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Going-up theorems for simultaneous Diophantine approximation

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ABSTRACT. We establish several new inequalities linking classical exponents of Diophantine approximation associated to a real vector $\underline{\xi} = (\xi, \xi^2, \dots, \xi^N)$, in various dimensions N. We thereby obtain variants, and partly refinements, of recent results of Badziahin and Bugeaud. We further implicitly recover inequalities of Bugeaud and Laurent as special cases, with new proofs. Similar estimates concerning general real vectors (not on the Veronese curve) with $\mathbb{Q}$-linearly independent coordinates are addressed as well. Our method is based on Minkowski's Second Convex Body Theorem, applied in the framework of parametric geometry of numbers introduced by Schmidt and Summerer. We also frequently employ Mahler's Duality result on polar convex bodies.

CONTENTS

1. Introduction and outline

Let $N \geq 1$ be an integer and $\underline{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$. We denote by $\lambda_N(\underline{\xi})$ the ordinary exponent of simultaneous approximation, defined as the supremum of real λ such that

$$1 \leq |x| \leq X, \qquad \max_{1\leq j\leq N} |\xi_j x - y_j| \leq X^{-\lambda}, \tag{1}$$

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has a solution $(x, y_1, \dots, y_N) \in \mathbb{Z}^{N+1}$ for arbitrarily large values of X. Let the ordinary exponent of linear form approximation $w_N(\underline{\xi})$ be the supremum of real w such that

$$\max_{1\le j\le N} |x_j| \le X, \qquad |x_0 + \xi_1 x_1 + \cdots + \xi_N x_N| \le X^{-w} \tag{2}$$

has a solution $(x_0, \dots, x_N) \in \mathbb{Z}^{N+1}$ for arbitrarily large X. Similarly, let the uniform exponents $\widehat{\lambda}_N(\underline{\xi})$ and $\widehat{w}_N(\underline{\xi})$ respectively be given as the respective suprema such that (1) and (2) have a solutions for *all* large X. Dirichlet's Theorem implies for any $\underline{\xi} \in \mathbb{R}^N$

$$\lambda_N(\underline{\xi}) \ge \widehat{\lambda}_N(\underline{\xi}) \ge \frac{1}{N}, \qquad w_N(\underline{\xi}) \ge \widehat{w}_N(\underline{\xi}) \ge N. \tag{3}$$

This paper is primarily concerned with the special case $\underline{\xi} = (\xi, \xi^2, \dots, \xi^N)$ for $\xi \in \mathbb{R}$, that is points on a Veronese curve. We then denote the exponents $\lambda_N(\underline{\xi}), w_N(\underline{\xi})$ simply by $\lambda_N(\xi), w_N(\xi)$ respectively, and likewise the respective uniform exponents by $\widehat{\lambda}_N(\xi), \widehat{w}_N(\xi)$. [1] Thereby, we see that any real ξ gives rise to four sequences of exponents

$$(\lambda_N(\xi))_{N\ge1}, \qquad (\widehat{\lambda}_N(\xi))_{N\ge1}, \qquad (w_N(\xi))_{N\ge1}, \qquad (\widehat{w}_N(\xi))_{N\ge1}. \tag{4}$$

Clearly the exponents $\lambda_N(\xi), \widehat{\lambda}_N(\xi)$ are non-increasing with N whereas the exponents $w_N(\xi), \widehat{w}_N(\xi)$ form non-decreasing sequences. Ordinary exponents may take the value $+\infty$, whereas uniform exponents turn out to be always less than twice the trivial lower bounds in (3), see Remarks 1, 4 below for refinements. Only for $N = 2$ numbers satisfying $\widehat{\lambda}_N(\xi) > 1/N$ or $\widehat{w}_N(\xi) > N$ have been found, see Roy [29], [30], Fischler [18], Bugeaud, Laurent [10] and Poels [28].

Investigation of these exponents with emphasis on Veronese curves is partly motivated by well-known connections to the problem of approximation to real numbers by algebraic numbers (integers) related to Wirsing's problem, see Wirsing [45], Davenport, Schmidt [16] and Badziahin, Schleischitz [3]. The main purpose of this paper is to establish new inequalities interconnecting these exponents, in various dimensions. Concretely, in Sections 2, 3 we establish several lower bounds for $\lambda_k(\xi)$ in terms of various exponents of index $n \le k$, and compare them. Thereby, we complement a recent paper by Badziahin and Bugeaud [2], as well as previous work of the author, especially [35], [41]. As a byproduct we further find new proofs of transference inequalities by Bugeaud, Laurent [12]. Section 4 treats analogous topics for general $\mathbb{Q}$-linearly independent vectors. Estimates are naturally weaker here and it is included rather for sake of completeness and to motivate a comprehensive conjecture. In Section 5

[1] We believe that this slight abuse of notation will improve readability of this paper, but want to remark that other notions for the exponents with respect to general $\underline{\xi}$ in $\mathbb{R}^N$, like $\omega_N(\underline{\xi}), \widehat{\omega}_N(\underline{\xi})$ and $\omega_N^*(\underline{\xi}), \widehat{\omega}_N^*(\underline{\xi})$, are more common. The exponents on the Veronese curve are denoted as in the standard literature.

we introduce parametric geometry of numbers, a key tool in the proofs of the main results carried out in Sections 6, 7. Finally in Section 8 we provide short proofs of the results from Section 4.

2. Relations between exponents of simultaneous approximation

2.1. Going-up Theorems for the sequence $(\lambda_N(\xi))_{N\geq 1}$. We want to understand relations between the exponents $\lambda_N(\xi)$ associated to real ξ in various dimensions N, thereby to draw information on the *joint spectrum* of the first sequence in (4), i.e. all possible sequences $(\lambda_1(\xi), \lambda_2(\xi), \ldots)$ induced by transcendental real ξ. Bugeaud [8] was the first to study this topic in detail. Among other results, he established the inequalities

$$\lambda_{nk}(\xi) \geq \frac{\lambda_k(\xi) - n + 1}{n}, \tag{5}$$

valid for positive integers k, n and any real number ξ. A generalization of (5) conjectured by the author in [35] was proved by Badziahin and Bugeaud [2].

Theorem 2.1 (Badziahin, Bugeaud)**.** *For any real number ξ and integers $k \geq n \geq 1$ we have the estimate*

$$\lambda_k(\xi) \geq \frac{n\lambda_n(\xi) + n - k}{k}. \tag{6}$$

In the special case $\lambda_n(\xi) > 1$ it had been known before, and if even $\lambda_k(\xi) > 1$ then there is in fact equality, see [35, Corollary 1.10]. In particular the estimate is sharp in certain cases. It is tempting to believe that it is best possible for all reasonable parameters (i.e. if the bound becomes at least $1/k$). If the bound in (6) is less than 1, this leaves some freedom for $\lambda_k(\xi)$. However, when considering all large k simultaneously, stringent restrictions on the joint spectrum were given in [41].

We refine Theorem 2.1 by means of introducing uniform exponents. We further include an alternative bound that is sometimes stronger.

Theorem 2.2. *Let $k \geq n \geq 1$ be integers. For any real ξ we have*

$$\lambda_k(\xi) \geq \frac{(n-1)\lambda_n(\xi) + (k-n)\widehat{\lambda}_n(\xi) + n - k}{(n-k)\widehat{\lambda}_n(\xi) + k - 1}. \tag{7}$$

Moreover, we have

$$\lambda_k(\xi) \geq \frac{(n-1)\lambda_n(\xi) + (k-1)\widehat{\lambda}_n(\xi) + n - k}{(n-1)\lambda_n(\xi) - (k-1)\widehat{\lambda}_n(\xi) + n + k - 2}. \tag{8}$$

Remark 1. *For every $n \geq 1$ and transcendental real ξ we have $\widehat{\lambda}_n(\xi) \leq \frac{2}{n+1}$, in fact*

$$\widehat{\lambda}_3(\xi) < 0.4246, \quad \widehat{\lambda}_n(\xi) \leq \frac{2}{n+1} \text{ if } n \text{ odd}, \quad \widehat{\lambda}_n(\xi) < \frac{2}{n+1} \text{ if } n \text{ even}, \tag{9}$$

follows from Roy [31]*, Laurent* [25] *and Schleischitz* [41, Section 4] *respectively. See also* [16]*,* [36]*. Badziahin recently announced further small improvements for even n, in a paper in preparation. If ξ satisfies $\lambda_n(\xi) > 1$, then $\widehat{\lambda}_n(\xi) = 1/n$, see* [35]*.*

Remark 2. *We could derive from* (8) *with $\widehat{\lambda}_n(\xi) \geq 1/n$ that $\lambda_k(\xi) \geq (n\lambda_n(\xi) + n - k + 1)/(n\lambda_n(\xi) + k + n - 1)$. However, this turns out not to be of interest as it never both exceeds the bound in Theorem* 2.1 *and $1/k$.*

For $\widehat{\lambda}_n(\xi) = 1/n$ the claim (7) becomes just Theorem 2.1, otherwise we get a stronger result. Thereby in particular we provide a new proof of Theorem 2.1 that is significantly different from the one given in [2], and from the proofs of (5), (6). Our proof is based on Minkowski's Theorem that we apply in the formalism of parametric geometry of numbers, and Mahler's Theorem on polar convex bodies. In particular, a novelty in our approach are its close ties to the dual linear form problem intorduced in (2).

We enclose a few more remarks on (7), (8). In view of the last claim in Remark 1, in (7) we need $\lambda_n(\xi) \leq 1$ for an improvement of Theorem 2.1. As indicated above, by (7) equality in (6) implies the identities

$$\widehat{\lambda}_i(\xi) = \frac{1}{i}, \qquad\qquad n \leq i \leq k,$$

a new result in this generality. If $\lambda_k(\xi) > 1$, it is already implied by [35, Theorem 1.12], in fact its proof shows the analogous claim up to $i = 2k - 1$. The special case $n = 2$ is of particular interest, as only then numbers ξ satisfying $\widehat{\lambda}_n(\xi) > 1/n$ have been found. Classical examples are extremal numbers as defined by Roy [29] and Sturmian continued fractions, see Bugeaud, Laurent [10], we omit definitions here. See also Poels [28]. For $n = 2, k = 3$ and ξ an extremal number, the identities from [29], [37]

$$\lambda_2(\xi) = 1,\ \widehat{\lambda}_2(\xi) = \frac{\sqrt{5}-1}{2} = 0.6180\ldots,\ \lambda_3(\xi) = \frac{1}{\sqrt{5}} = 0.4472\ldots, \tag{10}$$

induce equality in both (7) and (8). While some extremal numbers are Sturmian continued fractions, the identity does not extend to other Sturmian continued fractions ξ, nor does it to larger values of k. For $n = 2, k = 4$ and ξ an extremal number, (8) still provides a non-trivial bound that reads

$$\lambda_4(\xi) \geq \frac{6\sqrt{5}-5}{31} = 0.2715\ldots > \frac{1}{4}. \tag{11}$$

However, a stronger bound $\lambda_4(\xi) \geq (\sqrt{5}-1)/4 = 0.3090\ldots$ with conjectured identity was established in [37]. Moreover, we may alternatively derive (11) from (20) below.

Combining (8) with an inequality of Jarník [21] that reads

$$\lambda_2(\xi) \geq \frac{\widehat{\lambda}_2(\xi)^2}{1 - \widehat{\lambda}_2(\xi)} \tag{12}$$

we can formulate a bound for λ_k only in terms of $\widehat{\lambda}_2$ that is not trivial when $k \in \{3, 4\}$.

Corollary 2.3. *For any ξ we have*

$$\lambda_3(\xi) \geq \frac{-\widehat{\lambda}_2(\xi)^2 + 3\widehat{\lambda}_2(\xi) - 1}{3\widehat{\lambda}_2(\xi)^2 - 5\widehat{\lambda}_2(\xi) + 3} \tag{13}$$

and

$$\lambda_4(\xi) \geq \frac{-2\widehat{\lambda}_2(\xi)^2 + 5\widehat{\lambda}_2(\xi) - 2}{4\widehat{\lambda}_2(\xi)^2 - 7\widehat{\lambda}_2(\xi) + 4}. \tag{14}$$

If ξ is an extremal number, (13) becomes an identity again according to (10), and (14) becomes (11). The bound (13) exceeds $1/3$ if $\widehat{\lambda}_2(\xi) > (7 - \sqrt{13})/6 = 0.5657\ldots$, and (14) exceeds $1/4$ for $\widehat{\lambda}_2(\xi) > (9 - \sqrt{17})/8 = 0.6096\ldots$, just slightly below the maximum possible value in (10) obtained for extremal numbers [29]. Corollary 2.3 is stronger than what can be derived from combining (7) with (12). Similar bounds for $\lambda_k(\xi)$ in terms of $\widehat{\lambda}_n(\xi)$ can be obtained for $n > 2$ as well via the implicit estimates (45) below that originate in [27] and generalize (12), but their formulation becomes cumbersome.

