Model ∞-categories III: the fundamental theorem

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ABSTRACT. We prove that a model structure on a relative ∞-category $(\mathcal{M}, \mathbf{W})$ gives an efficient and computable way of accessing the hom-spaces $\hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x, y)$ in the localization. More precisely, we show that when the source $x \in \mathcal{M}$ is *cofibrant* and the target $y \in \mathcal{M}$ is *fibrant*, then this hom-space is a "quotient" of the hom-space $\hom_{\mathcal{M}}(x, y)$ by either of a *left homotopy relation* or a *right homotopy relation*.

CONTENTS

0. Introduction

0.1. Model ∞-categories. A *relative ∞-category* is a pair $(\mathcal{M}, \mathbf{W})$ of an ∞-category $\mathcal{M}$ and a subcategory $\mathbf{W} \subset \mathcal{M}$ containing all the equivalences, called the subcategory of *weak equivalences*. Freely inverting the weak equivalences, we obtain the *localization* of this relative ∞-category, namely the initial functor

$$\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$$

from $\mathcal{M}$ which sends all maps in $\mathbf{W}$ to equivalences. In general, it is extremely difficult to access the localization. The purpose of this paper is to show that the additional data of a *model structure* on $(\mathcal{M}, \mathbf{W})$ makes it far easier: we prove the following ***fundamental theorem of model ∞-categories***.[1]

Theorem (1.9). *Suppose that $\mathcal{M}$ is a model ∞-category. Then, for any cofibrant object $x \in \mathcal{M}^c$ and any fibrant object $y \in \mathcal{M}^f$, the induced map*

$$\hom_{\mathcal{M}}(x, y) \to \hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x, y)$$

on hom-spaces is a π_0-surjection. Moreover, this becomes an equivalence upon imposing either of a "left homotopy relation" or a "right homotopy relation" on the source (see definition 1.7).

We view this result – and the framework of model ∞-categories more generally – as providing a theory of **resolutions** which is native to the ∞-categorical setting. To explain this perspective, let us recall Quillen's classical theory of model categories, in which for instance

- replacing a topological space by a CW complex constitutes a *cofibrant resolution* – that is, a choice of representative which is "good for mapping out of" – of its underlying object of $\mathcal{T}\mathrm{op}[\mathbf{W}_{\mathrm{w.h.e.}}^{-1}]$ (i.e. its underlying weak homotopy type), while
- replacing an R-module by a complex of injectives constitutes a *fibrant resolution* – that is, a choice of representative which is "good for mapping into" – of its underlying object of $\mathrm{Ch}(R)[\mathbf{W}_{\mathrm{q.i.}}^{-1}]$.

Thus, a model structure on a relative (1- or ∞-)category $(\mathcal{M}, \mathbf{W})$ provides *simultaneously compatible* choices of objects of $\mathcal{M}$ which are "good for mapping out of" and "good for mapping into" with respect to the corresponding localization $\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$.

A prototypical example of this phenomenon arises from the interplay of ***left and right derived functors*** (in the classical model-categorical sense), i.e. of left and right adjoint functors of ∞-categories. For instance,

- in a *left localization* adjunction $\mathcal{C} \rightleftarrows \mathrm{L}\mathcal{C}$, we can think of the subcategory $\mathrm{L}\mathcal{C} \subset \mathcal{C}$ as that of the "fibrant" objects, while *every* object is "cofibrant", while dually
- in a *right localization* adjunction $\mathrm{R}\mathcal{C} \rightleftarrows \mathcal{C}$, we can think of the subcategory $\mathrm{R}\mathcal{C} \subset \mathcal{C}$ as that of the "cofibrant" objects, while *every* object is "fibrant".[2]

As a model structure generally has neither all its objects cofibrant nor all its objects fibrant, it can therefore be seen as a *simultaneous generalization* of the notions of left localization and right localization.

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[1]For the precise definition a model ∞-category, we refer the reader to [12, §1]. However, for the present discussion, it suffices to observe that it is simply a direct generalization of the standard definition of a model category.

[2]See [12, Examples 2.12 and 2.17] for more details on such model structures.

Remark 0.1. Indeed, this observation encompasses one of the most important examples of a model ∞-category, which was in fact the original motivation for their theory.

Suppose we are given a presentable ∞-category $\mathcal{C}$ along with a set $\mathcal{G}$ of generators which we assume (without real loss of generality) to be closed under finite coproducts. Then, the corresponding **nonabelian derived ∞-category** is the ∞-category $\mathcal{P}_\Sigma(\mathcal{G}) = \mathrm{Fun}_\Sigma(\mathcal{G}^{op}, \mathcal{S})$ of those presheaves on $\mathcal{G}$ that take finite coproducts in $\mathcal{G}$ to finite products in $\mathcal{S}$. This admits a canonical projection

$$
\begin{array}{ccc}
 & & s\mathcal{C} \\
 & \overset{\hom_{\mathcal{C}}^{\mathrm{lw}}(=,-)}{\swarrow} & \\
s(\mathcal{P}_\Sigma(\mathcal{G})) & & \\
 & \underset{|-|}{\searrow} & \\
 & & \mathcal{P}_\Sigma(\mathcal{G}),
\end{array}
$$

the composition of the (restricted) levelwise Yoneda embedding (a *right* adjoint) followed by (pointwise) geometric realization (a *left* adjoint): given a simplicial object $Y_\bullet \in s\mathcal{C}$ and a generator $S^\beta \in \mathcal{G}$, this composite is given by

$$
\begin{array}{ccc}
 & & Y_\bullet \\
 & \swarrow & \\
\hom_{\mathcal{C}}^{\mathrm{lw}}(S^\beta, Y_\bullet) & & \\
 & \searrow & \\
 & & \left|\hom_{\mathcal{C}}^{\mathrm{lw}}(S^\beta, Y_\bullet)\right|,
\end{array}
$$

where we use the abbreviation "lw" to denote "levelwise". In fact, this composite is a *free* localization (but neither a left nor a right localization): denoting by $\mathbf{W}_{\mathrm{res}} \subset s\mathcal{C}$ the subcategory spanned by those maps which it inverts, it induces an equivalence

$$s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \xrightarrow{\sim} \mathcal{P}_\Sigma(\mathcal{G}).$$

In future work, we will provide a **resolution model structure** on the ∞-category $s\mathcal{C}$ in order to organize computations in the nonabelian derived ∞-category $\mathcal{P}_\Sigma(\mathcal{G})$. (The resolution model structure on the ∞-category $s\mathcal{C}$, which might also be called an "E^2 model structure", is based on work of Dwyer–Kan–Stover and Bousfield (see [5] and [3], resp.).)

Remark 0.2. In turn, the original motivation for the resolution model structure was provided by *Goerss–Hopkins obstruction theory* (see [12, §0.3]). However, the nonabelian derived ∞-category also features prominently for instance in Barwick's universal characterization of *algebraic K-theory* (see [1]), as well as in his theory of *spectral Mackey functors* (which provide an ∞-categorical model for genuine equivariant spectra) (see [2]).

0.2. Conventions. The model ∞-categories papers share many key ideas; thus, rather than have the same results appear repeatedly in multiple places, we have chosen to liberally cross-reference between them. To this end, we introduce the following "code names".

title	reference	code
Model ∞-categories I: some pleasant properties of the ∞-category of simplicial spaces	[12]	S
The universality of the Rezk nerve	[13]	N
On the Grothendieck construction for ∞-categories	[14]	G
Hammocks and fractions in relative ∞-categories	[15]	H
Model ∞-categories II: Quillen adjunctions	[16]	Q
Model ∞-categories III: the fundamental theorem	n/a	M

Thus, for instance, to refer to [12, Theorem 4.4], we will simply write Theorem S.4.4. (The letters are meant to be mnemonical: they stand for "simplicial space", "nerve", "Grothendieck", "hammock", "Quillen", and "model", respectively.)

We take quasicategories as our preferred model for ∞-categories, and in general we adhere to the notation and terminology of [7] and [8]. In fact, our references to these two works will be frequent enough that it will be convenient for us to adopt Lurie's convention and use the code names T and A for them, respectively.

However, we work invariantly to the greatest possible extent: that is, we primarily work *within the ∞-category of ∞-categories*. Thus, for instance, we will omit all technical uses of the word "essential", e.g. we will use the term *unique* in situations where one might otherwise say "essentially unique" (i.e. parametrized by a contractible space). For a full treatment of this philosophy as well as a complete elaboration of our conventions, we refer the interested reader to §S.A. The casual reader should feel free to skip this on a first reading; on the other hand, the careful reader may find it useful to peruse that section before reading the present paper. For the reader's convenience, we also provide a complete index of the notation that is used throughout this sequence of papers in §S.B.

0.3. Outline. We now provide a more detailed outline of the contents of this paper.

- In §1, we give a precise statement of the **fundamental theorem of model ∞-categories** (1.9). This involves the notions of a *cylinder object* $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ and a *path object* $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ for our chosen source and target objects $x, y \in \mathcal{M}$, which generalize their corresponding model 1-categorical namesakes and play analogous roles thereto.

- In §2, we prove that the spaces of *left homotopy classes of maps* (defined in terms of a cylinder object $\mathrm{cyl}^\bullet(x)$) and of *right homotopy classes of maps* (defined in terms of a path object $\mathrm{path}_\bullet(y)$) are both equivalent to a more symmetric bisimplicial colimit (defined in terms of both $\mathrm{cyl}^\bullet(x)$ and $\mathrm{path}_\bullet(y)$).

- In §3, we prove that it suffices to consider the case that our cylinder and path objects are *special*.

- In §4, we digress to introduce *model diagrams*, which corepresent diagrams in a model ∞-category $\mathcal{M}$ of a specified type (i.e. whose constituent morphisms can be required to be contained in (one or more of) the various defining subcategories $\mathbf{W}, \mathbf{C}, \mathbf{F} \subset \mathcal{M}$).

- In §5, we prove that when our cylinder and path objects are both special, the bisimplicial colimit of §2 is equivalent to the groupoid completion of a certain ∞-category $\underline{\tilde{\mathbf{3}}}(x,y)$ of *special three-arrow zigzags* from x to y.

- In §6, we prove that the inclusion $\underline{\tilde{\mathbf{3}}}(x,y) \hookrightarrow \underline{\mathbf{3}}(x,y)$ into the ∞-category of (all) three-arrow zigzags from x to y induces an equivalence on groupoid completions.

- In §7, we prove that the inclusion $\underline{\mathbf{3}}(x,y) \hookrightarrow \underline{\mathbf{7}}(x,y)$ into a certain ∞-category of *seven-arrow zigzags* from x to y induces an equivalence on groupoid completions.

- In §8, in order to access the hom-spaces in the localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$, we prove that the *Rezk nerve* $\mathrm{N}_\infty^{\mathrm{R}}(\mathcal{M}, \mathbf{W})$ (see §N.3) of (the underlying relative ∞-category of) a model ∞-category is a Segal space. (By the local universal property of the Rezk nerve (Theorem N.3.8), this Segal space necessarily presents the localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$.)

- In §9, we prove that the groupoid completion $\underline{\mathbf{7}}(x,y)^{\mathrm{gpd}}$ of the ∞-category of seven-arrow zigzags from x to y is equivalent to the hom-space $\hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y)$.

- In §10, using the fundamental theorem of model ∞-categories (1.9), we prove that the Rezk nerve $\mathrm{N}_\infty^{\mathrm{R}}(\mathcal{M}, \mathbf{W})$ is in fact a *complete* Segal space.

As this paper is the culmination of its series, we would also like to take this opportunity to extend our thanks to the people who have most influenced the entire project, without whom it certainly could never have come into existence: Bill Dwyer and Dans Kan and Quillen, for the model-categorical foundations; André Joyal and Jacob Lurie, for the ∞-categorical foundations; Zhen Lin Low, for countless exceedingly helpful conversations; Eric Peterson, for his tireless, dedicated, and impressive TeX support, and for listening patiently to far too many all-too-elaborate "here's where I'm stuck" monologues; David Ayala, for somehow making it through every last one of these papers and providing numerous insightful comments and suggestions, and for providing much-needed encouragement through the tail end of the writing process; Clark Barwick, for many fruitful conversations (including the one that led to the realization that there should exist a notion of "model ∞-categories" in the first place!), and for his consistent enthusiasm for this project; Peter Teichner, for his trust in allowing free rein to explore, for his generous support and provision for so many extended visits to so many institutions around the world, and for his constant advocacy on our behalf; and, lastly and absolutely essentially, Katherine de Kleer, for her boundless love, for her ample patience, and for bringing beauty and excitement into each and every day.

1. The fundamental theorem of model ∞-categories

Given an ∞-category $\mathcal{M}$ equipped with a subcategory $\mathbf{W} \subset \mathcal{M}$, the primary purpose of extending these data to a model structure is to obtain an efficient and computable presentation of the hom-spaces in the localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$. In this section, we work towards a precise statement of this presentation, which comprises the *fundamental theorem of model ∞-categories* (1.9).

A key feature of a model structure is that it allows one to say what it means for two maps in $\mathcal{M}$ to be "homotopic", that is, to become equivalent (in the ∞-categorical sense) upon application of the localization functor $\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$. Classically, to pass to the homotopy category of a relative 1-category (i.e. to its 1-categorical localization), one simply *identifies* maps that are homotopic. In keeping with the core philosophy of higher category theory, we will instead want to *remember* these homotopies, and then of course we'll also want to keep track of the higher homotopies between them.

In the theory of model 1-categories, to abstractify the notion of a "homotopy" between maps from an object x to an object y, one introduces the dual notions of *cylinder objects* and *path objects*. In the ∞-categorical setting, at first glance it might seem that it will suffice to take cylinder and path objects to be as they were before (namely, as certain factorizations of the fold and diagonal maps, respectively): we'll recover a space of maps from a cylinder object for x to y, and we might hope that these spaces will keep track of higher homotopies for us. However, this is not necessarily the case:

it might be that a particular homotopy *between* homotopies only exists after passing to a cylinder object on the cylinders themselves. Of course, it is not possible to guarantee that this process will terminate at some finite stage, and so we must allow for an infinite sequence of such maneuvers.

Although the geometric intuition here no longer corresponds to mere cylinders and paths, we nevertheless recycle the terminology.

Definition 1.1. Let $\mathcal{M}$ be a model ∞-category. A ***cylinder object*** for an object $x \in \mathcal{M}$ is a cosimplicial object $\mathrm{cyl}^{\bullet}(x) \in c\mathcal{M}$ equipped with an equivalence $x \simeq \mathrm{cyl}^0(x)$, such that

- the codegeneracy maps $\mathrm{cyl}^n(x) \xrightarrow{\sigma^i} \mathrm{cyl}^{n-1}(x)$ are all in $\mathbf{W}$, and

- the latching maps $\mathrm{L}_n \, \mathrm{cyl}^{\bullet}(x) \to \mathrm{cyl}^n(x)$ are in $\mathbf{C}$ for all $n \geq 1$.

The cylinder object is called ***special*** if the codegeneracy maps are all also in $\mathbf{F}$ and the matching maps $\mathrm{cyl}^n(x) \to \mathrm{M}_n \, \mathrm{cyl}^{\bullet}(x)$ are in $\mathbf{W} \cap \mathbf{F}$ for all $n \geq 1$. We will use the notation $_\sigma\mathrm{cyl}^{\bullet}(x) \in c\mathcal{M}$ to denote a special cylinder object for $x \in \mathcal{M}$.

Dually, a ***path object*** for an object $y \in \mathcal{M}$ is a simplicial object $\mathrm{path}_{\bullet}(y) \in s\mathcal{M}$ equipped with an equivalence $y \simeq \mathrm{path}_0(y)$, such that

- the degeneracy maps $\mathrm{path}_n(y) \xrightarrow{\sigma_i} \mathrm{path}_{n+1}(y)$ are all in $\mathbf{W}$, and

- the matching maps $\mathrm{path}_n(y) \to \mathrm{M}_n \, \mathrm{path}_{\bullet}(y)$ are in $\mathbf{F}$ for all $n \geq 1$.

The path object is called ***special*** if the degeneracy maps are all also in $\mathbf{C}$ and the latching maps $\mathrm{L}_n \, \mathrm{path}_{\bullet}(y) \to \mathrm{path}_n(y)$ are in $\mathbf{W} \cap \mathbf{C}$ for all $n \geq 1$. We will use the notation $_\sigma\mathrm{path}_{\bullet}(y) \in s\mathcal{M}$ to denote a special path object for $y \in \mathcal{M}$.

Remark 1.2. Restricting a cylinder object $\mathrm{cyl}^{\bullet}(x) \in c\mathcal{M}$ to the subcategory $\mathbf{\Delta}_{\leq 1} \subset \mathbf{\Delta}$ and employing the identification $x \simeq \mathrm{cyl}^0(x)$, we recover the classical notion of a cylinder object, i.e. a factorization

$$x \sqcup x \rightarrowtail \mathrm{cyl}^1(x) \xrightarrow{\approx} x$$

of the fold map; the specialness condition then restricts to the single requirement that the weak equivalence $\mathrm{cyl}^1(x) \xrightarrow{\approx} x$ also be a fibration. In particular, if $\mathrm{ho}(\mathcal{M})$ is a model category – recall from Example S.2.11 that this will be the case as long as $\mathrm{ho}(\mathcal{M})$ satisfies limit axiom $\mathrm{M}_{\infty}1$ (i.e. is finitely bicomplete), e.g. if $\mathcal{M}$ is itself a 1-category –, then a cylinder object $\mathrm{cyl}^{\bullet}(x) \in c\mathcal{M}$ for $x \in \mathcal{M}$ gives rise to a cylinder object for $x \in \mathrm{ho}(\mathcal{M})$ in the classical sense. Of course, dual observations apply to path objects.

Remark 1.3. One might think of a cylinder object as a "cofibrant $\mathbf{W}$-cohypercover", and dually of a path object as a "fibrant $\mathbf{W}$-hypercover". Indeed, if $x \in \mathcal{M}^c$ then a cylinder object $\mathrm{cyl}^{\bullet}(x) \in c\mathcal{M}$ defines a cofibrant

replacement

$$\varnothing_{c\mathcal{M}} \rightarrowtail \mathrm{cyl}^\bullet(x) \xrightarrow{\approx} \mathrm{const}(x)$$

in $c\mathcal{M}_{\mathrm{Reedy}}$, and dually if $y \in \mathcal{M}^f$ then a path object $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ defines a fibrant replacement

$$\mathrm{const}(y) \xrightarrow{\approx} \mathrm{path}_\bullet(y) \twoheadrightarrow \mathrm{pt}_{s\mathcal{M}}$$

in $s\mathcal{M}_{\mathrm{Reedy}}$.[3] Note, however, that under definition 1.1, not every such co/fibrant replacement defines a cylinder/path object, simply because of our requirements that the 0th objects remain unchanged. In turn, we have made this requirement so that remark 1.2 is true, i.e. so that our definition recovers the classical one.

By contrast, in [4, 4.3], Dwyer–Kan introduce the notions of "co/simplicial resolutions" of objects in a model category (with the "special" condition appearing in [4, Remark 6.8]). These are functionally equivalent to our cylinder and path objects; the biggest difference is just that the 0th object of one of their resolutions is required to be a co/fibrant replacement of the original object. Of course, we'll ultimately only care about cylinder objects for cofibrant objects and path objects for fibrant objects, and on the other hand they eventually reduce their proofs to the case of co/simplicial resolutions in which this replacement map is the identity (so that in particular the original object is co/fibrant). Thus, in the end the difference is almost entirely aesthetic.

Remark 1.4. Since definition 1.1 is somewhat involved, here we collect the intuition and/or justification behind each of the pieces of the definition, focusing on (special) path objects.

- A path object is supposed to be a sort of simplicial resolution. Thus, the first demand we should place on this simplicial object is that it be "homotopically constant", i.e. its structure maps should be weak equivalences. This is accomplished by the requirement that the degeneracy maps lie in $\mathbf{W} \subset \mathcal{M}$.

