at least*

$$g_{ds}(K) \geq \max_{\omega \in S^1 \setminus \{1\}} |\sigma_\omega(K)|.$$

Let $\Delta_K(t)$ denote the Alexander polynomial of K. A classical lower bound for the slice genus is that for every $\omega \in S^1$ such that $\Delta_K(\omega) \neq 0$, we have $|\sigma_\omega(K)| \leq 2g_4(K)$ [4]. It follows that $|\sigma_\omega(K)| \leq g_{ds}(K)$ for these ω. Our theorem refines this, since it also applies when ω is a root of the Alexander polynomial of K. Given a slice knot K, in other words a knot with $g_4(K) = 0$, and for $\omega \in S^1$ such that $\Delta_K(\omega) \neq 0$, we have $\sigma_\omega(K) = 0$. Therefore the classical bound contains no information on the doubly slice genus for slice knots.

On the other hand, for every $\nu \in S^1 \setminus \{1\}$ that is the root of some Alexander polynomial there exists a slice knot K for which $\sigma_\omega(K)$ is nontrivial exactly at $\omega = \nu, \overline{\nu}$ [1, Corollary 2.1]. For any $N \in \mathbb{N}$, Theorem 1.1 applied to the N-fold connected sum of such a knot with itself immediately produces a slice knot with doubly slice genus at least N, recovering a theorem of Chen [2], which we discuss below. In the following result we obtain a refinement of such examples.

Theorem 1.2. *For each $N \in \mathbb{N}$ there exists a slice knot K_N with $g_{ds}(K_N) = N$. In fact, we may take $K_N = \#^N J$, the N-fold connected sum of J with itself for some*

$$J \in \left\{ \begin{array}{l} 8_{20},\ 10_{87},\ 10_{140},\ 11a28,\ 11a58,\ 11a165,\ 12a189,\ 12a377, \\ 12a979,\ 12n56,\ 12n57,\ \ 12n62,\ 12n66,\ 12n87,\ 12n106, \\ 12n288,\ 12n501,\ 12n504,\ 12n582,\ 12n670,\ 12n721 \end{array} \right\}.$$

Here we use the notation of KnotInfo [9].

Proof. The 21 knots listed are slice knots, found by searching the KnotInfo tables, of at most 12 crossings, whose ω-signature equals 1 for some $\omega \in S^1$

with $\Delta_J(\omega) = 0$. As the lower bound of Theorem 1.1 is additive under connected sum we therefore have $g_{ds}(K_N) \geq N$.

We will show in Proposition 4.2 that each of these knots admits a slice disc on which the radial Morse function has two minima and one saddle point i.e. J arises from one band move on the 2-component unlink. The following theorem of Clayton McDonald therefore shows that each of the knots J has doubly slice genus at most 1, and that K_N therefore has $g_{ds}(K_N) \leq N$. $\square$

Theorem 1.3 (McDonald [10, Theorem 3.2])**.** *Let $K \subset S^3$ be a knot and let Σ be a smoothly embedded surface in D^4 such that the radial Morse function restricts to a Morse function on Σ with b saddle points and no maxima. Then $g_{ds}(K) \leq b$.*

Corollary 1.4 (to Theorem 1.2)**.** *Let M, N be nonnegative integers with M even and $M \leq N$. There exists a knot K with $M = 2g_4(K)$ and $N = g_{ds}(K)$.*

Proof. Let J be the mirror image of the knot 5_2. This has $g_4(J) = g_3(J) = g_{ds}(J) = 1$, and $\sigma_\omega(J) = 2$ for $\omega := e^{\pi i/3}$, which is not a root of the Alexander polynomial. The knot $L := 8_{20}$ has $g_4(L) = 0$, but $\sigma_\omega(L) = 1$ and $g_{ds}(L) = 1$. Taking

$$K := \left(\#^{M/2} J\right) \# \left(\#^{N-M} L\right)$$

yields a knot with $2g_4(K) \leq M$ and $g_{ds}(K) \leq N$. Then $\sigma_\omega(K) = N$, so $g_{ds}(K) = N$ by Theorem 1.1. Since $|\sigma_\rho(K)| \leq 2g_4(K)$ except for finitely many values of $\rho \in S^1$, the averaged signature function defined by

$$\overline{\sigma}_{e^{i\pi\theta}}(K) := \frac{1}{2}\left(\lim_{\varphi \to \theta^+} \sigma_{e^{i\pi\varphi}}(K) + \lim_{\varphi \to \theta^-} \sigma_{e^{i\pi\varphi}}(K)\right)$$

satisfies $|\overline{\sigma}_\rho(K)| \leq 2g_4(K)$ for all $\rho \in S^1$. Then $\overline{\sigma}_\omega(K) = M$ so $2g_4(K) = M$. $\square$

Remark 1.5. KnotInfo does not provide an explanation for the computations of signature signature functions that we use in Theorem 1.1, so a brief discussion is in order. When computing the signature function at roots of the Alexander polynomial, one must take a little more care than with computations away from the roots. Nevertheless the signatures can be evaluated as follows. Let V be a Seifert matrix for K and calculate, by hand or with a computer algebra package, the set of roots for $\Delta_K = \det(tV - V^T)$ on the unit circle, in the form of algebraic numbers. For each such root, ω say, find the eigenvalues of $(1-\omega)V + (1-\omega^{-1})V^T$. In order to compute the signature one only needs to know whether each eigenvalue is positive, negative, or zero, so just evaluating the roots as decimal approximations will usually enable one to determine which of these three options is pertinent. In principle there might arise the problem that one is not sure how to categorise an eigenvalue that the computer tells us is very close to zero: is it actually 0 but appears different due to rounding errors? Similarly if the evaluation is claimed to be 0, it could be in reality a very small nonzero eigenvalue that the number of decimal places stored by the computer cannot distinguish from 0. However this issue does not occur for the low crossing number knots we looked at. In addition, as ω is a root of Δ_K, the matrix we are studying *must* have at least one 0 eigenvalue, and in practice the computer algebra package identified this eigenvalue precisely.

