2-categorical opfibrations, Quillen's Theorem B, and $S^{-1}S$

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ABSTRACT. In this paper we show that the strict and lax pullbacks of a 2-categorical opfibration along an arbitrary 2-functor are homotopy equivalent. We give two applications. First, we show that the strict fibers of an opfibration model the homotopy fibers. This is a version of Quillen's Theorem B amenable to applications. Second, we compute the E^2 page of a homology spectral sequence associated to an opfibration and apply this machinery to a 2-categorical construction of $S^{-1}S$. We show that if S is a symmetric monoidal 2-groupoid with faithful translations then $S^{-1}S$ models the group completion of S.

CONTENTS

1. Introduction

Fibrations and opfibrations of 1-categories, also known as Grothendieck fibrations and opfibrations (originally fibered and cofibered categories), were introduced in [SGA1, Exp. VI] for the application of categorical algebra to descent problems. If $P : C \to D$ is a fibration of 1-categories, then there is a natural adjunction between $P^{-1}(x)$ and the comma category $x \downarrow P$. This makes P^{-1} a contravariant pseudofunctor from C to Cat and moreover gives a homotopy equivalence between the classifying spaces of $P^{-1}(x)$ and $x \downarrow P$. Dually, if P is an opfibration, then P^{-1} is a covariant pseudofunctor to Cat and there is a homotopy equivalence on classifying spaces between $P^{-1}(x)$ and $P \downarrow x$.

In this paper we study both homotopical and categorical features of opfibrations for 2-categories and lax comma objects for 2-functors. The former

are dual to the fibrations of Buckley [Buc14], and the latter are due to Gray [Gra80a, Gra80b]. In Section 2 we recall the relevant definitions and prove the following result as Theorem 2.34.

Theorem 1.1. *Suppose*

$$C \xrightarrow{\ P\ } D \xleftarrow{\ F\ } B$$

is a cospan of 2-categories and all 2-cells of D are invertible. If P is an opfibration then the inclusion

$$i: \ \mathrm{pb}(P, F) \to P{\downarrow}F,$$

from the pullback of P and F to the lax comma object $P{\downarrow}F$, induces a homotopy equivalence on classifying spaces.

We go on to give two applications of this result, outlined in Sections 1.1 and 1.2 below.

1.1. Quillen's Theorem B for opfibrations of 2-categories. Our first application is related to Quillen's Theorems A and B [Qui73]. Theorem A states that under suitable conditions, if the comma objects of a functor between categories are assumed to be contractible, then the functor induces a homotopy equivalence on classifying spaces. It can be seen as a special case of Quillen's Theorem B, which says that the comma objects of a functor model its homotopy fibers.

Versions of Theorems A and/or B for higher categories have appeared in many papers. For Theorem A, this includes Bullejos-Cegarra [BC03] (Theorem A for 2-categories) and Ara-Maltsiniotis [AM18, AM20] (Theorem A for strict ∞-categories). For Theorem B, this includes Cegarra [Ceg11] (Theorem B for 2-categories), Calvo-Cegarra-Heredia [CCH14] (Theorem B for bicategories), and Ara [Ara19] (Theorem B for strict ∞-categories).

Combining the 2-categorical version of Quillen's Theorem B (see Theorem 2.39 below) with Theorem 1.1 gives the following form of Quillen's Theorem B, which appears as Corollary 2.41. This is a form that more closely resembles the version used in *K*-theoretic applications.

Corollary 1.2. *Assume that all 2-cells of D are invertible, and suppose $P: \ C \to D$ is an opfibration such that, for every 1-cell $x \to y$ in D the 2-functors $P^{-1}(x) \to P^{-1}(y)$ induced by base change are homotopy equivalences on classifying spaces. Then for each $x \in C$ the sequence*

$$P^{-1}(x) \to C \to D$$

induces a homotopy fiber sequence on classifying spaces.

Chiche [Chi14] defines a notion of pre-(coop)fibration for which the conclusion of Theorem 1.1 holds in the special case in which *P* is the inclusion of a single object. However the comparison between our opfibrations and the definitions of Chiche is nontrivial and beyond the scope of this work.

Two-dimensional (op)fibrations in the sense we study here have appeared in work of Hermida [Her99] (for 2-categories), Bakovic [Bak07] (for bicategories),

and Buckley [Buc14] (with additional results for both 2-categories and bicate-gories). However, a result like Theorem 1.1 has not previously appeared, and therefore nor has Corollary 1.2. Identifying the conditions that

(*a*) allow one to model homotopy fibers via strict fibers and

(*b*) occur in applications of interest

was a major motivation for all of the results in this paper.

Striking this balance allows us to analyze the homology spectral sequence of an opfibration in Section 3. We apply the theory of lax comma objects and opfibrations to give an explicit description of the E^2 page in the following result, which appears as Theorem 3.25.

Theorem 1.3. *If $P: C \to D$ is an opfibration and all 2-cells of D are invertible, then there is a spectral sequence*

$$E^2_{p,q} = H_p\big(D\,;\,\mathscr{H}_q P^{-1}\big) \Rightarrow H_{p+q}(C)$$

where $\mathscr{H}P^{-1}$ denotes the local coefficient system given by homology of the fibers of P.

The development of the homology spectral sequence for a 2-functor in Section 3 generalizes the corresponding theory for 1-categorical opfibrations, and might be conceptually familiar.

1.2. Group completion and the $S^{-1}S$ construction. Our second applica-tion relates to Quillen's categorical model for topological group completion. The primary difference between an arbitrary E_∞ space and the zeroth space of an Ω-spectrum occurs at the level of path components. The action of an E_∞ operad on a space X induces a multiplication on $\pi_0 X$, and the fact that the op-erad is E_∞ equips $\pi_0 X$ with the structure of a commutative monoid. If $Y = Y_0$ is the zeroth space of an Ω-spectrum $\{Y_n\}$, then the isomorphism of monoids $\pi_0 Y \cong \pi_n Y_n$ for any $n > 1$ shows that $\pi_0 Y$ is actually an abelian group. We say that an E_∞ space X is *grouplike* if $\pi_0 X$ is a group. *Topological group completion* [Bar61, BP72, May74, Seg74, MS76] is a process that

- universally completes the commutative monoid $\pi_0 X$ to an abelian group,
- similarly completes the homology of X,
- but in the process radically changes the higher homotopy groups of X.

For a symmetric monoidal category S with invertible morphisms and faithful translations, Quillen's $S^{-1}S$ construction [Gray76] provides a categorical model for the topological group completion of the classifying space $BS = |NS|$. In the case that S is a commutative monoid treated as a discrete symmetric monoidal category, $S^{-1}S$ is equivalent (as a symmetric monoidal category) to the algebraic group completion S^{gp}—the target of a universal monoid homomorphism from S into an abelian group. In this sense, topological group completion generalizes algebraic group completion.

The 1-categorical $S^{-1}S$ construction is crucial for the definition and study of higher algebraic K-theory. In the case that R is a commutative ring and $S =$

$\coprod_n GL_n(R)$, then the main result of [Gray76] provides a homotopy equivalence

$$B(S^{-1}S) \xrightarrow{\simeq} K_0(R) \times BGL(R)^+. \tag{1.4}$$

In the low-dimensional cases $n = 1, 2$, the categorical structure of $S^{-1}S$ provides direct algebraic access to the K-theory groups $K_n(R) = \pi_n BGL(R)^+$ for $n > 0$.