- On the other hand, a path object should also be "good for mapping into" (as discussed in remark 1.3). This fibrancy-like property is encoded by the requirement that the matching maps lie in $\mathbf{F} \subset \mathcal{M}$. (By the dual of lemma 2.2 (whose proof uses (the dual of) this condition), when $y \in \mathcal{M}$ is fibrant then so are all the objects $\mathrm{path}_n(y) \in \mathcal{M}$, for any path object $\mathrm{path}_\bullet(y) \in s\mathcal{M}$.)

- The first condition for the specialness of $\mathrm{path}_\bullet(y)$ – that the degeneracy maps are (acyclic) cofibrations – guarantees that for each $n \geq 0$, the unique structure map $y \simeq \mathrm{path}_0(y) \to \mathrm{path}_n(y)$ is also

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[3]Since the object $[0] \in \mathbf{\Delta}$ is terminal we obtain an adjunction $(-)^0 : c\mathcal{M} \rightleftarrows \mathcal{M} : \mathrm{const}$, via which the equivalence $\mathrm{cyl}^0(x) \xrightarrow{\sim} x$ in $\mathcal{M}$ determines a map $\mathrm{cyl}^\bullet(x) \to \mathrm{const}(x)$ in $c\mathcal{M}$; the map $\mathrm{const}(y) \to \mathrm{path}_\bullet(y)$ arises dually.

a cofibration. This is necessary for lemma 5.2 to even make sense, and also appears in the proof of the *factorization lemma* (4.24).

- The second condition for the specialness of $\mathrm{path}_\bullet(y)$ – that the latching maps be acyclic cofibrations – guarantees that special path objects are "weakly initial" among all path objects (in a sense made precise in lemma 3.2(2)).

Of course, these notions are only useful because of the following existence result.

Proposition 1.5. *Let $\mathcal{M}$ be a model ∞-category.*

(1) Every object of $\mathcal{M}$ admits a special cylinder object.

(2) Every object of $\mathcal{M}$ admits a special path object.

Proof. We only prove part (2); part (1) will then follow by duality. So, suppose we are given any object $y \in \mathcal{M}$. First, set $\mathrm{path}_0(y) = y$. Then, we inductively define $\mathrm{path}_n(y)$ by taking a factorization

$$
\mathrm{L}_n\,\mathrm{path}_\bullet(y) \xrightarrow{\hspace{3cm}} \mathrm{M}_n\,\mathrm{path}_\bullet(y)
$$
$$
\mathrm{path}_n(y)
$$

of the canonical map using factorization axiom $\mathrm{M}_\infty 5$.[4] As observed in Remark Q.1.15, this procedure suffices to define a simplicial object $\mathrm{path}_\bullet(y) \in s\mathcal{M}$.

Now, by construction, above degree 0 the latching maps are all in $\mathbf{W} \cap \mathbf{C}$ while the matching maps are all in $\mathbf{F}$. Thus, it only remains to check that the degeneracy maps are all in $\mathbf{W} \cap \mathbf{C}$. For this, note that for any $n \geq 0$, every degeneracy map $\mathrm{path}_n(y) \xrightarrow{\sigma_i} \mathrm{path}_{n+1}(y)$ factors canonically as a composite

$$
\mathrm{path}_n(y) \to \mathrm{L}_{n+1}\,\mathrm{path}_\bullet(y) \xrightarrow{\approx} \mathrm{path}_{n+1}(y)
$$

in $\mathcal{M}$, where the first map is the inclusion into the colimit at the object

$$
([n]^\circ \xrightarrow{\sigma_i} [n+1]^\circ) \in \partial\left(\overrightarrow{\boldsymbol{\Delta}^{op}}_{/[n+1]^\circ}\right).
$$

So, it suffices to show that this first map is also in $\mathbf{W} \cap \mathbf{C}$. This follows from applying lemma 1.6 to the data of

- the model ∞-category $\mathcal{M}$,

- the Reedy category $\partial\left(\overrightarrow{\boldsymbol{\Delta}^{op}}_{/[n+1]^\circ}\right)$,

- the maximal object $([n]^\circ \xrightarrow{\sigma_i} [n+1]^\circ) \in \partial\left(\overrightarrow{\boldsymbol{\Delta}^{op}}_{/[n+1]^\circ}\right)$, and

---

[4]At $n = 1$, the map $\mathrm{L}_1\,\mathrm{path}_\bullet(y) \to \mathrm{M}_1\,\mathrm{path}_\bullet(y)$ is just the diagonal map $y \to y \times y$.

- the composite functor

$$\partial\left(\overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ}\right) \hookrightarrow \overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ} \to \overrightarrow{\mathbf{\Delta}^{op}} \hookrightarrow \mathbf{\Delta}^{op} \xrightarrow{\mathrm{path}_\bullet(y)} \mathcal{M}.$$

Indeed, $\partial\left(\overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ}\right)$ is a Reedy category equal to its own direct subcategory by Lemma Q.1.28(1)(a), and it is clearly a poset. Moreover, our composite functor satisfies the hypothesis of lemma 1.6 by Lemma Q.1.28(1)(b); in fact, all the latching maps are acyclic cofibrations except for possibly the one at the initial object

$$([0]^\circ \to [n+1]^\circ) \in \partial\left(\overrightarrow{\mathbf{\Delta}^{op}}_{/[n+1]^\circ}\right).$$

Therefore, the degeneracy map $\mathrm{path}_n(y) \xrightarrow{\sigma_i} \mathrm{path}_{n+1}(y)$ is indeed an acyclic cofibration, and hence the object $\mathrm{path}_n(y) \in s\mathcal{M}$ defines a special path object for an arbitrary object $y \in \mathcal{M}$.   □

The proof of proposition 1.5 relies on the following result.

Lemma 1.6. *Let $\mathcal{M}$ be a model ∞-category, let $\mathcal{C}$ be a Reedy poset which is equal to its own direct subcategory, and let $m \in \mathcal{C}$ be a maximal element. Suppose that $\mathcal{C} \xrightarrow{F} \mathcal{M}$ is a functor such that for any $c \in \mathcal{C}$ which is incomparable to $m \in \mathcal{C}$ (i.e. such that $\hom_{\mathcal{C}}(c, m) = \varnothing_{\mathrm{Set}}$), the latching map $\mathrm{L}_c F \to F(c)$ lies in $(\mathbf{W} \cap \mathbf{C}) \subset \mathcal{M}$. Then, the induced map $F(m) \to \mathrm{colim}_{\mathcal{C}}(F)$ also lies in $(\mathbf{W} \cap \mathbf{C}) \subset \mathcal{M}$.*

Proof. We begin by observing that for any object $c \in \mathcal{C}$, the forgetful map $\mathcal{C}_{/c} \to \mathcal{C}$ is actually the inclusion of a full subposet. Now, writing $\mathcal{C}' = (\mathcal{C} \backslash \{m\}) \subset \mathcal{C}$, it is easy to see that we have a pushout square

$$
\begin{array}{ccc}
\partial(\mathcal{C}_{/m}) & \longrightarrow & \mathcal{C}_{/m} \\
\downarrow & & \downarrow \\
\mathcal{C}' & \longrightarrow & \mathcal{C}
\end{array}
$$

in $\mathcal{C}at_\infty$ of inclusions of full subposets. By Proposition T.4.4.2.2, this induces a pushout square

$$
\begin{array}{ccc}
\mathrm{L}_m F & \longrightarrow & F(m) \\
\downarrow & & \downarrow \\
\mathrm{colim}_{\mathcal{C}'}(F) & \longrightarrow & \mathrm{colim}_{\mathcal{C}}(F)
\end{array}
$$

in $\mathcal{M}$ (where the colimits all exist by limit axiom $\mathrm{M}_\infty 1$, and where we simply write F again for its restriction to any subposet of $\mathcal{C}$).[5] Thus, it suffices to

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[5]In the statement of Proposition T.4.4.2.2, note that the requirement that one of the maps be a monomorphism (i.e. a cofibration in $s\mathcal{S}et_{\mathrm{Joyal}}$) guarantees that this pushout is indeed a homotopy pushout in $s\mathcal{S}et_{\mathrm{Joyal}}$ (by the left properness of $s\mathcal{S}et_{\mathrm{Joyal}}$, or alternatively by the Reedy trick).

show that the map $\mathrm{L}_m F \to \mathrm{colim}_{\mathcal{C}'}(F)$ lies in $(\mathbf{W} \cap \mathbf{C}) \subset \mathcal{M}$, since this subcategory is closed under pushouts.

For this, let us choose an ordering

$$\mathcal{C}' \backslash \partial(\mathcal{C}_{/m}) = \{c_1, \ldots, c_k\}$$

such that for every $1 \leq i \leq k$ the object c_i is minimal in the full subposet $\{c_i, \ldots, c_k\} \subset \mathcal{C}.$[6] Let us write

$$\mathcal{C}_i = (\partial(\mathcal{C}_{/m}) \cup \{c_1, \ldots, c_i\}) \subset \mathcal{C}'$$

for the full subposet, setting $\mathcal{C}_0 = \partial(\mathcal{C}_{/m})$ for notational convenience, so that we have the chain of inclusions

$$\partial(\mathcal{C}_{/m}) = \mathcal{C}_0 \subset \cdots \subset \mathcal{C}_k = \mathcal{C}'.$$

Our requirement on the ordering of the objects c_i guarantees that we have

$$\partial(\mathcal{C}_{/c_i}) \subset \mathcal{C}_{i-1},$$

and from here it is not hard to see that in fact we have a pushout square

$$
\begin{array}{ccc}
\partial(\mathcal{C}_{/c_i}) & \longrightarrow & \mathcal{C}_{i-1} \\
\downarrow & & \downarrow \\
\mathcal{C}_{/c_i} & \longrightarrow & \mathcal{C}_i
\end{array}
$$

in $\mathcal{C}\mathrm{at}_\infty$ for all $1 \leq i \leq k$, from which by again applying Proposition T.4.4.2.2 we obtain a pushout square

$$
\begin{array}{ccc}
\mathrm{L}_{c_i} F & \longrightarrow & \mathrm{colim}_{\mathcal{C}_{i-1}}(F) \\
\downarrow & & \downarrow \\
F(c_i) & \longrightarrow & \mathrm{colim}_{\mathcal{C}_i}(F)
\end{array}
$$

in $\mathcal{M}$. But since $\hom_{\mathcal{C}}(c_i, m) = \varnothing_{\mathcal{S}\mathrm{et}}$ by assumption, our hypotheses imply that the map $\mathrm{L}_{c_i} F \to F(c_i)$ lies in $(\mathbf{W} \cap \mathbf{C}) \subset \mathcal{M}$; since this subcategory is closed under pushouts, it follows that it contains the map $\mathrm{colim}_{\mathcal{C}_{i-1}}(F) \to \mathrm{colim}_{\mathcal{C}_i}(F)$ as well. Thus, we have obtained the map $\mathrm{L}_m F \to \mathrm{colim}_{\mathcal{C}'}(F)$ as a composite

$$\mathrm{L}_m F = \mathrm{colim}_{\partial(\mathcal{C}_{/m})}(F) = \mathrm{colim}_{\mathcal{C}_0}(F) \overset{\approx}{\rightarrowtail} \cdots \overset{\approx}{\rightarrowtail} \mathrm{colim}_{\mathcal{C}_k}(F) = \mathrm{colim}_{\mathcal{C}'}(F)$$

of acyclic cofibrations in $\mathcal{M}$, so it is itself an acyclic cofibration. This proves the claim. $\qquad \square$

---

[6]If the Reedy structure on $\mathcal{C}$ is induced by a degree function $\mathrm{N}(\mathcal{C})_0 \overset{\deg}{\longrightarrow} \mathbb{N}$ (which must be possible by its finiteness), then this can be accomplished simply by requiring that $\deg(c_i) \leq \deg(c_{i+1})$ for all $1 \leq i < k$.

Now that we have shown that (special) cylinder and path objects always exist, we come to the following key definitions. These should be expected: taking the *quotient* by a relation in a 1-topos corresponds to taking the *geometric realization* of a simplicial object in an ∞-topos. (Among these, *equivalence* relations then correspond to ∞-*groupoid* objects (see Definition T.6.1.2.7).)

Definition 1.7. Let $\mathcal{M}$ be a model ∞-category, and let $x, y \in \mathcal{M}$. We define the space of ***left homotopy classes of maps*** from x to y with respect to a given cylinder object $\mathrm{cyl}^\bullet(x)$ for x to be

$$\hom_{\mathcal{M}}^{\overset{l}{\sim}}(x, y) = \left| \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(x), y) \right|.$$

Dually, we define the space of ***right homotopy classes of maps*** from x to y with respect to a given path object $\mathrm{path}_\bullet(y)$ for y to be

$$\hom_{\mathcal{M}}^{\overset{r}{\sim}}(x, y) = \left| \hom_{\mathcal{M}}^{\mathrm{lw}}(x, \mathrm{path}_\bullet(y)) \right|.$$

A priori these spaces depend on the choices of cylinder or path objects, but we nevertheless suppress them from the notation.

Remark 1.8. Note that $\hom_{\mathcal{M}}^{\mathrm{lw}}(x, \mathrm{path}_\bullet(y))$ is not itself an ∞-groupoid object in $\mathcal{S}$. To ask for this would be too strict: it would not allow for the "homotopies between homotopies" that we sought at the beginning of this section. (Correspondingly, by Yoneda's lemma this would also imply that $\mathrm{path}_\bullet(y)$ is itself an ∞-groupoid object in $\mathcal{M}$, which is clearly a far stronger condition than the "fibrant $\mathbf{W}$-hypercover" heuristic of remark 1.3 would dictate.)

We can now state the ***fundamental theorem of model ∞-categories***, which says that under the expected co/fibrancy hypotheses, the spaces of left and right homotopy classes of maps both compute the hom-space in the localization.

Theorem 1.9. *Let $\mathcal{M}$ be a model ∞-category, suppose that $x \in \mathcal{M}^c$ is cofibrant and $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ is any cylinder object for x, and suppose that $y \in \mathcal{M}^f$ is fibrant and $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ is any path object for y. Then there is a diagram of equivalences*

$$\hom_{\mathcal{M}}^{\overset{l}{\sim}}(x, y) \xrightarrow{\ \sim\ } \left\| \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_\bullet(y)) \right\| \xleftarrow{\ \sim\ } \hom_{\mathcal{M}}^{\overset{r}{\sim}}(x, y)$$

$$\Big\downarrow^{\wr}$$

$$\hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x, y)$$

in $\mathcal{S}$.

Proof. The horizontal equivalences are proved as proposition 2.1(3) and its dual. By proposition 3.4, it suffices to assume that both $\mathrm{cyl}^\bullet(x)$ and $\mathrm{path}_\bullet(y)$

are special. The vertical equivalence is then obtained as the composite of the equivalences

$$\left\|\hom_{\mathcal{M}}^{\mathrm{lw}}(_{\sigma}\mathrm{cyl}^{\bullet}(x),\,_{\sigma}\mathrm{path}_{\bullet}(y))\right\| \simeq \underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}} \simeq \underline{\mathbf{3}}(x,y)^{\mathrm{gpd}}$$

$$\simeq \underline{\mathbf{7}}(x,y)^{\mathrm{gpd}} \simeq \hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y)$$

(where the as-yet-undefined objects of which will be explained in notation 4.10 and definition 4.15) which are respectively proved as Propositions 5.1 (and 3.4), 6.1, 7.1, and 9.1. □

Remark 1.10. The proof of the fundamental theorem of model ∞-categories (1.9) roughly follows that of [4, Proposition 4.4] (and specifically the fix given in [9, §7] for [4, 7.2(iii)]). Speaking ahistorically, the main difference is that we have replaced the ultimate appeal to the hammock localization as providing a model for the hom-space $\hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y)$ with an appeal to the (∞-categorical) Rezk nerve $\mathrm{N}_{\infty}^{\mathrm{R}}(\mathcal{M},\mathbf{W})$, which we will prove (as proposition 8.1) likewise provides a model for this hom-space (by the local universal property of the Rezk nerve (Theorem N.3.8)).

An easy consequence of the fundamental theorem of model ∞-categories (1.9) is its "homotopy" version.

Corollary 1.11. *Let $\mathcal{M}$ be a model ∞-category, suppose that $x \in \mathcal{M}^c$ is cofibrant and $\mathrm{cyl}^{\bullet}(x) \in c\mathcal{M}$ is any cylinder object for x, and suppose that $y \in \mathcal{M}^f$ is fibrant and $\mathrm{path}_{\bullet}(y) \in s\mathcal{M}$ is any path object for y. Then there is a diagram of isomorphisms*

$$\left(\frac{[x,y]_{\mathcal{M}}}{[\mathrm{cyl}^1(x),y]_{\mathcal{M}}}\right) \xrightarrow{\ \sim\ } [x,y]_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]} \xleftarrow{\ \sim\ } \left(\frac{[x,y]_{\mathcal{M}}}{[x,\mathrm{path}_1(y)]_{\mathcal{M}}}\right)$$

in $\mathcal{S}\mathrm{et}$.

Proof. Observe that we have a commutative square

$$\begin{array}{ccc} s\mathcal{S} & \xrightarrow{\ \pi_0^{\mathrm{lw}}\ } & s\mathcal{S}\mathrm{et} \\ {\scriptstyle\mathrm{colim}_{\boldsymbol{\Delta}^{op}}^{\mathcal{S}}(-)}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathrm{colim}_{\boldsymbol{\Delta}^{op}}^{\mathcal{S}\mathrm{et}}(-)} \\ \mathcal{S} & \xrightarrow[\ \pi_0\ ]{} & \mathcal{S}\mathrm{et} \end{array}$$

in $\mathcal{C}\mathrm{at}_{\infty}$, since all four functors are left adjoints and the resulting composite right adjoints coincide. The claim now follows immediately from theorem 1.9. □

Remark 1.12. In the particular case that $\mathcal{M}$ is a model 1-category, we obtain equivalences $\mathrm{ho}(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$ and $\mathrm{ho}(\mathcal{M}[\![\mathbf{W}^{-1}]\!]) \xrightarrow{\sim} \mathcal{M}[\mathbf{W}^{-1}]$. Hence, corollary 1.11 specializes to recover the classical fundamental theorem of model categories (see e.g. [6, Theorems 7.4.9 and 8.3.9]).

Remark 1.13. In contrast with remark 1.8, the proof of [6, Theorem 7.4.9] carries over without essential change to show that in the situation of corollary 1.11, the diagram

$$[\mathrm{cyl}^1(x), y]_{\mathcal{M}} \qquad [x, \mathrm{path}_1(y)]_{\mathcal{M}}$$

$$[x, y]_{\mathcal{M}}$$

does define a pair of equal equivalence relations (in $\mathcal{S}et$).

2. The equivalence $\mathrm{hom}_{\mathcal{M}}^{\overset{\iota}{\sim}}(x, y) \simeq \big\|\mathrm{hom}_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_\bullet(y))\big\|$

Without first setting up any additional scaffolding, we can immediately prove the horizontal equivalences of theorem 1.9. The following result is an analog of [4, Proposition 6.2, Corollary 6.4, and Corollary 6.5].

Proposition 2.1. *Let $\mathcal{M}$ be a model ∞-category, suppose that $x \in \mathcal{M}^c$ is cofibrant, and let $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ be any cylinder object for x.*

(1) *The functor*

$$\mathcal{M} \xrightarrow{\mathrm{hom}_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(x), -)} s\mathcal{S}$$

sends $(\mathbf{W} \cap \mathbf{F}) \subset \mathcal{M}$ into $(\mathbf{W} \cap \mathbf{F})_{\mathrm{KQ}} \subset s\mathcal{S}$.