Context from previous work. A knot K is *doubly slice* if $g_{ds}(K) = 0$, and the doubly slice genus is a measure of how far a knot is from being doubly slice. The first detailed study of doubly slice knots, and the related algebra, was made by Sumners [13]. Further foundational algebraic studies, related to the work in this article, are those of Stoltzfus [12] and Levine [7]. We collect some previously known invariance properties of the signature function of a knot as motivation for Theorem 1.1.

(1) For every $\omega \in S^1$, the signature $\sigma_\omega(K) \in \mathbb{Z}$ is a knot invariant.
(2) If $\omega \in S^1$ is such that there is a polynomial $\Delta \in \mathbb{Z}[t, t^{-1}]$ with $\Delta(1) = 1$ and $\Delta(\omega) = 0$, then we call ω a *Knotennullstelle*; cf. [11]. If ω is not a Knotennullstelle, then $\sigma_\omega(K)$ is a concordance invariant and $|\sigma_\omega(K)| \leq 2g_4(K)$.
(3) For all $\omega \in S^1$, the averaged signature $\overline{\sigma}_\omega(K)$ defined in the previous proof is a concordance invariant, and $|\overline{\sigma}_\omega(K)| \leq 2g_4(K)$.
(4) For ω not a root of the Alexander polynomial of K, $\sigma_\omega(K) = \overline{\sigma}_\omega(K)$. The functions may differ at roots of Δ_K [1].

The observation motivating Theorem 1.1 is that while signatures of slice knots vanish away from roots of the Alexander polynomial, this is in general does not hold at roots; e.g. [1]. It was known that such signatures provide obstructions to a knot being doubly slice [7]. The signatures at roots of the Alexander polynomial may be be computed via the intersection form of a suitable 4-manifold with boundary the zero-framed surgery on the knot. Roughly speaking, our proof of Theorem 1.1 connects the size of the intersection form with the genus of a doubly slice surface, and shows that the knot signature, for every $\omega \in S^1$, is a lower bound for the size of the intersection form.

Instead of the doubly slice genus, a different measure of the failure of a knot to be doubly slice was studied by Cherry Kearton [5]. Given a slice knot K, he considered the minimal complex dimension of $H_1(S^4 \setminus J; \mathbb{C}[t, t^{-1}])$ among all knotted 2-spheres $J \subset S^4$ with cross-section K. He gave lower bounds for his invariant arising from signature obstructions. The signatures he considered are the (p, i)-signatures of the Blanchfield form (see [7]), and it is known that these signatures can be used to compute the ω-signatures of K [7, Theorem 2.3], tempting one to imagine a connection to the results of this paper. But despite the similar flavour of the invariants he uses, Kearton's complexity measure appears to be independent of the doubly slice genus, so there is no clear dependency between his work and ours.

This article was partly inspired by work of Wenzhao Chen [2], who ingeniously applied Casson-Gordon invariants to show that for every $N \in \mathbb{N}$, there is a slice knot K with $g_{ds}(K) \geq N$. In particular he proved that $g_{ds}(K) - 2g_4(K)$ can be arbitrarily large. Casson-Gordon invariants rely on the existence of interesting metabelian representations of the knot group $\pi_1(S^3 \setminus K)$ and are thus less basic than the ω-signatures in this paper, which can be thought of as arising from the abelianisation of the knot group $\pi_1(S^3 \setminus K) \to \mathbb{Z}$. While our method refines Chen's theorem, with a more elementary invariant, we cannot recover Chen's examples. These examples, as the original Casson-Gordon examples, are constructed using the Stevedore's knot. With rational coefficients the Stevedore's knot shares a Seifert matrix with 9_{46}, which is doubly slice. This means Chen's examples have hyperbolic Seifert matrices over the rational numbers, and so for all $\omega \in S^1 \setminus \{1\}$ the ω-signature of his knots vanish.

Outline. The paper is organised as follows. In Section 2 we recall the signature defect invariants of a 3-manifold with a map to $B\mathbb{Z}$, associated with a cobounding 4-manifold. We equate the signature defect invariant with ω-signatures. In Section 3 we use this to prove Theorem 1.1. In Section 4 we establish the upper bounds for the examples listed in Theorem 1.2.

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2. Signature defects

Let R be either the ring $\mathbb{C}$ with the involution given by complex conjugation, or the ring of finite complex Laurent polynomials $\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[t, t^{-1}]$ with involution given by $\sum a_k t^k \mapsto \sum \overline{a_k} t^{-k}$. An *$R$-module* will mean a left R-module unless otherwise stated, and $\overline{P}$ will denote the use of the involution to switch a left R-module P to a right R-module, or vice-versa.