As our second application, we obtain Theorem 1.5 below, generalizing the construction and results of [Gray76] to the case in which S is a symmetric monoidal bicategory. We assume without loss of generality that S is in a strict form known as a permutative Gray monoid. In Section 4.1 we recall this notion and the result that each symmetric monoidal bicategory is equivalent, via a symmetric monoidal pseudofunctor, to a permutative Gray monoid (Theorem 4.7). For simplicity we state the main result in these terms.

Theorem 1.5. *Let S be a permutative Gray monoid. There is a symmetric monoidal 2-category $S^{-1}S$ and a symmetric monoidal 2-functor*

$$i : S \to S^{-1}S$$

with the following properties.

  i. *The classifying space of $S^{-1}S$ is grouplike.*
  ii. *If S is, moreover, a 2-groupoid with faithful translations, then i is a group completion on classifying spaces.*

Claim (*i*) is proved in Theorem 4.28 by direct analysis of the construction $S^{-1}S$. Claim (*ii*) follows by applying Theorem 1.3 to the opfibration

$$S^{-1}S \to S^{-1} * .$$

Warning 1.6. As it is an easy source of confusion, we emphasize that our construction concerns symmetric monoidal bicategories and not symmetric bimonoidal categories. The latter are 1-dimensional categories with two symmetric monoidal structures, one of which distributes over the other. As pointed out by Thomason [Tho80], the classical $S^{-1}S$ construction on a symmetric bimonoidal category fails to have a bimonoidal structure.

1.3. Relation to other forms of group-completion. We relate Theorem 1.5 back to the program in [GJO19]. Recall that a stable P_2-equivalence is a symmetric monoidal functor inducing an isomorphism on stable homotopy groups in dimensions 0, 1, and 2. A Picard 2-category (Definition 4.14) is a symmetric monoidal 2-category in which all objects and morphisms are invertible in the weak sense. In [GJO19] we proved that every symmetric monoidal bicategory is stably P_2-equivalent to a strict Picard 2-category. This strict Picard 2-category is a categorical model for the Postnikov 2-truncation of the spectrum associated to the original symmetric monoidal bicategory. The construction in [GJO19] involves a long zigzag of stable P_2-equivalences, and employs a topological group completion of E_∞-spaces. One outcome of the present paper is an alternate construction of the 2-truncation in simpler and purely categorical terms. The

caveat is that such a construction is only available under the additional assumption of faithful translations. We do not know a general method for imposing the condition of faithful translations on a general symmetric monoidal category or 2-category while preserving its stable homotopy type, other than the full machinery developed in [GJO19].

Open Problem 1.7. Given a symmetric monoidal (2-)category M, find a purely algebraic functorial construction of a symmetric monoidal (2-)category M^{ft} with faithful translations together with a (perhaps only pseudo or lax) natural stable equivalence $M \to M^{ft}$.

In the presence of faithful translations, we can use the results in this paper and [GJO19] to construct an explicit 2-truncation as follows. If S is a permutative Gray monoid which is a 2-groupoid with faithful translations, then $S \to S^{-1}S$ is a homotopy group completion by Theorem 1.5. Section 5.1 of [GJO19] defines a 2-groupoidification functor WN such that $WN(S^{-1}S)$ is a symmetric monoidal 2-groupoid which receives a natural stable P_2-equivalence $S^{-1}S \to WN(S^{-1}S)$. We note that $WN(S^{-1}S)$ is a symmetric monoidal 2-category but generally not a permutative Gray monoid even when S is assumed to be so. Nevertheless, the composite $S \to S^{-1}S \to WN(S^{-1}S)$ is a stable P_2-equivalence, and $WN(S^{-1}S)$ is a Picard 2-category by construction. Therefore the classifying space of $WN(S^{-1}S)$ is the stable Postnikov 2-truncation of BS.

We point out that there is another, purely algebraic, notion of group completion one might study for symmetric monoidal categories or 2-categories. We want to distinguish between the fundamentally homotopic notion and the purely algebraic one. The algebraic group completion can be constructed formally via factorization system theory. The category of groups sits as a reflective subcategory of the category of monoids, and the group completion of a monoid is given by applying the left adjoint to this inclusion. This left adjoint can be seen as an application of the small object argument for the weak factorization system cofibrantly generated by the single map $\mathbb{N} \to \mathbb{Z}$, and so the group completion of a monoid is then the fibrant replacement for this factorization system.

The algebraic group completion for symmetric monoidal 1-categories, respectively 2-categories, is given by applying an analogous left adjoint. It is more complicated to write down explicitly and even less computable. Such a completion takes values in Picard categories, respectively Picard 2-categories. From the perspective of algebraic weak factorization systems [GT06, BG16a, BG16b] this construction must exist and is given, in dimension two, by finding generating maps i_0, i_1, i_2 such that lifting against the unique map to the terminal 2-category with respect to i_α exhibits all α-dimensional cells as invertible. The codomain of i_0 is then the free Picard 2-category generated by a single object, and the authors will give an explicit model for this object in a future paper. In contrast, the group completion $S^{-1}S$ we study below is defined in a way that is unexpected from a purely categorical perspective, but nevertheless models the topological group completion and requires less intense 2-categorical machinery to construct.

Outline. The rest of this paper is organized as follows. In Section 2 we provide 2-categorical background, define what it means for a 2-functor to be an opfibration, and establish Theorem 1.1 and Corollary 1.2. Section 3 contains the construction of a spectral sequence for an opfibration, and we analyze the local coefficients arising from the lax comma object and strict pullback. In Section 4 we give the necessary background on symmetric monoidal 2-categories and construct the 2-categorical version of $S^{-1}S$. We then prove part (*ii*) of Theorem 1.5 via a collapsing spectral sequence argument generalizing that of [Gray76].

2. Opfibrations and Quillen's Theorem B

This section has two primary goals: to introduce the 2-categorical notion of opfibration, and to study its homotopy-theoretic properties. We begin with 2-categorical background in Section 2.1 and then in Section 2.2 give the definition of opfibration. We define the lax and oplax comma objects for a cospan of 2-functors in Section 2.3. Section 2.4 is the technical heart of this paper; its main result is Theorem 2.34, which compares the strict pullback and lax comma object for an opfibration. We apply Theorem 2.34 at the end of Section 2.4 to prove Corollary 2.41: the strict fibers of an opfibration model the homotopy fibers on classifying spaces. This is Quillen's Theorem B for opfibrations. Looking ahead, we will also apply Theorem 2.34 in Section 3 to identify the E^2 page of the homology spectral sequence for an opfibration. That, in turn, is essential for our study of $S^{-1}S$ in Section 4.

2.1. 2-categorical definitions and conventions. To fix terminology and notation, we give the following overview of our conventions. We refer the reader to [Bén67, Lac10, JY21] for further background on 2-categories and bicategories.