2.2. Comparison (7) **vs** (8)**.** We discuss when (8) both improves on (7) and exceeds the trivial bound $1/k$. In view of Remark 1 and Remark 2, we may assume $\widehat{\lambda}_n(\xi) > 1/n$ and $\lambda_n(\xi) \leq 1$. We take into account the estimates (9) and distinguish 3 cases.

- Case 1: $k = 2n - 1$. If $\lambda_n(\xi) = 1$, the bounds in (8) and (7) coincide, regardless of the value of $\widehat{\lambda}_n(\xi)$. If $\lambda_n(\xi) < 1$, then the bound (8) is stronger for any $\widehat{\lambda}_n(\xi)$. Moreover, (8) is non-trivial if

$$\widehat{\lambda}_n(\xi) > \frac{n + 1 + (1 - n)\lambda_n(\xi)}{2n}. \tag{15}$$

  Thus for instance if $\lambda_n(\xi) = 1$ and $\widehat{\lambda}_n(\xi) > 1/n$, both are guaranteed. As we decrease $\lambda_n(\xi)$, condition (15) on $\widehat{\lambda}_n(\xi)$ becomes more stringent, if $\lambda_n(\xi) \leq 1 - \frac{2}{n-1}$ then $\widehat{\lambda}_n(\xi) \geq 2/n$ which contradicts (9). So for (8) to be interesting, we require

$$\lambda_n(\xi) \in (1 - \frac{2}{n - 1}, 1],$$

  in fact a slightly larger lower bound can be stated.
- Case 2: $k < 2n - 1$. The claim (8) turns out stronger than (7) as soon as

$$\widehat{\lambda}_n(\xi) > \frac{(1 - n)\lambda_n(\xi) + k - n}{k - 2n + 1}, \tag{16}$$

  (or $\lambda_n(\xi) < \frac{k-n}{n-1}$ but this is of no interest here) and non-trivial if

$$\widehat{\lambda}_n(\xi) > \frac{(1 - n)\lambda_n(\xi) + k - n + 2}{k + 1}. \tag{17}$$

The bound in (16) rises with respect to $\lambda_n(\xi)$ whereas (17) decays, and they coincide for $\lambda_n(\xi) = (k-n+1)/n < 1$ which yields $1/n$ in both expressions. Thus as soon as $\lambda_n(\xi) \in ((k-n+1)/n, 1]$ we only have to satisfy (16), i.e. $\widehat{\lambda}_n(\xi) > c > 1/n$ where c depends on $\lambda_n(\xi)$, and have both requirements met. It can be checked that here we do not get any restrictions from (9) under our assumptions above.

- Case 3: $k > 2n-1$. Then for improving (7), conversely to (16) we need

$$\widehat{\lambda}_n(\xi) < \frac{(1-n)\lambda_n(\xi) + k - n}{k-2n+1}, \tag{18}$$

whereas the condition (17) for non-triviality remains unchanged. A discussion of when (17), (18) can occur simultaneously, upon taking into account $\lambda_n(\xi) \leq 1$ and (9), after some calculation finally implies the necessary conditions

$$k = 2n, \qquad n \neq 3, \qquad 1 - \frac{n+2}{n^2-n} < \lambda_n(\xi) \leq 1. \tag{19}$$

We notice that for $n = 2$, $k = 4$ and extremal numbers ξ, condition (19) is satisfied and an improvement is indeed obtained, see (11).

3. Relations involving simultaneous and linear form exponents

3.1. Mixed properties. Now we want to find relations that also contain linear form exponents $w_N(\xi), \widehat{w}_N(\xi)$. The following relation was already implictly derived in [41] and obtained with a different proof and explicitly formulated by Badziahin, Bugeaud [2].

Theorem 3.1 (Badziahin, Bugeaud; Schleischitz)**.** *Let $k \geq n \geq 1$ be integers and ξ be a real number. We have*

$$\lambda_k(\xi) \geq \frac{w_n(\xi) - k + n}{(n-1)w_n(\xi) + k}. \tag{20}$$

The bound exceeds $1/k$ iff $w_n(\xi) > k$. For consequences of Theorem 3.1 regarding the Hausdorff dimensions of the level sets $\{\xi \in \mathbb{R} : \lambda_N(\xi) \geq \lambda\}$ and $\{\xi \in \mathbb{R} : \lambda_N(\xi) = \lambda\}$ for $\lambda \in [1/N, \infty]$, see [2]. In this note we put no emphasis on metrical aspects, see however Section 3.2. We remark that combining (20) for $n = 2$ and $k \in \{3, 4\}$ with Jarník's identity [23] and another estimate of Jarník [20] given as

$$w_2(\xi) \geq \widehat{w}_2(\xi)^2 - \widehat{w}_2(\xi), \qquad \widehat{w}_2(\xi) = \frac{1}{1-\widehat{\lambda}_2(\xi)} \tag{21}$$

yields another proof of Corollary 2.3. In particular, for $n = 2, k = 3$, Theorem 3.1 is again sharp when ξ is an extremal number, and also for any Sturmian continued fraction ξ as follows from [39]. We complement Theorem 3.1 with inequalities containing uniform exponents again. Our first estimate reads as follows.

Theorem 3.2. *Let $k \geq n \geq 2$ be integers and ξ be a real number. We have*

$$\lambda_k(\xi) \geq \frac{w_n(\xi)\widehat{w}_n(\xi) - w_n(\xi) + (n-k)\widehat{w}_n(\xi)}{(n-2)w_n(\xi)\widehat{w}_n(\xi) + w_n(\xi) + (k-1)\widehat{w}_n(\xi)}. \tag{22}$$

The special case $k = n$ simplifies to an estimate of Bugeaud and Laurent [12], i.e.

$$\lambda_k(\xi) \geq \frac{(\widehat{w}_k(\xi) - 1)w_k(\xi)}{((k-2)\widehat{w}_k(\xi) + 1)w_k + (k-1)\widehat{w}_k(\xi)}, \qquad k \geq 2. \tag{23}$$

See also [11], [26], and also Schmidt and Summerer [43] for another proof of (23). As in [12], [26], [43], then our proof applies to the more general setting of $\mathbb{Q}$-linearly independent $\{1, \xi_1, \dots, \xi_k\}$, so with respect to the exponents $\lambda_k(\underline{\xi}), w_k(\underline{\xi}), \widehat{w}_k(\underline{\xi})$ from Section 1.

We compare (22) with (20). In contrast to (7), here we only improve on Badziahin, Bugeaud in certain cases. As we explain below, it turns out this may happen in the cases

$$\underline{\text{Case 1}}: \; n \geq 3, \quad w_n(\xi) = w_{n-1}(\xi), \qquad \underline{\text{Case 2}}: \; n \geq 3, \quad n+1 \leq k \leq 2n-2. \tag{24}$$

A short calculation shows that our new result is stronger than Theorem 3.1 as soon as

$$\widehat{w}_n(\xi) > \frac{n w_n(\xi)}{w_n(\xi) - k + n}. \tag{25}$$

We elaborate on how restrictive this estimate is. First we notice that (25) enables the trivial condition $\widehat{w}_n(\xi) \leq w_n(\xi)$ as soon as $w_n(\xi) > k$ (a slightly more restrictive bound was obtained in [27]), which we impose anyway for a nontrivial estimate. Assume ξ satisfies

$$w_n(\xi) > w_{n-1}(\xi). \tag{26}$$

Then another restriction comes from the reverse estimate of the form

$$\widehat{w}_n(\xi) \leq \frac{n w_n(\xi)}{w_n(\xi) - n + 1} \tag{27}$$

of [13, Theorem 2.2]. Hence, according to (25) in this case Theorem 3.2 may improve on (20) only if $k < 2n - 1$, while it at best confirms the same bound if $k = 2n - 1$. It can be shown that for $n = 2$ and $\widehat{w}_2(\xi) > 2$ we have (26) automatically satisfied, see Proposition 5.2 below. Combination leaves the cases (24) open for potential improvement. Since the existence of ξ with $\widehat{w}_n(\xi) > n$ for any $n > 2$ is at present unproved, we cannot yet provide numbers for which Theorem 3.2 improves Theorem 3.1.

Let now $n = 2$ and ξ be a Sturmian continued fraction. This setup induces equality in (27) as can be seen from the main result of [10]. Then for $k = 3$ we once more obtain the correct value of $\lambda_3(\xi)$ from [39] as a lower bound, so Theorem 3.2 is sharp in certain cases as well. For $k > 3$ the bound (20) is stronger than (22).

In the case $w_n(\xi) = \infty$, Theorem 3.2 yields $\lambda_k(\xi) \geq (n-1)^{-1}$, confirming a partial claim of [41, Theorem 2.1] stating that if $w_{n-1}(\xi) < \infty$, i.e. ξ is a U_n-number in Mahler's classification, then $\lambda_k(\xi) = (n-1)^{-1}$ for large enough k. On the other hand, in case of $\widehat{\omega}_n(\xi) > n$ the bound (22) will exceed $1/(n-1)$ for every k. This leads to a new proof that U_n-numbers satisfy $\widehat{\omega}_n(\xi) = n$, already obtained in [13, Corollary 2.5]. Adamczewski, Bugeaud [1] showed the related claim that $\widehat{\omega}_n(\xi) > n$ for some $n \geq 1$ implies ξ is no U_k-number with $k > n$ (nor $k = n$ as pointed out above), see also Roy [33] when $n = 2$. We finally remark that $\widehat{\omega}_k(\xi) \leq k + n - 1$ holds for any U_n-number ξ and every $k \geq 1$ by [13, Corollary 2.5], for $n \leq k \leq 2n-2$ see also the bound from [40, Corollary 2.3].

In some cases, we can strengthen our estimates in Theorem 3.2 upon the additional assumption (26) on ξ.

Theorem 3.3. *Let k, n be integers with $2 \leq n \leq k \leq 2n-2$. Assume ξ is a real number and satisfies the inequality* (26)*. Then*

$$\lambda_k(\xi) \geq \frac{w_n(\xi)\widehat{\omega}_n(\xi) + (n-k-1)w_n(\xi) + (n-k)\widehat{\omega}_n(\xi)}{(2n-k-2)w_n(\xi)\widehat{\omega}_n(\xi) + (k-n+1)w_n(\xi) + (n-1)\widehat{\omega}_n(\xi)}. \tag{28}$$

Remark 3. *A variant for $k > 2n-2$ turns out weaker than Theorem* 3.1.

If $k = n$ we again obtain formula (23) as from Theorem 3.2, so then condition (26) is not required. Otherwise (28) is stronger than (22). Assumption (26) may not be required for the conclusion, however in our proof as in [13] it guarantees some nice properties of the integer polynomials realizing the exponent $w_n(\xi)$, see Proposition 5.2 below. Theorem 3.3 improves on Theorem 3.1 upon the same condition (25), thus we require Case 2 of (24) and still cannot settle existence of numbers ξ where this happens.

Our last claim provides a lower bound for $\lambda_k(\xi)$ in terms of $\widehat{\omega}_n(\xi)$ only, if $n \leq k \leq 2n-2$. We also include a bound that arises as a hybrid with Theorem 3.3.