A CW pair of connected topological spaces (X, Y) is *over* $\mathbb{Z}$ if X is equipped with a homomorphism $\varphi\colon \pi_1(X) \to \mathbb{Z}$. We write (X, Y, φ) for these data, or (X, φ) if $Y = \emptyset$. Write $p\colon \widetilde{X} \to X$ for the cover corresponding to φ and $\widetilde{Y} = p^{-1}(Y)$ for the corresponding cover of Y. Given a map of rings with involution $\alpha\colon \mathbb{C}[\mathbb{Z}] \to R$, the ring R becomes an $(R, \mathbb{C}[\mathbb{Z}])$-bimodule, and there are associated twisted homology and cohomology modules over R

$$H_r(X,Y;\alpha) := H_r(R \otimes_\alpha C_*(\widetilde{X},\widetilde{Y};\mathbb{C})),$$
$$H^r(X,Y;\alpha) := H_r(\mathrm{Hom}_{\mathbb{C}[\mathbb{Z}]}(\overline{C_*(\widetilde{X},\widetilde{Y};\mathbb{C})},R)).$$

Note we are abusing notation in suppressing the particular φ being used, but for all applications in this article the choice of φ will be understood, so this should cause no confusion.

Setting α to be the identity map $\mathrm{Id}\colon \mathbb{C}[\mathbb{Z}] \to \mathbb{C}[\mathbb{Z}]$ returns the ordinary complex coefficient homology and complex coefficient cohomology with compact support of the cover $(\widetilde{X},\widetilde{Y})$. We denote these by $H_r(X,Y;\mathbb{C}[\mathbb{Z}])$ and $H^r(X,Y;\mathbb{C}[\mathbb{Z}])$ respectively.

For each $\omega \in S^1 \setminus \{1\}$ there is a map of rings with involution

$$\alpha_\omega\colon \mathbb{C}[t,t^{-1}] \to \mathbb{C}; \qquad \alpha_\omega(t) = \omega.$$

The map α_ω induces a $(\mathbb{C},\mathbb{C}[\mathbb{Z}])$-bimodule structure on $\mathbb{C}$ and we will write $\mathbb{C}_\omega$ when we wish to emphasise this structure is being used. We will write

$$H_r(X,Y;\mathbb{C}_\omega) := H_r(X,Y;\alpha_\omega), \qquad H^r(X,Y;\mathbb{C}_\omega) := H_r(X,Y;\alpha_\omega).$$

Now consider (X,φ) where X is a compact, oriented n-dimensional manifold with (possibly empty) boundary. Denote the Poincaré duality isomorphism by $PD\colon H^{n-k}(X;\mathbb{C}_\omega) \to H_k(X,\partial X;\mathbb{C}_\omega)$. Define a map of complex vector spaces

$$\begin{aligned}\lambda_\omega(X)\colon H_k(X;\mathbb{C}_\omega) \to H_k(X,\partial X;\mathbb{C}_\omega) &\xrightarrow{PD^{-1}} H^{n-k}(X;\mathbb{C}_\omega)\\ &\xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_{\mathbb{C}}(H_{n-k}(X;\mathbb{C}_\omega),\mathbb{C})},\end{aligned}$$

where ev denotes the *evaluation* map given by $\mathrm{ev}([f])([z\otimes x]) = z \cdot \overline{f(x)}$. The map $\lambda_\omega(X)$ determines a pairing

$$H_{n-k}(X;\mathbb{C}_\omega) \times H_k(X;\mathbb{C}_\omega) \to \mathbb{C}; \qquad (x,y) \mapsto \lambda_\omega(X)(y)(x),$$

which is hermitian and sesquilinear but in general is degenerate. In particular, when $n = 2k$, we may take the signature of this complex hermitian pairing, denoted $\sigma(\lambda_\omega(X)) \in \mathbb{Z}$.

Definition 2.1. For W a compact, oriented 4-manifold with (possibly empty) boundary, over $\mathbb{Z}$, the (middle dimensional) $\mathbb{C}_\omega$-coefficient *intersection form* is the hermitian sesquilinear form $(H_2(W;\mathbb{C}_\omega),\lambda_\omega(W))$.

Definition 2.2. Let (M,φ) be a closed, connected, oriented 3-manifold over $\mathbb{Z}$. A *null-bordism* of (M,φ) is a pair (W,ψ) consisting of a compact, connected, oriented 4-manifold W with boundary $\partial W = M$ and a homomorphism $\psi\colon \pi_1(W) \to \mathbb{Z}$ such that $\psi|_{\partial W} = \varphi$.

Given a null-bordism (W,ψ) of (M,φ), we define the *ω-signature defect*

$$\sigma_\omega(M) := \sigma(\lambda_\omega(W)) - \sigma(W).$$

(We are abusing notation in suppressing the particular φ and ψ.)

Proposition 2.3. *Given a closed, connected, oriented 3-manifold (M, φ) over $\mathbb{Z}$, and any $\omega \in S^1 \setminus \{1\}$, the ω-signature defect $\sigma_\omega(M)$ is defined and well-defined, independent of the choice (W, ψ).*

Proof. Because $\Omega_3(B\mathbb{Z}) = 0$, there always exists a null-bordism (W, ψ) for (M, φ). The proof that the resultant ω-signature defect is independent of the choice of (W, ψ) is a well-known Novikov additivity argument, as we now outline. First, write $i\colon H_2(M; \mathbb{C}_\omega) \to H_2(W; \mathbb{C}_\omega)$ for the inclusion induced map. The image of i lies in the kernel of $\lambda_\omega(W)$ by exactness of the long exact sequence of the pair (W, M). The restriction of $\lambda_\omega(W)$ to the quotient $H_2(W; \mathbb{C}_\omega)/i(H_2(M; \mathbb{C}_\omega))$ determines a nonsingular pairing [11, Proposition 5.3 (i)]. Thus the signature of $\lambda_\omega(W)$ and the signature of its restriction to $H_2(W; \mathbb{C}_\omega)/i(H_2(M; \mathbb{C}_\omega))$ agree. We now refer the reader to the proof of [11, Proposition 5.3 (ii)] for the completion of the argument. □

Example 2.4. The main example we are interested in is the closed, oriented 3-manifold M_K obtained by 0-framed Dehn surgery on S^3 along an oriented knot K. The orientation on the knot determines a natural map $\varphi_K\colon \pi_1(M_K) \to \mathbb{Z}$ via abelianisation.