(2) *The same functor sends $(\mathcal{M}^f \cap \mathbf{W}) \subset \mathcal{M}$ into $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$.*

(3) *If $y \in \mathcal{M}^f$ is fibrant, then for any path object $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ for y, the canonical map $\mathrm{const}(y) \to \mathrm{path}_\bullet(y)$ in $s\mathcal{M}$ induces an equivalence*

$$\big|\mathrm{hom}_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(x), y)\big| \xrightarrow{\sim} \big\|\mathrm{hom}_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_\bullet(y))\big\|.$$

Proof. To prove part (1), we use the criterion of Proposition S.7.2 (that $s\mathcal{S}_{\mathrm{KQ}}$ has a set of generating cofibrations given by the boundary inclusions $I_{\mathrm{KQ}} = \{\partial\Delta^n \to \Delta^n\}_{n \geq 0}$). First, note that to say that x is cofibrant is to say that the 0th latching map $\varnothing_{\mathcal{M}} \simeq \mathrm{L}_0\,\mathrm{cyl}^\bullet(x) \to \mathrm{cyl}^0(x) \simeq x$ of $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ is also a cofibration. Then, for any $n \geq 0$, suppose we are given an acyclic fibration $y \xrightarrow{\approx} z$ in $\mathcal{M}$ inducing the right map in any commutative square

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathrm{hom}_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(x), y) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{hom}_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(x), z) \end{array}$$

in $s\mathcal{S}$. This commutative square is equivalent data to that of a commutative square

$$
\begin{array}{ccc}
\mathrm{L}_n\,\mathrm{cyl}^\bullet(x) & \longrightarrow & y \\
\downarrow & & \downarrow \wr \\
\mathrm{cyl}^n(x) & \longrightarrow & z,
\end{array}
$$

in $\mathcal{M}$, and mooveover a lift in either one determines a lift in the other. But the latter admits a lift by lifting axiom $\mathrm{M}_\infty 4$. Hence, the induced map $\hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), y) \to \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), z)$ is indeed in $(\mathbf{W} \cap \mathbf{F})_{\mathrm{KQ}}$.

Next, part (2) follows immediately from part (1) and the dual of Kenny Brown's lemma (Q.3.5).

To prove part (3), note that all structure maps in any path object are weak equivalences, and note also that when y is fibrant, then any path object $\mathrm{path}_\bullet(y)$ consists of fibrant objects by the dual of lemma 2.2. Hence, using

- Fubini's theorem for colimits,
- part (2), and
- the fact that simplicial objects whose structure maps are equivalences must be constant,

we obtain the string of equivalences

$$
\begin{aligned}
\left\| \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_\bullet(y)) \right\| & \\
&= \mathrm{colim}_{([m]^\circ, [n]^\circ) \in \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}} \hom_{\mathcal{M}}(\mathrm{cyl}^m(x), \mathrm{path}_n(y)) \\
&\simeq \mathrm{colim}_{[n]^\circ \in \mathbf{\Delta}^{op}} \left(\mathrm{colim}_{[m]^\circ \in \mathbf{\Delta}^{op}} \hom_{\mathcal{M}}(\mathrm{cyl}^m(x), \mathrm{path}_n(y)) \right) \\
&= \mathrm{colim}_{[n]^\circ \in \mathbf{\Delta}^{op}} \left| \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_n(y)) \right| \\
&\simeq \left| \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_0(y)) \right| \\
&\simeq \left| \hom^{\mathrm{lw}}_{\mathcal{M}}(\mathrm{cyl}^\bullet(x), y) \right|,
\end{aligned}
$$

proving the claim. $\qquad\square$

We needed the following auxiliary result in the proof of proposition 2.1.

Lemma 2.2. *If $x \in \mathcal{M}^c$ is cofibrant, then for any cylinder object $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ for x, for every $n \geq 0$ the object $\mathrm{cyl}^n(x) \in \mathcal{M}$ is cofibrant.*

Proof. Since $\mathrm{cyl}^0(x) \simeq x$ by definition, the claim holds at $n = 0$ by assumption. For $n \geq 1$, by definition we have a cofibration $\mathrm{L}_n\,\mathrm{cyl}^\bullet(x) \rightarrowtail \mathrm{cyl}^n(x)$, so it suffices to show that the object $\mathrm{L}_n\,\mathrm{cyl}^\bullet(x) \in \mathcal{M}$ is cofibrant. We prove this by induction: at $n = 0$, we have $\mathrm{L}_0\,\mathrm{cyl}^\bullet(x) = \mathrm{cyl}^0(x) \sqcup \mathrm{cyl}^0(x) \simeq x \sqcup x$, which is cofibrant.

Now, recall that by definition,

$$
\mathrm{L}_n\,\mathrm{cyl}^\bullet(x) = \mathrm{colim}_{\partial\left(\overrightarrow{\mathbf{\Delta}}_{/[n]}\right)} \mathrm{cyl}^\bullet(x),
$$

i.e. the latching object is given by the colimit of the composite

$$\partial\left(\vec{\mathbf{\Delta}}_{/[n]}\right) \hookrightarrow \vec{\mathbf{\Delta}}_{/[n]} \to \vec{\mathbf{\Delta}} \hookrightarrow \mathbf{\Delta} \xrightarrow{\mathrm{cyl}^\bullet(x)} \mathcal{M}.$$

Now, by Lemma Q.1.28(1)(a), the latching category $\partial\left(\vec{\mathbf{\Delta}}_{/[n]}\right)$ admits a Reedy category structure with fibrant constants, so that we obtain a Quillen adjunction

$$\mathrm{colim} : \mathrm{Fun}\left(\partial\left(\vec{\mathbf{\Delta}}_{/[n]}\right),\mathcal{M}\right)_{\mathrm{Reedy}} \rightleftarrows \mathcal{M} : \mathrm{const}$$

(since $\mathcal{M}$ is finitely cocomplete by limit axiom $\mathrm{M}_\infty 1$). Thus, it suffices to check that the above composite defines a cofibrant object of

$$\mathrm{Fun}\left(\partial\left(\vec{\mathbf{\Delta}}_{/[n]}\right),\mathcal{M}\right)_{\mathrm{Reedy}}.$$

For this, given an object $([m] \hookrightarrow [n]) \in \partial\left(\vec{\mathbf{\Delta}}_{/[n]}\right)$, by Lemma Q.1.28(1)(b), its latching category is given by

$$\partial\left(\overrightarrow{\partial\left(\vec{\mathbf{\Delta}}_{/[n]}\right)}_{/([m]\hookrightarrow[n])}\right) \cong \partial\left(\vec{\mathbf{\Delta}}_{/[m]}\right).$$

Hence, the latching map of the above composite at this object simply reduces to the cofibration

$$\mathrm{L}_m\,\mathrm{cyl}^\bullet(x) \rightarrowtail \mathrm{cyl}^m(x).$$

Therefore, the above composite does indeed define a cofibrant object of $\mathrm{Fun}\left(\partial\left(\vec{\mathbf{\Delta}}_{/[n]}\right),\mathcal{M}\right)_{\mathrm{Reedy}}$, which proves the claim. $\square$

3. Reduction to the special case

In order to proceed with the string of equivalences in the proof of the fundamental theorem of model ∞-categories (1.9), we will need to be able to make the assumption that our cylinder and path objects are *special*. In this section, we therefore reduce to the special case.

Notation 3.1. Let $\mathcal{M}$ be a model ∞-category. For any $x \in \mathcal{M}$, we write

$$\{\mathrm{cyl}^\bullet(x)\} \subset \left(c\mathcal{M} \underset{(-)^0,\mathcal{M},x}{\times} \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}\right)$$

for the full subcategory on the cylinder objects for x, and we write

$$\{\mathrm{path}_\bullet(x)\} \subset \left(s\mathcal{M} \underset{(-)_0,\mathcal{M},x}{\times} \mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}\right)$$

for the full subcategory on the path objects for x.

We now have the following analog of [4, Propositions 6.9 and 6.10].

Lemma 3.2. *Suppose that $x \in \mathcal{M}$.*

(1) *Every special cylinder object $_\sigma \mathrm{cyl}^\bullet(x) \in \{\mathrm{cyl}^\bullet(x)\}$ is weakly terminal: any $\mathrm{cyl}^\bullet(x) \in \{\mathrm{cyl}^\bullet(x)\}$ admits a map*

$$\mathrm{cyl}^\bullet(x) \to {}_\sigma \mathrm{cyl}^\bullet(x)$$

in $\{\mathrm{cyl}^\bullet(x)\}$.

(2) *Every special path object $_\sigma \mathrm{path}_\bullet(x) \in \{\mathrm{path}_\bullet(x)\}$ is weakly initial: any $\mathrm{path}_\bullet(x) \in \{\mathrm{path}_\bullet(x)\}$ admits a map*

$$_\sigma \mathrm{path}_\bullet(x) \to \mathrm{path}_\bullet(x)$$

in $\{\mathrm{path}_\bullet(x)\}$.

Proof. We only prove the first of two dual statements. We will construct the map by induction. The given equivalences

$$\mathrm{cyl}^0(x) \simeq x \simeq {}_\sigma \mathrm{cyl}^0(x)$$

imply that there is a unique way to begin in degree 0. Then, assuming the map has been constructed up through degree $(n-1)$, definition 1.1 and lifting axiom $\mathrm{M}_\infty 4$ guarantee the existence of a lift in the commutative rectangle

$$
\begin{array}{ccc}
\mathrm{L}_n \, \mathrm{cyl}^\bullet(x) \longrightarrow \mathrm{L}_n \, {}_\sigma \mathrm{cyl}^\bullet(x) \dashrightarrow {}_\sigma \mathrm{cyl}^n(x) \\
\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \wr \\
\mathrm{cyl}^n(x) \longrightarrow {}_\sigma \mathrm{cyl}^n(x) \longrightarrow \mathrm{M}_n \, {}_\sigma \mathrm{cyl}^\bullet(x)
\end{array}
$$

in $\mathcal{M}$, which provides an extension of the map up through degree n. $\qquad\square$

Lemma 3.3. *Let $\mathcal{M}$ be a model ∞-category, let $x \in \mathcal{M}^c$ be cofibrant, let $y \in \mathcal{M}^f$ be fibrant, let $\mathrm{cyl}_1^\bullet(x) \to \mathrm{cyl}_2^\bullet(x)$ be a map in $\{\mathrm{cyl}^\bullet(x)\}$, and suppose that $\mathrm{path}_\bullet(y) \in \{\mathrm{path}_\bullet(y)\}$. Then the induced maps*

$$\left| \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}_2^\bullet(x), y) \right| \to \left| \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}_1^\bullet(x), y) \right|$$

and

$$\left\| \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}_2^\bullet(x), \mathrm{path}_\bullet(y)) \right\| \to \left\| \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}_1^\bullet(x), \mathrm{path}_\bullet(y)) \right\|$$

are equivalences in $\mathcal{S}$.

Proof. By proposition 2.1(3) and its dual, these data induce a commutative diagram

$$
\begin{array}{ccc}
\left| \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}_2^\bullet(x), y) \right| & \longrightarrow & \left| \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}_1^\bullet(x), y) \right| \\
\wr \downarrow & & \downarrow \wr \\
\left\| \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}_2^\bullet(x), \mathrm{path}_\bullet(y)) \right\| & \longrightarrow & \left\| \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}_1^\bullet(x), \mathrm{path}_\bullet(y)) \right\| \\
& \nwarrow \sim \qquad \sim \nearrow & \\
& \left| \hom_{\mathcal{M}}^{\mathrm{lw}}(x, \mathrm{path}_\bullet(y)) \right| &
\end{array}
$$

of equivalences in $\mathcal{S}$.   □

Proposition 3.4. *Let $\mathcal{M}$ be a model ∞-category, let $x, y \in \mathcal{M}$, let $\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ be a cylinder object for x, and let $\mathrm{path}_\bullet(y) \in s\mathcal{M}$ be a path object for y. Then there exist*

- *a map $\mathrm{cyl}^\bullet(x) \to {}_\sigma\mathrm{cyl}^\bullet(x)$ to a special cylinder object for x, and*

- *a map $\mathrm{path}_\bullet(y) \to {}_\sigma\mathrm{path}_\bullet(y)$ to a special path object for y,*

such that the induced square

$$\begin{array}{ccc} \hom_{\mathcal{M}}^{\mathrm{lw}}({}_\sigma\mathrm{cyl}^\bullet(x), {}_\sigma\mathrm{path}_\bullet(y)) & \longrightarrow & \hom_{\mathcal{M}}^{\mathrm{lw}}({}_\sigma\mathrm{cyl}^\bullet(x), \mathrm{path}_\bullet(y)) \\ \downarrow & & \downarrow \\ \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(x), {}_\sigma\mathrm{path}_\bullet(y)) & \longrightarrow & \hom_{\mathcal{M}}^{\mathrm{lw}}(\mathrm{cyl}^\bullet(x), \mathrm{path}_\bullet(y)) \end{array}$$

in $ss\mathcal{S}$ becomes a square of equivalences upon applying the colimit functor

$$ss\mathcal{S} \xrightarrow{\ \|-\|\ } \mathcal{S}.$$

Proof. The maps are obtained from lemma 3.2; the claim then follows from lemma 3.3.   □

4. Model diagrams and left homotopies

In the remainder of the proof of the fundamental theorem of model ∞-categories (1.9), it will be convenient to have a framework for corepresenting diagrams of a specified type in our model ∞-category $\mathcal{M}$. This leads to the notion of a *model ∞-diagram*, which we introduce and study in §4.1. Then, in §4.2, we specialize this setup to describe the data that thusly corepresents a "left homotopy" in the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$. (In fact, in order to be completely concrete and explicit we will further specialize to deal only with *model diagrams* (as opposed to model ∞-diagrams), since in the end this is all that we will need.)

4.1. Model diagrams. We will be interested in ∞-categories of diagrams of a specified shape inside of a model ∞-category. These are corepresented, in the following sense.

Definition 4.1. A ***model ∞-diagram*** is an ∞-category $\mathcal{D}$ equipped with three wide subcategories $\mathbf{W}, \mathbf{C}, \mathbf{F} \subset \mathcal{D}$. These assemble into the evident ∞-category, which we denote by Model_∞. Of course, a model ∞-category can be considered as a model ∞-diagram. A ***model diagram*** is a model ∞-diagram whose underlying ∞-category is a 1-category. These assemble into a full subcategory $\mathrm{Model} \subset \mathrm{Model}_\infty$.

Remark 4.2. We introduced *model diagrams* in [11, Definition 3.1], where we required that the subcategory of weak equivalences satisfy the two-out-of-three property. As this requirement is superfluous for our purposes, we

have omitted it from definition 4.1. (However, the wideness requirement is necessary: it guarantees that a map of model diagrams can take any map to an identity map, which in turn jibes with the requirement that the three defining subcategories of a model ∞-category be wide.)

Remark 4.3. A relative ∞-category $(\mathcal{R}, \mathbf{W})$ can be considered as a model ∞-diagram by taking $\mathbf{C} = \mathbf{F} = \mathcal{R}^{\simeq}$. In this way, we will identify $\mathcal{R}el\mathcal{C}at_{\infty} \subset \mathcal{M}odel_{\infty}$ and $\mathcal{R}el\mathcal{C}at \subset \mathcal{M}odel$ as full subcategories.[7]

Notation 4.4. In order to disambiguate our notation associated with various model ∞-diagrams, we will sometimes decorate them for clarity: for instance, we may write $(\mathcal{D}_1, \mathbf{W}_1, \mathbf{C}_1, \mathbf{F}_1)$ and $(\mathcal{D}_2, \mathbf{W}_2, \mathbf{C}_2, \mathbf{F}_2)$ to denote two arbitrary model ∞-diagrams. (This is consistent with both Notations S.1.2 and N.1.3.)

Remark 4.5. Among the axioms for a model ∞-category, all but limit axiom $\mathrm{M}_{\infty}1$ (so two-out-of-three axiom $\mathrm{M}_{\infty}2$, retract axiom $\mathrm{M}_{\infty}3$, lifting axiom $\mathrm{M}_{\infty}4$, and factorization axiom $\mathrm{M}_{\infty}5$) can be encoded by requiring that the underlying model ∞-diagram has the extension property with respect to certain maps of model diagrams.

Since we will be working with a model ∞-category with chosen source and target objects of interest, we also introduce the following variant.

Definition 4.6. A ***doubly-pointed model ∞-diagram*** is a model ∞-diagram $\mathcal{D}$ equipped with a map $\mathrm{pt}_{\mathcal{M}odel_{\infty}} \sqcup \mathrm{pt}_{\mathcal{M}odel_{\infty}} \to \mathcal{D}$. The two inclusions $\mathrm{pt}_{\mathcal{M}odel_{\infty}} \hookrightarrow \mathrm{pt}_{\mathcal{M}odel_{\infty}} \sqcup \mathrm{pt}_{\mathcal{M}odel_{\infty}}$ select objects $s, t \in \mathcal{D}$, which we call the ***source*** and ***target***; we will sometimes subscript these to remove ambiguity, e.g. as $s_{\mathcal{D}}$ and $t_{\mathcal{D}}$. These assemble into the evident ∞-category

$$(\mathcal{M}odel_{\infty})_{**} = (\mathcal{M}odel_{\infty})_{(\mathrm{pt}_{\mathcal{M}odel_{**}} \sqcup \mathrm{pt}_{\mathcal{M}odel_{**}})/} \cdot$$

Of course, there is a forgetful functor $(\mathcal{M}odel_{\infty})_{**} \to \mathcal{M}odel_{\infty}$. We will often implicitly consider a model ∞-diagram equipped with two chosen objects as a doubly-pointed model ∞-diagram. We write $\mathcal{M}odel_{**} \subset (\mathcal{M}odel_{\infty})_{**}$ for the full subcategory of ***doubly-pointed model diagrams***, i.e. of those doubly-pointed model ∞-diagrams whose underlying ∞-category is a 1-category.

Remark 4.7. Similarly to remark 4.3, we will consider $(\mathcal{R}el\mathcal{C}at_{\infty})_{**} \subset (\mathcal{M}odel_{\infty})_{**}$ and $\mathcal{R}el\mathcal{C}at_{**} \subset \mathcal{M}odel_{**}$ as full subcategories.

Notation 4.8. In order to simultaneously refer to the situations of unpointed and doubly-pointed model ∞-diagrams, we will use the notation $(\mathcal{M}odel_{\infty})_{(**)}$ (and similarly for other related notations). When we use this notation, we will mean for the entire statement to be interpreted either in

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[7]This inclusion exhibits $\mathcal{R}el\mathcal{C}at_{\infty}$ as a right localization of $\mathcal{M}odel_{\infty}$. In fact, $\mathcal{R}el\mathcal{C}at_{\infty}$ is also a *left* localization of $\mathcal{M}odel_{\infty}$ via the inclusion which sets both $\mathbf{C}$ and $\mathbf{F}$ to be the entire underlying ∞-category, but this latter inclusion will not play any role here.

the unpointed context or the doubly-pointed context. (This is consistent with Notation H.3.3.)

It will be useful to expand on Definition H.3.5 (in view of remark 4.7) in the following way.

Definition 4.9. We define a ***model word*** to be a (possibly empty) word $\underline{\mathbf{m}}$ in any of the symbols $\mathbf{A}$, $\mathbf{W}$, $\mathbf{C}$, $\mathbf{F}$, $(\mathbf{W} \cap \mathbf{C})$, $(\mathbf{W} \cap \mathbf{F})$ or any of their inverses. Of course, these naturally define doubly-pointed model diagrams; we continue to employ the convention set in Definition H.3.5 that we read our model words *forwards*, so that for instance the model word $\underline{\mathbf{m}} = [\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}]$ defines the doubly-pointed model diagram

$$s \rightarrowtail \bullet \overset{\approx}{\twoheadleftarrow} \bullet \longrightarrow t.$$

We denote this object by $\underline{\mathbf{m}} \in \mathrm{Model}_{**}$. Of course, via remark 4.7, we can consider any relative word as a model word.

Notation 4.10. Since they will appear repeatedly, we make the abbreviation $\underline{\tilde{\mathbf{3}}} = [(\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}; (\mathbf{W} \cap \mathbf{C})^{-1}]$ for the model word

$$s \overset{\approx}{\twoheadleftarrow} \bullet \longrightarrow \bullet \overset{\approx}{\leftarrowtail} t$$

(which is a variant of Notation H.4.2), and we make the abbreviation $\underline{\mathbf{7}} = [\mathbf{W}; \mathbf{W}^{-1}; \mathbf{W}; \mathbf{A}; \mathbf{W}; \mathbf{W}^{-1}; \mathbf{W}]$ for the model word (in fact, relative word)

$$s \overset{\approx}{\leftarrow} \bullet \overset{\approx}{\longrightarrow} \bullet \overset{\approx}{\leftarrow} \bullet \longrightarrow \bullet \overset{\approx}{\leftarrow} \bullet \overset{\approx}{\longrightarrow} \bullet \overset{\approx}{\leftarrow} t.$$

We now make rigorous "the ∞-category of (either unpointed or doubly-pointed) $\mathcal{D}$-shaped diagrams in $\mathcal{M}$ (and either natural transformations or natural weak equivalences between them)".

