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Genericity and rigidity for slow entropy transformations

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Abstract. The notion of slow entropy, both upper and lower slow entropy, was defined by Katok and Thouvenot as a more refined measure of complexity for dynamical systems, than the classical Kolmogorov-Sinai entropy. For any subexponential rate function $a_n(t)$, we prove there exists a generic class of invertible measure preserving systems such that the lower slow entropy is zero and the upper slow entropy is infinite. Also, given any subexponential rate $a_n(t)$, we show there exists a rigid, weak mixing, invertible system such that the lower slow entropy is infinite with respect to $a_n(t)$. This gives a general solution to a question on the existence of rigid transformations with positive polynomial upper slow entropy, Finally, we connect slow entropy with the notion of entropy convergence rate presented by Blume. In particular, we show slow entropy is a strictly stronger notion of complexity and give examples which have zero upper slow entropy, but also have an arbitrary sublinear positive entropy convergence rate.

Contents

1. Introduction

The notion of slow entropy was introduced by Katok and Thouvenot in [15] for amenable discrete group actions. It generalizes the classical notion of

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Kolmogorov-Sinai entropy [16, 19] for $\mathbb{Z}$-actions and gives a method for distinguishing the complexity of transformations with zero Kolmogorov-Sinai entropy.[1] The recent survey [14] gives a general account of several extensions of entropy, including a comprehensive background on slow entropy. Slow entropy has been computed for several examples including compact group rotations, Chacon-3 [7], the Thue-Morse system and the Rudin-Shapiro system. In [5], it is shown that the lower slow entropy of any rank-one transformation is less than or equal to 2. Also, in [6], it is shown there exist rank-one transformations with infinite upper slow entropy with respect to any polynomial. In [12], Kanigowski is able to get more precise upper bounds on slow entropy of local rank-one flows. Also, in [13], the authors obtain polynomial slow entropies for unipotent flows.

In [14], the following question is given:

Question 6.1.2. Is it possible to have the upper slow entropy for a rigid transformation positive with respect to $a_n(t) = n^t$?

We give a positive answer to this question. Given any subexponential rate, we show that a generic transformation has infinite upper slow entropy with respect to that rate. We say $a_n(t) > 0$, for $n \in \mathbb{N}$ and $t > 0$, is subexponential, if given $\beta > 1$ and $t > 0$, $\lim_{n\to\infty} \frac{a_n(t)}{\beta^n} = 0$. We will only consider monotone $a_n(t)$ such that $a_n(t) \geq a_n(s)$ for $t > s$. Let $(X, \mathcal{B}, \mu)$ be a standard probability space (i.e., isomorphic to $[0, 1]$ with Lebesgue measure). Also, let

$$\mathcal{M} = \{T : X \to X \mid T \text{ is invertible and preserves } \mu\}.$$

One of our three main results is the following.

Theorem 1.1. Let $a_n(t)$ be any subexponential rate function. There exists a dense G_δ subset $G \subset \mathcal{M}$ such that for each $T \in G$, the upper slow entropy of T is infinite with respect to $a_n(t)$.

Thus, the generic transformation answers question 6.1.2 in the affirmative, since the generic transformation is known to be weak mixing and rigid [10]. Our proof is constructive and provides a recipe for constructing rigid rank-ones with infinite upper slow entropy.

We show that there is a generic class of transformations such that the lower slow entropy is zero with respect to a given divergent rate.

Theorem 1.2. Suppose $a_n(t) \in \mathbb{R}$ is a rate such that for $t > 0$,

$$\lim_{n\to\infty} a_n(t) = \infty.$$

There exists a dense G_δ subset $G \subset \mathcal{M}$ such that for each $T \in G$, the lower slow entropy of T is zero with respect to $a_n(t)$.

[1] The Kolmogorov-Sinai entropy of a transformation T is referred to as the entropy of T.

This shows for any slow rate $a_n(t)$, the generic transformation has infinitely occurring time spans where the complexity is sublinear. This is due to "super" rigidity times for a typical tranformation. This raises the question of whether there exists an invertible rigid measure preserving transformation with infinite polynomial lower slow entropy. We answer this question by constructing examples with infinite subexponential lower slow entropy in section 5. This also answers question 6.1.2.

Theorem 1.3. There exists a family $\mathcal{F} \subset \mathcal{M}$ of rigid, weak mixing transformations such that given any subexponential rate $a_n(t)$, there exists a transformation in $\mathcal{F}$ which has infinite lower slow entropy with respect to $a_n(t)$.

In the final section, we give the connections with entropy convergence rate as defined by Frank Blume in [2].

2. Preliminaries

We describe the setup and then give a few lemmas used in the proofs of our main results.

2.1. Definitions. Given an alphabet $\alpha_1, \alpha_2, \ldots, \alpha_r$, a codeword of length n is a vector $w = \langle w_1, w_2, \ldots, w_n \rangle = \langle w_i \rangle_{i=1}^n$ such that $w_i \in \{\alpha_1, \ldots, \alpha_r\}$ for $1 \leq i \leq n$. Our codewords will be obtained from a measure preserving system $(X, \mathcal{B}, \mu, T)$ and finite partition $P = \{p_1, p_2, \ldots, p_r\}$. In this case, we will consider the alphabet to be $\{1, \ldots, r\}$. Given $x \in X$ and $n \in \mathbb{N}$, define the codeword $\vec{P}_n(x) = \langle w_i \rangle_{i=1}^n$ such that $T^{i-1}x \in p_{w_i}$. When using this notation, the transformation will be fixed.

Let w, w' be codewords of length n. The (normalized) Hamming distance is defined as:

$$d(w, w') = \frac{1}{n} \sum_{i=1}^{n} \left(1 - \delta_{w_i w_i'}\right).$$

Given a codeword w of length n and $\varepsilon > 0$, an ε-ball is the subset $V \subseteq \{1, \ldots, r\}^n$ such that $d(w, v) < \varepsilon$ for $v \in V$. We will denote the ε-ball as $B_\varepsilon(w)$. If given a transformation T and partition P, define

$$B_\varepsilon^{T,P}(w) = \{x \in X : d(\vec{P}_n(x), w) < \varepsilon\}.$$

Given $\varepsilon > 0$, $\delta > 0$, $n \in \mathbb{N}$, finite partition $P = \{p_1, p_2, \ldots, p_r\}$ and dynamical system $(X, \mathcal{B}, \mu, T)$, define $S_P(T, n, \varepsilon, \delta) = S$ as:

$$S = \min\left\{k : \exists v_1, \ldots, v_k \in \{1, \ldots, r\}^n \text{ such that } \mu\Big(\bigcup_{i=1}^{k} B_\varepsilon^{T,P}(v_i)\Big) \geq 1 - \delta\right\}.$$

Now we give the definition of upper and lower slow entropy for $\mathbb{Z}$-actions. For more general discrete amenable group actions, the interested reader may see the survey [14]. Also, in [11], slow entropy is used to construct infinite-measure preserving $\mathbb{Z}^2$-actions which cannot be realized as a group of diffeomorphisms of a compact manifold preserving a Borel measure. Let T be an invertible measure preserving transformation defined on a standard probability space $(X, \mathcal{B}, \mu)$. Let $a = \{a_n(t) : n \in \mathbb{N}, t > 0\}$ be a family of positive sequences monotone in t and such that $\lim_{n\to\infty} a_n(t) = \infty$ for $t > 0$. Define the upper (measure-theoretic) slow entropy of T with respect to a finite partition P as

$$\text{s-}\overline{\mathcal{H}}_a^{\mu}(T, P) = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \ \text{s-}\overline{\mathcal{H}}\big(\varepsilon, \delta, P\big),$$

$$\text{where} \ \ \text{s-}\overline{\mathcal{H}}\big(\varepsilon, \delta, P\big) = \begin{cases} \sup \overline{\mathcal{G}}(\varepsilon, \delta, P), & \text{if } \ \overline{\mathcal{G}}\big(\varepsilon, \delta, P\big) \neq \emptyset, \\ 0, & \text{if } \ \overline{\mathcal{G}}\big(\varepsilon, \delta, P\big) = \emptyset, \end{cases}$$

$$\text{and} \ \ \overline{\mathcal{G}}\big(\varepsilon, \delta, P\big) = \{t > 0 : \limsup_{n\to\infty} \frac{S_P(T, n, \varepsilon, \delta)}{a_n(t)} > 0\}.$$

The upper slow entropy of T with respect to $a_n(t)$ is defined as

$$\text{s-}\overline{\mathcal{H}}_a^{\mu}(T) = \sup_P \ \text{s-}\overline{\mathcal{H}}_a^{\mu}(T, P).$$

To define the lower slow entropy of T, replace $\limsup$ in the definition above with $\liminf$.

2.2. Supporting lemmas. Define the binary entropy function,

$$\mathcal{H}(x) = -x \log_2 (x) - \big(1 - x\big) \log_2 (1 - x).$$

We give some preliminary lemmas involving binary codewords and measurable partitions that are used in the main results.