- We write *2Cat* for the category of 2-categories and 2-functors.
- We write $[n]$ for the poset $0 \leq 1 \leq \cdots \leq n$ treated as a category or locally discrete 2-category.
- We write e for the unit object of a monoidal 2-category, use 1 for identity 1-cells and 2-cells, and omit subscripts on identities unless they add clarity.
- We usually denote the composition of 1-cells and vertical composition of 2-cells by juxtaposition, but use ∘ when necessary for either readability or clarity.
- We write the horizontal composition of appropriately composable 2-cells as $\alpha' * \alpha$. If either α or α' is an identity 2-cell we write

$$\alpha' * f = \alpha' * 1_f \text{ and } g * \alpha = 1_g * \alpha$$

for the whiskerings.

Definition 2.1. Let X, Y be 2-categories. A *lax functor* $F : X \to Y$ consists of

- a function on objects $F : \mathrm{ob}X \to \mathrm{ob}Y$,
- functors $X(x, x') \to Y(Fx, Fx')$ for all objects $x, x' \in X$,

- 2-cells $F_2 : Fg{\circ}Ff \Rightarrow F(g{\circ}f)$ for all composable pairs of 1-cells g and f, natural in both arguments, and
- 2-cells $F_0 : 1_{Fx} \Rightarrow F(1_x)$ for all objects $x \in X$.

These are subject to three axioms, one for composable triples of 1-cells and two for composing a 1-cell with an identity on either side.

An *oplax functor* $F : X \to Y$ consists of

- a function on objects $F : \mathrm{ob} X \to \mathrm{ob} Y$,
- functors $X(x, x') \to Y(Fx, Fx')$ for all objects $x, x' \in X$,
- 2-cells $F_2 : F(g{\circ}f) \Rightarrow Fg{\circ}Ff$ for all composable pairs of 1-cells g and f, natural in both arguments, and
- 2-cells $F_0 : F(1_x) \Rightarrow 1_{Fx}$ for all objects $x \in X$.

These are subject to three analogous axioms.

We say that a lax or oplax functor F is a *pseudofunctor* if the structure 2-cells F_2, F_0 are all isomorphisms. We say that a lax or oplax functor F is *normal* if the 2-cells F_0 are identities for all x. We note that for lax functors between 2-categories, the condition that F_0 is the identity implies that F_2 is also the identity when either f or g is an identity 1-cell, but should be required separately for a lax functor between bicategories.

Definition 2.2. Given 2-categories X and Y together with a pair of 2-functors $F, G : X \to Y$, a *lax transformation* $\alpha : F \Rightarrow G$, consists of

- 1-cells $\alpha_x : Fx \to Gx$ for all objects $x \in X$ and
- 2-cells $\alpha_f : Gf{\circ}\alpha_x \Rightarrow \alpha_y{\circ}Ff$ for all 1-cells $f : x \to y$ in X.

These are subject to axioms stating that α_f is natural in 2-cells $\delta : f \Rightarrow g$, that α_1 is the identity 2-cell, and that α_{gf} is the appropriate pasting of α_f and α_g.

A *pseudonatural transformation* α is a lax transformation such that α_f is an invertible 2-cell for all f. We say that α is *2-natural* if α_f is an identity 2-cell for all f. An *oplax transformation* $\alpha : F \Rightarrow G$, where $F, G : X \to Y$ are 2-functors, consists of

- 1-cells $\alpha_x : Fx \to Gx$ for all objects $x \in X$ and
- 2-cells $\alpha_f : \alpha_y{\circ}Ff \Rightarrow Gf{\circ}\alpha_x$

subject to analogous axioms.

Definition 2.3. A *modification* $\Gamma : \alpha \Rrightarrow \alpha'$, where α, α' are either both lax or oplax transformations with common source and target, consists of 2-cells $\Gamma_x : \alpha_x \Rightarrow \alpha'_x$ for all objects $x \in X$ subject to one axiom stating that the cells Γ_x, Γ_y are compatible with α_f, α'_f for all 1-cells $f : x \to y$.

Notation 2.4. Let X and Y be 2-categories.

- *i.* We write $[X, Y]$ to denote the 2-category of 2-functors, 2-natural transformations, and modifications from X to Y.
- *ii.* We write $\mathrm{Lax}(X, Y)$ to denote the 2-category of 2-functors, lax transformations and modifications from X to Y.
- *iii.* We write $\mathrm{Oplax}(X, Y)$ to denote the 2-category of 2-functors, oplax transformations and modifications from X to Y.

Definition 2.5. For a 2-category X we let NX denote the normal oplax nerve: its p-simplices are the normal oplax functors $[p] \to X$, with $[p]$ regarded as a locally discrete 2-category. We call NX the *nerve* of X, and its geometric realization $|NX|$ the *classifying space* of X.

This is one of several homotopy equivalent nerves for 2-categories described in [CCG10]. In Section 3 we will give an alternate equivalent description of NX using the oriented simplices of Street [Str87]. We make repeated use of the following result

Theorem 2.6 ([CCG10]). *The geometric realization of a lax or oplax natural transformation is a homotopy between the realizations of its source and target 2-functors.*

2.2. Opfibrations.

In this section we give the definition of an opfibration between 2-categories. Our definition is essentially the same as the notion of fibration between bicategories developed by Buckley [Buc14], but in the opfibrational form and in the special case that the bicategories and pseudofunctors involved are in fact 2-categories and 2-functors. Making those changes yields the following definitions.

Definition 2.7. Let $P : K \to L$ be a 2-functor. For 1-cells $g, h : x \to y$ in K, we say that a 2-cell $\gamma : g \Rightarrow h$ is *cartesian* with respect to P if it is a cartesian morphism in the category $K(x, y)$, i.e., for all 1-cells $k : x \to y$ in K, the following square is a pullback.

$$
\begin{array}{ccc}
K(x,y)(k,g) & \xrightarrow{\ P\ } & L(Px,Py)(Pk,Pg) \\
\downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle P\gamma_*} \\
K(x,y)(k,h) & \xrightarrow[\ P\ ]{} & L(Px,Py)(Pk,Ph)
\end{array}
$$

Explicitly, this means that for each $\psi : k \Rightarrow h$ and $\phi : Pk \Rightarrow Pg$ such that $P\psi = (P\gamma) \circ \phi$, there is a unique 2-cell $\phi^c : k \Rightarrow g$ such that $P\phi^c = \phi$ and $\gamma \circ \phi^c = \psi$.
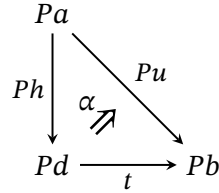
Definition 2.8. We say that a 2-functor $P : K \to L$ is a *local fibration* if the induced functor on hom-categories

$$
P : K(x,y) \to L(Px, Py)
$$

is a fibration for each $x, y \in K$. That is, each 2-cell $\alpha : f \Rightarrow P(g)$ in $L(Px, Py)$ has a cartesian lift $\widehat{\alpha} : \widehat{g} \Rightarrow g$ in $K(x, y)$, with $P\widehat{\alpha} = \alpha$.