The associated $\mathbb{C}[\mathbb{Z}]$-coefficient homology $H_*(M_K; \mathbb{C}[\mathbb{Z}])$ is torsion; that is there exists a Laurent polynomial $p \in \mathbb{C}[\mathbb{Z}]$ such that $p \cdot H_*(M_K; \mathbb{C}[\mathbb{Z}]) = 0$.

Example 2.5. Let $G \subset D^4$ be a properly embedded, connected genus g surface with one boundary component, homeomorphic to $\overline{\Sigma_g \setminus D^2} =: \Sigma_{g,1}$. Let νG be an open tubular neighbourhood extending an open tubular neighbourhood of the boundary knot $K \subset S^3$. Let H_g denote the 3-dimensional handlebody of genus g and let Σ_g be its boundary. By choosing a disc $D^2 \subset \partial H_g$, decompose the boundary of $H_g \times S^1$ as

$$\partial(H_g \times S^1) = (\Sigma_{g,1} \times S^1) \cup_{S^1 \times S^1} (D^2 \times S^1).$$

Glue the exterior of G to $H_g \times S^1$, along $\Sigma_{g,1} \times S^1$ to form

$$W := (D^4 \setminus \nu G) \cup_{G \times S^1} (H_g \times S^1),$$

a compact, connected, oriented 4-manifold with boundary M_K, the 0-surgery on K. Mayer-Vietoris calculations give

$$H_k(W; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z} & k = 1, \qquad \text{generated by a meridian of } G, \\ \mathbb{Z}^{2g} & k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the abelianisation of $\varphi\colon \pi_1(M_K) \to \mathbb{Z}$ extends to $\psi\colon \pi_1(W) \to \mathbb{Z}$ so that (W, ψ) is a null-bordism of (M_K, φ). Note that the homology is independent of the choice of identification of G with $\Sigma_{g,1} \subset \partial H_g$.

Lemma 2.6. *Let $K \subset S^3$ be an oriented knot and let M_K be the 0-surgery manifold. For any $\omega \in S^1 \setminus \{1\}$ and there is equality*

$$\sigma_\omega(M_K) = \sigma_\omega(K).$$

Proof. As the ω-signature is well-defined, independent of choice of null-bordism, it suffices to find a single null-bordism of M_K over $\mathbb{Z}$ such that the signature of the $\mathbb{C}_\omega$-coefficient intersection form agrees with $\sigma_\omega(K)$. Perform the construction of Example 2.5 on a pushed in Seifert surface F for K. In this case it was shown by Ko [6, pp. 538-9] (see also Cochran-Orr-Teichner [3, Lemma 5.4]) that in some basis the resultant $\mathbb{C}_\omega$-coefficient intersection form of W_F has matrix $(1-\omega)V + (1-\omega^{-1})V^T$, where V is a Seifert matrix associated to F. Moreover the ordinary signature $\sigma(W_F) = 0$, so the defect satisfies

$$\sigma_\omega(M_K) = \sigma(\lambda_\omega(W_F)) - \sigma(W_F) = \sigma_\omega(K). \qquad \square$$

3. A lower bound on g_{ds}

Let $K \subset S^3$ be an oriented knot, let $G_1, G_2 \subset D^4$ be locally flat, connected, compact, orientable, embedded surfaces with boundary K, such that $S = G_1 \cup_K G_2$ is an unknotted surface in S^4 of genus g.

Perform the construction described in Example 2.5 on each of G_1 and G_2 to obtain W_1 and W_2 respectively. Define

$$V := W_1 \cup_{M_K} -W_2.$$

Observe that $V = (S^4 \setminus \nu S) \cup_{\Sigma_g \times S^1} (H_g \times S^1)$, where H_g denotes the 3-dimensional handlebody of genus g and $\Sigma_g = \partial H_g$.

A straightforward Seifert-Van Kampen argument shows that $\pi_1(V) \cong \mathbb{Z}$. Various Mayer-Vietoris calculations give

$$H_k(V;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0, \\ \mathbb{Z} & k=1, \qquad \text{generated by a meridian of } \Sigma_g, \\ \mathbb{Z}^{2g} & k=2, \\ 0 & \text{otherwise.} \end{cases}$$

We now derive a series of technical lemmas we will use in the proof of Theorem 1.1

Lemma 3.1. *Let T be a finitely generated, torsion $\mathbb{C}[\mathbb{Z}]$-module, and let $\omega \in S^1 \setminus \{1\}$. Then* $\dim_{\mathbb{C}} \operatorname{Tor}_1^{\mathbb{C}[\mathbb{Z}]}(T, \mathbb{C}_\omega) = \dim_{\mathbb{C}}(\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} T)$.