Lemma 2.1. Suppose w_1, w_2 are binary words of length ℓ with Hamming distance $d = d(w_1, w_2) > 0$. Let $m \in \mathbb{N}$ and C be the set of all 2^m codewords consisting of all possible sequences of words w from $\{w_1, w_2\}$ of length $n = m\ell$. Given $\varepsilon, \theta > 0$ with $\frac{\varepsilon}{d} + \frac{1}{m} < \frac{1}{2}$, the minimum number S of ε-balls required to cover $1 - \theta$ of the words in C satisfies:

$$S \geq \big(1 - \theta\big) 2^{m\left(1 - \mathcal{H}(\frac{2\varepsilon}{d} + \frac{1}{m})\right)}.$$

Proof. The proof follows from a standard bound on the size of Hamming balls [17] (p.310). Suppose $v_1, \ldots, v_j$ are a minimum number of centers such that ε-balls $B_\varepsilon(v_i)$ cover at least $1 - \theta$ of codewords in C. For each i, choose $u_i \in B_\varepsilon(v_i)$. Thus, $B_{2\varepsilon}(u_i) \supseteq B_\varepsilon(v_i)$ and the 2ε-balls $B_{2\varepsilon}(u_i)$ cover at least $1 - \theta$ of the codewords in C.

This reduces the problem to a basic Hamming ball size question. Since all words are generated by w_1, w_2, we can map w_1 to 0 and w_2 to 1, and consider the number of Hamming balls needed to cover $1 - \theta$ of all binary words of length m. Thus, if at least $\lceil \frac{2\varepsilon}{d} m \rceil$ words differ, then the distance is greater than or equal to 2ε. Also, $\lceil \frac{2\varepsilon}{d} m \rceil \leq m\big(\frac{2\varepsilon}{d} + \frac{1}{m}\big)$. By [17](p.310), a Hamming ball of radius $\frac{2\varepsilon}{d}$ has a volume less than or equal to:

$$2^{m\mathcal{H}(\frac{2\varepsilon}{d} + \frac{1}{m})}.$$

Therefore, the minimum number of balls required to cover at least $(1-\theta)$ of the space is:

$$\big(1-\theta\big)2^{m\left(1-\mathcal{H}(\frac{2\varepsilon}{d}+\frac{1}{m})\right)}.$$

$\square$

Lemma 2.2. Suppose the setup is similar to Lemma 2.1 and there are two generating words w_1, w_2 of length ℓ with distance $d = d(w_1, w_2)$. Suppose C is the set of 2^m codewords consisting of all possible sequences of blocks of either w_1 or w_2. Suppose μ is a probability measure and A is a set of positive measure. [2] Let $\phi : A \to C$ be a measurable map such that $\mu(\{x \in A : \phi(x) = v\}) = \frac{\mu(A)}{2^m}$ for $v \in C$. Suppose $\psi : A \to C$ is a measurable map satisfying:

$$\mu\Big(\{x \in A : d(\psi(x), \phi(x)) < \eta\}\Big) > \big(1-\eta\big)\mu(A).$$

The minimum number S of ε-Hamming balls B such that

$$\mu\big(\{x \in A : \psi(x) \in B\}\big) \geq \big(1-\theta\big)\mu(A)$$

satisfies

$$S \geq \big(1-\theta-\eta\big)2^{m(1-\mathcal{H}(\frac{2(\varepsilon+\eta)}{d}+\frac{1}{m}))}.$$

Proof. Let $E = \{x \in A : d(u(x), v(x)) \geq \eta\}$. For $x \in A \setminus E$, $B_\varepsilon(\psi(x)) \subseteq B_{\varepsilon+\eta}(\phi(x))$. By applying Lemma 2.1 using the normalized probability measure $\mu(\cdot)/\mu(A)$,

$$\Big(1-\theta-\eta\Big)2^{m(1-\mathcal{H}(\frac{2(\varepsilon+\eta)}{d}+\frac{1}{m}))}$$

$(\varepsilon+\eta)$-balls are needed to cover $1-\theta-\eta$ of $\phi(x)$ words. Thus, the total number of ε-balls needed to cover $(1-\theta)$ mass of $\psi(x)$ words is at least:

$$\Big(1-\theta-\eta\Big)2^{m(1-\mathcal{H}(\frac{2(\varepsilon+\eta)}{d}+\frac{1}{m}))}.$$

$\square$

The following lemma is used in the proof of Proposition 4.2.

Lemma 2.3. Let $\eta > 0$ and $r, n \in \mathbb{N}$. Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system and b a set of positive measure such that $\hat{b} = \bigcup_{i=0}^{n-1} T^i b$ is a disjoint union (except for a set of measure zero). Suppose $P = \{p_1, p_2, \ldots, p_r\}$ and $Q = \{q_1, q_2, \ldots, q_r\}$ are partitions such that

$$\sum_{i=1}^{r} \mu\big((p_i \cap \hat{b}) \triangle (q_i \cap \hat{b})\big) < \eta^2 \mu(\hat{b}). \tag{2.1}$$

Then for $n \in \mathbb{N}$,

$$\mu\big(\{x \in b : d(\vec{P}_n(x), \vec{Q}_n(x)) < \eta\}\big) > \big(1-\eta\big)\mu(b).$$

[2]For our purposes, $A \subseteq X$ where $(X, \mathcal{B}, \mu)$ is a Lebesgue probability space.

Proof. Define

$$R_n = \bigvee_{i=0}^{n-1} T^{-i}\big(P \vee Q\big).$$

Define

$$A = \{p \in R_n \cap b : \#\{i : 0 \le i < n, T^i p \subset \bigcup_{j=1}^{r} \big(p_j \cap q_j\big)\} \le (1-\eta)n\}.$$

We show $\mu(A) < \eta\mu(b)$. Otherwise, for $p \in A$ and i such that $T^i p \subseteq p_j \cap q_k$ for $j \neq k$, this contributes $2\mu(p)$ to the sum (2.1). Thus, for $p \in A$, the number of such i gives measure greater than or equal to $2\mu(p)\eta n$. Adding up over all $p \in A$ gives measure greater than or equal to $2\eta\mu(b)\eta n > \eta^2\mu(\hat{b})$. For a.e. $x, y \in b \cap A^c$, $d(\vec{P}_n(x), \vec{Q}_n(x)) < \eta$ and this holds for $\mu(b \cap A^c) > (1-\eta)\mu(b)$. □

The following lemma is a more general version of Lemma 2.3 and used in multiple places throughout this paper. Given two ordered partitions $P = \langle p_1, p_2, \ldots, p_r\rangle$ and $Q = \langle q_1, q_2, \ldots, q_r\rangle$, let

$$D(P,Q) = \sum_{i=1}^{r} \mu(p_i \triangle q_i).$$

Lemma 2.4. Let $(X, \mathcal{B}, \mu, T)$ be ergodic. Let $\eta > 0$ and $r, n \in \mathbb{N}$. Suppose $P = \langle p_1, p_2, \ldots, p_r\rangle$ and $Q = \langle q_1, q_2, \ldots, q_r\rangle$ are ordered partitions such that

$$D(P,Q) < \eta^2. \tag{2.2}$$

Then for $n \in \mathbb{N}$,

$$\mu\big(\{x : d(\vec{P}_n(x), \vec{Q}_n(x)) < \eta\}\big) > 1 - \eta.$$

Proof. Let $\eta_1 \in \mathbb{R}$ such that

$$\sum_{i=1}^{r} \mu(p_i \triangle q_i) < \eta_1^2 < \eta^2. \tag{2.3}$$

Define

$$R_n = \bigvee_{i=0}^{n-1} T^{-i}\big(P \vee Q\big).$$

Let $\eta_0 < \frac{1}{2}$ and $C = \{I_0, \ldots, I_{n-1}\}$ be a Rohklin tower such that $\mu(\bigcup_{i=0}^{n-1} I_i) > 1 - \frac{\eta_0}{n}$. Define

$$A_0 = \{p \in R_n \cap I_0 : \#\{i : 0 \le i < n, T^i p \subset \cup_{j=1}^{r}\big(p_j \cap q_j\big)\} \le (1-\eta_1)n\}.$$

We show $\mu(A_0) < \eta_1\mu(I_0)$. Otherwise, for each i such that $T^i p \subseteq p_j \cap q_k$ for $j \neq k$, this contributes $2\mu(p)$ to the sum (2.3). Thus, for $p \in A_0$, the number of such i gives measure greater than $2\mu(p)\eta_1 n$. Adding up over all $p \in A_0$ gives measure greater than $2\eta_1\mu(I_0)\eta_1 n > 2\eta_1^2(1-\eta_0)$. For $x, y \in I_0 \cap A_0^c$,

$d(\vec{P}_n(x), \vec{Q}_n(x)) < \eta_1$ and this holds for $\mu(I_0 \cap A_0^c) > (1-\eta_1)\mu(I_0)$. By showing the analogous result for A_k defined as:

$$A_k = \{p \in R_n \cap I_k : \#\{i : 0 \leq i < n, T^i p \subset \cup_{j=1}^r (p_j \cap q_j)\} \leq (1-\eta_1)n\},$$

then $\mu(I_k \cap A_k^c) > (1-\eta_1)\mu(I_k)$. Hence,

$$\sum_{k=0}^{n-1} \mu(I_k \cap A_k^c) > (1-\eta_1)(1-\eta_0).$$

Therefore, since η_0 may be chosen arbitrarily small, our claim holds. □

2.3. Infinite rank. A result of Ferenczi [5] shows that the lower slow entropy of a rank-one transformation is less than or equal to 2 with respect to $a_n(t) = n^t$. Thus, our examples in section 5.4 are not rank-one and instead, have infinite rank. We will adapt the technique of independent cutting and stacking to construct rigid transformations with infinite lower slow entropy. Independent cutting and stacking was originally defined in [9,18]. A variation of this technique is used in [15] to obtain different types of important counterexamples. For a general guide on the cutting and stacking technique, see [8].