Notation 4.11. Recall from Notation N.1.6 that $\mathcal{R}el\mathcal{C}at_\infty$ is a cartesian closed symmetric monoidal ∞-category, with internal hom-object given by

$$\left(\mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathcal{R}el}, \mathrm{Fun}(\mathcal{R}_1, \mathcal{R}_2)^{\mathbf{W}} \right) \in \mathcal{R}el\mathcal{C}at_\infty$$

for $(\mathcal{R}_1, \mathbf{W}_1), (\mathcal{R}_2, \mathbf{W}_2) \in \mathcal{R}el\mathcal{C}at_\infty$. It is not hard to see that Model_∞ is enriched and tensored over $(\mathcal{R}el\mathcal{C}at_\infty, \times)$. Namely, for any

$$(\mathcal{D}_1, \mathbf{W}_1, \mathbf{C}_1, \mathbf{F}_1), (\mathcal{D}_2, \mathbf{W}_2, \mathbf{C}_2, \mathbf{F}_2) \in \mathrm{Model}_\infty,$$

we define

$$\left(\mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathrm{Model}}, \mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathbf{W}} \right) \in \mathcal{R}el\mathcal{C}at_\infty$$

by setting

$$\mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathrm{Model}} \subset \mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)$$

to be the full subcategory on those functors which send the subcategories $\mathbf{W}_1, \mathbf{C}_1, \mathbf{F}_1 \subset \mathcal{D}_1$ into $\mathbf{W}_2, \mathbf{C}_2, \mathbf{F}_2 \subset \mathcal{D}_2$ respectively, and setting

$$\mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathbf{W}} \subset \mathrm{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathrm{Model}}$$

to be the (generally non-full) subcategory on the natural weak equivalences; moreover, the tensoring is simply the cartesian product in $\mathcal{M}\text{odel}_\infty$ (composed with the inclusion $\mathcal{R}\text{el}\mathcal{C}\text{at}_\infty \subset \mathcal{M}\text{odel}_\infty$ of remark 4.3).

Notation 4.12. Similarly to Notations 4.11 and H.3.2, $(\mathcal{M}\text{odel}_\infty)_{**}$ is enriched and tensored over $(\mathcal{R}\text{el}\mathcal{C}\text{at}_\infty, \times)$. As for the enrichment, for any

$$(\mathcal{D}_1, \mathbf{W}_1, \mathbf{C}_1, \mathbf{F}_1), (\mathcal{D}_2, \mathbf{W}_2, \mathbf{C}_2, \mathbf{F}_2) \in (\mathcal{M}\text{odel}_\infty)_{**},$$

in analogy with Notation H.3.2 we define the object

$$\left(\text{Fun}_{**}(\mathcal{D}_1, \mathcal{D}_2)^{\text{Model}}, \text{Fun}_{**}(\mathcal{D}_1, \mathcal{D}_2)^{\mathbf{W}} \right) =$$

$$\lim \left(\begin{array}{c} \left(\text{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\text{Model}}, \text{Fun}(\mathcal{D}_1, \mathcal{D}_2)^{\mathbf{W}} \right) \\ \downarrow {\scriptstyle (\text{ev}_{s_1}, \text{ev}_{t_1})} \\ \text{pt}_{\mathcal{R}\text{el}\mathcal{C}\text{at}_\infty} \xrightarrow{\;\;(s_2, t_2)\;\;} (\mathcal{D}_2, \mathbf{W}_2) \times (\mathcal{D}_2, \mathbf{W}_2) \end{array} \right)$$

of $\mathcal{R}\text{el}\mathcal{C}\text{at}_\infty$ (where we write $s_1, t_1 \in \mathcal{D}_1$ and $s_2, t_2 \in \mathcal{D}_2$ to distinguish between the source and target objects). Then, the tensoring is obtained by taking $(\mathcal{R}, \mathbf{W}_\mathcal{R}) \in \mathcal{R}\text{el}\mathcal{C}\text{at}_\infty$ and $(\mathcal{D}, \mathbf{W}_\mathcal{D}, \mathbf{C}_\mathcal{D}, \mathbf{F}_\mathcal{D}) \in (\mathcal{M}\text{odel}_\infty)_{**}$ to the pushout

$$\text{colim} \left(\begin{array}{ccc} \mathcal{R} \times \{s, t\} & \longrightarrow & \mathcal{R} \times \mathcal{D} \\ \downarrow & & \\ \text{pt}_{\mathcal{M}\text{odel}_\infty} \times \{s, t\} & & \end{array} \right)$$

in $\mathcal{M}\text{odel}_\infty$, with its double-pointing given by the natural map from $\text{pt}_{\mathcal{M}\text{odel}_\infty} \sqcup \text{pt}_{\mathcal{M}\text{odel}_\infty} \simeq \text{pt}_{\mathcal{M}\text{odel}_\infty} \times \{s, t\}$.

Remark 4.13. While we are using the notation $\text{Fun}(-, -)^{\mathbf{W}}$ both in the context of relative ∞-categories and model ∞-diagrams, due to the identification $\mathcal{R}\text{el}\mathcal{C}\text{at}_\infty \subset \mathcal{M}\text{odel}_\infty$ of remark 4.3 this is actually *not* an abuse of notation. The notation $\text{Fun}_{**}(-, -)^{\mathbf{W}}$ is similarly unambiguous.

Notation 4.14. Similarly to Notation H.3.4, we will write

$$(\mathcal{M}\text{odel}_\infty)_{(**)} \times \mathcal{R}\text{el}\mathcal{C}\text{at}_\infty \xrightarrow{-\odot-} (\mathcal{M}\text{odel}_\infty)_{(**)}$$

to denote either tensoring of notation 4.11 or of notation 4.12 (using the convention of notation 4.8).

Corresponding to definition 4.9, we expand on Definition H.3.9 as follows.

Definition 4.15. Given a model ∞-diagram $\mathcal{M} \in \mathcal{M}\text{odel}_\infty$ (e.g. a model ∞-category) equipped with two chosen objects $x, y \in \mathcal{M}$, and given a model word $\underline{\mathbf{m}} \in \mathcal{M}\text{odel}_{**}$, we define the ∞-category of *zigzags* in $\mathcal{M}$ from x to y of type $\underline{\mathbf{m}}$ to be

$$\underline{\mathbf{m}}_{\mathcal{M}}(x, y) = \text{Fun}_{**}(\underline{\mathbf{m}}, \mathcal{M})^{\mathbf{W}}.$$

If the model ∞-diagram $\mathcal{M}$ is clear from context, we will simply write $\underline{\mathbf{m}}(x, y)$.

Definition 4.16. For any model ∞-diagram $\mathcal{M}$ and any objects $x, y \in \mathcal{M}$, we will refer to

$$\underline{\tilde{\mathbf{3}}}(x, y) = \mathrm{Fun}_{**}(\tilde{\mathbf{3}}, \mathcal{M})^{\mathbf{W}} \in \mathcal{C}\mathrm{at}_\infty$$

as the ∞-category of ***special three-arrow zigzags*** in $\mathcal{M}$ from x to y (which is a variant of Definition H.4.3), and we will refer to

$$\underline{\mathbf{7}}(x, y) = \mathrm{Fun}_{**}(\underline{\mathbf{7}}, \mathcal{M})^{\mathbf{W}} \in \mathcal{C}\mathrm{at}_\infty$$

as the ∞-category of ***seven-arrow zigzags*** in $\mathcal{M}$ from x to y.

Now, the reason we are interested in the tensorings of notation 4.14 is the following construction.

Notation 4.17. We define a functor

$$(\mathcal{M}\mathrm{odel}_\infty)_{(**)} \xrightarrow{c^{\bullet}_{(**)}} c(\mathcal{M}\mathrm{odel}_\infty)_{(**)}$$
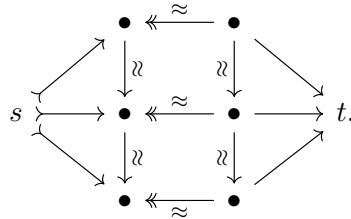
by setting

$$c^{\bullet}_{(**)}\mathcal{D} = \mathcal{D} \odot [\bullet]_{\mathbf{W}}$$

for any $\mathcal{D} \in (\mathcal{M}\mathrm{odel}_\infty)_{(**)}$ (where $[\bullet]_{\mathbf{W}}$ denotes the composite $\boldsymbol{\Delta} \hookrightarrow \mathcal{C}\mathrm{at} \xrightarrow{\mathrm{max}} \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \hookrightarrow \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$). Of course, this restricts to a functor

$$\mathcal{M}\mathrm{odel}_{(**)} \xrightarrow{c^{\bullet}_{(**)}} c\mathcal{M}\mathrm{odel}_{(**)}.$$

Example 4.18. If we consider $[\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}] \in \mathcal{M}\mathrm{odel}_{**}$, then $[\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}] \odot [2]_{\mathbf{W}} \in \mathcal{M}\mathrm{odel}_{**}$ is given by

On the other hand, if we consider $[\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}] \in \mathcal{M}\mathrm{odel}$, then $[\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}] \odot [2]_{\mathbf{W}} \in \mathcal{M}\mathrm{odel}$ is given by
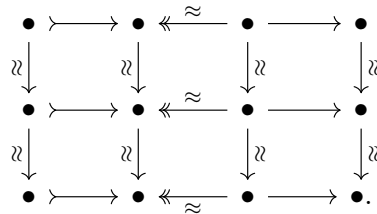
In turn, notation 4.17 is itself useful for the following reason.

Lemma 4.19. *For any* $\mathcal{D}, \mathcal{M} \in (\mathrm{Model}_\infty)_{(**)}$, *we have an equivalence*

$$\mathrm{hom}^{\mathrm{lw}}_{(\mathrm{Model}_\infty)_{(**)}}(c^\bullet_{(**)}\mathcal{D}, \mathcal{M}) \simeq \mathrm{N}_\infty \left(\mathrm{Fun}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathbf{W}} \right)$$

in $s\mathcal{S}$ *which is natural in both variables.*

Proof. For any $n \geq 0$ we have a composite equivalence

$$\mathrm{N}_\infty \left(\mathrm{Fun}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathbf{W}} \right)_n = \mathrm{hom}_{\mathcal{C}\mathrm{at}_\infty} \left([n], \mathrm{Fun}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathbf{W}} \right)$$

$$\simeq \mathrm{hom}_{\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty} \left([n]_{\mathbf{W}}, \left(\mathrm{Fun}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathcal{M}\mathrm{odel}}, \mathrm{Fun}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathbf{W}} \right) \right)$$

$$\simeq \mathrm{hom}_{(\mathrm{Model}_\infty)_{(**)}}(\mathcal{D} \odot [n]_{\mathbf{W}}, \mathcal{M})$$

$$= \mathrm{hom}_{(\mathrm{Model}_\infty)_{(**)}}(c^n_{(**)}\mathcal{D}, \mathcal{M})$$

which clearly commutes with the simplicial structure maps on both sides. $\square$

We now introduce slightly more elaborate versions of the concepts we have been exploring – an ∞-categorical version of [11, Variant 3.3] – which will be used in the proofs of proposition 6.1, proposition 7.1, and lemma 8.2.

Definition 4.20. A **decorated model ∞-diagram** is a model ∞-diagram with some subdiagrams decorated as colimit or limit diagrams. For instance, if we define $\mathcal{D}$ to be the "walking pullback square", then for any other model ∞-diagram $\mathcal{M}$, we let $\mathrm{hom}^{*}_{\mathcal{M}\mathrm{odel}_\infty}(\mathcal{D}, \mathcal{M}) \subset \mathrm{hom}_{\mathcal{M}\mathrm{odel}_\infty}(\mathcal{D}, \mathcal{M})$, $\mathrm{Fun}^{*}(\mathcal{D}, \mathcal{M})^{\mathcal{M}\mathrm{odel}} \subset \mathrm{Fun}(\mathcal{D}, \mathcal{M})^{\mathcal{M}\mathrm{odel}}$, and $\mathrm{Fun}^{*}(\mathcal{D}, \mathcal{M})^{\mathbf{W}} \subset \mathrm{Fun}(\mathcal{D}, \mathcal{M})^{\mathbf{W}}$ denote the subobjects spanned by those morphisms $\mathcal{D} \to \mathcal{M}$ of model ∞-diagrams which select a pullback square in $\mathcal{M}$. Of course, we define a **doubly-pointed decorated model ∞-diagram** similarly.

In fact, we will only use this variant in the doubly-pointed case, and then only for pushout and pullback squares. So, in the interest of easing our TikZographical burden, we will simply superscript these model diagrams with "p.o." and/or "p.b." as appropriate; the question of which square we are referring to is fully disambiguated by the fact that our pushouts will only be of acyclic cofibrations while our pullbacks will only be of acyclic fibrations.

Note that the constructions

$$\mathrm{hom}^{*}_{(\mathrm{Model}_\infty)_{(**)}}(\mathcal{D}, \mathcal{M}) \in \mathcal{S} \quad \text{and} \quad \mathrm{Fun}^{*}_{(**)}(\mathcal{D}, \mathcal{M})^{\mathbf{W}} \in \mathcal{C}\mathrm{at}_\infty$$

are not generally functorial in the target $\mathcal{M}$. On the other hand, they are functorial for *some* maps in the source $\mathcal{D}$. We will refer to such maps as **decoration-respecting**. These define an ∞-category $(\mathrm{Model}_\infty)^{*}_{(**)}$. (Note the distinction between $\mathrm{hom}_{(\mathrm{Model}_\infty)^{*}_{(**)}}(-, -)$ and $\mathrm{hom}^{*}_{(\mathrm{Model}_\infty)_{(**)}}(-, -)$.) We consider $(\mathrm{Model}_\infty)_{(**)} \subset (\mathrm{Model}_\infty)^{*}_{(**)}$ simply by considering undecorated model ∞-diagrams as being trivially decorated. We will not need a general theory for understanding which maps of decorated model diagrams are decoration-respecting; rather, it will suffice to observe once and for all

that given a square which is decorated as a pushout or pullback square, it is decoration-respecting to either

- take it to another similarly decorated square, or
- collapse it onto a single edge (since a commutative square in which two parallel edges are equivalences is both a pushout and a pullback).

Note that if the source of a map of decorated model ∞-diagrams is actually undecorated, then the map is automatically decoration-respecting; in other words, we must only check that maps in which the *source* is decorated are decoration-respecting.

Remark 4.21. Of course, adding in definition 4.20 allows us to also demand finite bicompleteness of a model ∞-diagram via lifting conditions, and hence all of the axioms for a model ∞-diagram to be a model ∞-category can now be encoded in this language (recall remark 4.5).

We will need the following analog of Lemma H.4.5 for model ∞-diagrams.

Lemma 4.22. *Given a pair of maps $\mathcal{D}_1 \rightrightarrows \mathcal{D}_2$ in $(\mathrm{Model}_\infty)^{\circledast}_{(**)}$, a morphism between them in $\mathrm{Fun}^{\circledast}_{(**)}(\mathcal{D}_1, \mathcal{D}_2)^{\mathbf{W}}$ induces, for any $\mathcal{M} \in (\mathrm{Model}_\infty)_{(**)}$, a natural transformation between the two induced functors*

$$\mathrm{Fun}^{\circledast}_{(**)}(\mathcal{D}_2, \mathcal{M})^{\mathbf{W}} \rightrightarrows \mathrm{Fun}^{\circledast}_{(**)}(\mathcal{D}_1, \mathcal{M})^{\mathbf{W}}.$$

Proof. It is not hard to see that the proof of Lemma H.4.5 carries over without essential change (this time using the enrichment of $(\mathrm{Model}_\infty)_{(**)}$ over $\mathcal{R}el\mathcal{C}at_\infty$). $\qquad\square$

In order to state the final result of this subsection, we need to introduce a bit of notation.

Notation 4.23. For any objects $x, y \in \mathcal{M}$, we denote

- by

$$\mathbf{W}_{x\updownarrow} \subset \mathbf{W}_{x/}$$

the full subcategory on those objects $(x \overset{\approx}{\rightarrowtail} z) \in \mathbf{W}_{x/}$ whose structure map is a cofibration,

- by

$$\mathbf{W}_{\downarrow y} \subset \mathbf{W}_{/y}$$

the full subcategory on those objects $(z \overset{\approx}{\twoheadrightarrow} y) \in \mathbf{W}_{/y}$ whose structure map is a fibration, and

- by

$$\mathbf{W}_{x\updownarrow\downarrow y} = \mathbf{W}_{x\updownarrow} \times_{\mathbf{W}} \mathbf{W}_{\downarrow y} \subset \mathbf{W}_{x//y}$$

the full subcategory on those objects $(x \overset{\approx}{\rightarrowtail} z \overset{\approx}{\twoheadrightarrow} y) \in \mathbf{W}_{x//y}$ whose structure maps are respectively a cofibration and fibration (as indicated).

We now give an extremely useful result, an analog of [4, 8.1], which will appear in the proofs of proposition 6.1, proposition 7.1, and lemma 8.2. We refer to it as the **factorization lemma**.

Lemma 4.24. *Let $\mathcal{M}$ be a model ∞-category, and let $x, y \in \mathcal{M}$. For any model words $\underline{\mathbf{m}}$ and $\underline{\mathbf{n}}$, applying $\mathrm{Fun}_{**}(-, \mathcal{M})^{\mathbf{W}}$ to the evident inclusion*

$$
\left(s \xrightarrow{\ \underline{\mathbf{m}}\ } \bullet \xleftarrow{\ \approx\ } \bullet \xrightarrow{\ \underline{\mathbf{n}}\ } t \right) \to \left(s \xrightarrow{\ \underline{\mathbf{m}}\ } \bullet \xleftarrow{\ \approx\ } \bullet \xrightarrow{\ \underline{\mathbf{n}}\ } t \right)
$$

in Model_{**} induces a map in $\mathbf{W}_{\mathrm{Th}} \subset \mathrm{Cat}_\infty$.

Proof. We first observe that the target of this inclusion in Model_{**} is isomorphic to the model word

$$[\underline{\mathbf{m}}; (\mathbf{W} \cap \mathbf{F})^{-1}; (\mathbf{W} \cap \mathbf{C})^{-1}; \underline{\mathbf{n}}],$$

it is just drawn so that the "evident inclusion" is truly evident. So, the induced map can be expressed as

$$[\underline{\mathbf{m}}; (\mathbf{W} \cap \mathbf{F})^{-1}; (\mathbf{W} \cap \mathbf{C})^{-1}; \underline{\mathbf{n}}](x, y) \to [\underline{\mathbf{m}}; \mathbf{W}^{-1}; \underline{\mathbf{n}}](x, y).$$

To abbreviate notation, we will write this map in Cat_∞ simply as $\mathcal{C}_1 \to \mathcal{C}_2$.

Now, showing that the induced map $\mathcal{C}_1^{\mathrm{gpd}} \to \mathcal{C}_2^{\mathrm{gpd}}$ is an equivalence in $\mathcal{S}$ is equivalent to showing that the induced map $(\mathcal{C}_1^{op})^{\mathrm{gpd}} \to (\mathcal{C}_2^{op})^{\mathrm{gpd}}$ is an equivalence in $\mathcal{S}$, and for this by Proposition G.4.8 it suffices to show that the functor $\mathcal{C}_1^{op} \to \mathcal{C}_2^{op}$ is final. According to the characterization of Theorem A (G.4.10), this is equivalent to showing that for any object

$$f = \left(x \xrightarrow{\ \underline{\mathbf{m}}\ } x_1 \xleftarrow{\ \approx\ } y_1 \xrightarrow{\ \underline{\mathbf{n}}\ } y \right) \in \mathcal{C}_2,$$

the groupoid completion of the comma ∞-category

$$(\mathcal{C}_1)^{op} \underset{(\mathcal{C}_2)^{op}}{\times} ((\mathcal{C}_2)^{op})_{f^\circ/} \simeq \left(\mathcal{C}_1 \underset{\mathcal{C}_2}{\times} (\mathcal{C}_2)_{/f} \right)^{op}$$

is contractible, which is in turn equivalent to showing that the groupoid completion of the comma ∞-category

$$\mathcal{C}_3 = \mathcal{C}_1 \underset{\mathcal{C}_2}{\times} (\mathcal{C}_2)_{/f}$$

is contractible.