Proof. By the structure theorem for finitely generated modules over a principal ideal domain, there exists an injective map $A\colon P_1 \hookrightarrow P_0$ such that $T \cong P_0/A(P_1)$ and so that P_1, P_0 are free $\mathbb{C}[\mathbb{Z}]$-modules of the same rank. The functor $\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} -$ induces an exact sequence

$$\begin{aligned} &\operatorname{Tor}_1^{\mathbb{C}[\mathbb{Z}]}(P_0, \mathbb{C}_\omega) \to \operatorname{Tor}_1^{\mathbb{C}[\mathbb{Z}]}(T, \mathbb{C}_\omega) \to \mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} P_1 \\ &\qquad\qquad \xrightarrow{\operatorname{Id} \otimes A} \mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} P_0 \to \mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} T \to 0. \end{aligned}$$

The leftmost term is 0 because P_0 is free. As P_1 and P_0 have the same free rank, $\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} P_1$ and $\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} P_0$ have the same complex dimension. The sequence has vanishing Euler characteristic because it is exact, so the claimed result follows. $\square$

Lemma 3.2. *For a space X over $\mathbb{Z}$ with $H_0(X;\mathbb{C}[\mathbb{Z}]) \cong \mathbb{C}$, and for $\omega \in S^1 \setminus \{1\}$ we have*

$$H_0(X;\mathbb{C}_\omega) = 0 \quad \text{and} \quad H_1(X;\mathbb{C}_\omega) \cong \mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} H_1(X;\mathbb{C}[\mathbb{Z}]).$$

Proof. First, $\mathbb{C} \cong \mathbb{C}[t,t^{-1}]/(t-1)$ as a $\mathbb{C}[\mathbb{Z}]$-module, so that $\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C} = 0$ since $\omega \neq 1$. This immediately gives $\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} H_0(X;\mathbb{C}[\mathbb{Z}]) = 0$ and by Lemma 3.1 we also have $\operatorname{Tor}_1^{\mathbb{C}[\mathbb{Z}]}(H_0(X;\mathbb{C}[\mathbb{Z}]),\mathbb{C}_\omega) = 0$. The result now follows from the Universal Coefficient Theorem. □

Lemma 3.3. *With $V = W_1 \cup_{M_K} -W_2$ as described above and $\omega \in S^1 \setminus \{1\}$,*

$$H_1(V;\mathbb{C}_\omega) = 0, \qquad H_3(V;\mathbb{C}_\omega) = 0, \qquad \text{and} \qquad \dim_{\mathbb{C}} H_2(V;\mathbb{C}_\omega) = 2g,$$

so that the Mayer-Vietoris sequence for V with $\mathbb{C}_\omega$ coefficients becomes

$$\begin{aligned} 0 \to H_2(M_K) \to H_2(W_1) \oplus H_2(W_2) \to \mathbb{C}^{2g} \\ \to H_1(M_K) \to H_1(W_1) \oplus H_1(W_2) \to 0. \end{aligned}$$

Proof. Consider that $\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} H_1(V;\mathbb{C}[\mathbb{Z}]) = 0$ since $\pi_1(V) \cong \mathbb{Z}$ implies $H_1(V;\mathbb{C}[\mathbb{Z}]) = 0$. Since $H_0(V;\mathbb{C}[\mathbb{Z}]) \cong \mathbb{C}$ and $\omega \neq 1$, this combines with Lemma 3.2 to give $H_1(V;\mathbb{C}_\omega) = 0$.

Next, we have $H_3(V;\mathbb{C}_\omega) \cong H^1(V;\mathbb{C}_\omega)$ by Poincaré duality. By the Universal Coefficient Theorem for cohomology we obtain an isomorphism $H^1(V;\mathbb{C}_\omega) \cong \operatorname{Ext}^1_{\mathbb{C}[\mathbb{Z}]}(H_0(V;\mathbb{C}[\mathbb{Z}]),\mathbb{C}_\omega)$. The projective $\mathbb{C}[\mathbb{Z}]$-module resolution

$$0 \to \mathbb{C}[\mathbb{Z}] \xrightarrow{f} \mathbb{C}[\mathbb{Z}] \to H_0(V;\mathbb{C}[\mathbb{Z}]) \to 0,$$

where $f\colon p(t) \mapsto (t-1)p(t)$, can be used to compute

$$\begin{aligned} &\operatorname{Ext}^1_{\mathbb{C}[\mathbb{Z}]}(H_0(V;\mathbb{C}[\mathbb{Z}]),\mathbb{C}_\omega) \\ &\qquad = \operatorname{coker}(\operatorname{Hom}_{\mathbb{C}[\mathbb{Z}]}(\mathbb{C}[\mathbb{Z}],\mathbb{C}_\omega) \xrightarrow{f^*} \operatorname{Hom}_{\mathbb{C}[\mathbb{Z}]}(\mathbb{C}[\mathbb{Z}],\mathbb{C}_\omega)). \end{aligned}$$

But $f^*(\varphi) = (\omega - 1)\varphi$, and $\omega \neq 1$, so this module vanishes as required.