3. Generic class with zero lower slow entropy

Let $a_n(t)$ be a sequence of real numbers such that $a_n(t) \geq a_n(s)$ for $t > s$ and $\lim_{n\to\infty} a_n(t) = \infty$ for $t > 0$. For $N, t, M \in \mathbb{N}$ and any finite partition P, define

$$\underline{G}(N,t,M,P) = \left\{ T \in \mathcal{M} : \exists n > N, 0 < \delta < \frac{1}{M} \text{ such that } S_P(T,n,\delta,\delta) < \frac{a_n(\frac{1}{t})}{N} \right\}. \quad (3.1)$$

Proposition 3.1. For $N, t, M \in \mathbb{N}$ and finite partition P, $\underline{G}(N,t,M,P)$ is open in the weak topology on $\mathcal{M}$.

Proof. Let $T_0 \in \underline{G}(N,t,M,P)$ and $n > N$, $0 < \delta_0 < \frac{1}{M}$ be such that $S_P(T_0, n, \delta_0, \delta_0) < \frac{a_n(\frac{1}{t})}{N}$. Let $P_n = \bigvee_{i=0}^{n-1} T_0^{-i} P$. Choose $\delta_1 \in \mathbb{R}$ such that $\delta_0 < \delta_1 < \frac{1}{M}$. Let $\alpha = (\delta_1 - \delta_0)/2$. In the weak topology, choose an open set U containing T_0 such that for $T_1 \in U$, $0 \leq i < n$, and $p \in P_n$,

$$\mu(T_1 T_0^i p \triangle T_0^{i+1} p) \leq \left(\frac{\alpha}{n^2}\right)\mu(p). \quad (3.2)$$

We will prove inductively in j, for $p \in P_n$, that

$$\mu(T_0^j p \triangle T_1^j p) \leq \left(\frac{j\alpha}{n^2}\right)\mu(p). \quad (3.3)$$

The case $j = 1$ follows directly from (3.2):

$$\mu(T_0 p \triangle T_1 p) \leq \left(\frac{\alpha}{n^2}\right)\mu(p).$$

Also, the case $j = 0$ is trivial. Suppose equation (3.3) holds for $j = i$. Below shows it holds for $j = i + 1$:

$$\begin{aligned}\mu(T_0^{i+1}p\triangle T_1^{i+1}p) &\leq \mu(T_0^{i+1}p\triangle T_1T_0^ip) + \mu(T_1T_0^ip\triangle T_1^{i+1}p)\\ &= \mu(T_0^{i+1}p\triangle T_1T_0^ip) + \mu(T_0^ip\triangle T_1^ip)\\ &\leq (\frac{\alpha}{n^2})\mu(p) + (\frac{i\alpha}{n^2})\mu(p)\\ &= (\frac{(i+1)\alpha}{n^2})\mu(p).\end{aligned}$$

For $p \in P_n$, let

$$E_p = \bigcap_{i=0}^{n-1} T_1^{-i}T_0^ip.$$

Thus,

$$\begin{aligned}\mu\big(E_p\big) &\geq \mu(p) - \sum_{i=1}^{n-1}\mu(p\triangle T_1^{-i}T_0^ip)\\ &= \mu(p) - \sum_{i=1}^{n-1}\mu(T_1^ip\triangle T_0^ip)\\ &> (1-\alpha)\mu(p).\end{aligned}$$

Hence, if $E = \bigcup_{p\in P_n} E_p$, $\mu(E) > 1 - \alpha$. Each $x \in E$ has the same P-name under T_1 and T_0. Suppose $V \subseteq 2^{\{0,1\}^n}$ is such that $A_0 = \bigcup_{v\in V} B_{\delta_0}^{T_0}(v)$ satisfies $\mu(A_0) \geq 1 - \delta_0$ and

$$\operatorname{card}(V) < \frac{a_n(\frac{1}{t})}{N}.$$

Let $A_1 = \bigcup_{v\in V} B_{\delta_1}^{T_1}(v)$ and $A_1' = \bigcup_{v\in V} B_{\delta_0}^{T_1}(v)$. Since $\mu(A_0\triangle A_1') \leq \mu(E^c) < \alpha$, then

$$\mu(A_1) \geq \mu(A_1') > 1 - \delta_0 - \alpha > 1 - \delta_1.$$

Therefore, since $\operatorname{card}(V) < \frac{a_n(1/t)}{N}$, $S_P(T_1, n, \delta_1, \delta_1) < \frac{a_n(\frac{1}{t})}{N}$ and we are done. $\square$

Now we prove the density of the class $\underline{G}(N, t, M, P)$.

Proposition 3.2. For $N, t, M \in \mathbb{N}$ and finite partition P, $\underline{G}(N, t, M, P)$ is dense in the weak topology on $\mathcal{M}$.

Proof. Let $P = \{p_1, p_2, \ldots, p_r\}$ be the partition into r elements for $r \in \mathbb{N}$. We can discard elements with zero measure. Since rank-ones are dense in $\mathcal{M}$, let $T_0 \in \mathcal{M}$ be a rank-one transformation and let $\epsilon > 0$. Let $\delta < \frac{1}{M}$ and define $\eta = \min\{\delta^2, \epsilon\}$. Choose a rank-one column $\mathcal{C} = \{I_0, I_1, \ldots, I_{h-1}\}$ for T_0 such that

(1) $\mu(\bigcup_{i=0}^{h-1} I_i) > 1 - \frac{\eta}{2}$,

(2) $h > \frac{2}{\eta}$,

(3) there exist disjoint collections J_i such that $\mu(p_i \triangle \bigcup_{j\in J_i} I_j) < \frac{\eta}{4r}\mu(p_i)$.

Let $q_i = \bigcup_{j\in J_i} I_j$ and $Q = \{q_1, q_2, \ldots, q_r\}$. Now we show how to construct a transformation $T_1 \in \underline{G}(N,t,M,P)$. Since T_1 will differ by T_0 inside the top level or outside the column, then T_1 will be within ϵ of T_0. Choose $k_1 \in \mathbb{N}$ such that $k_1 h > N$ and for $n = k_1 h$,

$$a_n(\frac{1}{t}) > Nh.$$

Choose $k_2 \in \mathbb{N}$ such that $k_2 > \frac{2}{\eta}$. Cut column $\mathcal{C}$ into k_1k_2 columns of equal width and stack from left to right. Call this column $\mathcal{C}'$ which has height k_1k_2h. Let $A_1 = \{x : d(\vec{P}_n(x), \vec{Q}_n(x)) < \frac{\delta}{2}\}$. By Lemma 2.4, since

$$\sum_{i=1}^{r} \mu(p_i \triangle q_i) < \frac{\eta}{4} \leq \frac{\delta^2}{4},$$

then

$$\mu(A_1) > 1 - \frac{\delta}{2}.$$

Let A_2 be the union of levels in $\mathcal{C}'$ except for the top n levels. For $x \in A_2$, $\vec{Q}_n(x)$ gives at most h distinct vectors. Also, δ-balls centered at these words will cover $A_1 \cap A_2$. Precisely,

$$\bigcup_{x\in A_2} B_\delta^{T_1,Q}(\vec{Q}_n(x)) \supseteq A_1 \cap A_2.$$

Since $\mu(A_1 \cap A_2) > 1 - \delta$, then $S_P(T_1, n, \delta, \delta) \leq h < \frac{a_n(1/t)}{N}$. Therefore, we are done. $\square$

Theorem 3.3. Suppose $a_n(t) \in \mathbb{R}$ is such that for $t > s$, $a_n(t) \geq a_n(s)$ and for $t > 0$, $\lim_{n\to\infty} a_n(t) = \infty$. There exists a dense G_δ subset $G \subset \mathcal{M}$ such that for each $T \in G$, the lower slow entropy of T is zero with respect to $a_n(t)$.