For this, let us first choose a factorization $y_1 \xrightarrow{\approx} z_1 \xrightarrow{\approx} x_1$ in $\mathcal{M}$ using factorization axiom $\mathrm{M}_\infty 5$; we can consider this as defining an object $Z_1 = (y_1 \xrightarrow{\approx} z_1 \xrightarrow{\approx} x_1) \in \mathcal{M}_{y_1//x_1}$. Then, working in the model ∞-category $\mathcal{M}_{y_1//x_1}$ (see Example S.2.3), we apply proposition 1.5(2) to obtain a special path object $\mathrm{path}_\bullet(Z_1) \in s(\mathcal{M}_{y_1//x_1})$. Note that every constituent object $\mathrm{path}_n(Z_1) \in \mathcal{M}_{y_1//x_1}$ is in fact bifibrant: it is cofibrant since specialness implies that the unique structure map $Z_1 \simeq \mathrm{path}_0(Z_1) \to \mathrm{path}_n(Z_1)$ (a

composite of degeneracy maps) is an acyclic cofibration and Z_1 itself is cofibrant, and it is fibrant by the dual of lemma 2.2 since Z_1 itself is fibrant. Moreover, since $\mathbf{W}$ has the two-out-of-three property, it follows that in fact $\mathrm{path}_\bullet(Z_1) \in s(\mathbf{W}_{y_1 \rtimes \downarrow x_1})$.

Now, observe that there is a natural functor

$$\mathbf{W}_{y_1 \rtimes \downarrow x_1} \to \mathcal{C}_3$$

which takes an object $(y_1 \overset{\approx}{\rightarrowtail} w_1 \overset{\approx}{\twoheadrightarrow} x_1) \in \mathbf{W}_{y_1 \rtimes \downarrow x_1}$ to the object

$$\left(
\begin{array}{c}
x_1 \xleftarrow{\;\approx\;} w_1 \xleftarrow{\;\approx\;} y_1 \\
x \overset{\underline{\mathbf{m}}}{\diagup} \quad \wr \quad \wr \quad \diagdown \overset{\underline{\mathbf{n}}}{} y \\
\overline{\mathbf{m}} \diagdown \quad \downarrow \quad \downarrow \quad \diagup \overline{\mathbf{n}} \\
x_1 \xleftarrow{\;\approx\;} y_1
\end{array}
\right) \in \mathcal{C}_3$$

(in which diagram the bottom zigzag is the chosen object $f \in \mathcal{C}_2$ and the top zigzag (an object of $\mathcal{C}_1$) is obtained by simply splicing the zigzag $x_1 \overset{\approx}{\leftarrow} w_1 \overset{\approx}{\leftarrowtail} y_1$ into it, and all vertical weak equivalences (including those not pictured) are identity maps). Thus, we obtain a composite

$$\mathbf{\Delta}^{op} \xrightarrow{\;\mathrm{path}_\bullet(Z_1)\;} \mathbf{W}_{y_1 \rtimes \downarrow x_1} \to \mathcal{C}_3,$$

which we will again denote simply by $\mathrm{path}_\bullet(Z_1) \in s(\mathcal{C}_3)$. Since $(\mathbf{\Delta}^{op})^{\mathrm{gpd}} \simeq \mathrm{pt}_\mathcal{S}$ (as $\mathbf{\Delta}^{op}$ is sifted), again referring to Proposition G.4.8 we see that it suffices to show that this functor is final. Then, again referring to Theorem A (G.4.10), we see that this is equivalent to showing that for any object

$$g = \left(
\begin{array}{c}
x_2 \xleftarrow{\;\approx\;} z_2 \xleftarrow{\;\approx\;} y_2 \\
x \overset{\underline{\mathbf{m}}}{\diagup} \quad \wr \quad \wr \quad \diagdown \overset{\underline{\mathbf{n}}}{} y \\
\overline{\mathbf{m}} \diagdown \quad \downarrow \quad \downarrow \quad \diagup \overline{\mathbf{n}} \\
x_1 \xleftarrow{\;\approx\;} y_1
\end{array}
\right) \in \mathcal{C}_3$$

(in which diagram the bottom zigzag is again the chosen object $f \in \mathcal{C}_2$ but now the top zigzag is an arbitrary object of $\mathcal{C}_1$), the groupoid completion of the comma ∞-category

$$\mathcal{C}_4 = \mathbf{\Delta}^{op} \underset{\mathcal{C}_3}{\times} (\mathcal{C}_3)_{g/}$$

is contractible.

For this, let us define a simplicial space $Y \in s\mathcal{S}$ by setting

$$Y_\bullet = \mathrm{hom}^{\mathrm{lw}}_{\mathcal{C}_3}(g, \mathrm{path}_\bullet(Z_1)).$$

On the one hand, considering $Y \in s\mathcal{S} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S})$, we have an equivalence

$$\mathrm{srep}(Y) \simeq \mathrm{N}_\infty(\mathcal{C}_4)$$

in $s\mathcal{S}$: for any $n \geq 0$ we have an equivalence

$$\mathrm{srep}(Y)_n \simeq \coprod_{\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n} Y_{\alpha(0)}$$

$$= \coprod_{\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n} \hom_{\mathcal{C}_3}(g, \mathrm{path}_{\alpha(0)}(Z_1))$$

$$\simeq \mathrm{N}_\infty(\mathcal{C}_4)_n,$$

and it is not hard to see that these respect the structure maps of the two simplicial spaces. But on the other hand, unwinding the definitions we obtain an identification

$$Y_\bullet \simeq \lim \begin{pmatrix} & & \hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(z_2, \mathrm{path}_\bullet(Z_1)) \\ & & \downarrow \\ \mathrm{pt}_{s\mathcal{S}} & \longrightarrow & \hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(y_2, \mathrm{path}_\bullet(Z_1)) \end{pmatrix},$$

in which pullback

- we implicitly consider $\mathrm{path}_\bullet(Z_1) \in s(\mathbf{W}_{/x_1})$ via the evident forgetful functor $\mathbf{W}_{y_1 \Downarrow x_1} \to \mathbf{W}_{/x_1}$,

- the vertical map is given by levelwise precomposition with $y_2 \overset{\approx}{\rightarrowtail} z_2$, and

- the horizontal map is given by the composite

$$\mathrm{pt}_{s\mathcal{S}} \to \hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(z_1, \mathrm{path}_\bullet(Z_1)) \to$$
$$\hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(y_1, \mathrm{path}_\bullet(Z_1)) \to \hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(y_2, \mathrm{path}_\bullet(Z_1))$$

  of the canonical point of $\hom^{\mathrm{lw}}_{\mathbf{W}_{/x_1}}(z_1, \mathrm{path}_\bullet(Z_1))$ followed by the maps induced by precomposition with the composite $y_2 \overset{\approx}{\to} y_1 \overset{\approx}{\rightarrowtail} z_1$.

Considering $\mathcal{M}_{/x_1}$ as a model ∞-category (again see Example S.2.3), the simplicial object $\mathrm{path}_\bullet(Z_1) \in s(\mathcal{M}_{/x_1})$ defines a path object for the fibrant object $z_1 \in (\mathcal{M}_{/x_1})^f$. Thus, by the dual of proposition 2.1(1), the vertical map in this pullback lies in $(\mathbf{W} \cap \mathbf{F})_{\mathrm{KQ}} \subset s\mathcal{S}$. Hence, by Proposition S.6.5 (and Proposition S.7.2) it follows that $|Y_\bullet| \simeq \mathrm{pt}_{\mathcal{S}}$. Finally, combining the two equivalences we have just obtained with the Bousfield–Kan colimit formula (Theorem G.5.8) and Proposition N.2.4, we obtain the string of equivalences

$$\mathrm{pt}_{\mathcal{S}} \simeq |Y_\bullet| \simeq |\mathrm{srep}(Y)_\bullet| \simeq |\mathrm{N}_\infty(\mathcal{C}_4)_\bullet| \simeq (\mathcal{C}_4)^{\mathrm{gpd}},$$

which completes the proof. $\square$

4.2. Left homotopies. Given two parallel maps $\mathcal{D}_1^\bullet \rightrightarrows \mathcal{D}_2^\bullet$ in $c\mathrm{Model}_{(**)}$, and any $\mathcal{M} \in \mathrm{Model}_{(**)}$, applying the functor

$$c\mathrm{Model}_{(**)} \xrightarrow{\hom^{\mathrm{lw}}_{(\mathrm{Model}_\infty)_{(**)}}(-,\mathcal{M})} s\mathcal{S}$$

yields two parallel maps

$$\hom^{\mathrm{lw}}_{(\mathrm{Model}_\infty)_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{M}) \rightrightarrows \hom^{\mathrm{lw}}_{(\mathrm{Model}_\infty)_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{M})$$

in $s\mathcal{S}$. We will be interested in explicitly describing additional data which causes these maps become equivalent upon geometric realization. This motivates the following definition.

Definition 4.25. Given two parallel maps $f, g \in \hom_{s\mathcal{S}}(Y, Z)$, a **_left homotopy_** from f to g (in the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$) is a map $h \in \hom_{s\mathcal{S}}(Y \times \Delta^1, Z)$ fitting into a commutative diagram

$$Y \xrightarrow{\sim} Y \times \Delta^{\{0\}} \longrightarrow Y \times \Delta^1 \longleftarrow Y \times \Delta^{\{1\}} \xleftarrow{\sim} Y$$

with f, h, g mapping to Z.

in $s\mathcal{S}$.

Of course, this comes with the following expected result.

Lemma 4.26. _A left homotopy $Y \times \Delta^1 \to Z$ in $s\mathcal{S}_{\mathrm{KQ}}$ between two parallel maps $Y \rightrightarrows Z$ in $s\mathcal{S}$ induces an equivalence between the two induced parallel maps $|Y| \rightrightarrows |Z|$ in $\mathcal{S}$._

Proof. The maps $Y \simeq Y \times \Delta^{\{i\}} \to Y \times \Delta^1$ are in $\mathbf{W}_{\mathrm{KQ}}$ since geometric realization (as a sifted colimit) commutes with finite products. Hence, the diagram

$$Y \xrightarrow{\sim} Y \times \Delta^{\{0\}} \xrightarrow{\approx} Y \times \Delta^1 \xleftarrow{\approx} Y \times \Delta^{\{1\}} \xleftarrow{\sim} Y$$

mapping to Z.

in $s\mathcal{S}_{\mathrm{KQ}}$ induces, upon geometric realization, the diagram

$$|Y| \xrightarrow{\sim} |Y \times \Delta^{\{0\}}| \xrightarrow{\sim} |Y \times \Delta^1| \xleftarrow{\sim} |Y \times \Delta^{\{1\}}| \xleftarrow{\sim} |Y|$$

mapping to $|Z|$.

in $\mathcal{S}$, which selects the desired equivalence between the two induced maps $|Y| \rightrightarrows |Z|$. $\square$

In our cases of interest, the left homotopy between two parallel maps
$$\hom^{\mathrm{lw}}_{(\mathrm{Model}_\infty)_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{M}) \rightrightarrows \hom^{\mathrm{lw}}_{(\mathrm{Model}_\infty)_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{M})$$
will be natural in the variable $\mathcal{M} \in (\mathrm{Model}_\infty)_{(**)}$. By Yoneda's lemma, the data of such a left homotopy itself will be corepresentable by some additional data relating $\mathcal{D}_1^\bullet$ and $\mathcal{D}_2^\bullet$. This leads us to the following definition.

Definition 4.27. Given $\varphi^\bullet, \psi^\bullet \in \hom_{c\mathrm{Model}_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{D}_2^\bullet)$, a **left homotopy corepresentation** from $\varphi^\bullet$ to $\psi^\bullet$ is a family of maps
$$\{h_n^i \in \hom_{\mathrm{Model}_{(**)}}(\mathcal{D}_1^{n+1}, \mathcal{D}_2^n)\}_{0 \leq i \leq n \geq 0}$$
satisfying the identities
$$h_n^0 \delta^0 = \varphi^n$$
$$h_n^n \delta^{n+1} = \psi^n$$
$$h_n^j \delta^i = \begin{cases} \delta^i h_{n-1}^{j-1}, & i < j \\ h_n^{j-1} \delta^i, & i = j \neq 0 \\ \delta^{i-1} h_{n-1}^j, & i > j+1 \end{cases}$$
$$h_n^j \sigma^i = \begin{cases} \sigma^j h_{n+1}^{j+1}, & i \leq j \\ \sigma^{i-1} h_{n+1}^j, & i > j. \end{cases}$$

Remark 4.28. These identities are nothing but the duals of those defining a "simplicial homotopy" in the classical sense (see e.g. [10, Definitions 5.1]).

Then, we have the following expected result.

Lemma 4.29. *Fix some* $\varphi^\bullet, \psi^\bullet \in \hom_{c\mathrm{Model}_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{D}_2^\bullet)$. *Then, giving a left homotopy corepresentation*
$$\{h_n^i \in \hom_{\mathrm{Model}_{(**)}}(\mathcal{D}_1^{n+1}, \mathcal{D}_2^n)\}_{0 \leq i \leq n \geq 0}$$
from $\varphi^\bullet$ *to* $\psi^\bullet$ *is equivalent to giving a left homotopy*
$$\hom^{\mathrm{lw}}_{(\mathrm{Model}_\infty)_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{M}) \times \Delta^1 \to \hom^{\mathrm{lw}}_{(\mathrm{Model}_\infty)_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{M})$$
from $\hom^{\mathrm{lw}}_{(\mathrm{Model}_\infty)_{(**)}}(\varphi^\bullet, \mathcal{M})$ *to* $\hom^{\mathrm{lw}}_{(\mathrm{Model}_\infty)_{(**)}}(\psi^\bullet, \mathcal{M})$ *which is natural in the variable* $\mathcal{M} \in (\mathrm{Model}_\infty)_{(**)}$.

Proof. Suppose we have such a natural left homotopy. If we apply it to $\mathcal{D}_2^n$, the natural map
$$\Delta^n \to \hom^{\mathrm{lw}}_{\mathrm{Model}_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{D}_2^n)$$
in $s\mathcal{S}$ corresponding to $\mathrm{id}_{\mathcal{D}_2^n}$ gives rise to the composite map
$$\Delta^n \times \Delta^1 \to \hom^{\mathrm{lw}}_{\mathrm{Model}_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{D}_2^n) \times \Delta^1 \to \hom^{\mathrm{lw}}_{\mathrm{Model}_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{D}_2^n).$$
Evaluating this at the $n + 1$ nondegenerate $(n + 1)$-simplices of $\Delta^n \times \Delta^1$ and ranging over all $n \geq 0$ yields the maps defining the left homotopy

corepresentation; that these satisfy the identities follows from applying the natural left homotopy to the cosimplicial structure maps of $\mathcal{D}_2^\bullet \in c\mathcal{M}\mathrm{odel}_{(**)}$.

Conversely, given a left homotopy representation, we define a natural left homotopy given in level n by the map

$$\mathrm{hom}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_2^n, \mathcal{M}) \times (\Delta^1)_n$$

$$\simeq \coprod_{(\Delta^1)_n} \mathrm{hom}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_2^n, \mathcal{M}) \to \mathrm{hom}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_1^n, \mathcal{M})$$

which, on the summand corresponding to the element of $(\Delta^1)_n \cong \mathrm{hom}_{\boldsymbol{\Delta}}([n],[1])$ associated to the decomposition

$$[n] = \{0, \ldots, n-i\} \sqcup \{(n+1) - i, \ldots, n\}$$

(for $i \in \{0, \ldots, n+1\}$), is corepresented by the map

$$\begin{cases} \varphi^n = h_n^0 \delta^0, & i = 0 \\ h_n^{i-1} \delta^i = h_n^i \delta^n, & 0 < i < n+1 \\ \psi^n = h_n^n \delta^{n+1}, & i = n+1 \end{cases}$$

in $\mathrm{hom}_{\mathcal{M}\mathrm{odel}_{(**)}}(\mathcal{D}_1^n, \mathcal{D}_2^n)$; that these do indeed define a left homotopy follows from the fact that our choices here are induced by the simplicial structure maps of $\Delta^1 \in s\mathcal{S}\mathrm{et} \subset s\mathcal{S}$. $\square$

Definition 4.30. In the situation of lemma 4.29, we refer to an induced map

$$\mathrm{hom}^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_2^\bullet, \mathcal{M}) \times \Delta^1 \to \mathrm{hom}^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{(**)}}(\mathcal{D}_1^\bullet, \mathcal{M})$$

as a ***corepresented left homotopy*** (in the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$) associated to the left homotopy corepresentation.

5. The equivalence $\left\|\mathrm{hom}^{\mathrm{lw}}_{\mathcal{M}}(\sigma\mathrm{cyl}^\bullet(x), \sigma\mathrm{path}_\bullet(y))\right\| \simeq \underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}}$

We now proceed with an analog of [9, Proposition 7.3].

Proposition 5.1. *Suppose we have $x, y \in \mathcal{M}$ with x cofibrant and y fibrant, and let $\sigma\mathrm{cyl}^\bullet(x) \in c\mathcal{M}$ and $\sigma\mathrm{path}_\bullet(y) \in s\mathcal{M}$ be a special cylinder object for x and a special path object for y, respectively. Then*

$$\left\|\mathrm{hom}^{\mathrm{lw}}_{\mathcal{M}}(\sigma\mathrm{cyl}^\bullet(x), \sigma\mathrm{path}_\bullet(y))\right\| \simeq \underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}}.$$

Proof. To prove the claim, we construct a commutative diagram

$$\begin{array}{ccccc} M_\bullet & \longrightarrow & Q_\bullet & \longleftarrow & P_\bullet \\ \downarrow & \nearrow & \uparrow & \nwarrow & \downarrow \\ N_\bullet & \longleftarrow & P_\bullet & \longrightarrow & P_\bullet \times \Delta^1 \end{array}$$

in $s\mathcal{S}$ whose maps are all in $\mathbf{W}_{\mathrm{KQ}}$, such that

$$|M_\bullet| \simeq \left\|\mathrm{hom}^{\mathrm{lw}}_{\mathcal{M}}(\sigma\mathrm{cyl}^\bullet(x), \sigma\mathrm{path}_\bullet(y))\right\|$$

and

$$|Q_\bullet| \simeq \tilde{\underline{\mathbf{3}}}(x,y)^{\mathrm{gpd}}.$$

We first define the simplicial spaces of the diagram. Certain auxiliary definitions will appear superfluous, but they will be used later in the proof.