Definition 2.9. We say that a 1-cell $h : a \to d$ in K is *opcartesian* with respect to P if the following two conditions hold.
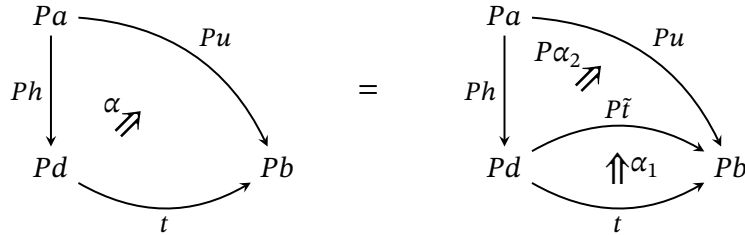
i. Suppose given a 1-cell $u : a \to b$ in K, a 1-cell $t : Pd \to Pb$, and an 2-cell isomorphism α as shown below.

$$
\begin{array}{ccc}
Pa & & \\
\Big\downarrow{\scriptstyle Ph} & \overset{\alpha}{\nearrow} & \searrow{\scriptstyle Pu} \\
Pd & \xrightarrow{\ t\ } & Pb
\end{array}
$$

Then there exists a 1-cell $\tilde{t} : d \to b$ in K and 2-cell isomorphisms

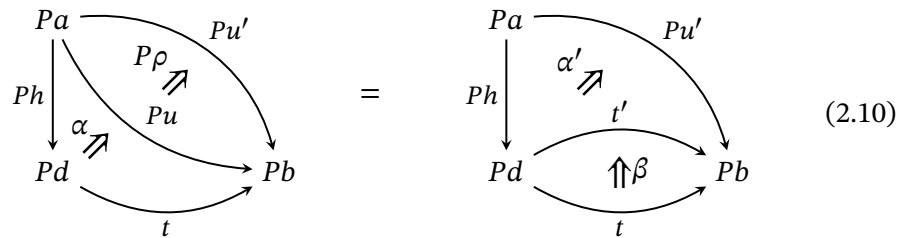$$\alpha_1 : t \cong P\tilde{t}, \quad \alpha_2 : \tilde{t}h \cong u$$

such that the following equality holds.

$$
\begin{array}{ccc}
Pa \xrightarrow{\ Pu\ } & & Pa \xrightarrow{\ Pu\ } \\
{\scriptstyle Ph}\big\downarrow\ \overset{\alpha}{\nearrow} & = & {\scriptstyle Ph}\big\downarrow\ \overset{P\alpha_2}{\nearrow}\ {\scriptstyle P\tilde{t}} \\
Pd \xrightarrow{\ t\ } Pb & & Pd\ \ \Uparrow\alpha_1\ \ Pb
\end{array}
$$

We say that $(\tilde{t}, \alpha_1, \alpha_2)$ is a *lift* of (u, t, α).

ii. Suppose given the following:
- a 2-cell $\beta : t \Rightarrow t'$;
- a pair of 1-cells $u, u' : a \to b$ in K with a 2-cell $\rho : u \Rightarrow u'$ between them; and
- 2-cell isomorphisms $\alpha : t \circ Ph \cong Pu$ and $\alpha' : t' \circ Ph \cong Pu'$, and lifts $(\tilde{t}, \alpha_1, \alpha_2), (\tilde{t}', \alpha'_1, \alpha'_2)$, of (u, t, α) and (u', t', α') respectively, such that the following equality holds.

$$
\begin{array}{ccccc}
Pa \xrightarrow{\ Pu'\ } & & & Pa \xrightarrow{\ Pu'\ } & \\
{\scriptstyle Ph}\big\downarrow\ \overset{P\rho}{\nearrow}\ \overset{Pu}{\overset{\alpha}{\nearrow}} & = & & {\scriptstyle Ph}\big\downarrow\ \overset{\alpha'}{\nearrow}\ {\scriptstyle t'} & \quad\text{(2.10)} \\
Pd \xrightarrow{\ t\ } Pb & & & Pd\ \ \Uparrow\beta\ \ Pb & \\
\end{array}
$$

Then there exists a unique 2-cell $\tilde{\beta} : \tilde{t} \Rightarrow \tilde{t}'$ such that the following two equalities hold

$$(2.11)$$

$$(2.12)$$

We give a characterization of opcartesian 1-cells via bipullbacks.

Lemma 2.13 ([Buc14, Proposition 3.1.2]). *A 1-cell $h : a \to d$ in K is opcartesian with respect to P if and only if the commutative square*

is a bipullback in $\mathcal{C}at$ for all objects $b \in K$.

Definitions 2.7 through 2.9 combine to give the following definition of opfibration for 2-functors.

Definition 2.14. A 2-functor $P : K \to L$ is an *opfibration* if the following conditions hold:

   i. 1-cells $f : P(x) \to z$ have opcartesian lifts $\widehat{f} : x \to \widehat{x}$, with $P\widehat{f} = f$,
  ii. P is a local fibration, and
 iii. horizontal composites of cartesian 2-cells are cartesian.

Remark 2.15. The previous definition mixes the variances of the lifting properties: 1-cells have opcartesian lifts, while 2-cells have cartesian ones. Any other mix of variances can be obtained by applying either op (reversing the direction of 1-cells) or co (reversing the direction of 2-cells) to both the source and target. We have chosen this set of variances with the particular application of Section 4.3 in mind.

2.3. Lax and oplax comma objects. In this section we define and study two comma object constructions that we call, respectively, the lax comma object $F{\downarrow}G$ (see Definition 2.16) and the oplax comma object $F{\downarrow}^{\mathrm{op}}G$ (see Definition 2.21). In each case, the input is a pair of 2-functors F, G and the output is a 2-category together with a universal lax, respectively oplax, transformation. These are special cases of a more general theory of lax limits developed by Gray [Gra80a, Gra80b].

Definition 2.16. Given a cospan of 2-functors

$$
\begin{array}{ccc}
 & & X \\
 & & \downarrow{\scriptstyle F} \\
Z & \xrightarrow{\ G\ } & Y
\end{array}
$$

we define the *lax comma object* $F{\downarrow}G$ as the following 2-category.

- An object of $F{\downarrow}G$ consists of a triple $[x, f, z]$ where x is an object of X, z is an object of Z, and $f : F(x) \to G(z)$ is a 1-cell in Y.
- A 1-cell from $[x, f, z]$ to $[x', f', z']$ is given by a triple $[s, \alpha, t]$ where $s : x \to x'$ is a 1-cell in X, $t : z \to z'$ is a 1-cell in Z, and α shown below is a 2-cell in Y.

$$
\begin{array}{ccc}
Fx & \xrightarrow{\ f\ } & Gz \\
{\scriptstyle Fs}\downarrow & \overset{\Swarrow}{\alpha} & \downarrow{\scriptstyle Gt} \\
Fx' & \xrightarrow{\ f'\ } & Gz'
\end{array}
$$

- A 2-cell from $[s, \alpha, t]$ to $[s', \alpha', t']$ is a pair $[\phi, \gamma]$ where $\phi : s \Rightarrow s'$ is a 2-cell in X and $\gamma : t \Rightarrow t'$ is a 2-cell in Z, subject to the following equality of pastings below.

$$
\begin{array}{ccc}
Fx & \xrightarrow{\ f\ } & Gz \\
{\scriptstyle Fs'}\left(\overset{F\phi}{\Leftarrow}\right){\scriptstyle Fs} \quad \overset{\Swarrow}{\alpha} & & \Big){\scriptstyle Gt} \\
Fx' & \xrightarrow{\ f'\ } & Gz'
\end{array}
\quad = \quad
\begin{array}{ccc}
Fx & \xrightarrow{\ f\ } & Gz \\
{\scriptstyle Fs'}\Big(\quad \overset{\Swarrow}{\alpha'} \quad {\scriptstyle Gt'}\left(\overset{G\gamma}{\Leftarrow}\right){\scriptstyle Gt} \\
Fx' & \xrightarrow{\ f'\ } & Gz'
\end{array}
\quad (2.17)
$$

Composition of 1-cells is given by composing the 1-cell components and vertically pasting the 2-cell components. Composition of 2-cells is given componentwise.