Proof. Let P_L be a sequence of nontrivial measurable partitions such that for each $k \in \mathbb{N}$, the collection $\{P_L : L \in \mathbb{N}\}$ is dense in the class of all measurable partitions with k nontrivial elements. By Proposition 3.2, for $N,t,M,L \in \mathbb{N}$, the set $\underline{G}(N,t,M,P_L)$ is dense, and also open by Proposition 3.1. Thus,

$$G = \bigcap_{L=1}^{\infty}\bigcap_{t=1}^{\infty}\bigcap_{M=1}^{\infty}\bigcap_{N=1}^{\infty} \underline{G}(N,t,M,P_L)$$

is a dense G_δ. Given a nontrivial measurable partition P and $t, M \in \mathbb{N}$, choose $L \in \mathbb{N}$ such that

$$D(P, P_L) < \frac{1}{9M^2}.$$

For $T \in \bigcap_{N=1}^{\infty} \underline{G}(N, t, 3M, P_L)$,

$$\liminf_{n\to\infty} \frac{S_{P_L}(T, n, \frac{1}{3M}, \frac{1}{3M})}{a_n(\frac{1}{t})} = 0.$$

By Lemma 2.4, $S_P(T, n, \frac{1}{M}, \frac{1}{M}) \leq S_{P_L}(T, n, \frac{1}{3M}, \frac{1}{3M})$. Therefore, for $T \in G$, the lower slow entropy is zero with respect to $a_n(t)$. $\square$

Corollary 3.4. In the weak topology, the generic transformation in $\mathcal{M}$ is rigid, weak mixing, rank-one and has zero polynomial lower slow entropy.

4. Generic class with infinite upper slow entropy

The transformations in this section are constructed by including alternating stages of cutting and stacking. Suppose T is representated by a single Rokhlin column $\mathcal{C}$ of height h.

4.1. Two approximately independent words. Cut column $\mathcal{C}$ into two subcolumns $\mathcal{C}_1$ and $\mathcal{C}_2$ of equal width. Given $k \in \mathbb{N}$, cut $\mathcal{C}_1$ into k subcolumns of equal width, stack from left to right, and place k spacers on top. Cut $\mathcal{C}_2$ into k subcolumns of equal width and place a single spacer on top of each subcolumn, then stack from left to right. After this stage, there are two columns of height $k(h+1)$.

4.2. Independent cutting and stacking. Independent cutting and stacking is defined similar to [18]. As opposed to [18], here it is not necessary to use columns of different heights, since weak mixing is generic and we are establishing a generic class of transformations. Also, in section 5, we include a weak mixing stage which allows all columns to have the same height and facilitates counting of codewords. Given two columns $\mathcal{C}_1$ and $\mathcal{C}_2$ of height h, and $s \in \mathbb{N}$, independent cutting and stacking the columns s times produces 2^{2^s} columns, each with height $2^s h$.

4.3. Infinite upper slow entropy. Let $P = \{p_1, p_2\}$ be a nontrivial measurable 2-set partition. We construct a dense G_δ for the case where $\mu(p_1) = \frac{1}{2}$, although a similar procedure will handle the more general case where $0 < \mu(p_1) < 1$. Let $a_n(t)$ be a sequence of real numbers with subexponential growth. In particular, for every $t, \beta > 1$, $\lim_{n\to\infty} \frac{a_n(t)}{\beta^n} = 0$. For $M, N, t \in \mathbb{N}$, define

$$\overline{G}(M, N, t, P) = \\ \{T \in \mathcal{M} : \exists n > N \text{ and } \delta > \frac{1}{M} \text{ such that } S_P(T, n, \delta, \delta) > a_n(t)\}. \quad (4.1)$$

Proposition 4.1. For $M, N, t \in \mathbb{N}$, the set $\overline{G}(M, N, t, P)$ is open in the weak topology on $\mathcal{M}$.

Proof. Let $T_0 \in \overline{G}(M, N, t, P)$ and $n > N$, $\delta_0 > \frac{1}{M}$ be such that

$$S_P(T_0, n, \delta_0, \delta_0) > a_n(t).$$

Let $P_n = \bigvee_{i=0}^{n-1} T_0^{-i} P$. The elements of P_n of positive measure correspond to the various P-names of length n. For almost every $x, y \in X$, x and y have the same P-name under T_0, if and only if $x, y \in p$ for some $p \in P_n$. Choose $\delta_1 \in \mathbb{R}$ such that $\frac{1}{M} < \delta_1 < \delta_0$. Let $\alpha = \big(\delta_0 - \delta_1\big)/2$. In the weak topology, choose an open set U containing T_0 such that for $T_1 \in U$, $0 \le i < n$, and $p \in P_n$,

$$\mu(T_1 T_0^i p \triangle T_0^{i+1} p) \le \big(\frac{\alpha}{n^2}\big)\mu(p). \tag{4.2}$$

We will prove inductively in j, for $p \in P_n$, that

$$\mu(T_0^j p \triangle T_1^j p) \le \big(\frac{j\alpha}{n^2}\big)\mu(p). \tag{4.3}$$

The case $j = 1$ follows directly from (4.2):

$$\mu(T_0 p \triangle T_1 p) \le \big(\frac{\alpha}{n^2}\big)\mu(p).$$

Also, the case $j = 0$ is trivial. Suppose equation (4.3) holds for $j = i$. Below shows it holds for $j = i + 1$:

$$\begin{aligned}
\mu(T_0^{i+1} p \triangle T_1^{i+1} p) &\le \mu(T_0^{i+1} p \triangle T_1 T_0^i p) + \mu(T_1 T_0^i p \triangle T_1^{i+1} p) \\
&= \mu(T_0^{i+1} p \triangle T_1 T_0^i p) + \mu(T_0^i p \triangle T_1^i p) \\
&\le \big(\frac{\alpha}{n^2}\big)\mu(p) + \big(\frac{i\alpha}{n^2}\big)\mu(p) \\
&= \big(\frac{(i+1)\alpha}{n^2}\big)\mu(p).
\end{aligned}$$

For $p \in P_n$, let

$$E_p = \bigcap_{i=0}^{n-1} T_1^{-i} T_0^i p.$$

Thus,

$$\begin{aligned}
\mu\big(E_p\big) &\ge \mu(p) - \sum_{i=1}^{n-1} \mu(p \triangle T_1^{-i} T_0^i p) \\
&= \mu(p) - \sum_{i=1}^{n-1} \mu(T_1^i p \triangle T_0^i p) \\
&> (1 - \alpha)\mu(p).
\end{aligned}$$

Hence, if $E = \bigcup_{p \in P_n} E_p$, $\mu(E) > 1 - \alpha$. Each $x \in E$ has the same P-name under T_1 and T_0. Suppose $V \subseteq 2^{\{0,1\}^n}$ is such that $A_1 = \bigcup_{v \in V} B_{\delta_1}^{T_1}(v)$

satisfies $\mu(A_1) \geq 1-\delta_1$. Let $A_0 = \bigcup_{v\in V} B_{\delta_0}^{T_0}(v)$ and $A_0' = \bigcup_{v\in V} B_{\delta_1}^{T_0}(v)$. Since $\mu(A_1 \triangle A_0') \leq \mu(E^c) < \alpha$, then

$$\mu(A_0) \geq \mu(A_0') > 1-\delta_1-\alpha > 1-\delta_0.$$

Therefore, since $\mu(A_0) \geq 1-\delta_0$, then $\text{card}(V) > a_n(t)$ and we are done. $\square$

Our density result follows.

Proposition 4.2. For sufficiently large M, and all $N, t \in \mathbb{N}$, the set $\overline{G}(M,N,t,P)$ is dense in the weak topology on $\mathcal{M}$.

Proof. It will be sufficient to consider $M \geq 1000$. This will allow us to choose $\delta < \frac{1}{900}$. Also, d in Lemma 2.2 can be chosen $d \geq \frac{1}{100}$. The value m in Lemma 2.2 will equal 2^r for some r. It will not be difficult to choose r such that $2^r > 100$. Also, let $0 < \eta < 10^{-5}$. Thus,

$$\mathcal{H}\big(\frac{2(\delta+\eta)}{d} + 2^{-r}\big) < \mathcal{H}\big(\frac{2}{9} + \frac{1}{500} + \frac{1}{100}\big) < \mathcal{H}\big(\frac{1}{4}\big) < \frac{7}{8}.$$

Since rank-ones are dense in $\mathcal{M}$, let $T_0 \in \mathcal{M}$ be a rank-one transformation. Let $\epsilon > 0$. We will show there exists T_1 within ϵ of T_0 in the weak topology. It is sufficient to construct T_1 such that $\mu(\{x : T_1(x) \neq T_0(x)\}) < \epsilon$. We can reset $\epsilon = \min\{\epsilon, \eta\}$. Let $N, t \in \mathbb{N}$ and $p = p_1$. Since T_0 is ergodic, we can choose $K \in \mathbb{N}$ such that for $k \geq K$,

$$\int_X |\frac{1}{k}\sum_{i=0}^{k-1} I_p(T_0^i x) - \mu(p)| d\mu < \frac{\eta^2}{6}. \tag{4.4}$$

Choose a rank-one column $\mathcal{C} = \{I_0, I_1, \ldots, I_h\}$ for T_0 such that

(1) $\mu(\bigcup_{i=0}^{h-2} I_i) > 1-\epsilon$,
(2) $h > N$,
(3) there exists a collection J such that $\mu(p \triangle \bigcup_{j\in J} I_j) < \frac{\eta^2}{6}\mu(p)$.