- We begin by defining the object $M_\bullet \in s\mathcal{S}$ by

$$M_\bullet = \mathrm{srep}\left(\boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \xrightarrow{\mathrm{hom}_{\mathcal{M}}(\sigma\,\mathrm{cyl}^\bullet(x),\sigma\,\mathrm{path}_\bullet(y))} \mathcal{S}\right)_\bullet.$$

By the Bousfield–Kan colimit formula (Theorem G.5.8), we have that

$$|M_\bullet| \simeq \left\|\mathrm{hom}_{\mathcal{M}}^{\mathrm{lw}}(\sigma\,\mathrm{cyl}^\bullet(x),\sigma\,\mathrm{path}_\bullet(y))\right\|,$$

as desired. Note that, since $[n] \in \mathcal{C}\mathrm{at}$ and $\boldsymbol{\Delta} \times \boldsymbol{\Delta}^{op} \in \mathcal{C}\mathrm{at}$ are gaunt, up to making the identification

$$\mathrm{hom}_{\mathcal{C}\mathrm{at}_\infty}([n], \boldsymbol{\Delta}^{op}) \simeq \mathrm{hom}_{\mathcal{C}\mathrm{at}_\infty}([n]^{op}, \boldsymbol{\Delta}^{op}) \simeq \mathrm{hom}_{\mathcal{C}\mathrm{at}_\infty}([n], \boldsymbol{\Delta}),$$

we have that

$$M_n \simeq \mathrm{colim}_{(\alpha,\beta)\in\mathrm{hom}_{\mathcal{C}\mathrm{at}_\infty}([n],\boldsymbol{\Delta}\times\boldsymbol{\Delta}^{op})} \mathrm{hom}_{\mathcal{M}}(\sigma\,\mathrm{cyl}^{\alpha(n)}(x),\sigma\,\mathrm{path}_{\beta(0)}(y))$$

$$\simeq \coprod_{(\alpha,\beta)\in\mathrm{N}(\boldsymbol{\Delta})_n\times\mathrm{N}(\boldsymbol{\Delta}^{op})_n} \mathrm{hom}_{\mathcal{M}}(\sigma\,\mathrm{cyl}^{\alpha(n)}(x),\sigma\,\mathrm{path}_{\beta(0)}(y)).$$

- We define the objects $N_\bullet, Q_\bullet, P_\bullet \in s\mathcal{S}$ simultaneously, as follows. For any $m, n \geq 0$, let $\underline{\mathbf{p}}^{m,n}$ denote the doubly-pointed model diagram
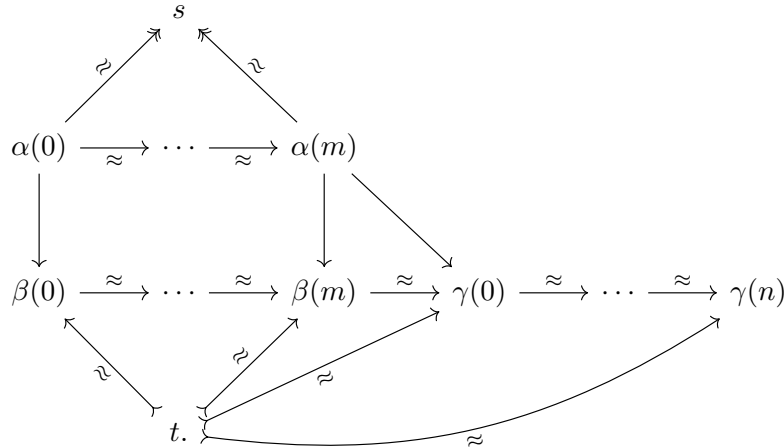
Moreover, let $\underline{\mathbf{n}}^{m,n} \subset \underline{\mathbf{p}}^{m,n}$ denote the full subcategory on the objects $\{s, t, \alpha(i), \gamma(j)\}_{0 \leq i \leq m, 0 \leq j \leq n}$ and let $\underline{\mathbf{q}}^{m,n} \subset \underline{\mathbf{p}}^{m,n}$ denote the full subcategory on the objects $\{s, t, \alpha(i), \beta(j)\}_{0 \leq i,j \leq m}$, both considered as doubly-pointed model diagrams in the evident way. Let us use

the placeholders $Y \in \{N, Q, P\}$ and $\underline{\mathbf{y}} \in \{\underline{\mathbf{n}}, \underline{\mathbf{q}}, \underline{\mathbf{p}}\}$. Then, the various objects $\underline{\mathbf{y}}^{m,n} \in \mathcal{M}\mathrm{odel}_{**}$ assemble into the evident bicosimplicial object $\underline{\mathbf{y}}^{\bullet\bullet} \in c\mathcal{M}\mathrm{odel}_{**}$, and we auxiliarily define

$$Y_{\bullet\bullet} = \hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{**}}(\underline{\mathbf{y}}^{\bullet\bullet}, \mathcal{M}) \in ss\mathcal{S}.$$

Then, we define $\underline{\mathbf{y}}^{\bullet} = \mathrm{diag}^*(\underline{\mathbf{y}}^{\bullet\bullet}) \in c\mathcal{M}\mathrm{odel}_{**}$, and we set

$$Y_{\bullet} = \hom^{\mathrm{lw}}_{(\mathcal{M}\mathrm{odel}_\infty)_{**}}(\underline{\mathbf{y}}^{\bullet}, \mathcal{M}) \in s\mathcal{S},$$

so that $Y_{\bullet} \simeq \mathrm{diag}^*(Y_{\bullet\bullet})$.

We now provide alternative identifications of the simplicial spaces $N_{\bullet}$ and $Q_{\bullet}$.

– As for $N_{\bullet}$, we clearly have

$$N_n \simeq \mathrm{colim}_{(\alpha,\gamma)\in\hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \mathbf{W}_{\downarrow x}\times\mathbf{W}_{y\downarrow}\right)} \hom_{\mathcal{M}}(\alpha(n), \gamma(0)).$$

Moreover, examining the structure maps of $N_{\bullet} \in s\mathcal{S}$, we see that up to making the identification

$$\hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \left(\mathbf{W}_{\downarrow x}\right)^{op}\right) \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n]^{op}, \left(\mathbf{W}_{\downarrow x}\right)^{op}\right) \simeq \hom_{\mathcal{C}\mathrm{at}_\infty}\left([n], \mathbf{W}_{\downarrow x}\right),$$

we have that

$$N_{\bullet} \simeq \mathrm{srep}\left(\left(\mathbf{W}_{\downarrow x}\right)^{op} \times \mathbf{W}_{y\downarrow} \xrightarrow{\left((x'\xrightarrow{\approx}x)^{\circ},(y\xrightarrow{\approx}y')\right)\mapsto\hom_{\mathcal{M}}(x',y')} \mathcal{S}\right)_{\bullet}.$$

– As for $Q_{\bullet}$, note first of all that $\underline{\mathbf{q}}^{m,n} \in \mathcal{M}\mathrm{odel}_{**}$ (and hence $Q_{m,n} \in \mathcal{S}$) is independent of n. Moreover, since we have an evident isomorphism $\underline{\mathbf{q}}^{\bullet} \cong c^{\bullet}_{**}\underline{\tilde{\mathbf{3}}}$ in $c\mathcal{M}\mathrm{odel}_{**}$ – indeed, the only difference is that we have named the intermediate objects of the constituent model diagrams of $\underline{\mathbf{q}}^{\bullet} \in c\mathcal{M}\mathrm{odel}_{**}$ – it follows from lemma 4.19 that

$$Q_{\bullet} \simeq \mathrm{N}_\infty(\underline{\tilde{\mathbf{3}}}(x,y))_{\bullet}.$$

Hence, Proposition N.2.4 this implies that

$$|Q_{\bullet}| \simeq \underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}},$$

as desired.

Finally, we observe that since $\mathbf{\Delta}^{op} \xrightarrow{\mathrm{diag}} \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ is final (as $\mathbf{\Delta}^{op}$ is sifted), then by Fubini's theorem for colimits, continuing to use the placeholder $Y \in \{N, Q, P\}$ we have an identification

$$\begin{aligned}
|Y_{\bullet}| &\simeq \|Y_{\bullet\bullet}\| \\
&= \mathrm{colim}_{([m]^{\circ},[n]^{\circ})\in\mathbf{\Delta}^{op}\times\mathbf{\Delta}^{op}} Y_{m,n} \\
&\simeq \mathrm{colim}_{[n]^{\circ}\in\mathbf{\Delta}^{op}}\left(\mathrm{colim}_{[m]^{\circ}\in\mathbf{\Delta}^{op}} Y_{m,n}\right) \\
&= \mathrm{colim}_{[n]^{\circ}\in\mathbf{\Delta}^{op}} |Y_{\bullet,n}|,
\end{aligned}$$

and similarly we have an identification

$$|Y_\bullet| \simeq \operatorname{colim}_{[m]^\circ \in \mathbf{\Delta}^{op}} |Y_{m,\bullet}|.$$

We now define the maps in the diagram, and along the way we show that the subdiagram

$$
\begin{array}{ccccc}
M_\bullet & & Q_\bullet & \longleftarrow & P_\bullet \\
\downarrow & & & & \downarrow \\
N_\bullet & \longleftarrow & P_\bullet & \longrightarrow & P_\bullet \times \Delta^1
\end{array}
$$

lies in $\mathbf{W}_{\mathrm{KQ}}$, which suffices to prove that the entire diagram is in $\mathbf{W}_{\mathrm{KQ}}$ by the two-out-of-three property.[8]

- We have a commutative diagram

$$
\begin{array}{ccc}
\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} & \xrightarrow{([m]^\circ,[n]^\circ) \mapsto \left((\sigma\mathrm{cyl}^m(x) \xrightarrow{\approx} x)^\circ, (y \xrightarrow{\approx} \sigma\mathrm{path}_n(y)) \right)} & \left(\mathbf{W}_{\downarrow x} \right)^{op} \times \mathbf{W}_{y\downarrow} \\
& \searrow_{\hom_{\mathcal{M}}^{\mathrm{lw}}(\sigma\mathrm{cyl}^\bullet(x),\sigma\mathrm{path}_\bullet(y))} \quad \swarrow_{\left((x' \xrightarrow{\approx} x)^\circ, (y \xrightarrow{\approx} y') \right) \mapsto \hom_{\mathcal{M}}(x',y')} & \\
& \mathcal{S} &
\end{array}
$$

in $\mathcal{C}\mathrm{at}_\infty$; considering this as a map in $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{S}}$, we obtain the map $M_\bullet \to N_\bullet$ from Proposition G.5.13(2). The upper map in this diagram is the product of two functors which are each final, the second by lemma 5.2 and the first by the opposite of its dual. Hence, this functor is itself final by Proposition G.4.9. Thus, the map $M_\bullet \to N_\bullet$ is in $\mathbf{W}_{\mathrm{KQ}}$ by the Bousfield–Kan colimit formula (Theorem G.5.8).

- The map $N_\bullet \to Q_\bullet$ is corepresented by the morphism in

$$\hom_{c\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{q}}^\bullet, \underline{\mathbf{n}}^\bullet)$$

given in level n by the unique functor satisfying $\alpha(i) \mapsto \alpha(i)$ and $\beta(i) \mapsto \gamma(i)$. (Note that there are composite morphisms $\alpha(i) \to \beta(i)$ implicit in the diagram defining $\underline{\mathbf{n}}^n$.)

- The map $M_\bullet \to Q_\bullet$ is the composition $M_\bullet \to N_\bullet \to Q_\bullet$.

- The map $P_\bullet \to N_\bullet$ is corepresented by the morphism in

$$\hom_{c\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{n}}^\bullet, \underline{\mathbf{p}}^\bullet)$$

which is simply the defining inclusion in each level. Note that this is obtained by applying $cc\mathcal{M}\mathrm{odel}_{**} \xrightarrow{\mathrm{diag}^*} c\mathcal{M}\mathrm{odel}_{**}$ to the morphism in

---

[8]Of course, really it would already have sufficed to obtain the zigzag $M_\bullet \to N_\bullet \leftarrow P_\bullet \to Q_\bullet$ of maps in $\mathbf{W}_{\mathrm{KQ}}$, but this proof is almost no more work and has the added benefit of showing that the map inducing the equivalence is the expected one.

$\mathrm{hom}_{cc\mathrm{Model}_{**}}(\underline{\mathbf{n}}^{\bullet\bullet}, \underline{\mathbf{p}}^{\bullet\bullet})$ which is again simply the defining inclusion in each bidegree. This latter map corepresents a map $P_{\bullet\bullet} \to N_{\bullet\bullet}$ in $ss\mathcal{S}$, from which the map $P_\bullet \to N_\bullet$ in $s\mathcal{S}$ is therefore obtained by applying $ss\mathcal{S} \xrightarrow{\mathrm{diag}^*} s\mathcal{S}$.

Now, since

$$|P_\bullet| \simeq \mathrm{colim}_{[n]^\circ \in \mathbf{\Delta}^{op}} |P_{\bullet,n}|, \quad |N_\bullet| \simeq \mathrm{colim}_{[n]^\circ \in \mathbf{\Delta}^{op}} |N_{\bullet,n}|,$$

to prove that the map $P_\bullet \to N_\bullet$ is in $\mathbf{W}_{\mathrm{KQ}}$, it suffices to prove that for each $[n]^\circ \in \mathbf{\Delta}^{op}$, the map $|P_{\bullet,n}| \to |N_{\bullet,n}|$ is an equivalence in $\mathcal{S}$, i.e. that the map $P_{\bullet,n} \to N_{\bullet,n}$ is in $\mathbf{W}_{\mathrm{KQ}}$.

To see this, we construct an inverse up to left homotopy in $s\mathcal{S}_{\mathrm{KQ}}$ for this map. This is corepresented by the map in

$$\mathrm{hom}_{c\mathrm{Model}_{**}}(\underline{\mathbf{p}}^{\bullet,n}, \underline{\mathbf{n}}^{\bullet,n})$$

given in level m by the unique functor satisfying $\alpha(i) \mapsto \alpha(i)$, $\beta(i) \mapsto \gamma(0)$, and $\gamma(i) \mapsto \gamma(i)$. As the resulting composite map $\underline{\mathbf{n}}^{\bullet,n} \to \underline{\mathbf{p}}^{\bullet,n} \to \underline{\mathbf{n}}^{\bullet,n}$ in $c\mathrm{Model}_{**}$ is the identity, it follows that the corepresented composite map $N_{\bullet,n} \to P_{\bullet,n} \to N_{\bullet,n}$ is also the identity.

On the other hand, the composite map $\underline{\mathbf{p}}^{\bullet,n} \to \underline{\mathbf{n}}^{\bullet,n} \to \underline{\mathbf{p}}^{\bullet,n}$ is not equal to the identity. However, it suffices to give a left homotopy corepresentation

$$\{\underline{\mathbf{p}}h^i_m \in \mathrm{hom}_{\mathrm{Model}_{**}}(\underline{\mathbf{p}}^{m+1,n}, \underline{\mathbf{p}}^{m,n})\}_{0 \le i \le m \ge 0}$$

from this composite to $\mathrm{id}_{\underline{\mathbf{p}}^{\bullet,n}}$, which we define by taking $\underline{\mathbf{p}}h^i_m$ to be the unique functor satisfying

$$\alpha(j) \mapsto \begin{cases} \alpha(j), & j \le i \\ \alpha(j-1), & j > i \end{cases}$$

$$\beta(j) \mapsto \begin{cases} \beta(j), & j \le i \\ \gamma(0), & j > i \end{cases}$$

$$\gamma(j) \mapsto \gamma(j).$$

(It is tedious but straightforward to verify that these formulas do indeed define such a left homotopy corepresentation.) By lemma 4.29 this gives us a left homotopy in $s\mathcal{S}_{\mathrm{KQ}}$ from the corepresented composite map $P_{\bullet,n} \to N_{\bullet,n} \to P_{\bullet,n}$ to $\mathrm{id}_{P_{\bullet,n}}$, and so by lemma 4.26 this corepresented composite map becomes equivalent upon geometric realization to $\mathrm{id}_{|P_{\bullet,n}|}$. Thus, the map $P_{\bullet,n} \to N_{\bullet,n}$ does indeed lie in $\mathbf{W}_{\mathrm{KQ}}$ for all $[n]^\circ \in \mathbf{\Delta}^{op}$, so that the map $P_\bullet \to N_\bullet$ lies in $\mathbf{W}_{\mathrm{KQ}}$ as well.

- The vertical map $P_\bullet \to Q_\bullet$ is of course given by the composition $P_\bullet \to N_\bullet \to Q_\bullet$. More explicitly, it is corepresented by the morphism in $\mathrm{hom}_{c\mathrm{Model}_{**}}(\underline{\mathbf{q}}^\bullet, \underline{\mathbf{p}}^\bullet)$ given in level n by the unique functor satisfying $\alpha(i) \mapsto \alpha(i)$ and $\beta(i) \mapsto \gamma(i)$.

- The horizontal map $P_\bullet \to Q_\bullet$ is corepresented by the morphism in $\hom_{c\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{q}}^\bullet, \underline{\mathbf{p}}^\bullet)$ which is simply the the defining inclusion in each level. Note that this is obtained by applying $cc\mathcal{M}\mathrm{odel}_{**} \xrightarrow{\mathrm{diag}^*} c\mathcal{M}\mathrm{odel}_{**}$ to the morphism in $\hom_{cc\mathcal{M}\mathrm{odel}_{**}}(\underline{\mathbf{q}}^{\bullet\bullet}, \underline{\mathbf{p}}^{\bullet\bullet})$ which is again simply the defining inclusion in each bidegree. This latter map corepresents a map $P_{\bullet\bullet} \to Q_{\bullet\bullet}$ in $ss\mathcal{S}$, from which the horizontal map $P_\bullet \to Q_\bullet$ in $s\mathcal{S}$ is therefore obtained by applying $ss\mathcal{S} \xrightarrow{\mathrm{diag}^*} s\mathcal{S}$.

  Now, since

  $$|P_\bullet| \simeq \mathrm{colim}_{[m]^\circ \in \mathbf{\Delta}^{op}} |P_{m,\bullet}|, \quad |Q_\bullet| \simeq \mathrm{colim}_{[m]^\circ \in \mathbf{\Delta}^{op}} |Q_{m,\bullet}|,$$

  to prove that the horizontal map $P_\bullet \to Q_\bullet$ is in $\mathbf{W}_{\mathrm{KQ}}$, it suffices to prove that for each $[m]^\circ \in \mathbf{\Delta}^{op}$, the map $|P_{m,\bullet}| \to |Q_{m,\bullet}| \simeq Q_m$ is an equivalence in $\mathcal{S}$ (where the given equivalence comes from the fact that $Q_{m,\bullet} \simeq \mathrm{const}(Q_m)$).

  Via the map $P_{m,\bullet} \to Q_{m,\bullet} \simeq \mathrm{const}(Q_m)$, we can consider $P_{m,\bullet}$ as a simplicial object

  $$\mathbf{\Delta}^{op} \xrightarrow{P_{m,\bullet}} \mathcal{S}_{/Q_m};$$

  moreover, $|P_{m,\bullet}|$ is still its colimit in this ∞-category since colimits in $\mathcal{S}_{/Q_m}$ are created in $\mathcal{S}$. Now, we have a composite equivalence

  $$\mathrm{Fun}(Q_m, \mathcal{S}) \xrightarrow[\sim]{\mathrm{Gr}} \mathcal{LF}\mathrm{ib}(Q_m) \simeq \mathcal{S}_{/Q_m}$$

  (recall Remark G.1.5), under which the above simplicial object corresponds to a simplicial object

  $$\mathbf{\Delta}^{op} \xrightarrow{\mathrm{Gr}^{-1}(P_{m,\bullet})} \mathrm{Fun}(Q_m, \mathcal{S}).$$

  Hence, to show that $|P_{m,\bullet}| \in \mathcal{S}_{/Q_m}$ is a terminal object (i.e. to show that $|P_{m,\bullet}| \xrightarrow{\sim} Q_m$), it suffices to obtain an equivalence

  $$|\mathrm{Gr}^{-1}(P_{m,\bullet})| \simeq \mathrm{pt}_{\mathrm{Fun}(Q_m,\mathcal{S})}.$$

  As colimits in $\mathrm{Fun}(Q_m, \mathcal{S})$ are computed pointwise, for this it suffices to show that for any point $q \in Q_m$, we have

  $$|\mathrm{Gr}^{-1}(P_{m,\bullet})(q)| \simeq \mathrm{pt}_{\mathcal{S}}.$$

  Moreover, the naturality of the Grothendieck construction implies that we can identify the constituent simplicial spaces of this geometric realization as

  $$\mathrm{Gr}^{-1}(P_{m,n})(q) \simeq \lim \left(\begin{array}{ccc} & & P_{m,n} \\ & & \downarrow \\ \mathrm{pt}_{\mathcal{S}} & \xrightarrow{q} & Q_m \end{array} \right)$$

for all $n \geq 0$ in a way compatible with the simplicial structure maps; in other words, we have an equivalence

$$
\mathrm{Gr}^{-1}(P_{m,\bullet})(q) \simeq \lim \left(\begin{array}{c} P_{m,\bullet} \\ \downarrow \\ \mathrm{pt}_{s\mathcal{S}} \xrightarrow[\mathrm{const}(q)]{} \mathrm{const}(Q_m) \end{array} \right)
$$

in $s\mathcal{S}$.