Remark 2.18. When X, Y and Z are considered as strict ∞-categories, the lax comma ∞-category of [AM20, §6] is in fact a 2-category, and coincides with the lax comma object just defined. Several of the results presented below can be seen as special cases of results in [AM20].

Proposition 2.19. *The lax comma object $F{\downarrow}G$ is equipped with projection 2-functors p_X and p_Z together with a lax natural transformation π as shown below.*

$$
\begin{array}{ccc}
F{\downarrow}G & \xrightarrow{\;p_X\;} & X \\
{\scriptstyle p_Z}\downarrow & \overset{\Swarrow}{\pi} & \downarrow{\scriptstyle F} \\
Z & \xrightarrow[G]{} & Y
\end{array}
$$

These data are universal in the following sense. Let K be a 2-category, R and Q be 2-functors, and λ be a lax natural transformation as shown below.

$$
\begin{array}{ccc}
K & \xrightarrow{\;R\;} & X \\
{\scriptstyle Q}\downarrow & \overset{\Swarrow}{\lambda} & \downarrow{\scriptstyle F} \\
Z & \xrightarrow[G]{} & Y
\end{array}
$$

Then there exists a unique 2-functor $h\colon K \to F{\downarrow}G$ such that $R = p_X{\circ}h$, $Q = p_Z{\circ}h$, and the following equality of pasting diagrams holds.

Proof. The 2-functor $p_X\colon F{\downarrow}G \to X$ is given by projection onto the first coordinate; the 2-functor $p_Z\colon F{\downarrow}G \to Z$ is given by projection to the final coordinate. The component of π at an object $[x, f, z]$ is f. The laxity of π at a 1-cell $[s, \alpha, t]$ is α.

Now we turn to the universal property. Given K, R, Q, and λ as in the statement, we define a 2-functor $h\colon K \to F{\downarrow}G$ as follows:

- $k \mapsto [Rk, \lambda_k, Qk]$;
- $(m\colon k \to k') \mapsto [Rm, \lambda_m, Qm]$; and
- $(\mu\colon m \Rightarrow m') \mapsto [R\mu, Q\mu]$.

Note that the first and last coordinate are determined by the compatibility with p_X and p_Z, respectively, while compatibility with π determines the middle coordinate for objects and 2-cells, proving uniqueness. $\qquad\square$

The 2-categorical dualities op (reversing 1-cells) and co (reversing 2-cells) provide the following isomorphism of lax comma objects.

Lemma 2.20. *For any 2-functors F, G, there is an isomorphism*

$$(G{\downarrow}F)^{\mathrm{coop}} \cong F^{\mathrm{coop}}{\downarrow}G^{\mathrm{coop}}$$

induced by the universal properties.

Proof. We use the identification $\mathrm{Lax}(D, E) = \mathrm{Lax}(D^{\mathrm{coop}}, E^{\mathrm{coop}})^{\mathrm{coop}}$. The result follows by applying coop to the square defining $G{\downarrow}F$. □

Definition 2.21. Given a cospan of 2-functors

$$\begin{array}{ccc} & & X \\ & & \downarrow F \\ Z & \xrightarrow{G} & Y \end{array}$$

we define the *oplax comma object $F{\downarrow}^{\mathrm{op}}G$* as follows.

- An object of $F{\downarrow}^{\mathrm{op}}G$ consists of a triple $[x, f, z]$ where x is an object of X, z is an object of Z, and $f : F(x) \to G(z)$ is a 1-cell in Y. In other words, the objects are the same as those of $F{\downarrow}G$.
- A 1-cell from $[x, f, z]$ to $[x', f', z']$ is given by a triple $[s, \alpha, t]$ where $s : x \to x'$ is a 1-cell in X, $t : z \to z'$ in Z, and $\alpha : f' {\circ} Fs \Rightarrow Gt {\circ} f$ is a 2-cell in Y. Note that the direction of the 2-cell is opposite to the one in $F{\downarrow}G$.
- A 2-cell from $[s, \alpha, t]$ to $[s', \alpha', t']$ is a pair $[\phi, \gamma]$ where $\phi : s \Rightarrow s'$ is a 2-cell in X and $\gamma : t \Rightarrow t'$ is a 2-cell in Z subject to the an equality of pasting diagrams, analogous to Display (2.17).

We need the explicit descriptions of both the lax and oplax comma objects, but we note that they are formally dual.

Lemma 2.22. *For any 2-functors F, G, there is an isomorphism*

$$(G{\downarrow}F)^{\mathrm{op}} \cong F^{\mathrm{op}}{\downarrow}^{\mathrm{op}}G^{\mathrm{op}}$$

induced by the universal properties.

Proof. We use the identification $\mathrm{Oplax}(D, E) = \mathrm{Lax}(D^{\mathrm{op}}, E^{\mathrm{op}})^{\mathrm{op}}$. The result follows by applying op to the square defining $G{\downarrow}F$. □

The following result is dual to Proposition 2.19 and left to the reader.

Proposition 2.23. *The oplax comma object $F{\downarrow}^{\mathrm{op}}G$ is equipped with projection 2-functors p_X and p_Z together with an oplax natural transformation π as shown below.*

$$\begin{array}{ccc} F{\downarrow}^{\mathrm{op}}G & \xrightarrow{p_X} & X \\ p_Z \downarrow & \overset{\pi}{\Leftarrow} & \downarrow F \\ Z & \xrightarrow{G} & Y \end{array}$$

These data are universal in the following sense. Let K be a 2-category, R and Q be 2-functors, and λ be an oplax natural transformation as shown below.

$$
\begin{array}{ccc}
K & \xrightarrow{\ R\ } & X \\
Q \downarrow & \overset{\lambda}{\Leftarrow} & \downarrow F \\
Z & \xrightarrow{\ G\ } & Y
\end{array}
$$

Then there exists a unique 2-functor $h : K \to F{\downarrow}^{\mathrm{op}}G$ such that $R = p_X \circ h$, $Q = p_Z \circ h$, and the following equality of pasting diagrams holds.

We need a few key results about the (op)lax comma objects.

Lemma 2.24. *For any 2-functor $G : E \to D$, the projection*

$$
p_E : 1_D {\downarrow} G \to E
$$

is a homotopy equivalence on classifying spaces.