Let $q = \bigcup_{j\in J} I_j$ and $Q = \{q, q^c\}$. Since $\mu(p) = 1/2$, we can assume $1/3 < \mu(q) < 2/3$. Thus,

$$\begin{aligned}\int_X |\frac{1}{k}\sum_{i=0}^{k-1} I_q(T_0^i x) - \mu(q)| d\mu &\leq \int_X |\frac{1}{k}\sum_{i=0}^{k-1} \big(I_q(T_0^i x) - I_p(T_0^i x)\big)| d\mu \\ &+ \int_X |\frac{1}{k}\sum_{i=0}^{k-1} I_p(T_0^i x) - \mu(p)| d\mu + |\mu(p) - \mu(q)| \\ &< \mu(q \triangle p) + \frac{\eta^2}{6} + \frac{\eta^2}{6} < \frac{\eta^2}{2}.\end{aligned}$$

The transformation will apply independent cutting and stacking using two words who have a significant distance. The words are pure with respect to a subset of intervals $I_j, j \in J$. Since we are counting balls using the partition P, we will apply Lemma 2.2 to covers of P-names.

Now we show how to construct a transformation $T_1 \in \overline{G}(M, N, t, P)$ such that $d_w(T_0, T_1) < \epsilon$. Cut column $\mathcal{C}_1 = \{I_0, I_1, \ldots, I_{h-1}\}$ into 2 subcolumns of equal width, labeled as $\mathcal{C}_{1,0}$ and $\mathcal{C}_{1,1}$. For $k = 2(K + 2)$ as above, cut each subcolumn into k subcolumns of equal width. For $\mathcal{C}_{1,0}$, stack from left to right, and then place k spacers on top. For $\mathcal{C}_{1,1}$, place a spacer on top, and then stack from left to right. The auxillary level I_h may be used to add these spacers. Call these columns $\mathcal{C}_{2,0}$ and $\mathcal{C}_{2,1}$ respectively.

Before proceeding with the construction, let us demonstrate the usefulness of these two blocks. Suppose a copy of $\mathcal{C}_{2,0}$ is shifted by j and we measure the overlap with two catenated unshifted copies of $\mathcal{C}_{2,1}$. The shifted copy of $\mathcal{C}_{2,0}$ will have an overlap of at least $(K + 2)$ copies of $\mathcal{C}_{1,0}$ with one of the copies of $\mathcal{C}_{2,1}$. If $j \le (K+2)(h+1)$, then consider the overlap with the first copy, otherwise consider the larger overlap with the second copy. Assume $j \le (K + 2)(h + 1)$. The other case is handled similarly. For h sufficiently large, there will be an overlap of at least K full copies of $\mathcal{C}_{1,0}$ with a copy of $\mathcal{C}_{1,1}$ which we call $\mathcal{C}'$. Also, the copies will be distributed equally with shifts $j', j' + 1, j' + 2, \ldots, j' + K - 1$ for some j'. Since $T_1(x) = T_0(x)$ for $x \in \mathcal{C}'$, if $\alpha = \mu(q)$,

$$\begin{aligned}
\mu\big(T_1^j q \cap q^c \cap \mathcal{C}'\big) &\ge \frac{\mu(\mathcal{C}')}{k} \sum_{i=0}^{K-1} \mu(T_1^{j+i} q \cap q^c) \\
&= \frac{\mu(\mathcal{C}')K}{k} \frac{1}{K} \sum_{i=0}^{K-1} \mu(T_1^{j+i} q \cap q^c) \\
&\ge \frac{\mu(\mathcal{C}')}{6} \Big(\frac{1}{K} \sum_{i=0}^{K-1} \mu(T_1^{j+i} q \cap q^c) \Big) \\
&\ge \frac{\mu(\mathcal{C}')}{6} \Big(\frac{1}{K} \sum_{i=0}^{K-1} \mu(T_1^{j+i} q \cap q^c) - \alpha(1-\alpha) \Big) + \frac{\mu(\mathcal{C}')\alpha(1-\alpha)}{6} \\
&> \mu(\mathcal{C}') \Big(\frac{\alpha(1-\alpha)}{6} - \frac{\eta^2}{6} \Big) > \mu(\mathcal{C}') \Big(\frac{1}{54} - \frac{\eta^2}{6} \Big) > \frac{\mu(\mathcal{C}')}{100}.
\end{aligned}$$

Let $\ell = k(h + 1)$ and $\beta = 2^{\frac{1}{8k(h+1)}}$. Choose $r \in \mathbb{N}$ such that for $n = 2^r k(h + 1)$,

$$\beta^n > 2a_n(t).$$

Choose $s \in \mathbb{N}$, $s > r$ such that $2^{r-s} < \eta^2$.

By inequality (4.4), distance between shifts of the P-word formed from $\mathcal{C}_{1,1}$ and the P-word formed from $\mathcal{C}_{1,0}$ will be bounded away from 0. Both columns have length $k(h + 1)$. Remark: we cut into k, so that any two shifted blocks will have at least K sub-blocks that overlap.

Apply independent cutting and stacking to both columns, s number of times. This will cause the number of P-names of length $k(h + 1)$ to grow exponentially in s. (Note, a Chacon-2 similar stage could be inserted to guarantee weak mixing, but this is not needed, since weak mixing is generic.) There will be 2^{2^s} columns, each of height $2^s k(h+1)$. We will consider P-names of length n where n is large compared to $k(h+1)$, but small relative to $2^s k(h+1)$. For most points $y \in X$, we get a P-name by taking x in the base of a column and some j such that $y = T_1^j x$. We can get a P-name for y by forming the vector $v = \langle I_p(T^{j+i}x)\rangle_{i=0}^{n-1}$. Let b be the set containing the bases of all columns of height $2^s k(h+1)$. Define $b_j = T_1^j b$ and $\vec{b_j} = \{\langle I_p(T_1^{j+i}x)\rangle_{i=0}^{n-1} | x \in b, 0 \leq i < n\}$. Suppose $S = S_P(T_1, n, \delta, \delta)$ is the minimal number of δ-balls such that the union of measure is at least $1-\delta$. Let $v_1, v_2, \ldots, v_S$ be codewords of length n such that $B_\delta^{T_1,P}(v_i)$ cover at least measure $1-\delta$. Let $B = \bigcup_{i=1}^S B_\delta^{T_1,P}(v_i)$. Thus,

$$\mu\Big(B \cap (\bigcup_{i=0}^{n-1} \bigcup_{j=0}^{2^{s-r}-2} b_{jn+i})\Big) > 1 - \delta - \epsilon - 2^{r-s}.$$

There exists i_0 such that

$$\mu\Big(B \cap (\bigcup_{j=0}^{2^{s-r}-2} b_{jn+i_0})\Big) > \frac{1}{n}\big(1 - \delta - \epsilon - 2^{r-s}\big)$$

Let $D = \bigcup_{j=0}^{2^{s-r}-2} b_{jn+i_0}$. Then

$$\begin{aligned}
\mu\Big(B \cap D\Big) &> \frac{\mu(D)}{n\mu(D)}\big(1 - \delta - \epsilon - 2^{r-s}\big) \\
&\geq \frac{(1 - \delta - \epsilon - 2^{r-s})}{(1 - \epsilon - 2^{r-s})}\mu(D) \\
&= \big(1 - \frac{\delta}{1 - \epsilon - 2^{r-s}}\big)\mu(D).
\end{aligned}$$

Hence,

$$\mu(D \setminus B) < \big(\frac{\delta}{1 - \epsilon - 2^{r-s}}\big)\mu(D).$$

Let

$$J_1 = \{j : 0 \leq j \leq 2^{s-r} - 2, \mu(b_{jn+i_0} \setminus B) < \frac{2\delta}{1 - \epsilon - 2^{r-s}}\mu(b_{jn+i_0})\}.$$

Let $\hat{b}_j = \bigcup_{i=0}^{n-1} T^i b_j$. Note

$$\sum_{j=0}^{2^{s-r}-2} \mu(q \cap \cup_{j=0}^{2^{s-r}-2} \hat{b}_{jn+i_0} \triangle p \cap \cup_{j=0}^{2^{s-r}-2} \hat{b}_{jn+i_0}) < \frac{\eta^2}{6}.$$

Let

$$J_2 = \{j : 0 \leq j \leq 2^{s-r} - 2, \mu(q \cap \hat{b}_{jn+i_0} \triangle p \cap \hat{b}_{jn+i_0}) < \frac{\eta^2}{3(2^{s-r}-1)}\}.$$

Both $|J_1| > \frac{1}{2}\big(2^{s-r}-1\big)$ and $|J_2| > \frac{1}{2}\big(2^{s-r}-1\big)$. Thus, there exists $j_0 \in J_1 \cap J_2$.