Theorem 3.4. *Let $k \geq n \geq 2$ be integers and ξ be a transcendental real number. Then*

$$\lambda_k(\xi) \geq \frac{\widehat{\omega}_n(\xi) + 2n - 2k - 1}{(2n-k-2)\widehat{\omega}_n(\xi) + k}, \qquad \text{if } k \leq 2n-2. \tag{29}$$

In fact we have the stronger bound

$$\lambda_k(\xi) \geq \min\left\{\Theta\,,\ \frac{\widehat{\omega}_n(\xi) + 2n - 2k - 2}{(2n-k-3)\widehat{\omega}_n(\xi) + k}\right\}, \qquad \text{if } k \leq 2n-3, \tag{30}$$

where Θ denotes the bound in (28)*.*

Remark 4. *The proof method of Theorem* 3.4 *also provides a new proof of*

$$\widehat{\omega}_n(\xi) \leq 2n-1, \qquad n \geq 1, \tag{31}$$

which is due to Davenport and Schmidt [16]*, corresponding to the case $k = 2n-1$. See* [13], [38] *for slightly stronger bounds, and* [16], [29] *for $n = 2$. Unfortunately, combining* (29) *with German's estimates* (88) *below turns out not to give an interesting relation between λ_k and $\widehat{\lambda}_n$ in view of* (9)*.*

The bound (29) is non-trivial, i.e. gives $\lambda_k(\xi) > 1/k$, as soon as $\widehat{w}_n(\xi) > k$. So we restrict to this case in the sequel (which requires $n \geq 3$ if $k > n$). The bound (29) is smaller than both expressions in (30) if $w_n(\xi) > \widehat{w}_n(\xi)$, in particular weaker than the conditional bound (28). We compare (29) with the unconditional Theorem 3.1 and Theorem 3.2. A short computation shows that it improves Theorem 3.1 if

$$w_n(\xi) < \frac{(2n-k)\widehat{w}_n(\xi) - k}{2n-1-\widehat{w}_n(\xi)}. \tag{32}$$

For $\widehat{w}_n(\xi) = k$ the right hand side gives k as well, and it exceeds $\widehat{w}_n(\xi)$ if $\widehat{w}_n(\xi) > k$. Marnat, Moshchevitin [27] generalized (21) by improving the trivial estimate $w_n(\xi) \geq \widehat{w}_n(\xi)$ for $n \geq 2$, which plays against (32). Nevertheless, (29) is potentially of interest in many cases. Assume $1 < \alpha < \beta < 2$ are fixed, and for large n choose $k = \alpha n + o(n)$ and $\widehat{w}_n(\xi) = \beta n + o(n)$. It can be checked that then [27] only gives a lower bound $w_n(\xi)/\widehat{w}_n(\xi) \geq 1 + o(1)$ as $n \to \infty$. When neglecting lower order terms, we see that for $w_n(\xi)/\widehat{w}_n(\xi) < (2-\alpha)/(2-\beta) + o(1)$ as $n \to \infty$, property (32) will be satisfied. This leaves a non-empty interval $(1 + o(1), (2-\alpha)/(2-\beta) + o(1))$ for the ratio, for large n. See also Example 3.5 below.

Inequality (29) improves Theorem 3.2 as soon as

$$\frac{w_n(\xi)}{\widehat{w}_n(\xi)} < \Psi(\widehat{w}_n(\xi)), \tag{33}$$

where

$$\Psi(\widehat{w}_n(\xi)) = \frac{(3kn - k^2 - k - 2n^2 + 2n - 1)\widehat{w}_n(\xi) + kn - k^2 + k - 2n + 1}{(n-k)\widehat{w}_n(\xi)^2 + (3n - 2k - 1 - 2n^2 + 2kn)\widehat{w}_n(\xi) + k + 1 - 2n}.$$

Despite [27] recalled above, the scenario that (33) holds for many ξ and k, n is likely. With α, β as above, for $w_n(\xi)/\widehat{w}_n(\xi) < (3\alpha - \alpha^2 - 2)/(\beta - \alpha\beta + 2\alpha - 2) + o(1)$ as $n \to \infty$, we satisfy (33). We want to mention that (33) requires $k > n$, indeed for $k = n$ the bound in (29) becomes (23) if $w_n(\xi) = \widehat{w}_n(\xi)$ and is weaker otherwise. The next example illustrates a hypothetical scenario where Theorem 3.4 is reasonably strong.

Example 3.5. *Let $n = 10, k = 13$ and assume ξ is a real number satisfying $\widehat{w}_{10}(\xi) = 14$. Then [27] gives $w_{10}(\xi) \geq 15.0190\ldots$. The right hand sides of (32) and (33) become 17 and 16.1875 …, respectively. Hence $\lambda_{13}(\xi) \geq 7/83 = 0.0843\ldots$ from Theorem 3.4 improves on both Theorem 3.1 and Theorem 3.2 if $w_{10}(\xi) \in (15.0190, 16.1875)$.*

For $k = 2n - 2$, the bound (29) becomes an easy affine function

$$\lambda_{2n-2}(\xi) \geq \frac{\widehat{w}_n(\xi) - 2n + 3}{2n - 2}.$$

This may be of interest for $3 \leq n \leq 9$. If $n \geq 10$ then $\widehat{w}_n(\xi) \leq 2n - 2$ for any ξ was established in [38] (see also [13]), and the bound becomes trivial. For $n = 2$ the implied bound $\lambda_2(\xi) \geq (\widehat{w}_2(\xi) - 1)/2$ is weaker than $\lambda_2(\xi) \geq$

$\widehat{w}_2(\xi) - 2 + \widehat{w}_2(\xi)^{-1}$ derived from Jarník's identity (21) and (12). We close this section with an asymptotic result.

Corollary 3.6. *Let ξ be a real transcendental number and write*

$$\overline{\widehat{w}}(\xi) = \limsup_{n\to\infty} \frac{\widehat{w}_n(\xi)}{n}, \qquad \overline{\lambda}(\xi) = \limsup_{n\to\infty} n\lambda_n(\xi).$$

Then

$$\overline{\lambda}(\xi) \geq \frac{\left(2-\sqrt{2-\overline{\widehat{w}}(\xi)}\right)\cdot\left(\overline{\widehat{w}}(\xi)+2\sqrt{2-\overline{\widehat{w}}(\xi)}-2\right)}{\overline{\widehat{w}}(\xi)\sqrt{2-\overline{\widehat{w}}(\xi)}}.$$

The according estimate with respect to the lower limits holds as well.

The bound as a function of $\overline{\widehat{w}}(\xi)$ induces an increasing bijection of the interval $[1,2]$ onto itself, upon taking the left-sided limit if $\overline{\widehat{w}}(\xi) = 2$. It can be seen complementary to $\overline{\lambda}(\xi) \geq (\overline{w}(\xi)+1)^2/(4\overline{w}(\xi))$ from [41, Theorem 2.1], for $\overline{w}(\xi)$ defined analogously with respect to ordinary exponents. The latter estimate can be derived from Theorem 3.1.

Proof. Choose $k = \lfloor (2-\sqrt{2-\widehat{w}_n(\xi)/n})\cdot n \rfloor$ in (29) and look at the dominant terms as $n\to\infty$, we skip details. □

3.2. Metrical considerations. As a small metrical application of our results, we discuss the problem of estimating the Hausdorff dimensions of

$$\{\xi : \widehat{w}_n(\xi) \geq \widehat{w}\}, \qquad \widehat{w} \in [n, 2n-1).$$

For simplicity we deal with a normalized problem and consider $n\to\infty$. A well-known metric result of Bernik [6] immediately yields the trivial bound

$$\dim\left\{\xi : \frac{\widehat{w}_n(\xi)}{n} \geq \beta\right\} \leq \dim\left\{\xi : \frac{w_n(\xi)}{n} \geq \beta\right\} \leq \frac{1}{\beta} + o(1), \;\; \beta \in [1,2], \tag{34}$$

as $n\to\infty$. The estimates from [27] also do not improve this asymptotic relation. While (34) seems a very crude estimate, nothing better seems currently available.

Upon suitable choice of k, the inclusion

$$\{\xi : \widehat{w}_n(\xi) \geq \widehat{w}\} \subseteq \left\{\xi : \lambda_k(\xi) \geq \frac{\widehat{w}+2n-2k-1}{(2n-k-2)\widehat{w}+k}\right\}, \quad \widehat{w} > n, \tag{35}$$

induced by (29) may have potential to improve (34), at least in certain parameter ranges for β. Unfortunately, no reasonable upper bounds for the dimensions of level sets $\{\xi : \lambda_k(\xi) \geq \lambda\}$ for $\lambda \in [1/k, 2/k]$ are yet available that we would require for this avenue. However, we want to give in to some speculation. Assume Beresnevich's [4] lower bound

$$\dim\{\xi \in \mathbb{R} : \lambda_k(\xi) \geq \lambda\} \geq \frac{k+1}{\lambda+1} - (k-1), \qquad \lambda \in \left[\frac{1}{k}, \frac{3}{2k-1}\right], \tag{36}$$

is an identity (as conjectured by him and proved for $k = 2$ and very recently in some smaller interval for arbitrary k by Beresnevich and Yang in [5]) and the reverse estimate extends to $\lambda \in [1/k, c/k]$ for some c close to 2. Then choosing $k = \lfloor (2-\sqrt{2-\beta})n \rfloor$ in order to maximize the expression $k\lambda_k(\xi)$, indeed it turns out via (35) we improve (34) for $\beta \in (\frac{17}{9}, 2-\epsilon)$ with small ϵ, for n large enough. We believe our assumption is reasonable, in particular it agrees with the lower bound

$$\dim\{\xi \in \mathbb{R} : \lambda_k(\xi)\} \geq \max_{1\leq N\leq k} \left\{ \frac{(N+1)(1-(N-1)\lambda)}{(k-N+1)(1+\lambda)} \right\}, \quad \lambda \geq \frac{1}{n}, \tag{37}$$

from [2, Theorem 2.3] (also obtained in [41]) is ϵ is small enough. If (37) is a good approximation to the true value, we can even extend the above interval for β, in case of a hypothetical equality in (37) (that however contradicts (36) for $\lambda \leq 3/(2k-1) \approx (3/2)k^{-1}$) a calculation verifies we get a stronger bound for every $\beta \in [1, 2]$. Roughly speaking, Theorem 3.4 shows that not both (34) and (37) can be sharp.

4. The $\mathbb{Q}$-linearly independent case

For sake of completeness, we want to formulate similar going-up principles for the case of $\mathbb{Q}$-linearly independent real vectors. In this situation we consider extensions of a given real vector, or equivalently projections of infinite vectors $\underline{\xi} \in \mathbb{R}^{\mathbb{N}}$ to its first N coordinates, and compare the exponents of approximation as N increases (note that this is a very different setup than the going-up principles for fixed N that relate the so-called intermediate exponents, as for instance in [11]). If $\underline{\xi} = (\xi, \xi^2, \xi^3, \ldots)$ we are in the situation of Sections 2 and 3. In the general setting, all results will be considerably weaker, as may be expected, and the proofs are considerably shorter and easier when directly applying well-known transference inequalities. The hidden work in proving these preliminaries appears to some extent in our proofs for results of Sections 2, 3, we elaborate a little more on this issue in Section 8. Our first result resembles (7).

Theorem 4.1. *Let $k \geq n \geq 1$ be integers and $\underline{\xi} = (\xi_1, \xi_2, \ldots)$ be an infinite vector of real numbers. For $N \geq 1$, denote by $\underline{\xi}_N = (\xi_1, \ldots, \xi_N)$ the projection of $\underline{\xi}$ to the first N entries. Assume that $\{1, \xi_1, \ldots, \xi_k\}$ is $\mathbb{Q}$-linearly independent. Then*

$$\lambda_k(\underline{\xi}_k) \geq \frac{(n-1)\lambda_n(\underline{\xi}_n) + \widehat{\lambda}_n(\underline{\xi}_n) + n - 2}{(k-1)(n-1)\lambda_n(\underline{\xi}_n) - \widehat{\lambda}_n(\underline{\xi}_n) + kn - n - k + 2}. \tag{38}$$

Moreover,

$$\lambda_k(\underline{\xi}_k) \geq \frac{(A-1)B}{((N-2)A+1)B + (N-1)A} \tag{39}$$

with

$$A = \frac{(n-1)(k-1)^2}{nk - k - 2n + 3 - \widehat{\lambda}_n(\underline{\xi}_n)}, \qquad B = \frac{(n-1)\lambda_n(\underline{\xi}_n) + \widehat{\lambda}_n(\underline{\xi}_n) + n - 2}{1 - \widehat{\lambda}_n(\underline{\xi}_n)}.$$

When expanded by inserting for A, B, the bound (39) becomes a lengthy expression that we omit to state explicitly. It exceeds (38) as soon as $\widehat{\lambda}_n(\underline{\xi}_n) > (k-n+1)/n$, which relies on the fact that we use Theorem 4.3 below in the proof. Since $\widehat{\lambda}_n(\underline{\xi}_n) \geq 1/n$, as a corollary of (38) we obtain a variant that resembles Theorem 2.1.