Using the integral homology of V, we compute the Euler characteristic $\chi(V) = 2g$. We shall compute it again with $\mathbb{C}_\omega$-coefficients in order to find the dimension of $H_2(V;\mathbb{C}_\omega)$. By Lemma 3.2 we have $H_0(V;\mathbb{C}_\omega) = 0$, so also $H_4(V;\mathbb{C}_\omega) = 0$ by Poincaré duality and the Universal Coefficient Theorem. Therefore $H_i(V;\mathbb{C}_\omega) = 0$ for $i \neq 2$, and we have

$$2g = \chi(V) = \chi^{\mathbb{C}_\omega}(V) = \dim_{\mathbb{C}} H_2(V;\mathbb{C}_\omega). \qquad \square$$

Lemma 3.4. *For $i = 1, 2$ there is equality*

$$\begin{aligned} &\dim_{\mathbb{C}} \operatorname{Im}(H_2(M_K;\mathbb{C}_\omega) \to H_2(W_i;\mathbb{C}_\omega)) \\ &\qquad = \dim_{\mathbb{C}} H_2(M_K;\mathbb{C}_\omega) - \dim_{\mathbb{C}} H_1(W_i;\mathbb{C}_\omega). \end{aligned}$$

Proof. The map $H_1(M_K;\mathbb{C}_\omega) \to H_1(W_i;\mathbb{C}_\omega)$ is surjective by Lemma 3.3. This implies $H_1(W_i, M_K;\mathbb{C}_\omega) = 0$, since $H_0(M_K;\mathbb{C}_\omega) = 0$ by Lemma 3.2. Therefore

$$H_3(W_i;\mathbb{C}_\omega) \cong H^1(W_i, M_K;\mathbb{C}_\omega) \cong H_1(W_i, M_K;\mathbb{C}_\omega) = 0$$

by Poincaré duality and the Universal Coefficient Theorem. For the same reasons, we have

$$H_3(W_i, M_K;\mathbb{C}_\omega) \cong H^1(W_i;\mathbb{C}_\omega) \cong H_1(W_i;\mathbb{C}_\omega).$$

Since $H_3(W_i;\mathbb{C}_\omega) = 0$, the long exact sequence of the pair (W_i, M_K) takes the form

$$0 \to H_3(W_i, M_K;\mathbb{C}_\omega) \to H_2(M_K;\mathbb{C}_\omega) \to H_2(W_i;\mathbb{C}_\omega) \to \cdots .$$

We deduce that

$$\begin{aligned}
&\dim_{\mathbb{C}} \operatorname{Im}(H_2(M_K;\mathbb{C}_\omega) \to H_2(W_i;\mathbb{C}_\omega)) \\
&\qquad = \dim_{\mathbb{C}} H_2(M_K;\mathbb{C}_\omega) - \dim_{\mathbb{C}} H_3(W_i, M_K;\mathbb{C}_\omega) \\
&\qquad\qquad = \dim_{\mathbb{C}} H_2(M_K;\mathbb{C}_\omega) - \dim_{\mathbb{C}} H_1(W_i;\mathbb{C}_\omega).
\end{aligned}$$

as desired. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Fix $\omega \in S^1 \setminus \{1\}$ and let W_1, W_2 be as above. Define for $i = 1, 2$,

$$\begin{aligned}
\beta &:= \dim_{\mathbb{C}} H_2(M_K;\mathbb{C}_\omega), \\
n_i &:= \dim_{\mathbb{C}}(\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} H_2(W_i;\mathbb{C}[\mathbb{Z}])), \\
m_i &:= \dim_{\mathbb{C}} \operatorname{Tor}_1^{\mathbb{C}[\mathbb{Z}]}(H_1(W_i;\mathbb{C}[\mathbb{Z}]), \mathbb{C}_\omega).
\end{aligned}$$

By the Universal Coefficient Theorem

$$n_i + m_i = \dim_{\mathbb{C}} H_2(W_i;\mathbb{C}_\omega) \quad \text{and} \quad \beta = \dim_{\mathbb{C}} \operatorname{Tor}_1^{\mathbb{C}[\mathbb{Z}]}(H_1(M_K;\mathbb{C}[\mathbb{Z}]), \mathbb{C}_\omega),$$

where the latter equality also uses the fact that $H_2(M_K;\mathbb{C}[\mathbb{Z}]) = 0$.

The module $H_1(M_K;\mathbb{C}[\mathbb{Z}])$ is $\mathbb{C}[\mathbb{Z}]$-torsion. As $H_1(V;\mathbb{C}[\mathbb{Z}]) = 0$, the map $H_1(M_K;\mathbb{C}[\mathbb{Z}]) \to H_1(W_1;\mathbb{C}[\mathbb{Z}]) \oplus H_1(W_2;\mathbb{C}[\mathbb{Z}])$ in the Mayer-Vietoris sequence is surjective. Hence $H_1(W_i;\mathbb{C}[\mathbb{Z}])$ is torsion for $i = 1, 2$. By Lemma 3.1 we deduce

$$\begin{aligned}
\beta &= \dim_{\mathbb{C}}(\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} H_1(M_K;\mathbb{C}[\mathbb{Z}])), \\
m_i &= \dim_{\mathbb{C}}(\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} H_1(W_i;\mathbb{C}[\mathbb{Z}])).
\end{aligned}$$

For each of the spaces $X = M_K, W_1, W_2$, Lemma 3.2 implies $\mathbb{C}_\omega \otimes_{\mathbb{C}[\mathbb{Z}]} H_1(X;\mathbb{C}[\mathbb{Z}]) \cong H_1(X;\mathbb{C}_\omega)$ so that we furthermore obtain

$$\begin{aligned}
\beta &= \dim_{\mathbb{C}} H_1(M_K;\mathbb{C}_\omega), \\
m_i &= \dim_{\mathbb{C}} H_1(W_i;\mathbb{C}_\omega).
\end{aligned}$$