Now, by definition $Q_m = \mathrm{hom}_{(\mathrm{Model}_\infty)_{**}}(\underline{\mathbf{q}}^m, \mathcal{M})$, and so our point $q \in Q_m$ corresponds to some map $\underline{\mathbf{q}}^m \xrightarrow{q'} \mathcal{M}$ in $(\mathrm{Model}_\infty)_{**}$. Via this map we can consider $\mathcal{M} \in ((\mathrm{Model}_\infty)_{**})_{\underline{\mathbf{q}}^m/}$, and it is not hard to see that we have equivalences

$$
\lim \left(\begin{array}{c} P_{m,\bullet} \\ \downarrow \\ \mathrm{pt}_{s\mathcal{S}} \xrightarrow[\mathrm{const}(q)]{} \mathrm{const}(Q_m) \end{array} \right) \simeq
$$

$$
\mathrm{hom}^{\mathrm{lw}}_{((\mathrm{Model}_\infty)_{**})_{\underline{\mathbf{q}}^m/}}(\underline{\mathbf{p}}^{m,\bullet}, \mathcal{M}) \simeq \mathrm{N}_\infty \left(\left(\mathbf{W}_{y\backslash} \right)_{(y \xrightarrow{\approx} q'(\beta(i)))/} \right).
$$

But this last simplicial space is the nerve of an ∞-category with an initial object, so it has contractible geometric realization by Proposition N.2.4 and the opposite of Corollary G.4.11. Thus, we have shown that $|P_{m,\bullet}| \xrightarrow{\sim} Q_m$, which as we have seen implies that $|P_\bullet| \xrightarrow{\sim} |Q_\bullet|$, i.e. that $P_\bullet \to Q_\bullet$ lies in $\mathbf{W}_{\mathrm{KQ}}$.

- The maps $P_\bullet \to P_\bullet \times \Delta^1$ are given by

$$
P_\bullet \simeq P_\bullet \times \Delta^{\{i\}} \to P_\bullet \times \Delta^1,
$$

where we take $i = 0$ for the horizontal map and $i = 1$ for the vertical map. These lie in $\mathbf{W}_{\mathrm{KQ}}$ since the geometric realization functor $|-| : s\mathcal{S} \to \mathcal{S}$ (as a sifted colimit) commutes with finite products.

- The map $P_\bullet \times \Delta^1 \to Q_\bullet$ is the corepresented left homotopy associated to the left homotopy corepresentation

$$
\{\{_{\underline{\mathbf{q}}}h_n^i \in \mathrm{hom}_{\mathrm{Model}_{**}}(\underline{\mathbf{q}}^{n+1}, \underline{\mathbf{p}}^n)\}_{0 \leq i \leq n}\}_{n \geq 0}
$$

given by defining $_{\underline{\mathbf{q}}}h_n^i$ to be the unique functor satisfying

$$
\alpha(j) \mapsto \left\{ \begin{array}{ll} \alpha(j), & j \leq i \\ \alpha(j-1), & j > i \end{array} \right.
$$

$$
\beta(j) \mapsto \left\{ \begin{array}{ll} \beta(j), & j \leq i \\ \gamma(j), & j > i. \end{array} \right.
$$

(It is tedious but straightforward to verify that these formulas do indeed define a suitable left homotopy corepresentation.)

Thus, we have exhibited the above original commutative diagram in $s\mathcal{S}$ and shown that it lies entirely in $\mathbf{W}_{\mathrm{KQ}}$. In particular, it follows that $|M_\bullet| \xrightarrow{\sim} |Q_\bullet|$, i.e. that

$$\left\| \hom_{\mathcal{M}}^{\mathrm{lw}}({}_\sigma\mathrm{cyl}^\bullet(x), {}_\sigma\mathrm{path}_\bullet(y)) \right\| \xrightarrow{\sim} \underline{\tilde{\mathbf{3}}}(x,y)^{\mathrm{gpd}},$$

as desired. $\square$

We now prove an auxiliary result which was needed in the proof of proposition 5.1, an analog of [4, Proposition 6.11].[9]

Lemma 5.2. *If $y \in \mathcal{M}^f$ is fibrant and ${}_\sigma\mathrm{path}_\bullet(y) \in s\mathcal{M}$ is any special path object for y, then the functor*

$$\mathbf{\Delta}^{op} \to \mathbf{W}_{y\backslash}$$

$$[n]^\circ \mapsto (y \overset{\approx}{\rightarrowtail} {}_\sigma\mathrm{path}_n(y))$$

is final.

Proof. According to the characterization of Theorem A (G.4.10), it suffices to show that for any object $(y \overset{\approx}{\rightarrowtail} z) \in \mathbf{W}_{y\backslash}$, the groupoid completion of the comma ∞-category

$$\mathbf{\Delta}^{op} \underset{\mathbf{W}_{y\backslash}}{\times} \left(\mathbf{W}_{y\backslash} \right)_{(y \overset{\approx}{\rightarrowtail} z)/}$$

is contractible.

First of all, note that the chosen equivalence $y \simeq {}_\sigma\mathrm{path}_0(y)$ endows the object $\hom_{\mathcal{M}}^{\mathrm{lw}}(y, {}_\sigma\mathrm{path}_\bullet(y)) \in s\mathcal{S}$ with a canonical basepoint $\mathrm{pt}_{s\mathcal{S}} \to \hom_{\mathcal{M}}^{\mathrm{lw}}(y, {}_\sigma\mathrm{path}_\bullet(y))$. The dual of proposition 2.1(1) implies that the map

$$\hom_{\mathcal{M}}^{\mathrm{lw}}(z, {}_\sigma\mathrm{path}_\bullet(y)) \to \hom_{\mathcal{M}}^{\mathrm{lw}}(y, {}_\sigma\mathrm{path}_\bullet(y))$$

is in $(\mathbf{W} \cap \mathbf{F})_{\mathrm{KQ}}$, which implies (by Proposition S.6.5) that its fiber over that basepoint has contractible geometric realization. As fibers (being limits) in $s\mathcal{S} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S})$ are computed objectwise, this fiber is given in level n by

$$\hom_{\left(\mathbf{W}_{y\backslash}\right)} \left(y \overset{\approx}{\rightarrowtail} z, y \overset{\approx}{\rightarrowtail} {}_\sigma\mathrm{path}_n(y) \right).$$

---

[9]The proof of [4, Proposition 6.11] contains a mild but rather confusing typo. There, it is claimed that a certain category is isomorphic to the homotopy colimit of a simplicial set, which is then claimed to have the same homotopy type as another simplicial set. In fact, it is the *nerve* of the category which is *isomorphic* to the first simplicial set itself (without saying "homotopy colimit"), and then this simplicial set is equivalent to the other simplicial set because the latter is the nerve of the category of simplices of the former. This last statement can be seen as coming from the fact that there are two ways to take the homotopy colimit of a simplicial set: either by taking its usual geometric realization, or by taking the geometric realization of its simplicial replacement.

(Note that the inclusions $\mathbf{W}_{y\backslash} \subset \mathbf{W}_{y/} \subset \mathcal{M}_{y/}$ are both inclusions of full subcategories (the latter by the two-out-of-three property).) By the Bousfield–Kan colimit formula (Theorem G.5.8), the geometric realization of this simplicial space is equivalent to the geometric realization of its simplicial replacement when considered in $s\mathcal{S} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S})$. In level n, this simplicial replacement is given by

$$\coprod_{\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n} \hom_{\left(\mathbf{W}_{y\backslash}\right)} \left(y \xrightarrow{\approx} z, y \xrightarrow{\approx} {}_{\sigma}\mathrm{path}_{\alpha(0)}(y) \right).$$

We claim that this latter simplicial space is precisely the nerve of the comma ∞-category

$$\mathbf{\Delta}^{op} \underset{\mathbf{W}_{y\backslash}}{\times} (\mathbf{W}_{y\backslash})_{(y \xrightarrow{\approx} z)/}.$$

To see this, observe that

$$\mathrm{N}_\infty \left(\mathbf{\Delta}^{op} \underset{\mathbf{W}_{y\backslash}}{\times} (\mathbf{W}_{y\backslash})_{(y \xrightarrow{\approx} z)/} \right)_n = \hom_{\mathcal{C}\mathrm{at}_\infty} \left([n], \mathbf{\Delta}^{op} \underset{\mathbf{W}_{y\backslash}}{\times} \left(\mathbf{W}_{y\backslash}\right)_{(y \xrightarrow{\approx} z)/} \right)$$

$$\simeq \lim \left(\begin{array}{ccc} & & \hom_{\mathcal{C}\mathrm{at}_\infty} \left([n], \left(\mathbf{W}_{y\backslash}\right)_{(y \xrightarrow{\approx} z)/} \right) \\ & & \downarrow \\ \hom_{\mathcal{C}\mathrm{at}_\infty} ([n], \mathbf{\Delta}^{op}) & \longrightarrow & \hom_{\mathcal{C}\mathrm{at}_\infty} \left([n], \mathbf{W}_{y\backslash} \right) \end{array} \right).$$

Since $\hom_{\mathcal{C}\mathrm{at}_\infty}([n], \mathbf{\Delta}^{op}) \simeq \mathrm{N}(\mathbf{\Delta}^{op})_n$ is discrete, this pullback is equivalent to a coproduct over its elements of the corresponding fibers. Over the element $\alpha \in \mathrm{N}(\mathbf{\Delta}^{op})_n$, this fiber is

$$\lim \left(\begin{array}{ccc} & & \hom_{\mathcal{C}\mathrm{at}_\infty} \left([n], \left(\mathbf{W}_{y\backslash}\right)_{(y \xrightarrow{\approx} z)/} \right) \\ & & \Big\downarrow \\ \left\{ [n] \xrightarrow{\alpha} \mathbf{\Delta}^{op} \to \mathbf{W}_{y\backslash} \right\} & & \\ & \searrow & \\ & & \hom_{\mathcal{C}\mathrm{at}_\infty} \left([n], \mathbf{W}_{y\backslash} \right) \end{array} \right)$$

$$\simeq \lim \left(\begin{array}{c} \{(y \xrightarrow{\approx} z)\} \\ \hom_{\mathcal{C}at_\infty}\left(\{(-1) \to \cdots \to n\}, \mathbf{W}_{y\downdownarrows}\right) \\ \left\{[n] \xrightarrow{\alpha} \boldsymbol{\Delta}^{op} \to \mathbf{W}_{y\downdownarrows}\right\} \qquad \hom_{\mathcal{C}at_\infty}\left(\{(-1)\}, \mathbf{W}_{y\downdownarrows}\right) \\ \hom_{\mathcal{C}at_\infty}\left([n], \mathbf{W}_{y\downdownarrows}\right) \end{array} \right)$$

$$\simeq \lim \left(\begin{array}{c} \{(y \xrightarrow{\approx} z)\} \\ \hom_{\mathcal{C}at_\infty}\left(\{(-1) \to 0\}, \mathbf{W}_{y\downdownarrows}\right) \\ \left\{(y \xrightarrow{\approx} {}_\sigma \mathrm{path}_{\alpha(0)}(y))\right\} \qquad \hom_{\mathcal{C}at_\infty}\left(\{(-1)\}, \mathbf{W}_{y\downdownarrows}\right) \\ \hom_{\mathcal{C}at_\infty}\left([0], \mathbf{W}_{y\downdownarrows}\right) \end{array} \right)$$

$$\simeq \hom_{\left(\mathbf{W}_{y\downdownarrows}\right)}\left(y \xrightarrow{\approx} z, y \xrightarrow{\approx} {}_\sigma \mathrm{path}_{\alpha(0)}(y)\right).$$

Moreover, it is clear that the structure maps of this simplicial space agree with those of the above simplicial replacement: both are ultimately induced by the structure maps of ${}_\sigma \mathrm{path}_\bullet(y) \in s\mathcal{M}$. So, these are indeed equivalent simplicial spaces.

We have just shown that the geometric realization of the complete Segal space

$$\mathrm{N}_\infty \left(\boldsymbol{\Delta}^{op} \underset{\mathbf{W}_{y\downdownarrows}}{\times} (\mathbf{W}_{y\downdownarrows})_{(y \xrightarrow{\approx} z)/} \right)$$

is contractible. Thus, by Proposition N.2.4, the groupoid completion

$$\left(\boldsymbol{\Delta}^{op} \underset{\mathbf{W}_{d\downdownarrows}}{\times} (\mathbf{W}_{d\downdownarrows})_{(d \xrightarrow{\approx} d')/} \right)^{\mathrm{gpd}}$$

is indeed contractible. $\qquad \square$

6. The equivalence $\underline{\tilde{\mathbf{3}}}(x, y)^{\mathbf{gpd}} \simeq \underline{\mathbf{3}}(x, y)^{\mathbf{gpd}}$

We now prove that the ∞-category of three-arrow zigzags from x to y has equivalent groupoid completion to that of its subcategory of special three-arrow zigzags.

Proposition 6.1. *For any model ∞-category $\mathcal{M}$ and any $x, y \in \mathcal{M}$, the unique map $\underline{\mathbf{3}} \to \underline{\tilde{\mathbf{3}}}$ in Model_{**} induces an equivalence*

$$\underline{\tilde{\mathbf{3}}}(x, y)^{\mathrm{gpd}} \xrightarrow{\sim} \underline{\mathbf{3}}(x, y)^{\mathrm{gpd}}$$

on groupoid completions of ∞-categories of zigzags in $\mathcal{M}$ from x to y.

Proof. We apply the functor $\left(\mathrm{Fun}^{*}_{**}(-, \mathcal{M})^{\mathbf{W}}\right)^{\mathrm{gpd}}$ to the sequence of maps in Model^{*}_{**} given in the proof of [11, Proposition 3.11(1)] (which factors the unique map $\underline{\mathbf{3}} \to \underline{\tilde{\mathbf{3}}}$ in Model_{**}). To show that the induced maps in $\mathcal{S}$ are all equivalences, the arguments given there generalize as follows.

- To show that the maps φ_1 and φ_4 defined there induce equivalences in $\mathcal{S}$, we replace the appeal to [11, Lemma 3.9(1)] with an appeal to the factorization lemma (4.24).

- The maps φ_2 and φ_5 defined there even induce equivalences in $\mathcal{C}\mathrm{at}_\infty$ upon application of $\mathrm{Fun}^{*}_{**}(-, \mathcal{M})^{\mathbf{W}}$; to see this, we use the argument given in the proof of proposition 7.1 for why the maps φ_2, φ_4, φ_9, and φ_{11} (of that proof) have this same property.

- To show that the maps φ_3 and φ_6 defined there induce equivalences in $\mathcal{S}$, we use the argument given in the proof of proposition 7.1 for why the maps φ_7 and φ_{14} (of that proof) have this same property.

Thus, we obtain the desired equivalence $\underline{\tilde{\mathbf{3}}}(x, y)^{\mathrm{gpd}} \simeq \underline{\mathbf{3}}(x, y)^{\mathrm{gpd}}$ in $\mathcal{S}$.   □

7. The equivalence $\underline{\mathbf{3}}(x, y)^{\mathbf{gpd}} \simeq \underline{\mathbf{7}}(x, y)^{\mathbf{gpd}}$

We now prove that the ∞-categories of three-arrow zigzags and seven-arrow zigzags from x to y have equivalent groupoid completions.

Proposition 7.1. *If $\mathcal{M}$ is a model ∞-category, then for any $x, y \in \mathcal{M}$, the map $\underline{\mathbf{7}} \to \underline{\mathbf{3}}$ in Model_{**} given by collapsing the middle four instances of $\mathbf{W}^{\pm}$ induces an equivalence*

$$\underline{\mathbf{3}}(x, y)^{\mathrm{gpd}} \xrightarrow{\sim} \underline{\mathbf{7}}(x, y)^{\mathrm{gpd}}$$

on groupoid completions of ∞-categories of zigzags in $\mathcal{M}$ from x to y.

Proof. In essence, we use the factorization lemma (4.24) to remove each instance of $\mathbf{W}^{-1}$ in $\underline{\mathbf{7}}$ which is adjacent to the unique instance of $\mathbf{A}$, and then we "compose out" the remaining instances of $\mathbf{W}$. To be precise, we define a diagram

$$\underline{\mathbf{7}} \xrightarrow{\varphi_1} \mathcal{I}_1 \xrightarrow{\varphi_2} \mathcal{I}_2 \xleftarrow{\varphi_3} \mathcal{I}_3 \xrightarrow{\varphi_4} \mathcal{I}_4 \xleftarrow{\varphi_5} \mathcal{I}_5 \xleftarrow{\varphi_6} \mathcal{I}_6 \xleftarrow{\varphi_7} \mathcal{I}_7 \xrightarrow{\varphi_8}$$

$$\mathcal{I}_8 \xrightarrow{\varphi_9} \mathcal{I}_9 \xleftarrow{\varphi_{10}} \mathcal{I}_{10} \xrightarrow{\varphi_{11}} \mathcal{I}_{11} \xrightarrow{\varphi_{12}} \mathcal{I}_{12} \xleftarrow{\varphi_{13}} \mathcal{I}_{13} \xleftarrow{\varphi_{14}} \underline{\mathbf{3}}$$

in $\mathcal{M}\mathrm{odel}^{*}_{**}$, given by

$$\underline{\mathbf{7}} = \left(s \xleftarrow{\approx} \bullet \xrightarrow{\approx} \bullet \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \bullet \xrightarrow{\approx} \bullet \xleftarrow{\approx} t \right)$$

$\xrightarrow{\varphi_1}$

$\xrightarrow{\varphi_2}$ (p.b.)

$\xleftarrow{\varphi_3}$

$\xrightarrow{\varphi_4}$ (p.o.)

$\xleftarrow{\varphi_5}$

$$\xleftarrow{\varphi_6} \left(s \xleftarrow{\approx} \bullet \xrightarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \bullet \xrightarrow{\approx} \bullet \xleftarrow{\approx} t \right)$$

$$\xleftarrow{\varphi_7} \left(s \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \bullet \xrightarrow{\approx} \bullet \xleftarrow{\approx} t \right)$$

$\xrightarrow{\varphi_8}$

$\xrightarrow{\varphi_9}$ (p.b.)

$\xleftarrow{\varphi_{10}}$

$$\xrightarrow{\varphi_{11}} \left(\begin{array}{c} s \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} t \\ \bullet \longrightarrow \bullet \xrightarrow{\approx} \bullet \end{array} \right)^{\mathrm{p.o.}}$$

$$\xleftarrow{\varphi_{12}} \left(\begin{array}{c} s \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} t \\ \bullet \longrightarrow \bullet \xrightarrow{\approx} \bullet \end{array} \right)$$

$$\xleftarrow{\varphi_{13}} \left(s \xleftarrow{\approx} \bullet \longrightarrow \bullet \xrightarrow{\approx} \bullet \xleftarrow{\approx} t \right)$$

$$\xleftarrow{\varphi_{14}} \left(s \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} t \right) = \underline{\mathbf{3}},$$

where all maps are the completely evident inclusions, except that

- φ_6 and φ_{13} are the "lower inclusions" (whose images omit any objects in the upper rows that are the source or target of a drawn-in diagonal arrow – note that there are certain "hidden" diagonal maps in $\mathcal{I}_5$ and $\mathcal{I}_{12}$, which are only composites of drawn-in arrows), and

- φ_7 and φ_{14} are obtained by taking the unique copy of $\mathbf{A}$ onto the composite $[\mathbf{W}; \mathbf{A}]$ or $[\mathbf{A}; \mathbf{W}]$, respectively.

We claim that this induces a diagram of equivalences in $\mathcal{S}$ upon application of $\left(\mathrm{Fun}^*_{**}(-, \mathcal{M})^{\mathbf{W}} \right)^{\mathrm{gpd}}$. The arguments can be grouped as follows.

- The maps φ_1 and φ_8 induce equivalences in $\mathcal{S}$ by the factorization lemma (4.24).

- The maps φ_2, φ_4, φ_9, and φ_{11} actually even induce equivalences in $\mathcal{C}\mathrm{at}_\infty$ upon application of $\mathrm{Fun}^*_{**}(-, \mathcal{M})^{\mathbf{W}}$; this follows from the facts that

  – $\mathcal{M}$ is finitely bicomplete,

  – the subcategories $(\mathbf{W} \cap \mathbf{F}), (\mathbf{W} \cap \mathbf{C}) \subset \mathcal{M}$ are respectively closed under pullbacks and pushouts, and

  – the subcategory $\mathbf{W} \subset \mathcal{M}$ has the two-out-of-three property

  (see e.g. Proposition T.4.3.2.15).