Theorem 4.2. *Upon the assumptions of Theorem 4.1, assume*

$$\lambda_n(\underline{\xi}_n) > \frac{k-n+1}{n}.$$

Then

$$\lambda_k(\underline{\xi}_k) \geq \frac{n\lambda_n(\underline{\xi}_n) + n - 1}{n(k-1)(\lambda_n(\underline{\xi}_n) + 1) + 1} > \frac{1}{k}. \tag{40}$$

For $\lambda_n(\underline{\xi}_n) = (k-n+1)/n$ the right inequality of (40) would become an identity. We believe Theorem 4.2 is optimal in the general setting. We briefly talk about metric consequences, even though the metric theory with respect to the entire space is complete. It is known thanks to Jarník [22] (see also Dodson [17]) that for $\lambda \in [1/N, \infty]$ we have

$$\mathscr{D}_N(\lambda) := \dim\{\underline{\xi} \in \mathbb{R}^N : \lambda_N(\underline{\xi}) \geq \lambda\} = \frac{N+1}{\lambda+1}, \qquad N \geq 1. \tag{41}$$

Theorem 4.2 and the property $\dim(A \times B) \geq \dim(A) + \dim(B)$ of the Hausdorff dimension for A the set in (41) with $N = n$ and $B = \mathbb{R}^{k-n}$ implies

$$\mathscr{D}_k\Big(\frac{n\lambda + n - 1}{n(k-1)(\lambda+1)+1}\Big) \geq \mathscr{D}_n(\lambda) + k - n, \qquad \lambda \geq \frac{k - n(k-n) + 1}{n}. \tag{42}$$

Clearly, the estimate (42) can alternatively derived from (41). We calculate

$$\mathscr{D}_k\Big(\frac{n\lambda + n - 1}{n(k-1)(\lambda+1)+1}\Big) - (\mathscr{D}_n(\lambda) + k - n) = \frac{(\lambda n - k + n - 1)(kn-1)}{(1+\lambda)nk},$$

the right hand side is non-negative as soon as $\lambda \geq (k-n+1)/n$. We derive that there is equality in (42) precisely for $\lambda = (k-n+1)/n$ to obtain $\mathscr{D}_k(\frac{1}{k}) = k$. Hence, for larger λ, from a metrical point of view, the majority of vectors contributing to the left set of (42) is not coming from λ-approximable points in a projection to n coordinates. We next establish corresponding going-up results concerning the uniform exponents.

Theorem 4.3. *Keep the definitions and assumptions of Theorem 4.1. If we assume that*

$$\widehat{\lambda}_n(\underline{\xi}_n) > \frac{k-n+1}{n}, \tag{43}$$

then

$$\widehat{\lambda}_k(\underline{\xi}_k) \geq \frac{\widehat{\lambda}_n(\underline{\xi}_n) + n - 2}{(n-1)(k-1)} > \frac{1}{k}. \tag{44}$$

Theorem 4.3 is of no interest for Veronese curves as condition (43) contradicts (9) as soon as $k > n$. The spectrum of $\widehat{\lambda}_N$ among $\underline{\xi} \in \mathbb{R}^N$ that are $\mathbb{Q}$-linearly independent with $\{1\}$ equals $[1/N, 1]$, as follows for example from the constructions in [34, Theorem 2.5], or alternatively Roy's deep existence result [32]. Consequently the condition (43) can be satisfied for $2 \leq n \leq k \leq 2n-2$. Metrical implications in the spirit of (42) between sets

$$\widehat{\mathscr{D}}_N(\lambda) := \dim\{\underline{\xi} \in \mathbb{R}^N : \widehat{\lambda}_N(\underline{\xi}) \geq \lambda\}, \qquad \lambda \in [1/N, 1],$$

in various dimensions N follow, we omit explicitly stating them. If $N = 1$, then $\widehat{\lambda}_1(\xi) = 1$ for any irrational ξ, see [24]. For larger N, the problem of determining $\widehat{\mathscr{D}}_N(\lambda)$ is only solved in a paper in preparation for $N = 2$ by Das, Fishman, Simmons, Urbański [14], [15] and independently by Bugeaud, Cheung, Chevallier [9]. However, when taking $n = 2$ Theorem 4.3 does not provide new information on any value $\widehat{\mathscr{D}}_N(\lambda)$.

We believe that apart from obvious obstructions, the restrictions (40), (44) on sequences are sufficient for the projections of suitable $\underline{\xi} \in \mathbb{R}^{\mathbb{N}}$ to attain all values simultaneously.

Conjecture 4.4. *Let $(\lambda_N)_{N\geq 1}$ and $(\widehat{\lambda}_N)_{N\geq 1}$ be non-increasing sequences of reals satisfying $\widehat{\lambda}_N \geq 1/N$ for $N \geq 1$, the estimates*

$$\widehat{\lambda}_N + \frac{\widehat{\lambda}_N^2}{\lambda_N} + \cdots + \frac{\widehat{\lambda}_N^N}{\lambda_N^{N-1}} \leq 1, \qquad N \geq 1, \tag{45}$$

originating in [27] and for all $k \geq n \geq 1$ the relations

$$\lambda_k \geq \frac{n\lambda_n + n - 1}{n(k-1)(\lambda_n + 1) + 1}, \qquad \widehat{\lambda}_k \geq \frac{n + \widehat{\lambda}_n - 2}{(n-1)(k-1)}.$$

Then there is $\underline{\xi} \in \mathbb{R}^{\mathbb{N}}$ such that $\lambda_N(\underline{\xi}_N) = \lambda_N$ and $\widehat{\lambda}_N(\underline{\xi}_N) = \widehat{\lambda}_N$ for all $N \geq 1$.

This resembles the "main problem" formulated in [7, Section 3.4] regarding approximation to the Veronese curve, which however involves different types of exponents. Less audacious conjectures can be readily stated by considering only one type of exponents, i.e. either ordinary or uniform. We omit the formulation.

We close with a version of Theorem 3.2 for the $\mathbb{Q}$-linearly independent case, that is again considerably weaker but admits an easy deduction from classical transference principles.

Theorem 4.5. *Upon the assumptions of Theorem 4.1, we have*

$$\lambda_k(\underline{\xi}_k) \geq \frac{(\widehat{w}_n(\underline{\xi}_n) - 1)w_n(\underline{\xi}_n)}{((k-2)\widehat{w}_n(\underline{\xi}_n) + 1)w_n + (k-1)\widehat{w}_n(\underline{\xi}_n)}.$$

5. Parametric geometry of numbers and preliminary results

Our proofs are based on classical tools from geometry of numbers, in particular Minkowski's Convex Body Theorems. To simplify to some extent the slightly cumbersome calculations that appear, we work within the framework of parametric geometry of numbers introduced by Schmidt and Summerer in [42]. We slightly deviate from its original notation and put emphasis on the concrete estimates we require. We refer to [42, 43] for a more comprehensive introduction, see also Roy [32] for a different setup. Recall that the j-th successive minimum of a convex body K with respect to a lattice Λ is the minimum $\lambda > 0$ so that λK contains j linearly independent points of Λ.

5.1. Parametric functions. Let $N \geq 1$ an integer and $\underline{\xi} \in \mathbb{R}^N$ be given. Let $q > 0$ be a parameter and let $Q = e^q$. Define convex bodies

$$K(Q) = \{(z_0, \dots, z_N) : |z_0| \leq Q, \quad |z_1| \leq Q^{-1/N}, \dots, \quad |z_N| \leq Q^{-1/N}\},$$

and a lattice by

$$\Lambda_{\underline{\xi}} = \{(x, \xi_1 x - y_1, \dots, \xi_N x - y_N) : \ x, y_j \in \mathbb{Z}\}.$$

The successive minima of $K(Q)$ with respect to $\Lambda_{\underline{\xi}}$ contain important information on simultaneous rational approximation to $(\xi_1, \dots, \xi_N)$. For $1 \leq j \leq N+1$, denote by $\tau_{N,j}(Q)$ the j-th successive minimum and derive $\psi_{N,j}(Q)$ and $L_{N,j}(q)$ as in [42] via

$$\psi_{N,j}(Q) = \frac{\log \tau_{N,j}(Q)}{q}, \qquad L_{N,j}(q) = \log \tau_{N,j}(Q) = q\psi_{N,j}(Q).$$

The functions $L_{N,j}$ are piecewise linear with slopes among $\{-1, 1/N\}$, see [42].

The linear form problem corresponds to dual approximation problem, i.e. the successive minima problem with respect to the dual parametric convex bodies

$$K^*(Q) = \{\underline{y} \in \mathbb{R}^{N+1} : |\underline{y} \cdot \underline{z}| \leq 1, \underline{z} \in K(Q)\}$$

given in coordinates by

$$K^*(Q) = \{(y_0, \dots, y_N) \in \mathbb{R}^{N+1} : Q|y_0| + Q^{-N}|y_1| + \dots + Q^{-N}|y_N| \leq 1\},$$

and the dual lattice $\Lambda^*_{\underline{\xi}} = \{\underline{y} \in \mathbb{R}^{N+1} : \underline{y} \cdot \underline{z} \in \mathbb{Z}, \underline{z} \in \Lambda_{\xi}\}$, given as

$$\Lambda^*_{\underline{\xi}} = \{(x_0 + \xi_1 x_1 + \dots + \xi_N x_N, x_1, \dots, x_N) \in \mathbb{R}^{N+1} : x_j \in \mathbb{Z}\}.$$

Again, for $1 \leq j \leq N+1$, from successive minima with respect to $K^*(Q)$ and $\Lambda^*_{\underline{\xi}}$ we derive functions $\psi^*_{N,j}(Q)$ and $L^*_{N,j}(q)$ accordingly. Any $L^*_{N,j}(q)$ is locally induced by the function $L^{*}_{N,\underline{x}}(q)$ for some $\underline{x}=(x_0,x_1,\dots,x_N)\in\mathbb{Z}^{N+1}$ defined as

$$L^{*}_{N,\underline{x}}(q)=\max\left\{\log\|\underline{x}\|_{\infty}-\frac{q}{N},\ \log\langle\underline{x},\underline{\xi}\rangle_N+q\right\}, \tag{46}$$

where

$$\|\underline{x}\|_{\infty}=\max_{0\le i\le N}|x_i|, \qquad \langle\underline{x},\underline{\xi}\rangle_N=|x_0+\xi_1x_1+\cdots+\xi_Nx_N|.$$

The functions $L^{*}_{N,j}(q)$ therefore have slope among $\{1,-1/N\}$. For $j=1$, the value $L^{*}_{N,1}(q)$ just equals the minimum of $L^{*}_{N,\underline{x}}(q)$ over $\underline{x}\in\mathbb{Z}^{N+1}\setminus\{\underline{0}\}$. Also notice that for successive powers $\underline{\xi}=(\xi,\xi^2,\dots,\xi^N)$ the scalar product $\langle\underline{x},\underline{\xi}\rangle_N$ may be written $|P(\xi)|$ with $P\in\mathbb{Z}[T]$ of degree at most N. We close this section by defining the upper and lower limits

$$\underline{\psi}_{N,j}=\liminf_{Q\to\infty}\psi_{N,j}(Q), \qquad \overline{\psi}_{N,j}=\limsup_{Q\to\infty}\psi_{N,j}(Q),$$

and $\underline{\psi}^{*}_{N,j},\overline{\psi}^{*}_{N,j}$ accordingly that are linked to classical exponents, see next section.

5.2. Minkowski's Theorems, Mahler's duality, relation to classical exponents. Variants of Dirichlet's Theorem, or Minkowski's First Convex Body Theorem, imply $\psi_{N,1}(Q)<0$ and $L_{N,1}(q)<0$, as well as $\psi^{*}_{N,1}(Q)<0$ and $L^{*}_{N,1}(q)<0$, for all $q>0$. Minkowski's Second Convex Body Theorem yields

$$\left|\sum_{j=1}^{N+1}\psi_{N,j}(Q)\right|\le\frac{C_N}{q}, \qquad \left|\sum_{j=1}^{N+1}L_{N,j}(q)\right|\le C_N, \qquad q>0, \tag{47}$$

and similarly

$$\left|\sum_{j=1}^{N+1}\psi^{*}_{N,j}(Q)\right|\le\frac{C^{*}_N}{q}, \qquad \left|\sum_{j=1}^{N+1}L^{*}_{N,j}(q)\right|\le C^{*}_N, \qquad q>0, \tag{48}$$

for constants $C_N>0$ and $C^{*}_N>0$.

Our two approximation problems, simultaneous approximation and linear forms, are connected by Mahler's theorem on dual convex bodies. It implies

$$|\psi_{N,1}(Q)+\psi^{*}_{N,N+1}(Q)|\le\frac{c_N}{q}, \qquad |\psi^{*}_{N,1}(Q)+\psi_{N,N+1}(Q)|\le\frac{c_N}{q}, \tag{49}$$

for some constant $c_N>0$ independent from Q. In particular

$$\underline{\psi}_{N,1}=-\overline{\psi}^{*}_{N,N+1}, \qquad \overline{\psi}_{N,1}=-\underline{\psi}^{*}_{N,N+1}. \tag{50}$$

From (48) and (49) we obtain the two asymptotic identities

$$\sum_{j=1}^{N}\psi^{*}_{N,j}(Q)=\psi_{N,1}(Q)+O(q^{-1}),\ \sum_{j=1}^{N}\psi_{N,j}(Q)=\psi^{*}_{N,1}(Q)+O(q^{-1}). \tag{51}$$

From (48) one may readily derive [43, (1.11)], which reads in our notation

$$j\underline{\psi}_{N,j} + (N+1-j)\overline{\psi}_{N,N+1} \geq 0, \quad j\overline{\psi}_{N,j} + (N+1-j)\underline{\psi}_{N,N+1} \geq 0 \tag{52}$$

and similarly for $\psi_{N,j}^{*}$. For $j = 1$ we immediately deduce [43, (1.11)] that may be written

$$-\overline{\psi}_{N,N+1}^{*}(Q) \leq \frac{1}{N} \cdot \underline{\psi}_{N,1}^{*}(Q), \qquad -\overline{\psi}_{N,N+1}(Q) \leq \frac{1}{N} \cdot \underline{\psi}_{N,1}(Q). \tag{53}$$

In fact only the right estimates occur in [43], but the dual left inequalities admit an analogous proof.