Since $\mu(q \cap \hat{b}_{j_0 n+i_0} \triangle p \cap \hat{b}_{j_0 n+i_0}) < \eta^2 \mu(\hat{b}_{j_0 n+i_0})$, by Lemma 2.3,

$$\mu\big(\{x \in b_{j_0 n+i_0} : d(\vec{Q}_n(x), \vec{P}_n(x)) < \eta\}\big) > \big(1-\eta\big)\mu(b_{j_0 n+i_0}).$$

Therefore, by Lemma 2.2, the number of Hamming balls S satisfies:

$$\begin{aligned} S \geq & \big(1 - \frac{2\delta}{1-\epsilon-2^{r-s}} - \eta\big) 2^{2^r(1-\mathcal{H}(\frac{2(\delta+\eta)}{d}+2^{-r}))} \\ \geq & \big(1 - \frac{2\delta}{1-\epsilon-2^{r-s}} - \eta\big)\beta^n \\ > & 2\big(1 - \frac{2\delta}{1-\epsilon-2^{r-s}} - \eta\big)a_n(t) > a_n(t). \end{aligned}$$

Therefore, $S_P(T_1, n, \delta, \delta) > a_n(t)$ and $T_1 \in \overline{G}(M, N, t, P)$. □

Remark: It is clear from the proof of Proposition 4.2 that given $\gamma > 0$, there exists $M(\gamma)$ such that $\overline{G}(M, N, t, P)$ is dense for $M \geq M(\gamma)$ and 2-set partition P such that $\mathcal{H}(P) \geq \gamma$.

Theorem 4.3. Suppose $a_n(t)$ is such that for $t, \beta > 1$, $\lim_{n\to\infty} \frac{a_n(t)}{\beta^n} = 0$. There exists a dense G_δ subset $G \subset \mathcal{M}$ such that for each $T \in G$, the upper slow entropy of T is infinite with respect to $a_n(t)$.

Proof. By Proposition 4.2, for M sufficiently large, $\overline{G}(M, N, t, P)$ is dense for all $N, t \in \mathbb{N}$. Also, $\overline{G}(M, N, t, P)$ is open by Proposition 4.1. Thus,

$$G = \bigcap_{t=1}^{\infty} \bigcap_{N=1}^{\infty} \overline{G}(M, N, t, P)$$

is a dense G_δ. Fix $t \in \mathbb{N}$. For $T \in \bigcap_{N=1}^{\infty} \overline{G}(M, N, t, P)$,

$$\limsup_{n\to\infty} \frac{S_P(T, n, \frac{1}{M}, \frac{1}{M})}{a_n(t)} \geq 1.$$

Therefore, the upper slow entropy is ∞ with respect to $a_n(t)$. □

Corollary 4.4. In the weak topology, the generic transformation in $\mathcal{M}$ is rigid, weak mixing, rank-one and has infinite polynomial upper slow entropy.

The following corollary is a strengthening of Theorem 4.3.

Corollary 4.5. Given a rate function $b = b_n(t)$, let

$$\mathcal{G}_b = \{T \in \mathcal{M} : \text{s-}\overline{\mathcal{H}}^{\mu}_{b}(T, P) > 0 \text{ for every nontrivial finite measurable } P\}. \tag{4.5}$$

If $b_n(t)$ is subexponential, then $\mathcal{G}_b$ contains a dense G_δ class in $\mathcal{M}$.

Proof. It is sufficient to prove this corollary for 2-set partitions. Let P_L be a sequence of nontrivial measurable 2-set partitions such that the collection $\{P_L : L \in \mathbb{N}\}$ is dense in the class of all measurable 2-set partitions. By the proofs of Propositions 4.1 and 4.2, then for $N, t, L \in \mathbb{N}$ and M_L sufficiently large, the set $G(M, N, t, P_L)$ is open and dense in the weak topology for $M \geq M_L$. Thus,

$$G = \bigcap_{L=1}^{\infty} \bigcap_{M=M_L}^{\infty} \bigcap_{t=1}^{\infty} \bigcap_{N=1}^{\infty} \overline{G}(M, N, t, P_L)$$

is a dense G_δ. Also, M_L may be set equal to $M(\gamma)$ where $\gamma = \mathcal{H}(P_L)$. Actually, choose $M \geq M(\frac{\gamma}{2})$ such that if

$$D(P, Q) < \frac{1}{9M^2},$$

then $\mathcal{H}(Q) > \mathcal{H}(P) - \frac{\gamma}{2}$. Choose $L_0 \in \mathbb{N}$ such that $D(P, P_{L_0}) < \frac{1}{9M^2}$. Let $t > 0$. For $T \in \bigcap_{N=1}^{\infty} \overline{G}(M, N, t, P_{L_0})$,

$$\limsup_{n\to\infty} \frac{S_{P_{L_0}}(T, n, \frac{1}{M}, \frac{1}{M})}{b_n(t)} \geq 1.$$

By Lemma 2.4, $S_P(T, n, \frac{1}{3M}, \frac{1}{3M}) \geq S_{P_{L_0}}(T, n, \frac{1}{M}, \frac{1}{M})$. Therefore, for $T \in G$, s-$\overline{\mathcal{H}}_b^{\mu}(T, P) > 0$ which implies $G \subseteq \mathcal{G}_b$. $\square$

A transformation $T \in \mathcal{G}_b$ is said to have *strictly positive upper slow entropy with respect to b.*

5. Infinite lower slow entropy rigid transformations

All transformations are constructed on $[0, 1]$ with Lebesgue measure. The transformations with infinite lower slow entropy will be infinite rank transformations defined using three different types of stages. These stages will be repeated in sequence ad infinitum. If all stages are labeled $\mathcal{S}_i$ for $i = 0, 1, 2, \ldots$, then the stages break down as follows:

(1) $\mathcal{S}_{3i}$ will be an independent cutting and stacking stage;
(2) $\mathcal{S}_{3i+1}$ will be a weak mixing stage (similar to Chacon-2);
(3) $\mathcal{S}_{3i+2}$ will be a rigidity stage.

The transformation will be initialized with two columns $\mathcal{C}_{0,1}$ and $\mathcal{C}_{0,2}$ each containing a single interval. Let P be the 2-set partition containing each of $\mathcal{C}_{0,1}$ and $\mathcal{C}_{0,2}$. As the transformation is defined, we will add spacers infinitely often. Spacers will be unioned with the second set, so in general, the partition $P = \{C_{0,1}, X \setminus C_{0,1}\}$.

5.1. Independent cutting and stacking stage. To define the independent cutting and stacking stage, we use a sequence $s_k \in \mathbb{N}$ for $k = 0, 1, \ldots$ such that $s_k \to \infty$. For stage $\mathcal{S}_0$, independent cut and stack s_0 times beginning with columns $\mathcal{C}_{0,1}$ and $\mathcal{C}_{0,2}$. The number of columns is squared at each cut and stack stage, so after s_0 substages, there are $2^{2^{s_0}}$ columns each having height 2^{s_0}.

For the general stage $\mathcal{S}_{3k}$, suppose there are ℓ columns each having height h at the start of the stage. If we independent cut and stack s_k times, then there will be

$$2^{2^{s_k}}\ell$$

columns each having height $2^{s_k}h$.

5.2. Weak mixing stage. For the general weak mixing stage $\mathcal{S}_{3k+1}$, suppose there are ℓ columns of height h. Cut each column into 2 subcolumns of equal width, add a single spacer on the right subcolumn, and stack the right subcolumn on top of the left subcolumn. Thus, there will be ℓ columns of height $2h + 1$.

Suppose f is an eigenfunction with eigenvalue λ. Thus, $f(Tx) = \lambda f(x)$. There will be a weak mixing stage with refined columns such that f is nearly constant on most intervals. Let I be one such interval. Suppose $x, y \in I$ such that $T^h x \in I$ and $T^{h+1} y \in I$ and f is nearly constant on each of these four values. Since the following stage is a rigidity stage, it is not difficult to show points x and y exist with this property. Then

$$f(T^h x) = \lambda^h f(x) \approx f(T^{h+1} y) = \lambda^{h+1} f(y).$$

Since $f(x) \approx f(y)$, then $\lambda^h \approx \lambda^{h+1}$. Hence, $\lambda \approx 1$. In the limit, this shows that $\lambda = 1$ is the only eigenvalue, and since T will be ergodic, f is constant and T is weak mixing. This is the same argument used in the original proof of Chacon that the Chacon-2 transformation is weak mixing [4].