Proof. The universal property of the lax comma object, applied to the commutative square

$$
\begin{array}{ccc}
E & \xrightarrow{\ G\ } & D \\
1 \downarrow & & \downarrow 1 \\
E & \xrightarrow{\ G\ } & D
\end{array}
$$

gives an inclusion $J : E \to 1_D {\downarrow} G$ such that $p_E \circ J = 1_E$. It sends an object $z \in E$ to $[Gz, 1, z]$ and a 1-cell $g : x \to x'$ to $[Gg, 1, g]$. It is straightforward to verify that there is a lax transformation $\mu : 1 \Rightarrow J \circ p_E$ with components

$$
\begin{aligned}
\mu_{[x,f,z]} &= [f, 1, 1], \\
\mu_{[s,\alpha,t]} &= [\alpha, 1].
\end{aligned}
$$

Upon taking classifying spaces, these 2-functors and transformations give the desired homotopy equivalence (see Theorem 2.6). $\qquad\square$

Notation 2.25. Given an object k of a 2-category K, we let $\widehat{k}$ denote the 2-functor $* \to K$ which sends the unique object to k.

Notation 2.26. Given a 2-functor $F : C \to D$, we let ΔF denote any of the diagonal 2-functors $C \to [E, D]$, resp. $C \to \text{Lax}(E, D)$, $C \to \text{Oplax}(E, D)$, given by $\Delta F(c)(e) = F(c)$ for all cells $c \in C$ and $e \in E$.

Definition 2.27. Let E be a 2-category, and let $\text{I} \in E$ be an object. Consider the 2-functors

where u is the unique map. We say that I is an *oplax initial object* if there exists an oplax natural transformation

$$h : \widehat{\text{I}} \circ u \Rightarrow 1_E$$

such that the whiskering $h * \widehat{\text{I}}$ is the identity at $\widehat{\text{I}}$. Note that the 2-functor $u \circ \widehat{\text{I}}$ and the whiskering $u * h$ are necessarily identities. We say that $\text{T} \in E$ is an *oplax terminal object* if it is oplax initial in E^{op}.

The data of an oplax initial object consists of components $h_j : \text{I} \to j$ for each object $j \in E$ together with 2-cells h_f for each 1-cell $f : i \to j$ in E as in the following square.

These must satisfy that $h_{\text{I}} = 1_{\text{I}}$ and $h_{1_{\text{I}}} = 1_{1_{\text{I}}}$.

Lemma 2.28. *Consider the following cospan.*

If E has an oplax initial object I, then the lax comma objects

have homotopy equivalent classifying spaces.

Proof. By the universal property of Proposition 2.19 we have unique 2-functors d and e indicated below.

In these diagrams, the unlabeled regions commute and we write ev_{I} for evaluation at the object I of E. The unlabeled 2-cells are the structure 2-cells of the lax comma objects $\Delta F \downarrow \widehat{G}$ and $F \downarrow \widehat{G(\mathrm{I})}$. The 2-cell labeled $(\star)$ is the 2-natural transformation whose component at the unique object of $*$ is the oplax transformation Gh.

Explicit formulas for d and e are given as follows. An object of $F \downarrow \widehat{G(\mathrm{I})}$ is a pair $[c, f_{\mathrm{I}}]$ where $c \in C$ and f_{I} is a 1-cell $F(c) \to G(\mathrm{I})$. Then applying G to h we have an oplax transformation whose component at $j \in E$ is given by

$$((Gh)_* f_{\mathrm{I}})_j = G(h_j) \circ f_{\mathrm{I}} \colon F(c) \to G(\mathrm{I}) \to G(j).$$

Thus we obtain

$$d[c, f_{\mathrm{I}}] = [c, (Gh)_* f_{\mathrm{I}}]$$

on objects, with similar formulas on higher cells. On the other hand, an object of $\Delta F \downarrow \widehat{G}$ is a pair $[c, f]$ where $c \in C$ and f is an oplax transformation $\Delta F(c) \to G$. We obtain

$$e[c, f] = [c, f_{\mathrm{I}}].$$

The reader can easily verify that $e \circ d$ is the identity because both $u \circ \widehat{\mathrm{I}}$ and $h * \widehat{\mathrm{I}}$ are identities. For the other composite, we have

$$de[c, f] = d[c, f_{\mathrm{I}}] = [c, G(h)_* f_{\mathrm{I}}].$$

The oplaxity of f with respect to the 1-cells $h_j : \mathrm{I} \to j$ provides a modification

$$\gamma_{c,f} : G(h)_* f_{\mathrm{I}} \Rightarrow f,$$

so that the 1-cell $[1_c, \gamma_{c,f}] : [c, G(h)_* f_{\mathrm{I}}] \to [c, f]$ is the component of a 2-natural transformation $1 \Rightarrow de$. The equality $ed = 1$ and the 2-natural transformation $1 \Rightarrow de$ provide a homotopy equivalence on classifying spaces as desired (see Theorem 2.6). $\square$

Remark 2.29. The above argument can be rephrased in terms of the universal property of the lax comma object, with the relatively short proof above replaced by a sequence of large pasting diagrams.

Combining Lemmas 2.24 and 2.28 proves the following corollary.

Corollary 2.30. *If E has an oplax initial object I then $\Delta 1_D \downarrow \widehat{G}$ is contractible for any $G : E \to D$.*

Applying Lemma 2.22 to Lemma 2.28 and Corollary 2.30 yields the following dual result.

Corollary 2.31. *If E has an oplax terminal object T, then*

$$\widehat{G} \downarrow^{\mathrm{op}} \Delta F \simeq \widehat{G(\mathrm{T})} \downarrow^{\mathrm{op}} F.$$

In particular, if $F = 1_D$ then $\widehat{G} \downarrow^{\mathrm{op}} \Delta 1_D$ is contractible for any $G : E \to D$.

2.4. Comparison of pullbacks and lax comma objects. This section is devoted to a comparison of the lax comma object with the pullback of 2-categories.

Notation 2.32. Given a cospan of 2-categories

$$A \xrightarrow{F} C \xleftarrow{G} B$$

we let $\mathrm{pb}(F, G)$ denote the pullback. For $k = 0, 1, 2$, a k-cell w of the pullback consists of a pair (w_1, w_2) of k-cells with $w_1 \in A, w_2 \in B$ such that $F(w_1) = G(w_2)$.

Remark 2.33. We often refer to the pullback of 2-categories as constructed above as the *strict 2-pullback* to emphasize that this is the strictest possible notion amongst all of the various types of pullbacks of 2-functors one could construct.

Theorem 2.34. *Let $P : K \to L$ be an opfibration and $F : J \to L$ be any 2-functor. Assume that all of the 2-cells of L are invertible. Then the inclusion*

$$i : \mathrm{pb}(P, F) \to P \downarrow F$$

induces a homotopy equivalence on classifying spaces.

Proof. We will define a 2-functor

$$H: P{\downarrow}F \to \mathrm{pb}(P,F),$$

such that Hi is the identity on $\mathrm{pb}(P,F)$, together with a pseudonatural transformation $\eta: 1 \Rightarrow iH$. This implies that i and H induce inverse homotopy equivalences on classifying spaces (see Theorem 2.6).