In [42, Theorem 1.4], a fundamental link between the upper and lower limits on one side and the exponents from Section 1 on the other side is given via the identities

$$(1+\lambda_N(\underline{\xi}))(1+\underline{\psi}_{N,1}) = (1+\widehat{\lambda}_k(\underline{\xi}))(1+\overline{\psi}_{N,1}) = \frac{N+1}{N}, \tag{54}$$

and

$$(1+w_N(\underline{\xi}))\Big(\frac{1}{N}+\underline{\psi}_{N,1}^{*}\Big) = (1+\widehat{w}_N(\underline{\xi}))\Big(\frac{1}{N}+\overline{\psi}_{N,1}^{*}\Big) = \frac{N+1}{N}. \tag{55}$$

In fact we will often implicitly use parametric versions of (54), (55), stating that for any $1 \leq j \leq N+1$, a set of j linearly independent integer vectors inducing an exponent λ resp. w in (1) resp. (2) gives rise to q with the according identity linking λ with $\psi_{N,j}(q)$ resp. w with $\psi_{N,j}^{*}(q)$.

5.3. A transference lemma and an observation on minimal polynomials.

The following lemma stems from a simple calculation and will be frequently applied throughout our proofs. It describes the transformation of the functions $L_{n,\underline{x}}^{*}$ above induced by some $\underline{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$, into $L_{k,\underline{x}'}^{*}$ in some larger dimension $k > n$ upon setting $\underline{x}' = (x_0, \dots, x_n, 0, \dots, 0) \in \mathbb{Z}^{k+1}$. In the case of successive powers we easily gain some improvement by varying $\underline{x}'$ that turns out crucial.

Lemma 5.1. *Let $k \geq n \geq 1$ be integers. Further let $\underline{\xi} = (\xi_1, \dots, \xi_k)$ be a real vector and $\underline{\tilde{\xi}} = (\xi_1, \dots, \xi_n)$ the restriction of $\underline{\xi}$ to the first n components. Assume $\underline{x} = (x_0, x_1, \dots, x_n) \in \mathbb{Z}^{n+1}$ and $q > 0$ and ψ are parameters so that the function $L_{n,\underline{x}}^{*}$ associated to $\underline{\tilde{\xi}}$ and $\underline{x}$ satisfies*

$$L_{n,\underline{x}}^{*}(q) \leq \psi q.$$

Let

$$q' = q\frac{(n+1)k}{n(k+1)}, \qquad \psi' = \Phi_{k,n}(\psi) \tag{56}$$

where $\Phi_{k,n}$ is the affine function given as

$$\Phi_{k,n}(t) := (t-1)\frac{(k+1)n}{k(n+1)} + 1 = \frac{n(k+1)}{(n+1)k}t + \frac{k-n}{k(n+1)}. \tag{57}$$

Then for

$$\underline{x}' = (x_0, x_1, \dots, x_n, 0, 0, \dots, 0) \in \mathbb{Z}^{k+1}, \tag{58}$$

we have

$$L^*_{k,\underline{x}'}(q') \leq \psi' q'.$$

Moreover, if $\xi_j = \xi^j$ for $1 \leq j \leq n$ and some $\xi \in (0,1)$, then the same claim holds for any vector $\underline{x}' = \underline{x}'_i$ of the form

$$\underline{x}'_i = (0, \dots, 0, x_0, x_1, \dots, x_n, 0, 0, \dots, 0) \in \mathbb{Z}^{k+1}, \qquad 1 \leq i \leq k-n+1, \tag{59}$$

where in $\underline{x}'_i$ the coordinate x_0 is in position i.

Proof. First we treat the case of general vectors $\underline{\xi}$. Observe that obviously for $\underline{x}'$ as in (58) we have

$$\|\underline{x}\|_\infty = \|\underline{x}'\|_\infty, \qquad \langle \underline{x}', \underline{\xi} \rangle_k = \langle \underline{x}, \underline{\tilde{\xi}} \rangle_n.$$

Hence, according to (46) we have

$$L^*_{n,\underline{x}}(q) = \max\left\{\log \|\underline{x}\|_\infty - \frac{q}{n},\ \log \langle \underline{x}, \underline{\tilde{\xi}} \rangle_n + q\right\}$$

and

$$L^*_{k,\underline{x}'}(q') = \max\left\{\log \|\underline{x}\|_\infty - \frac{q'}{k},\ \log \langle \underline{x}, \underline{\tilde{\xi}} \rangle_n + q'\right\}.$$

Thus it suffices to check that for q', ψ' as given in (56), the inequalities

$$\log \|\underline{x}\|_\infty - \frac{q}{n} \leq q\psi, \qquad \log \langle \underline{x}, \underline{\tilde{\xi}} \rangle_n + q \leq q\psi$$

imply

$$\log \|\underline{x}\|_\infty - \frac{q'}{k} \leq q'\psi', \qquad \log \langle \underline{x}, \underline{\tilde{\xi}} \rangle_n + q' \leq q'\psi'.$$

We leave these elementary calculations to the reader.

Now take the special case $\xi_j = \xi^j$ for $1 \leq j \leq n$ and some $\xi \in (0,1)$. Then if we identify $\underline{x}$ with the polynomial $P(T) = x_0 + x_1 T + \cdots + x_n T^n$, we readily check that a right shift of $\underline{x}$ within $\underline{x}'$ corresponds to a multiplictation by T, so that $\underline{x}'_i$ corresponds to $T^{i-1}P(T)$ for $1 \leq i \leq k-n+1$. Since $\xi \in (0,1)$ we have $|\xi^j P(\xi)| \leq |P(\xi)|$ for $j \geq 0$, and thus again

$$\|\underline{x}\|_\infty = \|\underline{x}'_i\|_\infty, \qquad \langle \underline{x}'_i, \underline{\xi} \rangle_k \leq \langle \underline{x}, \underline{\tilde{\xi}} \rangle_n,$$

for any $1 \leq i \leq k-n+1$. The claim follows as above. □

We finish this section with a proposition that extends an observation of Wirsing [45, Hilfssatz 4]. It concerns the degrees of well approximating polynomials that play a role in the proofs below. It is unrelated to parametric geometry of numbers.

Proposition 5.2. *Let ξ be a transcendental real number, $n \geq 2$ an integer and $\epsilon > 0$. Then*

$$|P(\xi)| < H(P)^{-w_n(\xi)+\epsilon} \tag{60}$$

has infinitely many solutions in irreducible integer polynomials P of arbitrarily large height and degree at least $\lceil \widehat{w}_n(\xi)\rceil - n + 1$ and at most n. On the other hand, the inequality

$$|P(\xi)| < H(P)^{-\widehat{w}_n(\xi)+\epsilon} \tag{61}$$

only finitely many solutions in integer polynomials of degree at most $\lceil \widehat{w}_n(\xi)\rceil - n$ if ϵ is small enough. Moreover, if $w_n(\xi) > w_{n-1}(\xi)$ and ϵ is small enough, then there exist P irreducible of degree n satisfying (60) of arbitrarily large height.

Proof. By [45, Hilfssatz 4], we may choose irreducible integer polynomials P of degree at most n with property (60) of arbitrarily large height. The last, conditional claim follows immediately when taking $\epsilon = (w_n(\xi) - w_{n-1}(\xi))/2$ as then these polynomials cannot have degree smaller than n. For the other claims, we conclude by showing that the degree of polynomials P satisfying the weaker property (61) can be $\lceil \widehat{w}_n(\xi)\rceil - n$ or less only for finitely many P.

So let $m \in \{1, 2, \dots, n\}$ be the minimum integer so that (61) has infinitely many solutions in integer polynomials P of degree m or less. It was shown in [13, Theorem 2.3] that for any transcendental real ξ and any integers $m, n \geq 1$ we have

$$\min\{w_m(\xi), \widehat{w}_n(\xi)\} \leq m + n - 1. \tag{62}$$

Assume contrary to our claim that $m \leq \lceil \widehat{w}_n(\xi)\rceil - n$. Then $\widehat{w}_n(\xi) > \lceil \widehat{w}_n(\xi)\rceil - 1 \geq m + n - 1$, and from (62) we conclude $w_m(\xi) \leq m + n - 1$. On the other hand by definition of m we have $w_m(\xi) \geq \widehat{w}_n(\xi)$. Combining we get the contradiction

$$\widehat{w}_n(\xi) \leq w_m(\xi) \leq m + n - 1 < \widehat{w}_n(\xi).$$

Hence indeed $m \geq \lceil \widehat{w}_n(\xi)\rceil - n + 1$. □

6. Proofs of the mixed properties

We first prove the results of Section 3 as the proofs are a bit easier. For simplicity and improved readability, we will omit the argument ξ in the exponents $w_{.}, \widehat{w}_{.}, \lambda_{.}, \widehat{\lambda}_{.}$ in all proofs. Moreover, it will be throughout understood that ϵ_i derived from some initial $\epsilon > 0$ are positive and tend to 0 as ϵ does.

6.1. Proof of Theorem 3.2. Consider the combined graph of the linear form problem with respect to $(\xi, \xi^2, \dots, \xi^n)$. Let $\epsilon > 0$. By (55), at certain arbitrarily large $Q = e^q$ the first minimum satisfies

$$|\psi^*_{n,1}(Q) - \frac{n - w_n}{n(1 + w_n)}| = |\frac{L^*_{n,1}(q)}{q} - \frac{n - w_n}{n(1 + w_n)}| < \epsilon.$$

Let

$$\alpha^* = \frac{n - w_n}{n(1 + w_n)}. \tag{63}$$

We may assume that q is a local minimum of $L_{n,1}^*$. Let $s^* > 0$ be the smallest positive number such that $L_{n,1}^*(q+s^*) = L_{n,2}^*(q+s^*)$, so that $q+s^*$ is the first meeting point of first and second minimum functions to the right of q. Let $S^* = e^{s^*}$ and $Q^* = QS^* = e^{q+s^*}$. Then by (55) we have

$$\psi_{n,2}^*(Q^*) = \frac{L_{n,2}^*(q+s^*)}{q+s^*} \leq \frac{n-\widehat{\omega}_n}{n(1+\widehat{\omega}_n)} + \epsilon_1. \tag{64}$$

Let

$$\beta^* := \frac{n-\widehat{\omega}_n}{n(1+\widehat{\omega}_n)}.$$

Since every local maximum of $L_{n,1}^*$ is a local minimum of $L_{n,2}^*$, the function $L_{n,1}^*$ increases with slope $+1$ in the interval $[q, q+s^*]$. Thus we have $L_{n,2}^*(q+s^*) = L_{n,1}^*(q+s^*) = L_{n,1}^*(q) + s^*$ and we calculate

$$\frac{s^*}{q} = \frac{L_{n,2}^*(q+s^*)}{q} - \frac{L_{n,1}^*(q)}{q} \leq (\beta^*+\epsilon_1)\frac{q+s^*}{q} - \alpha^* + \epsilon = \beta^* - \alpha^* + \beta^*\frac{s^*}{q} + \epsilon_2$$

and solving for s^*/q thus

$$\frac{s^*}{q} \leq \frac{\beta^*-\alpha^*}{1-\beta^*} + \epsilon_3.$$

Since $L_{n,2}^*$ has slope at least $-1/n$, with (64) and inserting for α^* and β^* at once, we infer

$$\begin{aligned}
\psi_{n,2}^*(Q) = \frac{L_{n,2}^*(q)}{q} &\leq \frac{1}{q}\Big(L_{n,2}^*(q+s^*) + \frac{s^*}{n}\Big) = \frac{q+s^*}{q}\frac{L_{n,2}^*(q+s^*)}{q+s^*} + \frac{1}{n}\frac{s^*}{q} \\
&\leq \left(1 + \frac{\beta^*-\alpha^*}{1-\beta^*}\right)\beta^* + \frac{1}{n}\frac{\beta^*-\alpha^*}{1-\beta^*} + \epsilon_4 \\
&\leq \frac{\widehat{\omega}_n(n-\omega_n) + (n+1)(\omega_n-\widehat{\omega}_n)}{n\widehat{\omega}_n(1+\omega_n)} + \epsilon_5.
\end{aligned}$$

For simplicity let

$$\gamma^* = \frac{\widehat{\omega}_n(n-\omega_n) + (n+1)(\omega_n-\widehat{\omega}_n)}{n\widehat{\omega}_n(1+\omega_n)}. \tag{65}$$

Now we transition to dimension k. Let $\underline{x}_1, \underline{x}_2$ be the integer points inducing $L_{n,1}^*(q), L_{n,2}^*(q)$ according to (46) for our q above, respectively. We will implictily identify $\underline{x}_j = (x_{j,0}, \dots, x_{j,n})$ with polynomials $P_j(T) = x_{j,0} + x_{j,1}T + \cdots + x_{j,n}T^n$, for $j = 1, 2$. Say d is the exact degree of P_1, where $d \in \{1, 2, \dots, n\}$. Consider the set of $k-n+2$ polynomials

$$\mathscr{R} = \{R_1, \dots, R_{k-n+2}\} = \{P_1, T^{n-d+1}P_1, T^{n-d+2}, \dots, T^{k-d}P_1, P_2\}.$$

It consists of polynomials of degree at most k and we readily check $\mathscr{R}$ is linearly independent. Indeed P_1, P_2 are linearly independent and adding one by one the remaining polynomials from $T^{n-d+1}P_1$ up to $T^{k-d}P_1$ increases the dimension in each step because the new polynomial has larger degree than any polynomial that occurred before.