- Upon application of $\mathrm{Fun}^*_{**}(-, \mathcal{M})^{\mathbf{W}}$, the maps φ_3 and φ_{10} induce functors which admit left adjoints, and so they induce equivalences in $\mathcal{S}$ upon application of $\left(\mathrm{Fun}^*_{**}(-, \mathcal{M})^{\mathbf{W}} \right)^{\mathrm{gpd}}$ by Corollary N.1.28. Dually, the maps φ_5 and φ_{12} also induce equivalences in $\mathcal{S}$.

- The maps φ_6, φ_7, φ_{13}, and φ_{14} admit evident retractions ψ_6, ψ_7, ψ_{13}, and ψ_{14}, respectively. Moreover,

  - there are evident cospans of doubly-pointed natural weak equivalences connecting $\mathrm{id}_{\mathcal{J}_5}$ with $\varphi_6 \circ \psi_6$ and connecting $\mathrm{id}_{\mathcal{J}_{12}}$ with $\varphi_{13} \circ \psi_{13}$, and

  - there are evident doubly-pointed natural weak equivalences $\varphi_7 \circ \psi_7 \overset{\approx}{\to} \mathrm{id}_{\mathcal{J}_6}$ and $\mathrm{id}_{\mathcal{J}_{13}} \overset{\approx}{\to} \varphi_{14} \circ \psi_{14}$.

  Hence, by Lemmas 4.22 and N.1.26, these maps all induce equivalences in $\mathcal{S}$.

Thus, we obtain the desired equivalence $\underline{\mathbf{3}}(x,y)^{\mathrm{gpd}} \simeq \underline{\mathbf{7}}(x,y)^{\mathrm{gpd}}$ in $\mathcal{S}$ which, tracing back through the above zigzag in Model^*_{**}, it is clear is indeed induced by the asserted map $\underline{\mathbf{7}} \to \underline{\mathbf{3}}$ in Model_{**}. $\qquad\square$

8. Localization of model ∞-categories

So far, given a model ∞-category $\mathcal{M}$ and suitably co/fibrant objects $x, y \in \mathcal{M}$, we have related the spaces of left/right homotopy classes of maps from x to y to the groupoid completions of various ∞-categories of zigzags from x to y. However, in order to show that these are all actually equivalent to the space $\hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y)$ of maps from x to y in the localization $\mathcal{M}[\![\mathbf{W}^{-1}]\!]$, we must access this latter hom-space. This aim is one of the primary purposes of the local universal property of the Rezk nerve (Theorem N.3.8) and the calculus theorem (H.6.1), which we now bring to fruition. The following result will be strictly generalized by theorem 10.1, but the latter actually requires the full force of the fundamental theorem of ∞-categories (theorem 1.9). Thus, to avoid circularity, we prove only this weaker version first.

Proposition 8.1. *If $\mathcal{M}$ is a model ∞-category with underlying relative ∞-category $(\mathcal{M}, \mathbf{W})$, then $\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M}, \mathbf{W}) \in \mathcal{SS}$, and moreover the morphism $\mathrm{N}_\infty(\mathcal{M}) \to \mathrm{L}_{\mathcal{CSS}}(\mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M}, \mathbf{W}))$ in $\mathcal{CSS}$ corresponds to the morphism $\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$ in $\mathcal{C}\mathrm{at}_\infty$.*

Proof. The first claim is obtained by combining lemma 8.2 and the calculus theorem (H.6.1(1)), while the second claim follows from the local universal property of the Rezk nerve (Theorem N.3.8). $\qquad\square$

We now give an auxiliary result on which the proof of proposition 8.1 relies.

Lemma 8.2. *If $\mathcal{M}$ is a model ∞-category, then its underlying relative ∞-category $(\mathcal{M}, \mathbf{W})$ admits a homotopical three-arrow calculus.*

Proof. After choosing any pair of objects $x, y \in \mathcal{M}$, we apply the functor $\left(\mathrm{Fun}^*_{**}(-, \mathcal{M})^{\mathbf{W}}\right)^{\mathrm{gpd}}$ to the diagram in Model^*_{**} given in the proof of [11,

Proposition 3.16(1)]. To show that the induced maps in S are all equivalences, the arguments given there generalize as follows.

- To show that the map ρ_1 defined there induces an equivalence in S, we replace the appeal to [11, Lemma 3.9(1)] with an appeal to the factorization lemma (4.24).

- The map ρ_2 defined there even induces an equivalence in $\mathcal{C}\mathrm{at}_\infty$ upon application of $\mathrm{Fun}^*_{**}(-,\mathcal{M})^{\mathbf{W}}$; to see this, we repeatedly apply the argument given in the proof of proposition 7.1 for why the maps φ_2, φ_4, φ_9, and φ_{11} (of that proof) have this same property.

- The map ρ_3 defined there induces an equivalence in S in exactly the same manner; we replace the appeal to [11, Lemma 3.10] with an appeal to Lemmas 4.22 and N.1.26.

Thus, the underlying relative ∞-category $(\mathcal{M}, \mathbf{W})$ of the model ∞-category $\mathcal{M}$ does indeed admit a homotopical three-arrow calculus. $\qquad\square$

9. The equivalence $\underline{7}(x,y)^{\mathrm{gpd}} \simeq \hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y)$

In this section, we show that the groupoid completion of the ∞-category of seven-arrow zigzags from x to y is equivalent to the hom-space $\hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y)$, thus completing the string of equivalences in the proof of the fundamental theorem of model ∞-categories (1.9).

Proposition 9.1. *For any model ∞-category $\mathcal{M}$ and any $x,y \in \mathcal{M}$, we have a canonical equivalence*

$$\underline{7}(x,y)^{\mathrm{gpd}} \xrightarrow{\sim} \hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y).$$

Proof. First of all, by proposition 8.1 (and Remark H.2.6), we have

$$\hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y) \simeq \lim \left(\begin{array}{c} \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M},\mathbf{W})_1 \\ \downarrow {\scriptstyle (s,t)} \\ \mathrm{pt}_S \xrightarrow[(x,y)]{} \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M},\mathbf{W})_0 \times \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M},\mathbf{W})_0 \end{array}\right)$$

$$\simeq \lim \left(\begin{array}{ccc} & & \mathrm{pt}_S \\ & & \downarrow {\scriptstyle y} \\ \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M},\mathbf{W})_1 & \xrightarrow{\ t\ } & \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M},\mathbf{W})_0 \\ \downarrow {\scriptstyle s} & & \\ \mathrm{pt}_S \xrightarrow{\ x\ } \mathrm{N}^{\mathrm{R}}_\infty(\mathcal{M},\mathbf{W})_0 & & \end{array}\right)$$

$$= \lim \left(\begin{array}{ccc} & & \mathrm{pt}_{\mathcal{S}} \\ & & \downarrow{\scriptstyle y} \\ (\mathrm{Fun}([1],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} & \xrightarrow{\ t^{\mathrm{gpd}}\ } & (\mathrm{Fun}([0],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} \\ \downarrow{\scriptstyle s^{\mathrm{gpd}}} & & \\ \mathrm{pt}_{\mathcal{S}} \xrightarrow{\ x\ } (\mathrm{Fun}([0],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} & & \end{array} \right)$$

$$\simeq \lim \left(\begin{array}{ccc} & & (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{\mathrm{gpd}} \\ & & \downarrow{\scriptstyle y^{\mathrm{gpd}}} \\ (\mathrm{Fun}([1],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} & \xrightarrow{\ t^{\mathrm{gpd}}\ } & \mathbf{W}^{\mathrm{gpd}} \\ \downarrow{\scriptstyle s^{\mathrm{gpd}}} & & \\ (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{\mathrm{gpd}} \xrightarrow{\ x^{\mathrm{gpd}}\ } \mathbf{W}^{\mathrm{gpd}} & & \end{array} \right).$$

Note that this final limit is that of a diagram in $\mathcal{S}$ coming from a diagram in $\mathcal{C}\mathrm{at}_\infty$ via postcomposition with $(-)^{\mathrm{gpd}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$. We will compute this limit by first computing the pullback of the lower left cospan (defined by the maps x and s) and then computing the pullback of the resulting cospan; for both pullbacks we will appeal to Theorems B_n and C_n (G.4.23 and G.4.26), noting once and for all that $\mathbf{W}^{op}$ has property C_3 by Lemmas 9.2 and 8.2.

First of all, by Theorem C_n (G.4.26), the functor

$$(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{op} \xrightarrow{\ x^\circ\ } \mathbf{W}^{op}$$

has property B_3. Hence, by Theorem B_n (G.4.23), we have a homotopy pullback square

$$\begin{array}{ccc} (x^\circ((\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{op}) \downarrow_3 s^{op}((\mathrm{Fun}([1],\mathcal{M})^{\mathbf{W}})^{op})) & \xrightarrow{\ t\ } & (\mathrm{Fun}([1],\mathcal{M})^{\mathbf{W}})^{op} \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s^{op}} \\ (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{op} & \xrightarrow{\ x^\circ\ } & \mathbf{W}^{op} \end{array}$$

in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$; unwinding the definitions, we can identify the homotopy pullback as

$$(\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1}; \mathbf{W}; \mathbf{W}^{-1}; \mathbf{A}], \mathcal{M})^{\mathbf{W}})^{op},$$

where the object $x \in \mathcal{M}$ determines the pointing. As homotopy pullback squares in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$ are preserved under the involution $(-)^{op} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$, it follows that we have a pullback square

$$\begin{array}{ccc} (\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1}; \mathbf{W}; \mathbf{W}^{-1}; \mathbf{A}], \mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} & \longrightarrow & (\mathrm{Fun}([1],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} \\ \downarrow & & \downarrow{\scriptstyle s^{\mathrm{gpd}}} \\ (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{\mathrm{gpd}} & \xrightarrow{\ x^{\mathrm{gpd}}\ } & \mathbf{W}^{\mathrm{gpd}} \end{array}$$

in $\mathcal{S}$, and hence we can simplify the above limit computing $\hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y)$ to give the identification

$$\hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y) \simeq$$

$$\lim \left(\begin{array}{ccc} & & (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{\mathrm{gpd}} \\ & & \downarrow y \\ (\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A}],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} & \xrightarrow[t^{\mathrm{gpd}}]{} & \mathbf{W}^{\mathrm{gpd}} \end{array} \right).$$

Then, again by Theorem C_n (G.4.26), the functor

$$(\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A}],\mathcal{M})^{\mathbf{W}})^{op} \xrightarrow{t^{op}} \mathbf{W}^{op}$$

has property B_3, so that by Theorem B_n (G.4.23) we have a homotopy pullback square

$$\begin{array}{ccc} (t^{op}((\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A}],\mathcal{M})^{\mathbf{W}})^{op}) \downarrow_3 y^\circ((\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{op})) & \xrightarrow{t} & (\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})^{op} \\ s\downarrow & & \downarrow y^\circ \\ (\mathrm{Fun}_{*\circ}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A}],\mathcal{M})^{\mathbf{W}})^{op} & \xrightarrow[t^{op}]{} & \mathbf{W}^{op} \end{array}$$

in $(\mathcal{C}\mathrm{at}_\infty)_{\mathrm{Th}}$; this time, unwinding the definitions we can identify the homotopy pullback as

$$(\mathrm{Fun}_{**}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A};\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1}],\mathcal{M})^{\mathbf{W}})^{op},$$

where the objects $x,y \in \mathcal{M}$ determine the double-pointing. Hence we obtain an equivalence

$$\underline{7}(x,y)^{\mathrm{gpd}} = (\mathrm{Fun}_{**}([\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1};\mathbf{A};\mathbf{W}^{-1};\mathbf{W};\mathbf{W}^{-1}],\mathcal{M})^{\mathbf{W}})^{\mathrm{gpd}} \xrightarrow{\sim}$$
$$\hom_{\mathcal{M}[\![\mathbf{W}^{-1}]\!]}(x,y),$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now provide a result which was needed in the proof of proposition 9.1.

Lemma 9.2. *If $(\mathcal{R},\mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ admits a homotopical three-arrow calculus and $\mathbf{W} \subset \mathcal{R}$ has the two-out-of-three property, then $\mathbf{W}^{op}$ has property C_3.*

Proof. To show that $\mathbf{W}^{op}$ has property C_3, we must show that any functor $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \xrightarrow{r^\circ} \mathbf{W}^{op}$ (selecting an object $r^\circ \in \mathbf{W}^{op}$) has property B_3, i.e. that the induced functor

$$\mathbf{W}^{op} \xrightarrow{(r^\circ(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty})\downarrow_3 -)} \mathcal{C}\mathrm{at}_\infty$$

has property Q, i.e. that for any map $z^\circ \xrightarrow{\varphi^\circ} y^\circ$ in $\mathbf{W}^{op}$ (opposite to a map $z \xleftarrow{\varphi} y$ in $\mathbf{W}$), the induced map

$$(r^\circ(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}) \downarrow_3 z^\circ) \to (r^\circ(\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty}) \downarrow_3 y^\circ)$$

is in $\mathbf{W}_{\mathrm{Th}} \subset \mathcal{C}\mathrm{at}_\infty$. Unwinding the definitions, we can identify this map simply as the functor

$$\underline{\mathbf{3}}_{(\mathbf{W},\mathbf{W})}(r,z) \to \underline{\mathbf{3}}_{(\mathbf{W},\mathbf{W})}(r,y)$$

that postconcatenates a zigzag $r \xleftarrow{\approx} \bullet \xrightarrow{\approx} \bullet \xleftarrow{\approx} z$ with the map φ (considered as a $[\mathbf{W}^{-1}]$-shaped zigzag) and then composes the last two maps.[10] Thus, the nerve of the above map in $\mathcal{C}\mathrm{at}_\infty$ sits as the upper composite in a commutative square

$$
\begin{array}{ccccc}
\mathrm{N}_\infty(\underline{\mathbf{3}}(r,z)) & \longrightarrow & \mathrm{N}_\infty([\mathbf{W}^{-1};\mathbf{A};(\mathbf{W}^{-1})^{\circ 2}](r,y)) & \longrightarrow & \mathrm{N}_\infty(\underline{\mathbf{3}}(r,y)) \\
\Big\downarrow{\scriptstyle \approx} & & & & \Big\downarrow{\scriptstyle \approx} \\
\underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(r,z) & \xrightarrow[\chi^{\mathscr{L}^H(\mathbf{W},\mathbf{W})}_{r,z,y}(-,\varphi^{-1})]{\approx} & & & \underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(r,y)
\end{array}
$$

in $s\mathcal{S}_{\mathrm{KQ}}$, in which

- the lower map
  - is the evaluation of the composition map

$$\underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(y,z) \times \underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(z,r) \xrightarrow{\chi^{\mathscr{L}^H(\mathbf{W},\mathbf{W})}_{z,y,r}} \underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(y,r)$$

        in $\mathscr{L}^H(\mathbf{W},\mathbf{W}) \in \mathcal{C}\mathrm{at}_{s\mathcal{S}}$ (recall Definition H.2.9) at the point chosen by the composite

$$\mathrm{pt}_{s\mathcal{S}} \to \mathrm{N}_\infty([\mathbf{W}^{-1}](z,y)) \to \underline{\hom}_{\mathscr{L}^H(\mathbf{W},\mathbf{W})}(z,y)$$

        in which the first map is selected by φ and the second map is the defining inclusion into the colimit, and
  - lies in $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$ by Proposition H.5.8,
- the triangle commutes by the definition of the hammock simplicial space as a colimit over $\mathcal{Z}^{op}$ (see Definition H.3.17),
- the trapezoid commutes by the definition of composition in the hammock localization (see §H.5), and
- the vertical maps are in $\mathbf{W}_{\mathrm{KQ}}$ by the fundamental theorem of homotopical three-arrow calculi (H.4.4) since the relative ∞-category $(\mathbf{W},\mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ admits a homotopical three-arrow calculus by lemma 9.3.

The upper map is therefore also in $\mathbf{W}_{\mathrm{KQ}}$ since $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$ has the two-out-of-three property, and hence the result follows from Proposition N.2.4. $\qquad\square$

In the proof of lemma 9.2, we needed the following stability property of homotopical three-arrow calculi.

---

[10]Recall that $\underline{\mathbf{z}}_3 = (s \to \bullet \leftarrow \bullet \to t)$ (see Notation G.4.14) while $\underline{\mathbf{3}} = (s \xleftarrow{\approx} \bullet \to \bullet \xleftarrow{\approx} t)$, so there are *two* orientation-reversals going on here (counting the passage between $\mathbf{W}^{op}$ and $\mathbf{W}$), which cancel each other out.

Lemma 9.3. *If $(\mathcal{R}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ admits a homotopical three-arrow calculus and $\mathbf{W} \subset \mathcal{R}$ has the two-out-of-three property, then $(\mathbf{W}, \mathbf{W}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ also admits a homotopical three-arrow calculus.*

Proof. This follows directly from Definition H.4.1: if $\mathbf{W} \subset \mathcal{R}$ has the two-out-of-three property, then the vertical maps in the commutative square

$$\mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathbf{W})^{\mathbf{W}}$$

$$\mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{W}^{-1}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathbf{W})^{\mathbf{W}}$$

$$\mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathcal{R})^{\mathbf{W}}$$

$$\mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{W}^{-1}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathcal{R})^{\mathbf{W}}$$

induced by the map $(\mathbf{W}, \mathbf{W}) \to (\mathcal{R}, \mathbf{W})$ in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_\infty$ induce monomorphisms in $\mathcal{S}$ upon groupoid completion. $\square$

10. Localization of model ∞-categories, redux

For completeness, we include the following improvement of proposition 8.1, whose proof relies on the fundamental theorem of model ∞-categories (1.9).

Theorem 10.1. *If $\mathcal{M}$ is a model ∞-category with underlying relative ∞-category $(\mathcal{M}, \mathbf{W})$, then $\mathrm{N}_\infty^{\mathrm{R}}(\mathcal{M}, \mathbf{W}) \in \mathcal{C}\mathcal{S}\mathcal{S}$, and moreover the morphism $\mathrm{N}_\infty(\mathcal{M}) \to \mathrm{N}_\infty^{\mathrm{R}}(\mathcal{M}, \mathbf{W})$ in $\mathcal{C}\mathcal{S}\mathcal{S}$ corresponds to the morphism $\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$ in $\mathcal{C}\mathrm{at}_\infty$.*

Proof. In light of proposition 8.1, it only remains to show that $\mathrm{N}_\infty^{\mathrm{R}}(\mathcal{M}, \mathbf{W})$ is not just a Segal space, but is in fact complete. By the calculus theorem (H.6.1(2)), this follows from lemma 10.2 and the fact that $\mathbf{W} \subset \mathcal{M}$ satisfies the two-out-of-three property. $\square$

We needed the following result in the proof of theorem 10.1.

Lemma 10.2. *If $\mathcal{M}$ is a model ∞-category, then its underlying relative ∞-category $(\mathcal{M}, \mathbf{W})$ is saturated.*

Proof. We would like to show that the localization functor $\mathcal{M} \to \mathcal{M}[\![\mathbf{W}^{-1}]\!]$ creates the subcategory $\mathbf{W} \subset \mathcal{M}$. This is equivalent to showing that the functor $\mathrm{ho}(\mathcal{M}) \to \mathrm{ho}(\mathcal{M}[\![\mathbf{W}^{-1}]\!])$ creates the subcategory $\mathrm{ho}(\mathbf{W}) \subset \mathrm{ho}(\mathcal{M})$. For this, we must show that if a map $x \to y$ in $\mathrm{ho}(\mathcal{M})$ is taken to an isomorphism in $\mathrm{ho}(\mathcal{M}[\![\mathbf{W}^{-1}]\!])$, then it lies in the subcategory $\mathrm{ho}(\mathbf{W})$. By two-out-of-three axiom $\mathrm{M}_\infty 2$, it suffices to show this in the case that both objects $x, y \in \mathcal{M}^{cf} \subset \mathcal{M}$ are bifibrant. From here, with corollary 1.11 in hand, the proof runs identically to that of [6, Theorem 7.8.5]. $\square$

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