5.3. Rigidity stage. A sequence $r_k \in \mathbb{N}$ is used to control rigidity. Suppose at stage $\mathcal{S}_{3k+2}$, there are ℓ columns of height h. Cut each column into r_k subcolumns of equal width, and stack from left to right to obtain ℓ columns of height $r_k h$. Thus, for any set A that is a union of intervals from the ℓ colummns, then

$$\mu(T^h A \cap A) > \Big(\frac{r_k - 1}{r_k}\Big)\mu(A).$$

5.4. Infinite lower slow entropy. Our family $\mathcal{F}$ of transformations are parameterized by sequences $r_k \to \infty$ and $s_k \in \mathbb{N}$. Initialize $T \in \mathcal{F}$ using two columns of height 1. Independent cut and stack s_0 times. This produces $2^{2^{s_0}}$ columns of height 2^{s_0}. Let $h_1 = 2^{s_0}$ be the heights of these columns. Cut each column into two subcolumns of equal width, add a single spacer on the rightmost subcolumn and stack the right subcolumn on the left subcolumn. Then cut each of these columns of height $2h_1 + 1$ into r_1 subcolumns of equal width and stack from left to right. Independent cut and stack these

$2^{2^{s_0}}$ columns of height $r_1(2h_1+1)$ to form $2^{2^{s_0+s_1}}$ columns of height $h_2 = 2^{s_1}r_1(2h_1+1)$. In general, $h_k = 2^{s_{k-1}}r_{k-1}(2h_{k-1}+1)$. Also, let

$$\sigma_k = \sum_{i=0}^{k-1} s_i.$$

Remark: As long as $s_k > 0$ infinitely often, the transformations are ergodic due to the independent cutting and stacking stages. Thus, based on section 5.2, the transformations will be weak mixing. Rigidity will follow from a given sequence $r_k \to \infty$. The proof that the lower slow entropy is infinite is given next.

Theorem 5.1. Given any subexponential rate $a_n(t)$, there exists a rigid weak mixing system $T \in \mathcal{F}$ such that T has infinite lower slow entropy with respect to $a_n(t)$.

Proof. Let $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < \frac{1}{100}$. Suppose $r_k, t_k \in \mathbb{N}$ such that $\lim_{k\to\infty} r_k = \lim_{k\to\infty} t_k = \infty$. Let $h_0 = 1$ and $h_1 = 2^{s_0}$. Recall the formulas for h_{k+1} and σ_k,

$$h_{k+1} = 2^{s_k} r_k\big(2h_k + 1\big) \quad \text{and} \quad \sigma_k = \sum_{i=0}^{k-1} s_i.$$

We will specify the sequence s_k inductively based on σ_k, r_k, h_k and r_{k+1}. For sufficiently large k,

$$2^{2^{\sigma_k}\mathcal{H}(2\varepsilon+2^{-\sigma_k})} < 2^{2^{\sigma_k}\mathcal{H}(3\varepsilon)}.$$

Let

$$\alpha_k = \frac{2^{\sigma_k}\big(1-\mathcal{H}(3\varepsilon)\big)}{2^5 r_{k+1} r_k h_k} \quad \text{and} \quad \beta_k = 2^{\alpha_k}.$$

Choose $s_k \in \mathbb{N}$ such that for $n \geq 2^{s_k}$,

$$\beta_k^n > k a_n(t_k).$$

Let $0 < \delta < \frac{1}{100}$. Suppose $S \in \mathbb{N}$ and v_i, $1 \leq i \leq S$, are such that

$$\mu\Big(\bigcup_{i=1}^{S} B_\varepsilon^{T,P}(v_i)\Big) > 1 - \delta.$$

Suppose a large $n \in \mathbb{N}$ is chosen. There exists $k \in \mathbb{N}$ such that $2h_k \leq n < 2h_{k+1}$. We divide the proof into two cases:

(1) $2r_k(2h_k+1) \leq n < 2^{s_k+1} r_k(2h_k+1)$
(2) $2h_k \leq n < 2r_k(2h_k+1)$.

For case (1), there exist $\rho \in \mathbb{N}$ such that $1 \leq \rho \leq s_k$ and

$$2^{\rho} r_k(2h_k+1) \leq n < 2^{\rho+1} r_k(2h_k+1).$$

Our T is represented as a cutting and stacking construction with the number of columns tending to infinity. Fix a height H much larger than n and set of columns $\mathcal{C}$. Pick H such that $\frac{n}{H} < \frac{\eta}{2}$ and the complement of columns in

$\mathcal{C}$ is less that $\frac{\eta}{2}$ for small η. Let b be points in the bottom levels of $\mathcal{C}$. Also, let $b_j = T^j b$ for $0 \le j < H - n$.

Let $B = \bigcup_{i=1}^{S} B_{\varepsilon}^{T,P}(v_i)$. There exists $j < H - n$ such that

$$\mu(B \cap b_j) > \big(\frac{1-\delta-\eta}{1-\eta}\big)\mu(b_j) > (1-2\delta)\mu(b_j) \text{ for sufficiently small } \eta.$$

If $\mu(B_{\varepsilon}^{T,P}(v_i) \cap b_j) = 0$ for some i, then remove that Hamming ball from the list. Let $R \le S$ be the number of remaining Hamming balls and rename the vectors so that $B = \bigcup_{i=1}^{R} B_{\varepsilon}^{T,P}(v_i)$ satisfies $\mu(B \cap b_j) > (1-2\delta)\mu(b_j)$. Choose $x_i \in B_{\varepsilon}^{T,P}(v_i) \cap b_j$ such that $\mu(\{x \in B \cap b_j : \vec{P}_n(x) = \vec{P}_n(x_i)\}) > 0$. Let $w_i = \vec{P}_n(x_i)$. Thus, by the triangle inequality of the Hamming distance, $B_{2\varepsilon}(w_i) \supseteq B_{\varepsilon}(v_i)$ and hence,

$$B_{2\varepsilon}^{T,P}(\vec{P}_n(x_i)) \cap b_j \supseteq B_{\varepsilon}^{T,P}(v_i) \cap b_j.$$

This previous statement allows us to consider only balls with P-names as centers by doubling the radius. For a.e. point $x \in b_j$, the h_k long blocks, called $\mathcal{C}_k$, will align. Also, lower level blocks h_i for $i < k$ will align as well.

Since all h_k blocks and sub-blocks align for $x \in b_j$, we can view this problem as a standard estimate on hamming ball sizes for i.i.d. binary sequences. The spacers added over time diminish and will have little effect on the distributions and hamming distance. For repeating blocks to be within 2ε of a word w_i, it is necessary the sub-blocks to be within approximately 2ε. Note,

$$n \ge 2^{\rho} r_k(2h_k+1) = 2^{\rho} r_k\Big(2\big(2^{s_{k-1}} r_{k-1}(2h_{k-1}+1)\big)\Big) > 2^{s_{k-1}}.$$

Since we have approximated must but not all variables, we can make R at least,

$$R \ge \frac{1}{2}\big(1-2\delta\big)2^{2^{\rho}2^{\sigma_k}\big(1-\mathcal{H}(3\varepsilon)\big)}.$$

Hence,

$$\begin{aligned} 2^{2^{\rho}2^{\sigma_k}\big(1-\mathcal{H}(3\varepsilon)\big)} &= 2^{2^{\rho+s_{k-1}}2^{\sigma_{k-1}}\big(1-\mathcal{H}(3\varepsilon)\big)} \\ &= \Big(2^{\frac{(1-\mathcal{H}(3\varepsilon))2^{\sigma_{k-1}}}{32 r_k r_{k-1} h_{k-1}}}\Big)^{2^{\rho+s_{k-1}} 32 r_k r_{k-1} h_{k-1}} \\ &> \Big(2^{\alpha_{k-1}}\Big)^{2^{\rho} 8 r_k h_k} \\ &> \big(2^{\alpha_{k-1}}\big)^{2^{\rho+1} r_k(2h_k+1)} \\ &> \big(\beta_{k-1}\big)^{n} > (k-1)a_n(t_{k-1}). \end{aligned}$$

This handles case (1). Case (2) can be handled in a similar fashion, with focus on the blocks of length h_{k-1}. □

6. Slow entropy and the convergence rates of Blume

This section is devoted to comparing and contrasting the notion of slow entropy with the study of entropy convergence rates of Blume [2,3]. In [2], Blume proves that the set of all transformations T satisfying the property that for all nontrivial finite measurable partitions P, the entropy $\mathcal{H}(\bigvee_{i=0}^{n-1} T^{-i}P)$ of the n^{th} refinements converges in the limit superior faster to ∞ than a given sublinear rate a_n is residual with respect to the weak topology.