Now, for $1 \leq u \leq k-n+2$, the coefficient vector of R_u can be interpreted as a vector $\underline{x}_i'$ with some $i = i(u)$ as in (59), derived from putting $\underline{x} = \underline{x}_1$ if $1 \leq u \leq k-n+1$ and $\underline{x} = \underline{x}_2$ if $u = k-n+2$. For simplicity denote $L^*_{k,R_u}(q) = L^*_{k,\underline{x}_i'}(q)$ the functions in (46) upon this identification. Hence, with $\Phi_{k,n}$ from (57), from Lemma 5.1 we get that the first $k-n+1$ polynomials in $\mathscr{R}$ induce average slope $\Phi_{k,n}(\psi^*_{n,1}(Q))$ in $[0,q']$, and one more induces average slope $\Phi_{k,n}(\psi^*_{n,2}(Q))$ in $[0,q']$, at some transformed position $q' = (n+1)k/(n(k+1)) \cdot q$. Writing $Q' = e^{q'}$, in other words we establish

$$\psi^*_{k,k-n+1}(Q') = \frac{L^*_{k,k-n+1}(q')}{q'} \leq \min_{1\leq u\leq k-n+1} \frac{L^*_{k,R_u}(q')}{q'} \leq \Phi_{k,n}(\psi^*_{n,1}(Q))$$

and

$$\psi^*_{k,k-n+2}(Q') = \frac{L^*_{k,k-n+1}(q')}{q'} \leq \frac{L^*_{k,R_{k-n+2}}(q')}{q'} \leq \Phi_{k,n}(\psi^*_{n,2}(Q)).$$

Then $k+1-|\mathscr{R}| = k+1-(k-n+2) = n-1$ successive minima functions remain. Thus from (48) for the last function at Q' we derive

$$\begin{aligned}\psi^*_{k,k+1}(Q') &= \frac{L^*_{k,k+1}(q')}{q'} \\ &\geq -\frac{(k-n+1)\Phi_{k,n}(\psi^*_{n,1}(Q)) + \Phi_{k,n}(\psi^*_{n,2}(Q))}{n-1} - O(q'^{-1}),\end{aligned}$$

thus

$$\psi^*_{k,k+1}(Q') \geq -\frac{(k-n+1)\Phi_{k,n}(\alpha^*) + \Phi_{k,n}(\gamma^*)}{n-1} - \epsilon_6 - O(q'^{-1}).$$

From Mahler's duality (49), for $\psi_{k,1}(Q')$ the average slope in the successive minima diagram of the first successsive minimum in $[0,q']$ (with respect to $(\xi,\xi^2,\dots,\xi^k)$) we obtain

$$\begin{aligned}\psi_{k,1}(Q') &\leq -\psi^*_{k,k+1}(Q') + O(q'^{-1}) \\ &\leq \frac{(k-n+1)\Phi_{k,n}(\alpha^*) + \Phi_{k,n}(\gamma^*)}{n-1} + \epsilon_6 + O(q'^{-1}).\end{aligned} \tag{66}$$

Let $\epsilon > 0$. As $Q \to \infty$, with (54) for $N = k$, $j = 1$ applied to our estimate (66), and inserting for α^*, γ^* from (63), (65), after a lengthy computation we get a lower bound of the form

$$\lambda_k \geq \frac{w_n\widehat{w}_n - w_n + (n-k)\widehat{w}_n}{(n-2)w_n\widehat{w}_n + w_n + (k-1)\widehat{w}_n} - \epsilon_7.$$

Since ϵ_7 can be arbitrarily close to 0, the desired bound is obtained. The proof is complete.

The key point of the proof was to find a relatively large set $\mathscr{R}$ of linearly independent polynomials with small evaluation at ξ. For this we made extensive use of the fact that we work with successive powers of some ξ. The proofs of Theorems 3.3, 3.4 rely on the same principle.

6.2. Proof of Theorem 3.3. To improve our result upon condition (26), in the proof of Theorem 3.3 below the main step is to notice that in this case we can extend the polynomial set $\mathscr{R}$ from the proof of Theorem 3.2 and still guarantee that it remains linearly independent.

We verify (28) upon our assumption (26) and $k \leq 2n-2$. Let $\alpha^*, \beta^*, \gamma^*$ as in the proof of Theorem 3.2. Further take the same q and derived q'. In place of (66), we show the stronger estimate

$$\psi_{k,1}(Q') \leq \frac{(k-n+1)\Phi_{k,n}(\alpha^*) + (k-n+1)\Phi_{k,n}(\gamma^*)}{2n-1-k} + \epsilon_4 + O(q'^{-1}). \tag{67}$$

Observe that the denominator is positive by assumption. By our hypothesis (26) and Proposition 5.2, we may assume that the polynomial P_1 inducing $\psi^*_{n,1}(Q) \leq \alpha^* + o(1)$ is irreducible and of degree exactly n. In particular coprime to the polynomial P_2 inducing $\psi^*_{n,2}(Q) \leq \gamma^* + o(1)$. We claim that then the set of polynomials

$$\tilde{\mathscr{R}} = \{P_1, TP_1, \dots, T^{k-n}P_1, P_2, TP_2, \dots, T^{k-n}P_2\}$$

consists of polynomials of degree at most k, and is linearly independent. Indeed, otherwise if some non-trivial linear combination within $\tilde{\mathscr{R}}$ vanishes identically, we have a polynomial identity

$$P_1(T)U(T) = P_2(T)V(T)$$

with U, V integer polynomials, U of degree at most $k-n$ and V of degree at most $k-n \leq n-2 < n$. Thus P_1 has to divide either P_2 or V. Clearly it cannot divide V as P_1 has larger degree. However, it cannot divide P_2 either since P_1 is irreducible of degree n and P_2 has degree at most n and is not a scalar multiple of P_1. We obtain a contradiction and our claim is proved.

From the above argument, in the k-dimensional combined graph, with the same position q' as in the proof of Theorem 3.2, we now have $k-n+1$ polynomials inducing average slope essentially at most $\Phi_{k,n}(\alpha^*)$ in $[0, q']$, and further $k-n+1$ polynomials inducing average slope essentially at most $\Phi_{k,n}(\gamma^*)$ in $[0, q']$. Thus

$$\psi^*_{k,k-n+1}(Q') \leq \Phi_{k,n}(\alpha^*), \qquad \psi^*_{k,2(k-n+1)}(Q') \leq \Phi_{k,n}(\gamma^*).$$

Then $k+1-|\tilde{\mathscr{R}}| = k+1-2(k-n+1) = 2n-1-k \geq 1$ polynomials corresponding to successive minima remain. Using Mahler's duality as in the proof of Theorem 3.2, this obviously implies (67) as the sum of $\psi_{k,j}(Q')$ over $j = 1, 2, \dots, k+1$ is $O(q'^{-1})$. The rest of the proof is done analogously to Theorem 3.2, we skip the details and computation.

Remark 5. *Considering $k = 2n-1$, with α^*, γ^* in (63), (65), a similar argument implies $\Phi_{2n-1,n}(\alpha^*) + \Phi_{2n-1,n}(\gamma^*) \geq 0$, upon condition (26). This turns out to be* *equivalent to (27), again upon the same hypothesis. Thereby we have found a new proof of this fact that relies only on Minkowski's Second Convex Body Theorem.*

6.3. Proof of Theorem 3.4. Gelfond's Lemma states that for polynomials P, R of degree at most N we have $H(PR) \asymp_N H(P)H(R)$. In particular for any integer N there is some absolute $c(N) > 0$ so that

$$H(PR) > c(N) \cdot H(P)H(R) \geq c(N)H(P) \tag{68}$$

holds for all non-zero polynomials P, R of degree at most N. Using this property, the proof is similar to that of Theorem 3.3 again.

So let us prove Theorem 3.4 now. As recalled in Proposition 5.2, inequality (60) has solutions in irreducible integer polynomials P of degree $\leq n$ and arbitrarily large height. Let P_1 be such a polynomial. Let $c(n)$ as in (68) and put $M = (c(n)/2) \cdot H(P_1)$. By definition of $\widehat{\omega}_n(\xi)$ there is an integer polynomial P_2 of degree at most n with

$$H(P_2) \leq M, \qquad |P_2(\xi)| \leq M^{-\widehat{\omega}_n(\xi)+\epsilon/2}.$$

By construction P_2 is not a multiple of P_1, thus coprime with P_1. Hence we have found coprime P_1, P_2 with

$$\max_{i=1,2} H(P_i) \leq M, \qquad \max_{i=1,2} |P_i(\xi)| < M^{-\widehat{\omega}_n(\xi)+\epsilon}. \tag{69}$$

Identitfy P_1 as above with its coefficient vector $\underline{x} \in \mathbb{Z}^{n+1}$, so that we have $L^*_{n,P_1}(q) = L^*_{n,\underline{x}}(q)$ for the induced function from (46), and similarly for P_2. Now by (55) with $N = n$, estimates (69) induce parameters $Q = e^q$ with

$$\begin{aligned} \psi^*_{n,2}(Q) \leq \max_{i=1,2} \frac{L^*_{n,P_i}(q)}{q} &\leq \frac{n+1}{n}\frac{1}{1+\widehat{\omega}_n(\xi)} - \frac{1}{n} + \epsilon_1 \\ &= \frac{n-\widehat{\omega}_n(\xi)}{n(1+\widehat{\omega}_n(\xi))} + \epsilon_1. \end{aligned} \tag{70}$$

Let d be the degree of P_1. Next we claim that

$$\mathscr{R} := \{P_1(T), TP_1(T), \dots, T^{k-d}P_1(T), P_2(T), \dots, T^{\min\{d-1,k-n\}}P_2(T)\}$$

is a linearly independent set of integer polynomials of degree at most k. Since $d \leq n$ and $\deg P_2 \leq n$ as well, only the linear independence needs to be checked. Indeed, otherwise there would again be a polynomial identity $P_1(T)U(T) = P_2(T)V(T)$ with integer polynomials U, V of degrees at most $k-d$ and $d-1$ respectively, and a very similar argument as in the proof of Theorem 3.3 shows this is impossible. This proves the claim.