By Example 2.6 we have $|\sigma_\omega(K)| \leq \dim_{\mathbb{C}} H_2(W_i;\mathbb{C}_\omega)$. However, recall that the image of $H_2(M_K;\mathbb{C}_\omega) \to H_2(W_i;\mathbb{C}_\omega)$ lies in the kernel of $\lambda_\omega(W_i)$, so that moreover

$$\begin{aligned}|\sigma_\omega(K)| &\leq \dim_{\mathbb{C}} H_2(W_i;\mathbb{C}_\omega) - \dim_{\mathbb{C}} \operatorname{Im}(H_2(M_K;\mathbb{C}_\omega) \to H_2(W_i;\mathbb{C}_\omega)) \\ &= \dim_{\mathbb{C}} H_2(W_i;\mathbb{C}_\omega) - \big(\dim_{\mathbb{C}} H_2(M_K;\mathbb{C}_\omega) - \dim_{\mathbb{C}} H_1(W_i;\mathbb{C}_\omega)\big) \\ &= (n_i + m_i) - (\beta - m_i) \\ &= n_i + 2m_i - \beta,\end{aligned}$$

where in the second line we have used Lemma 3.4. Taking the sum for $i = 1,2$ we obtain:

$$2|\sigma_\omega(K)| \leq n_1 + n_2 + 2m_1 + 2m_2 - 2\beta. \tag{$*$}$$

We saw in Lemma 3.3 that $H_1(M_K;\mathbb{C}_\omega) \to H_1(W_1;\mathbb{C}_\omega) \oplus H_1(W_2;\mathbb{C}_\omega)$ is surjective, so that

$$m_1 + m_2 \leq \beta.$$

It follows that $2m_1 + 2m_2 - 2\beta \leq 0$, so combining this with $(*)$ we have

$$2|\sigma_\omega(K)| \leq n_1 + n_2 + 2m_1 + 2m_2 - 2\beta \leq n_1 + n_2. \tag{$\dagger$}$$

Finally, we calculate the Euler characteristic for the section of the Mayer-Vietoris sequence of $V = W_1 \cup_{M_K} -W_2$ obtained in Lemma 3.3 as

$$0 = \beta - (n_1 + m_1 + n_2 + m_2) + 2g - \beta + (m_1 + m_2),$$

so that $2g = n_1 + n_2$. Substituting into $(\dagger)$ yields $2|\sigma_\omega(K)| \leq 2g$ and hence $|\sigma_\omega(K)| \leq g$.

Since this is true for all $\omega \in S^1 \setminus \{1\}$ and all pairs of slice surfaces that glue to be unknotted, the claimed result follows. □

4. Examples of band moves

A *ribbon surface* for a knot $K \subset S^3$ is a smoothly embedded surface $\Sigma \subset D^4$ with $\partial\Sigma = K$, such that the radial function $D^4 \to [0,1]$ restricts to a Morse function on Σ whose critical points are of index either 0 or 1.

Definition 4.1. The *ribbon surface band number* $b(K)$ of a knot K is the minimal number of index 1 critical points, among all ribbon surfaces Σ for K.

The following proposition, combined with Theorem 1.3 of McDonald, gives the promised upper bounds on g_{ds} that complete the proof of Theorem 1.2.

Proposition 4.2. *The ribbon surface band number $b(J) = 1$ for each of the knots*

$$J \in \left\{ \begin{array}{l} 8_{20},\ 10_{87},\ 10_{140},\ 11a28,\ 11a58,\ 11a165,\ 12a189,\ 12a377, \\ 12a979,\ 12n56,\ 12n57,\ \ 12n62,\ 12n66,\ 12n87,\ 12n106, \\ 12n288,\ 12n501,\ 12n504,\ 12n582,\ 12n670,\ 12n721 \end{array} \right\}.$$