In order to define H, we note that P being an opfibration between 2-categories means that

(1) given a 1-cell $f: Px \to y$, we have an opcartesian lift $\widehat{f}: x \to \widehat{x}$; and
(2) given a 2-cell $\sigma: u \Rightarrow Pr$, we have a cartesian lift $\widehat{\sigma}: \widehat{r} \Rightarrow r$.

We now define H as an assignment on 0-, 1-, and 2-cells of $P{\downarrow}F$, using the notation of Definition 2.16.

- For an object $[x,f,z]$, apply (1) to $f: Px \to Fz$. This gives an opcartesian lift $\widehat{f}: x \to \widehat{x}$ such that $P\widehat{f} = f$ and hence $P\widehat{x} = Fz$. Therefore $(\widehat{x}, z)$ is an object of the pullback $\mathrm{pb}(P,F)$. Note that if f is an identity 1-cell, we take $\widehat{x} = x$ and $\widehat{f} = 1$. Define

$$H[x,f,z] = (\widehat{x},z).$$

- For a 1-cell $[s,\alpha,t]: [x,f,z] \to [x',f',z']$, recall that α is an invertible 2-cell $Ft{\circ}f \cong f'{\circ}Ps$. Since $f = P\widehat{f}$ and $f' = P\widehat{f}'$, we have that α is a 2-cell isomorphism as depicted below.

Thus, there exists a lift $(\widetilde{s}, \overline{\alpha}, \widetilde{\alpha})$ of $(\widehat{f}'s, Ft, \alpha)$; see Definition 2.9 ($i$). Explicitly, we have
  ∘ $\widetilde{s}: \widehat{x} \to \widehat{x}'$,
  ∘ $\overline{\alpha}: Ft \Rightarrow P\widetilde{s}$ invertible, and
  ∘ $\widetilde{\alpha}: \widetilde{s}\widehat{f} \Rightarrow \widehat{f}'s$ invertible
such that α is equal to the pasting of $\overline{\alpha}$ with $P\widetilde{\alpha}$. Next we apply (2) with $\sigma = \overline{\alpha}$ to get a cartesian lift $\widehat{\alpha}: \widehat{s} \Rightarrow \widetilde{s}$ of $\overline{\alpha}$ such that $P\widehat{\alpha} = \overline{\alpha}$ and hence $P\widehat{s} = Ft$. Note that $\widehat{\alpha}$ is invertible since $\overline{\alpha}$ is. Therefore $(\widehat{s}, t)$ is a 1-cell $(\widehat{x},z) \to (\widehat{x}',z')$ in the pullback $\mathrm{pb}(F,P)$.

We take specific lifts in two special cases. For an identity 1-cell

$$[1,1,1]: [x,f,z] \to [x,f,z],$$

we take $\widehat{s}$ to be the identity 1-cell on $\widehat{x}$. This can be done since $\widetilde{s} = 1_{\widehat{x}}$, $\overline{\alpha} = 1_{1_{P\widehat{x}}}$, and $\widetilde{\alpha} = 1_{\widehat{f}}$ is a valid lift for $(\widehat{f}, 1_{P\widehat{x}}, 1_{P\widehat{f}})$, and identity 2-cells are always cartesian.

For a 1-cell

$$[s, 1, t] \colon [x, 1, z] \to [x', 1, z'],$$

we take $\widehat{s} = \widetilde{s} = s$, with $\overline{\alpha}, \widetilde{\alpha}$ and $\widehat{\alpha}$ being the appropriate identity 2-cells. Define

$$H[s, \alpha, t] = (\widehat{s}, t).$$

- For a 2-cell $[\phi, \gamma] \colon [s, \alpha, t] \Rightarrow [s', \alpha', t']$, the condition given in Display (2.17) for $[\phi, \gamma]$ gives a version of Display (2.10). The data required for Definition 2.9 (ii) consists of
  - i. the opcartesian 1-cell h in the definition is $\widehat{f}$;
  - ii. the 2-cell $\beta \colon t \Rightarrow t'$ in the definition is $F\gamma \colon Ft \Rightarrow Ft'$;
  - iii. the 2-cell $\rho \colon u \Rightarrow u'$ is $\widehat{f}' * \phi \colon \widehat{f}'s \Rightarrow \widehat{f}'s'$;
  - iv. the isomorphism 2-cells α, α' are the 2-cells of the same names; and
  - v. the required lifts are $(\widetilde{s}, \overline{\alpha}, \widetilde{\alpha})$ and $(\widetilde{s}', \overline{\alpha}', \widetilde{\alpha}')$, respectively.

Thus we can apply Definition 2.9 to obtain a unique 2-cell of the form $\widetilde{\phi} \colon \widetilde{s} \Rightarrow \widetilde{s}'$ satisfying the appropriate versions of Displays (2.11) and (2.12). Using that $P\widehat{\alpha} = \overline{\alpha}$ and $P\widehat{\alpha}' = \overline{\alpha}'$, together with (2.11), we have the following equality:

$$P(\widetilde{\phi} \circ \widehat{\alpha}) = P\widetilde{\phi} \circ P\widehat{\alpha} = P\widetilde{\phi} \circ \overline{\alpha} = \overline{\alpha}' \circ F\gamma = P\widehat{\alpha}' \circ F\gamma. \qquad (2.35)$$

Since the 2-cell $\widehat{\alpha}'$ is cartesian, the pullback in Definition 2.7 applied to the pair $(\gamma, \widetilde{\phi})$ produces a unique 2-cell $\widehat{\phi} \colon \widehat{s} \Rightarrow \widehat{s}'$ such that both $P\widehat{\phi} = F\gamma$ and $\widehat{\alpha}'\widehat{\phi} = \widetilde{\phi}\widehat{\alpha}$. Therefore $(\widehat{\phi}, \gamma)$ is a 2-cell $(\widehat{s}, t) \Rightarrow (\widehat{s}', t')$ in $\mathrm{pb}(F, P)$. Define

$$H[\phi, \gamma] = (\widehat{\phi}, \gamma).$$

This completes the definition of H on cells. Note that our definition of H on 2-cells produces unique $\widetilde{\phi}$ and $\widehat{\phi}$. This uniqueness implies that H strictly preserves identity 2-cells and vertical composition of 2-cells.

We now construct the pseudofunctoriality constraints for H. We have specifically chosen the definition of H on 1-cells to strictly preserve identities. Given composable 1-cells

$$[s_1, \alpha_1, t_1] \colon [x, f, z] \to [x', f', z'] \quad \text{and} \quad [s_2, \alpha_2, t_2] \colon [x', f', z'] \to [x'', f'', z''],$$

we must construct a 2-cell isomorphism

$$[\widehat{s}_2, t_2] \circ [\widehat{s}_1, t_1] \cong [\widehat{s_2 \circ s_1}, t_2 \circ t_1].$$

The second coordinate will be the identity 2-cell, and we now produce a 2-cell isomorphism $\widehat{s}_2 \circ \widehat{s}_1 \cong \widehat{s_2 \circ s_1}$.