The entropy convergence rates studied by Blume are fundamentally different from the slow entropy formulation introduced by Katok and Thouvenot. In particular, given a sublinear rate $a_n \to \infty$ and subexponential rate $b_n(t)$, we give an outline of a cutting and stacking construction which has zero upper slow entropy with respect to $b_n(t)$, but still satisfies for every nontrivial finite measurable partition P:

$$\limsup_{n\to\infty} \frac{1}{a_n}\mathcal{H}\big(\bigvee_{i=0}^{n-1} T^{-i}P\big) = \infty.$$

We say $a = a_n$ is sublinear, if $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} \frac{a_n}{n} = 0$. Previously introduced by Blume [2], the class $ES(a)$ is defined as:

$$ES(a) = \{T \in \mathcal{M} : \limsup_{n\to\infty} \frac{1}{a_n}\mathcal{H}\big(\bigvee_{i=0}^{n-1} T^{-i}P\big) > 0 \text{ for all nontrivial finite } P\}.$$

Also, we show for any sublinear rate $a = a_n$, then for the subexponential rate $b = b_n(t) = 2^{ta_n}$, the following holds: [3]

$$\mathcal{G}_b \subsetneq ES(a).$$

This demonstrates that the slow entropy formulation provides a stricter measure of complexity for a dynamical system than the entropy convergence rate.

6.1. Infinite entropy convergence rate and zero lower slow entropy.

The following result distinguishes slow entropy from entropy convergence rate.

Proposition 6.1. Given a sublinear rate a_n and any subexponential rate $b = b_n(t)$, there exists an ergodic invertible measure preserving transformation T such that T has zero upper slow entropy with respect to $b_n(t)$, but for any nontrivial finite measurable partition P,

$$\limsup_{n\to\infty} \frac{1}{a_n}\mathcal{H}\big(\bigvee_{i=0}^{n-1} T^{-i}P\big) > 0.$$

Proof. It is sufficient to prove this for any nontrivial measurable 2-set partition $P = \{p, p^c\}$. First we describe the inductive step for constructing T. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be columns of height h_n. Cut $\mathcal{C}_1$ into $2(n+2)$ subcolumns

[3] The set $\mathcal{G}_b$ is defined in (4.5).

of equal width and cut $\mathcal{C}_2$ into 2 subcolumns of equal width. Swap the last subcolumn of $\mathcal{C}_1$ with the second subcolumn of $\mathcal{C}_2$. For the modified $\mathcal{C}_2$, cut into 2 subcolumns of equal width. Cut each of these subcolumns into k_n subcolumns of equal width and use spacers to build nearly independent words as described in section 4.1. Then apply independent cutting and stacking as described in section 4.2 to the two $\mathcal{C}_2$ subcolumns to create $2^{2^{s_n}}$ subcolumns of equal width. Each of these subcolumns will have height $2^{s_n}k_n(h_n+1)$. Stack these subcolumns to build a single column of height $2^{2^{s_n}}2^{s_n}k_n(h_n+1)$.

For the modified $\mathcal{C}_1$ column, cut into k_n subcolumns of equal width, stack from left to right, and add k_n spacers on top. Then cut this column into 2 subcolumns of equal width, stack the right subcolumn on top of the left subcolumn. Repeat this procedure a total of $2^{2^{s_n}+s_n}$ times. After this stage, we have produced two subcolumns of equal height,

$$h_{n+1} = 2^{2^{s_n}}2^{s_n}k_n(h_n+1).$$

This process can be initialized by taking any rank-one transformation and dividing a column of height h_1 into 2 subcolumns of equal width. After the first stage, there will be 2 subcolumns, one with width $\frac{2}{3}$ and the other with width $\frac{1}{3}$. After $n-1$ stages, there will be 2 subcolumns of height h_n, one with width $\frac{n}{n+1}$ and the other with width $\frac{1}{n+1}$.

For rapidly growing s_n, this transformation T will boost the entropy on the right column for each stage. In particular, given sublinear a_n, it is possible to choose s_n such that the right portion appears to be a positive entropy transformation, but scaled by a factor of $\frac{1}{n+1}$. More explicitly, if X_n represents the left column at stage n, and $U_n \subset X_n$ is the portion that is switched, then U_i will be approximately conditionally independent of U_j on X_n. Thus, by the 2nd Borel-Cantelli lemma, a.e. point will fall in U_i for infinitely many i. Moreover, for large s_n, near pairwise independence of U_i will lead to near pairwise independence between any set p and most U_i. In particular, we can choose s_n such that given $\gamma > 0$,

$$\Gamma_p = \{i \in \mathbb{N} : |\mu(p \cap U_i) - \mu(p)\mu(U_i)| < \gamma\mu(U_i)\},$$

satisfies

$$\sum_{i\in\Gamma_p} \mu(U_i) < \infty.$$

Since $\lim_{n\to\infty}\mu(X_n) = 1$, T will be ergodic. This will enable the creation of two nearly independent words similar to section 4.1. Thus, independent cutting and stacking raises the slow entropy on this portion (right column) of the space. However, using cut in half and stack on the left column lowers the slow entropy on most of the space. Hence, given $\delta > 0$, once the right column mass $\frac{1}{n+1}$ falls below δ, then it will be possible to use a sublinear number of P-names to cover the names produced by the left column. The scaled entropy calculation $\frac{1}{a_n}\mathcal{H}(\bigvee_{i=0}^{n-1} T^{-i}P)$ will include the rising entropy from the right column of mass $\frac{1}{n+1}$.

If sublinear a_n is specified and $b = b_n(t)$ such that $\lim_{n\to\infty} b_n(t) = \infty$ for all $t > 0$, then s_n can be chosen to increase rapidly such that for every nontrivial measurable 2-set partition P,

$$\text{s-}\overline{\mathcal{H}}_b^{\mu}(T, P) = 0$$

and

$$\limsup_{n\to\infty} \frac{1}{a_n}\mathcal{H}\big(\bigvee_{i=0}^{n-1} T^{-i}P\big) > 0.$$

Therefore, $T \in ES(a)$, but T has zero upper slow entropy with respect to $b_n(t)$.
Remark: $b_n(t)$ may be chosen with slow growth and is not necessarily 2^{ta_n}. □

6.2. Strictly positive upper slow entropy implies positive entropy convergence rate. It is straightforward to show that the rate function $b_n(t) = 2^{ta_n}$ is subexponential if and only if a_n is sublinear.

Proposition 6.2. Suppose a_n is sublinear and $b_n(t) = 2^{ta_n}$. If $T \in \mathcal{G}_b$, then for every nontrivial finite measurable partition P,

$$\limsup_{n\to\infty} \frac{1}{a_n}\mathcal{H}\big(\bigvee_{i=0}^{n-1} T^{-i}P\big) > 0.$$

Thus, $\mathcal{G}_b \subsetneq ES(a)$.

Proof. We prove the contrapositive. Suppose P is a finite measurable partition such that

$$\lim_{n\to\infty} \frac{1}{a_n}\mathcal{H}\big(\bigvee_{i=0}^{n-1} T^{-i}P\big) = 0.$$

Let $P_n = \bigvee_{i=0}^{n-1} T^{-i}P$. Let $V_n = \{p \in P_n : 0 < \mu(p) < 2^{-ta_n}\}$ and $V_n' = \{p \in P_n : \mu(p) \geq 2^{-ta_n}\}$. Thus, for n sufficiently large,

$$\begin{aligned}\mathcal{H}\big(P_n\big) &= -\sum_{p\in V_n} \mu(p)\log\mu(p) - \sum_{p\in V_n'} \mu(p)\log\mu(p) \\ &\geq -\sum_{p\in V_n} \mu(p)\log 2^{-ta_n} - \sum_{p\in V_n'} 2^{-ta_n}\log 2^{-ta_n} \\ &= ta_n\mu(V_n) + |V_n'|ta_n 2^{-ta_n}.\end{aligned}$$

This implies

$$\mu(V_n) \leq \frac{1}{ta_n}\mathcal{H}\big(P_n\big) \to 0, \text{ as } n \to \infty.$$

Hence, for $\delta > 0$ and n sufficiently large, $\mu(V_n) < \delta$, and

$$\frac{S_P(T, n, \delta, \delta)}{2^{ta_n}} \leq \frac{|V_n'|}{2^{ta_n}} \leq \frac{1}{ta_n}\mathcal{H}(P_n) \to 0, \text{ as } n \to \infty.$$

Since this is true for all $t > 0$ and $\delta > 0$, then s-$\overline{\mathcal{H}}_b^{\mu}(T, P) = 0$ and therefore $T \notin \mathcal{G}_b$. $\square$

The previous proposition combined with Corollary 4.5 gives an extension of Blume's Theorem 4.8 from [2]. Also, counterexamples from Proposition 6.1 demonstrate this is a nontrivial extension, since there exist $T \in ES(a)$ such that $T \notin \mathcal{G}_b$ for corresponding $b_n(t) = 2^{ta_n}$. Using the technique given in [1], given any rigidity sequence ρ_n for an ergodic invertible measure preserving transformation, it is possible to construct an ergodic invertible measure preserving transformation $T \in ES(a)$ which is rigid on ρ_n. It is an open question what families of rigidity sequences are realizable for transformations with infinite or positive lower slow entropy with respect to a given subexponential rate.

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