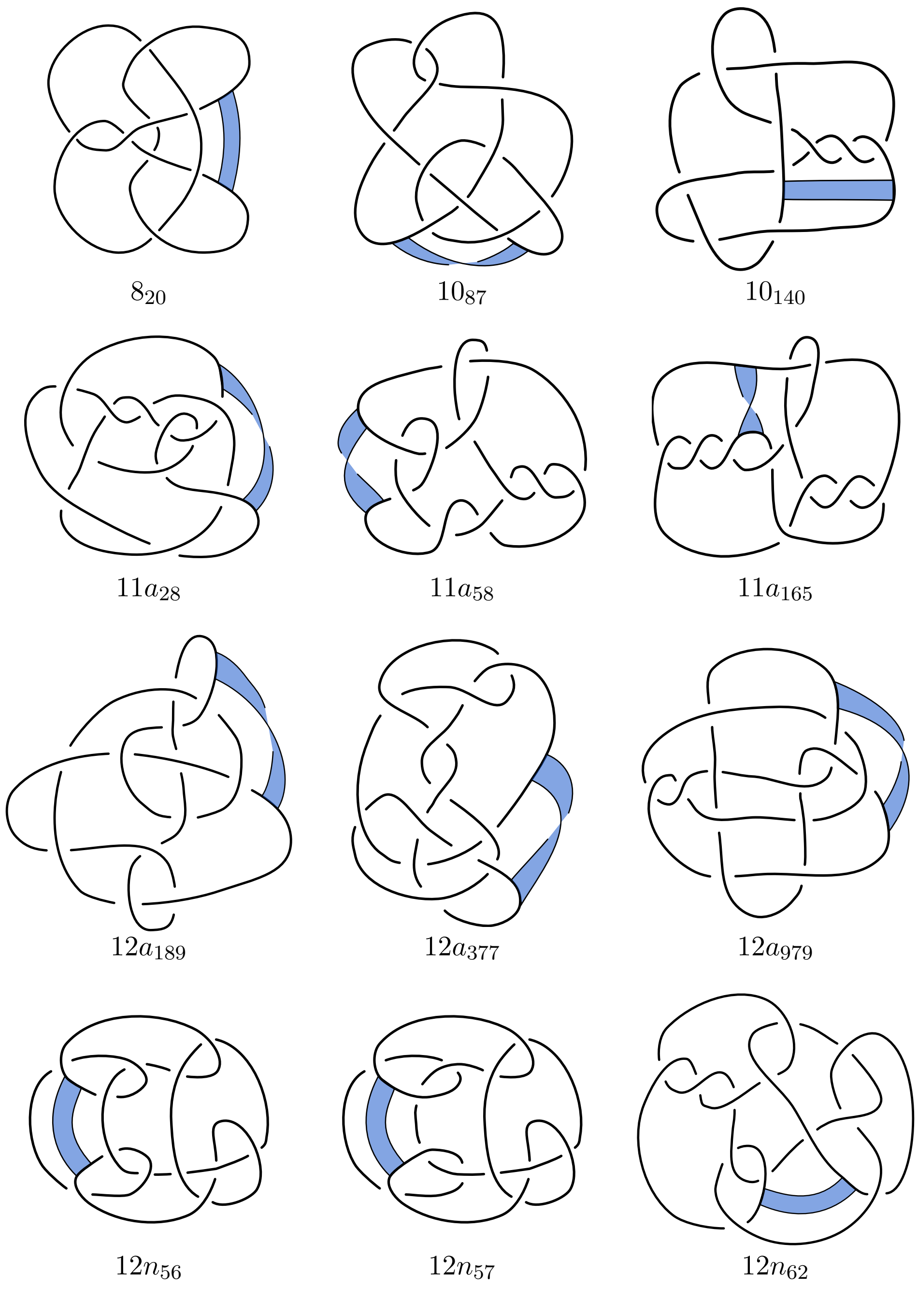

FIGURE 1. Band moves for the proof of Proposition 4.2.

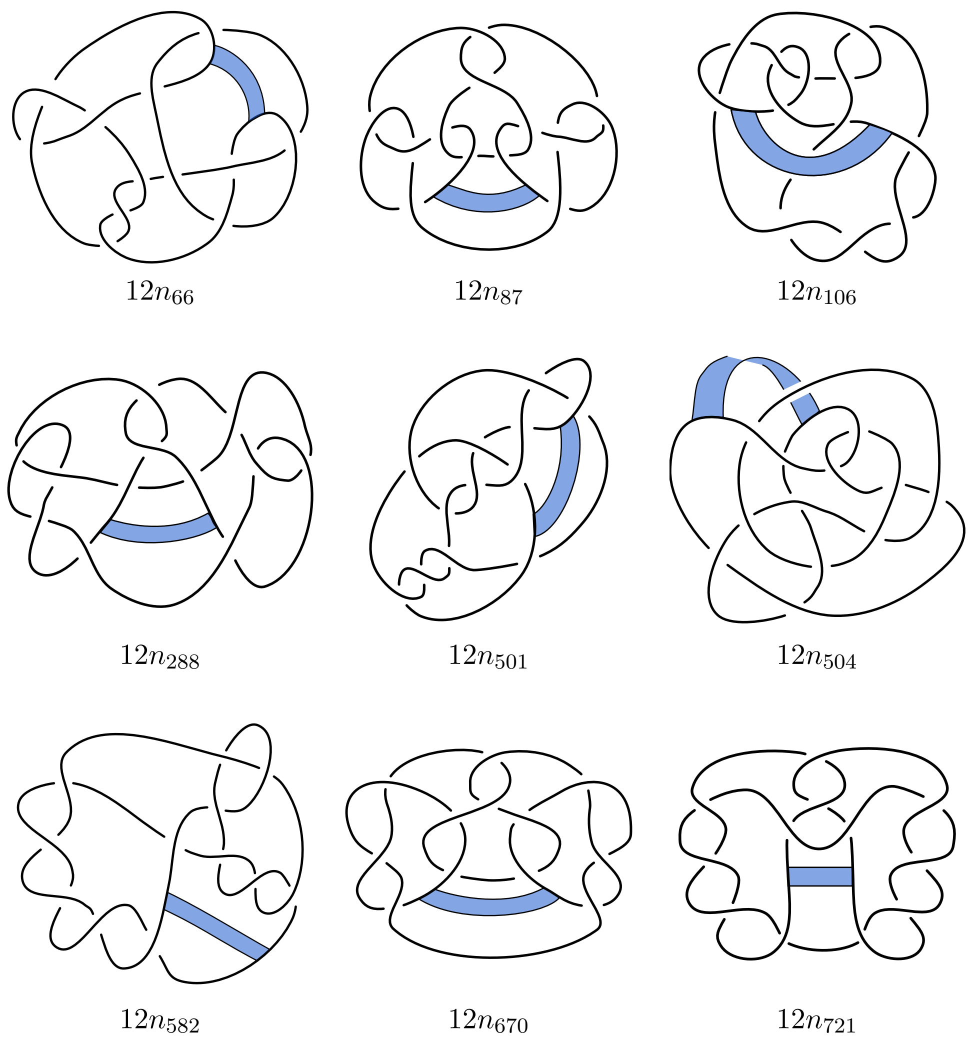

FIGURE 2. More band moves for the proof of Proposition 4.2.

Proof. It suffices to exhibit a single band move on J that produces a 2-component unlink. The required band moves are shown in the diagrams of Figure 1 and Figure 2. $\square$

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