Since all polynomials in $\mathscr{R}$ also have height $\leq M$ and evaluation at ξ of absolute value smaller than $M^{-\widehat{\omega}_n(\xi)+\epsilon}$ if we assume $\xi \in (0,1)$, we have found

$$h := |\mathscr{R}| = (k-d+1) + (\min\{d-1, k-n\}+1) = \min\{k+1, 2k+2-d-n\}$$

linearly independent integer polynomials $R_1, \dots, R_h$ of degree at most k and with

$$\max_{1\leq i\leq h} H(R_i) \leq M, \qquad \max_{1\leq i\leq h} |R_i(\xi)| < M^{-\widehat{\omega}_n(\xi)+\epsilon}.$$

Since $d \leq n$ and $k \leq 2n-2 < 2n-1$ we have $h \geq 2(k-n+1)$. Again we identify R_i with its coefficient vector $\underline{x}_i \in \mathbb{Z}^{k+1}$ and write $L_{k,R_i}^* = L_{k,\underline{x}_i}$. Then for the induced functions, Lemma 5.1 and (70) gives rise to positions $Q' = e^{q'}$ with

$$\begin{aligned}\psi_{k,h}^*(Q') &\leq \max_{1\leq i\leq h} \frac{L_{k,R_i}^*(q')}{q'} \leq \Phi_{k,n}\left(\max_{i=1,2} \frac{L_{n,P_i}^*(q)}{q}\right)\\ &\leq \Phi_{k,n}\left(\frac{n-\widehat{\omega}_n(\xi)}{n(1+\widehat{\omega}_n(\xi))}\right) + \epsilon_2 = \frac{n(k+1)}{(n+1)k}\frac{n-\widehat{\omega}_n(\xi)}{n(1+\widehat{\omega}_n(\xi))} + \frac{k-n}{k(n+1)} + \epsilon_2.\end{aligned}$$

Since there are arbitrarily large such Q' and ϵ can be taken arbitrarily small

$$\underline{\psi}_{k,h}^* \leq \frac{n(k+1)}{(n+1)k}\frac{n-\widehat{\omega}_n(\xi)}{n(1+\widehat{\omega}_n(\xi))} + \frac{k-n}{k(n+1)}. \tag{71}$$

Using (50) and (52) we can estimate

$$\underline{\psi}_{k,1} = -\overline{\psi}_{k,k+1}^* \leq \frac{h}{k+1-h}\underline{\psi}_{k,h}^*, \qquad \text{if } k \leq 2n-2. \tag{72}$$

The condition on k ensures $k+1-h > 0$. Inserting the bound for $\underline{\psi}_{k,h}^*$ from (71) and the worst case $h = 2(k-n+1)$ in (72) and applying (54), we get (29) after some calculation.

For (30), we notice that if the degree of P_1 above is $d = n$ then we can proceed as in the proof of Theorem 3.3 to get its bound (28). Otherwise $d \leq n-1$ and thus now $h \geq 2k-2n+3$, and as soon as $k \leq 2n-3$ we can proceed as above to obtain the other bound when using $h = 2k-2n+3$ in (72).

7. Proof of the going-up Theorem 2.2

Similar ideas as for the mixed inequalities are used to prove the estimates that contain only simultaneous approximation exponents $\lambda_N(\xi)$. However, roughly speaking, one more step of duality considerations between simultaneous and linear form approximation is required here. We apply the same notational simplifications as in Section 6.

7.1. Proof of (7). Let ξ be a real number. It follows from (54) that for any $\epsilon > 0$ there exist arbitrarily large parameters Q such that

$$|\psi_{n,1}(Q) - \frac{1-n\lambda_n}{n(1+\lambda_n)}| < \epsilon. \tag{73}$$

Consider such large Q fixed and let $q = \log Q$. When we transition to the linear form problem, together with (51) we infer

$$\psi_{n,1}^*(Q) + \cdots + \psi_{n,n}^*(Q) \leq \frac{1-n\lambda_n}{n(1+\lambda_n)} + \epsilon + O(q^{-1}). \tag{74}$$

We also want to bound $\psi^*_{n,1}(Q)$ from above. We could estimate it by the right hand side of (74) divided by n, which would turn out to reprove Theorem 2.1, but using the uniform exponent we find a better bound.

From (73) we obtain points $(q, L_{n,1}(q))$ with arbitrarily large q and the property

$$(-\alpha-\epsilon)q \le L_{n,1}(q) \le (-\alpha+\epsilon)q, \tag{75}$$

where we have put

$$-\alpha = \frac{1-n\lambda_n}{n(1+\lambda_n)}. \tag{76}$$

for simplicity. We can assume that $L_{n,1}$ has a local minimum at q. Then in some interval $[q-s,q]$ the function $L_{n,1}$ decays with slope -1. The switch point $q-s$, where $L_{n,1}$ changes slope from $1/n$ to -1, is where it meets the second minimum function $L_{n,2}$. At $q-s$, again from (54) we obtain

$$L_{n,1}(q-s) = L_{n,2}(q-s) \le \frac{1-n\widehat{\lambda}_n}{n(1+\widehat{\lambda}_n)}(q-s)+\epsilon_1 q.$$

Again let

$$-\beta := \frac{1-n\widehat{\lambda}_n}{n(1+\widehat{\lambda}_n)}.$$

Since $L_{n,1}$ decays with slope -1 in $[q-s,q]$, on the other hand by (75) we have

$$L_{n,1}(q-s) = L_{n,1}(q)+s = (-\alpha+\delta)q+s,$$

where $\delta\in(-\epsilon,\epsilon)$ is of small modulus. Equating the two expressions for $L_{n,1}(q-s)$, after some calculation we get

$$0 < s \le q\cdot\frac{\lambda_n-\widehat{\lambda}_n}{1+\lambda_n}+\epsilon_2 q.$$

As the second successive minimum has slope at most $1/n$ in $[q-s,q]$, inserting for s, at position q we get

$$L_{n,2}(q) \le L_{n,2}(q-s)+\frac{1}{n}s \le \frac{\lambda_n-(n+1)\widehat{\lambda}_n+1}{n(1+\lambda_n)}q+\epsilon_3 q.$$

Let

$$-\gamma := \frac{\lambda_n-(n+1)\widehat{\lambda}_n+1}{n(1+\lambda_n)}. \tag{77}$$

Now again consider the dual linear form problem with respect to the vector $(\xi,\xi^2,\dots,\xi^n)$. Recall the notation $q=\log Q$ and $\psi^*_{n,j}(Q)=L^*_{n,j}(q)/q$. By Mahler's duality (49), for the last two successive minima at position q we obtain

$$\psi^*_{n,n+1}(Q) = \frac{L^*_{n,n+1}(q)}{q} \ge -\frac{L_{n,1}(q)}{q}-O(q^{-1}) \ge (\alpha-\epsilon)-O(q^{-1}) \tag{78}$$

and

$$\psi^*_{n,n}(Q) = \frac{L^*_{n,n}(q)}{q} \geq -\frac{L_{n,2}(q)}{q} - O(q^{-1}) = (\gamma - \epsilon_3) - O(q^{-1}). \tag{79}$$

Since the sum of all $n+1$ successive minima functions $\psi^*_{n,j}$ at Q is $O(q^{-1})$ by (48), we have that

$$\sum_{j=1}^{n-1} \psi^*_{n,j}(Q) \leq -\psi^*_{n,n}(Q) - \psi^*_{n,n+1}(Q) + O(q^{-1}) \leq -\alpha - \gamma + \epsilon + \epsilon_3 + O(q^{-1}).$$

In particular

$$\psi^*_{n,1}(Q) \leq \frac{\sum_{j=1}^{n-1} \psi^*_{n,j}(Q)}{n-1} \leq \frac{-\alpha-\gamma}{n-1} + \epsilon_4 + O(q^{-1}). \tag{80}$$

This is the desired bound for $\psi^*_{n,1}(Q)$.

Now we transition to dimension $k \geq n$. Each of the pairs $(Q, \psi^*_{n,j}(Q))$ are induced by $L^*_{n,\underline{x}_j}$ as defined in (46) for some $\underline{x}_j = (x_{j,0}, \dots, x_{j,n})$. We identify each $\underline{x}_j$ with the polynomial $P_j(T) = x_{j,0} + x_{j,1}T + \cdots + x_{j,n}T^n$ again. Moreover $\mathscr{P} = \{P_1, \dots, P_n\}$ are linearly independent. Let d be the degree of P_1, that is the largest index with $x_{1,d} \neq 0$. Clearly $1 \leq d \leq n$. Starting from these polynomials we derive the ordered set of k polynomials

$$\mathscr{R} = \{R_1, \dots, R_k\} = \{P_1, T^{l-d+1}P_1, \dots, T^{k-d}P_1, P_2, P_3, \dots, P_n\}.$$

Any R_i has degree at most k. Furthermore it is easy to check that the linear independence of $\mathscr{P}$ implies that $\mathscr{R}$ is linearly independent as well, since starting with $\mathscr{P}$ and adding one by one the new polynomials in $\mathscr{R} \setminus \mathscr{P} = \{T^{n-d+1}P_1, \dots, T^{k-d}P_1\}$, the dimension of the span increases in each step because the new polynomial has strictly larger degree than all the previous polynomials, and thus does not lie in their span.

For simplicity now assume the typical case $d = n$, otherwise the correspondence to Lemma 5.1 in following argument has to be slightly altered, and the remainder of the proof remains unaffected anyway. Then the first $k-n$ polynomials $R_1, \dots, R_{k-n}$ correspond to vectors $\underline{x}'_i$ in (59) for $1 \leq i \leq k-n$ in Lemma 5.1 for $\underline{x} = \underline{x}_1$ the coefficient vector of P_1, and similarly R_{k-n+j} to $\underline{x}'$ in (58) for $\underline{x} = \underline{x}_j$ the coefficient vector of P_j as defined above, for $1 \leq j \leq n$ (so $\underline{x}_1$ appears $k-n+1$ times in total). We may assume $\xi \in (0,1)$ and apply Lemma 5.1 to each R_i. With $Q' = e^{q'}$ for q' in (56), from the linear independence of $\mathscr{R}$ we obtain

$$\psi^*_{k,j}(Q') \leq \Phi_{k,n}(\psi^*_{n,1}(Q)) = (\psi^*_{n,1}(Q) - 1)\frac{k+1}{k}\frac{n}{n+1} + 1, \; 1 \leq j \leq k-n,$$

$$\psi^*_{k,k-n+j}(Q') \leq \Phi_{n,k}(\psi^*_{n,j}(Q)) = (\psi^*_{n,j}(Q) - 1)\frac{k+1}{k}\frac{n}{n+1} + 1, \;\; 1 \leq j \leq n.$$

Summing over $j = 1, 2, \dots, k$ we infer

$$\sum_{j=1}^{k} \psi_{k,j}^{*}(Q') \leq (k-n)A + B,$$

where

$$A = (\psi_{n,1}^{*}(Q) - 1)\frac{(k+1)n}{k(n+1)} + 1, \quad B = \sum_{j=1}^{n} \left[(\psi_{n,j}^{*}(Q) - 1)\frac{(k+1)n}{k(n+1)} + 1 \right].$$

This can be equivalently written

$$\sum_{j=1}^{k} \psi_{k,j}^{*}(Q') \leq (k-n)\frac{(k+1)n}{k(n+1)}\psi_{n,1}^{*}(Q) + \frac{(k+1)n}{k(n+1)}\sum_{j=1}^{n} \psi_{n,j}^{*}(Q) + \frac{k-n}{n+1}.$$

Now we use the estimates (74) and (80) and inserting for α, γ from (76), (77) after some calculation we end up at

$$\sum_{j=1}^{k} \psi_{k,j}^{*}(Q') \leq \frac{k(1-n)\lambda_n + (kn + n - k^2 - k)\widehat{\lambda}_n + k^2 - kn + k - 1}{k(n-1)(\lambda_n + 1)} + \epsilon_5.$$

Together with (51) this implies for large Q we derive the estimate

$$\psi_{k,1}(Q') \leq \frac{k(1-n)\lambda_n + (kn + n - k^2 - k)\widehat{\lambda}_n + k^2 - kn + k - 1}{k(n-1)(\lambda_n + 1)} + \epsilon_6.$$

Since there are arbitrarily large Q and thus induced Q' with this property, we derive

$$\underline{\psi}_{k,1} \leq \frac{k(1-n)\lambda_n + (kn + n - k^2 - k)\widehat{\lambda}_n + k^2 - kn + k - 1}{k(n-1)(\lambda_n + 1)} + \epsilon_6.$$

Inserting in (54) we derive the desired estimate (7) after some calculation and $\epsilon \to 0$.

Remark 6. *From* (80) *when inserting for* α, γ *in* (76), (77) *and applying* (55) *we get a new proof of the inequality*

$$w_k(\xi) \geq \frac{(k-1)\lambda_k(\xi) + \widehat{\lambda}_k(\xi) + k - 2}{1 - \widehat{\lambda}_k(\xi)}, \qquad k \geq 2,$$

already obtained by Bugeaud, Laurent [12], *and with a different proof by Schmidt and Summerer* [43]. *Again our proof of this estimate, as in* [12] *and* [43], *extends to the general case of* $\mathbb{Q}$*-linearly independent* $\{1, \xi_1, \dots, \xi_k\}$.

The proof of (8) is very similar, with a slightly different strategy for estimation.

7.2. Proof of (8). We proceed precisely as in the proof of (7) up to (80). From (78), (79) and since the sum of all $n+1$ successive minima functions $\psi^*_{n,j}$ at Q is $O(q^{-1})$ by (48), we have that

$$\sum_{j=1}^{n-1} \psi^*_{n,j}(Q) \leq -\psi^*_{n,n}(Q) - \psi^*_{n,n+1}(Q) + O(q^{-1}) \leq -\alpha - \gamma + \epsilon_7 + O(q^{-1}). \quad (81)$$

We will use this in place of (74), and combine it again with (80).

Now we transition to dimension $k \geq n$. Let $\underline{x}_1, \dots, \underline{x}_{n-1}$ be the linearly independent integer vectors inducing $L^*_{n,j}(q)$ (or equivalently $\psi^*_{n,j}(Q)$) for $1 \leq j \leq n-1$ as above. They correspond to polynomials $P_j(T) = x_{j,0} + x_{j,1}T + \cdots + x_{j,n}T^n$. Let

$$\mathscr{P} = \{P_1, \dots, P_{n-1}\}.$$

Let d be the degree of P_1, that is the largest index with $x_{1,d} \neq 0$. Clearly $1 \leq d \leq n$. Starting from these polynomials we derive the ordered set of $k-1$ polynomials

$$\begin{aligned}\mathscr{R} &= \{R_1, \dots, R_{k-1}\} \\ &= \{P_1, T^{n-d+1}P_1, T^{n-d+2}P_1(T), \dots, T^{k-d}P_1, P_2, P_3, \dots, P_{n-1}\}.\end{aligned}$$

Any R_i has degree at most k. Furthermore it is easy to check that the linear independence of $\mathscr{P}$ implies that $\mathscr{R}$ is linearly independent as well. Indeed, starting with $\mathscr{P}$ and adding one by one the new polynomials in $\mathscr{R} \setminus \mathscr{P} = \{T^{n-d+1}P_1, \dots, T^{k-d}P_1\}$, the dimension of the span increases in each step because the new polynomial has strictly larger degree than all the previous polynomials, and thus does not lie in their span.

The polynomials $P_j \in \mathscr{P}$ give rise to points $\underline{x}'_1, \dots, \underline{x}'_{n-1}$ as in (58) via embedding them into $\mathbb{Z}^{k+1}$. Write $\psi^*_{k,P_j}(Q') = \psi^*_{k,\underline{x}'_j}(Q') = L^*_{k,\underline{x}'_j}(q')/q'$ with the functions $L^*_{k,\underline{x}'_j}$ as in (46) for the polynomial P_j above, and the corresponding notation for other polynomials. With $\Phi_{k,n}$ as in (57), from Lemma 5.1 and (81) we get some point $Q' = e^{q'}$ where we have

$$\begin{aligned}
\sum_{P\in\mathscr{P}} \psi^*_{k,P}(Q') &= \sum_{j=1}^{n-1} \psi^*_{k,P_j}(Q') \\
&= \sum_{j=1}^{n-1} \psi^*_{k,\underline{x}'_j}(Q') \\
&= \sum_{j=1}^{n-1} \Phi_{k,n}(\psi^*_{n,j}(Q)) \\
&= \Phi_{k,n}(\sum_{j=1}^{n-1} \psi^*_{n,j}(Q)) + (n-2)\frac{k-n}{k(n+1)} \\
&\leq \Phi_{k,n}(-\alpha-\gamma) + \epsilon_8 + (n-2)\frac{k-n}{k(n+1)} + O(q^{-1}).
\end{aligned}$$

Hereby we used the fact that $\Phi_{k,n}$ are affine functions with constant term $(k-n)/(k(n+1))$. Assume without loss of generality $\xi \in (0,1)$. Then, again by Lemma 5.1, for the remaining $(k-1)-(n-1) = k-n$ polynomials in $\mathscr{R}\setminus\mathscr{P}$ we obtain

$$\sum_{R\in\mathscr{R}\setminus\mathscr{P}} \psi^*_{k,R}(Q') \leq (k-n)\Phi_{k,n}(\psi^*_{n,1}(Q)).$$

The entire sum over $\mathscr{R} = \mathscr{P}\cup(\mathscr{R}\setminus\mathscr{P})$ is the sum of both left hand sides above, thus by (80) we infer

$$\begin{aligned}
\sum_{R\in\mathscr{R}} \psi^*_{k,R}(Q') &\leq \Phi_{k,n}(-\alpha-\gamma) + (k-n)\Phi_{k,n}(\psi^*_{n,1}(Q)) + \\
&\quad + (n-2)\frac{k-n}{k(n+1)} + \epsilon_9 + O(q'^{-1}) \\
&\leq \Phi_{k,n}(-\alpha-\gamma) + (k-n)\Phi_{k,n}(-\frac{\alpha+\gamma}{n-1}) + \\
&\quad + (n-2)\frac{k-n}{k(n+1)} + \epsilon_{10} + O(q'^{-1}).
\end{aligned}$$

As $\mathscr{R}$ is a linearly independent set of cardinality $k-1$ here, we may write this as

$$\sum_{j=1}^{k-1} \psi^*_{k,j}(Q') \leq \sum_{R\in\mathscr{R}} \psi^*_{k,R}(Q') \leq \tau + \epsilon_{10} + O(q^{-1}),$$

where inserting in $\Phi_{k,n}$ we calculate

$$\begin{aligned}
\tau := &\ \Phi_{k,n}(-\alpha-\gamma) + (k-n)\Phi_{k,n}(-\frac{\alpha+\gamma}{n-1}) + (n-2)\frac{k-n}{k(n+1)} \\
= &\ -(\alpha+\gamma)\frac{(k+1)(k-1)n}{(n+1)(n-1)k} + \frac{(k-1)(k-n)}{k(n+1)}.
\end{aligned}$$

Now for $\psi^{*}_{k,k+1}(Q')$, which is the average slope of the last minimum function $L^{*}_{k,k+1}$ in $[0,q']$, by (48) we obtain

$$\psi^{*}_{k,k+1}(Q') \geq -\frac{\sum_{j=1}^{k-1}\psi^{*}_{k,j}(Q')}{2} - O(q^{-1}) \geq -\frac{\tau}{2} - \epsilon_{11} - O(q^{-1}).$$

Again by Mahler's duality (49), for the first minimum of the simultaneous approximation problem in dimension k at q we get

$$\psi_{k,1}(Q') \leq -\psi^{*}_{k,k+1}(Q') + O(q'^{-1}) \leq \frac{\tau}{2} + \epsilon_{11} + O(q'^{-1}). \tag{82}$$

Using

$$\lambda_k(\xi) \geq \limsup_{Q'\to\infty} \frac{1-k\psi_{k,1}(Q')}{k+k\psi_{k,1}(Q')} \tag{83}$$

from (54) again, from (82) we get a lower bound for $\lambda_k(\xi)$ in terms of τ, which in turn depends only on α,γ. Inserting for α,γ from (76), (77) for large enough $q \geq q_0(\epsilon)$ as above, after a tidious calculation and rearrangement, we end up at

$$\lambda_k \geq \frac{(n-1)\lambda_n + (k-1)\widehat{\lambda}_n + n - k}{(n-1)\lambda_n - (k-1)\widehat{\lambda}_n + n + k - 2} - \epsilon_{12}. \tag{84}$$

Since we can choose ϵ arbitrarily small, the bound becomes as in the theorem.

8. Deduction of the results from Section 4

Let $N \geq 1$ be an integer. Assume $\{1,\zeta_1,\dots,\zeta_N\}$ is linearly independent over $\mathbb{Q}$ and write $\underline{\zeta} = (\zeta_1,\dots,\zeta_N)$. Khintchine's transference principle [24] states

$$\frac{w_N(\underline{\zeta})}{(N-1)w_N(\underline{\zeta})+N} \leq \lambda_N(\underline{\zeta}) \leq \frac{w_N(\underline{\zeta})-N+1}{N}. \tag{85}$$

We recall the refinements in terms of introducing uniform exponents

$$\lambda_N(\underline{\zeta}) \geq \frac{(\widehat{w}_N(\underline{\zeta})-1)w_N(\underline{\zeta})}{((N-2)\widehat{w}_N(\underline{\zeta})+1)w_N(\underline{\zeta})+(N-1)\widehat{w}_N(\underline{\zeta})} \tag{86}$$

and

$$w_N(\underline{\zeta}) \geq \frac{(N-1)\lambda_N(\underline{\xi}) + \widehat{\lambda}_N(\underline{\zeta}) + N - 2}{1-\widehat{\lambda}_N(\underline{\zeta})}, \tag{87}$$

already quoted below Theorem 3.2 and Remark 6, respectively. Considering only uniform exponents, German [19] showed

$$\frac{\widehat{w}_N(\underline{\zeta})-1}{(N-1)\widehat{w}_N(\underline{\zeta})} \leq \widehat{\lambda}_N(\underline{\zeta}) \leq \frac{\widehat{w}_N(\underline{\zeta})-N+1}{\widehat{w}_N(\underline{\zeta})}. \tag{88}$$

The estimates in (85), (88) are best possible, and (86), (87) at least for $N=2$ as well [12],[26]. In the Remark on page 80 in [42], a short proof of (85) that only uses parametric geometry of numbers is given. It resembles our proofs from Sections 6, 7 that implicitly recover the refined estimates (86), (87). For an alternative proof of (88) and its optimality based on parametric geometry of numbers, see [44]. It is worth noting that (88) is stronger than the analogue of (85) obtained from replacing ordinary by uniform exponents. In the proofs, we apply above estimates to finite dimensional projections of $\underline{\xi} \in \mathbb{R}^{\mathbb{N}}$.

Proof of Theorem 4.1. We start with the inequality (87) for $N = n$ and $\underline{\zeta} = \underline{\xi}_n$. On the other hand it is easy to see that for any $\underline{\xi} \in \mathbb{R}^{\mathbb{N}}$ and $k \geq n$

$$w_k(\underline{\xi}_k) \geq w_n(\underline{\xi}_n), \tag{89}$$

since for any vector $\underline{x} = (x_0, \dots, x_n)$ as in the definition of w_n taking $\underline{x}' = (x_0, \dots, x_n, 0, \dots, 0) \in \mathbb{Z}^{k+1}$ yields $\|\underline{x}\|_\infty = \|\underline{x}'\|_\infty$ and $\langle \underline{x}, \underline{\xi}_n \rangle_n = \langle \underline{x}', \underline{\xi}_k \rangle_k$. Combining yields

$$w_k(\underline{\xi}_k) \geq \frac{(n-1)\lambda_n(\underline{\xi}_n) + \widehat{\lambda}_n(\underline{\xi}_n) + n - 2}{1 - \widehat{\lambda}_n(\underline{\xi}_n)} = B, \tag{90}$$

with B as defined in the theorem. Now apply the left inequality from (85) with $N = k$, $\underline{\zeta} = \underline{\xi}_k$ to (90) to obtain the bound (38) after a short calculation.

For (39), we notice that (44) combined with the right estimate in (88) for $N = n$ and $\underline{\zeta} = \underline{\xi}_n$ yields

$$\widehat{w}_k(\underline{\xi}_k) \geq \frac{k-1}{1 - \frac{\widehat{\lambda}_n(\underline{\xi}_n) + n - 2}{(n-1)(k-1)}} = \frac{(n-1)(k-1)^2}{nk - k - 2n + 3 - \widehat{\lambda}_n(\underline{\xi}_n)} = A,$$

again with A as defined in the theorem. Inserting this and (90) in (86) with $N = k$ and $\underline{\zeta} = \underline{\xi}_k$ yields (39). □

Starting with (85) in the proof, instead of (38) we would have directly obtained Theorem 4.2. The proof of Theorem 4.3 relies solely on the inequalities in (88).

Proof of Theorem 4.3. Similar to (89) we have

$$\widehat{w}_k(\underline{\xi}_k) \geq \widehat{w}_n(\underline{\xi}_n).$$

Together with (88) for $N = n$ and $\underline{\zeta} = \underline{\xi}_n$ we infer

$$\widehat{w}_k(\underline{\xi}_k) \geq \widehat{w}_n(\underline{\xi}_n) \geq \frac{n-1}{1 - \widehat{\lambda}_n(\underline{\xi}_n)}.$$

Inserting in the left inequality of (88) with $N = k$ and $\underline{\zeta} = \underline{\xi}_k$ yields the claim. □

Proof of Theorem 4.5. We combine

$$w_k(\underline{\xi}_k) \geq w_n(\underline{\xi}_n), \qquad \widehat{w}_k(\underline{\xi}_k) \geq \widehat{w}_n(\underline{\xi}_n),$$

with (86) for $N = k$ and $\underline{\zeta} = \underline{\xi}_k$. □

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