We have $\widetilde{s}_2, \widetilde{s}_1$, and $\widetilde{s_2 \circ s_1}$ constructed as above. Let α_{21} denote the pasting of α_2 with α_1. Then

$$(\widetilde{s_2 \circ s_1}, \overline{\alpha}_{21}, \widetilde{\alpha}_{21}) \quad \text{and} \quad (\widetilde{s}_2 \circ \widetilde{s}_1, \overline{\alpha}_2 * \overline{\alpha}_1, (\widetilde{\alpha}_2 * s_1) \circ (\widetilde{s}_2 * \widetilde{\alpha}_1))$$

are two lifts of $(\widehat{f}'' s_2 s_1, F(t_2 t_1), \alpha_{21})$. Thus, by Definition 2.9 (ii) with ρ and β identity 2-cells, there exists a unique isomorphism $\delta \colon \widetilde{s}_2 \circ \widetilde{s}_1 \cong \widetilde{s_2 \circ s_1}$ compatible

with the rest of the data. We define the pseudofunctoriality constraint to have first coordinate

$$\widehat{s}_2 \circ \widehat{s}_1 \xLongrightarrow{\widehat{\alpha}_2 * \widehat{\alpha}_1} \widetilde{s}_2 \circ \widetilde{s}_1 \xLongrightarrow{\delta} \widetilde{s_2 \circ s_1} \xLongrightarrow{(\widehat{\alpha}_{21})^{-1}} \widehat{s_2 \circ s_1}.$$

Applying P to this composite gives the identity, so we have constructed a 2-cell in $\mathrm{pb}(P, F)$ as desired. The pseudofunctor axioms follow from the uniqueness of the 2-dimensional lifts. This concludes the construction of H as a pseudofunctor $P{\downarrow}F \to \mathrm{pb}(P, F)$.

We next consider the composite

$$\mathrm{pb}(P, F) \xhookrightarrow{i} P{\downarrow}F \xrightarrow{H} \mathrm{pb}(P, F).$$

On objects and 1-cells, this composite is the identity by construction. For 2-cells, one checks that if $P\phi = F\gamma$ then the definition of H on $[\phi, \gamma]$ gives the equalities $\phi = \widetilde{\phi} = \widehat{\phi}$ by uniqueness. The pseudofunctoriality constraints are constructed using the unique 2-dimensional lifts, which in this case are all identities, so the composite above is the identity 2-functor.

Now consider

$$P{\downarrow}F \xrightarrow{H} \mathrm{pb}(P, F) \xhookrightarrow{i} P{\downarrow}F.$$

On objects, this maps $[x, f, z]$ to $[\widehat{x}, 1, z]$, so we define the component of a pseudonatural transformation $\eta : 1 \Rightarrow iH$ to be the 1-cell $[\widehat{f}, 1_f, 1_z]$. For a 1-cell $[s, \alpha, t]$, the pseudonaturality constraint is given by

$$[\widetilde{\alpha} \circ (\widehat{\alpha} * \widehat{f}), 1_t] : [\widehat{s}\widehat{f}, 1_{Ft \circ f}, t] \cong [\widehat{f}'s, \alpha, t].$$

One can check that this is indeed a 2-cell in $P{\downarrow}F$. The pseudonaturality axiom reduces to the equality

$$\widetilde{\alpha}_{21} \circ (\delta * \widehat{f}) = (\widetilde{\alpha}_2 * s_1) \circ (\widetilde{s}_2 * \widetilde{\alpha}_1),$$

which is precisely the compatibility (2.12) for δ. $\square$

Remark 2.36. Our proof of Theorem 2.34 only makes use of invertibility for α and α'. Therefore the same argument also proves, without the assumption of invertible 2-cells in L, that there is a homotopy equivalence between the classifying spaces of the pseudo-comma object (defined via a pseudonatural transformation in the square, but with a universal property still up to isomorphism) and the pullback.

Remark 2.37. In the case of 1-categories, one has an adjunction between the pullback and comma object. The construction we have given in the proof of Theorem 2.34 can be extended to a biadjunction, but we leave that to the interested reader.

Now we apply Theorem 2.34 to give a version of Quillen's Theorem B for opfibrations of 2-categories. We use the following terms.

Definition 2.38. Suppose $F : C \to D$ is a 2-functor and $\phi : x \to y$ is a 1-cell in D.

   i. Let $F{\downarrow}\widehat{x}$, respectively $F{\downarrow}\widehat{y}$, denote the lax comma object for the cospan given by F and x, respectively y, regarded as a 2-functor $* \to D$.

   ii. Regard ϕ as a 2-natural transformation of 2-functors $* \to D$. Let

$$\phi_* : F{\downarrow}\widehat{x} \to F{\downarrow}\widehat{y}$$

denote the 2-functor given by the universal property described in Proposition 2.19 with λ given by the following pasting.

We call $\phi_* : F{\downarrow}\widehat{x} \to F{\downarrow}\widehat{y}$ the *base change* 2-functor induced by ϕ.

Theorem 2.39 ([Ceg11]). *Suppose $F : C \to D$ is a 2-functor such that for every 1-cell $\phi : x \to y$ in D the base change 2-functor $\phi_* : F{\downarrow}\widehat{x} \to F{\downarrow}\widehat{y}$ is a homotopy equivalence on classifying spaces. Then for each $x \in D$ and each $\dot{x} \in F^{-1}(x)$, the sequence*

$$F{\downarrow}\widehat{x} \to C \to D$$

induces a homotopy fiber sequence on classifying spaces with respect to the basepoints $[\dot{x}, 1] \in BF{\downarrow}\widehat{x}$, $\dot{x} \in BC$, and $x \in BD$.

Remark 2.40. The result in [Ceg11] is given and proved for $\widehat{x}{\downarrow}F$, with the base change maps going in the opposite direction. The version given here follows from the original, together with Lemma 2.20 and the fact that given a 2-category C, there is a natural homotopy equivalence $BC \simeq B(C^{\mathrm{coop}})$ (see [CCG10]).

   If F is an opfibration and the 2-cells in its codomain are invertible, then Theorem 2.34 shows that we can replace $F{\downarrow}\widehat{x}$ with $F^{-1}(x)$ to obtain the following result.

Corollary 2.41. *Suppose that all 2-cells in D are invertible, and $F : C \to D$ is an opfibration such that, for every 1-cell $\phi : x \to y$ in D the the base change 2-functor $\phi_* : F{\downarrow}\widehat{x} \to F{\downarrow}\widehat{y}$ is a homotopy equivalence on classifying spaces. Then for each $x \in D$ and each $\dot{x} \in F^{-1}(x)$, the sequence*

$$F^{-1}(x) \to C \to D$$

induces a homotopy fiber sequence on classifying spaces with respect to the basepoints $\dot{x} \in BF^{-1}(x)$, $\dot{x} \in BC$, and $x \in BD$.
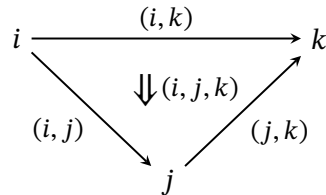
3. Homology spectral sequence for an opfibration

   In this section we let $F : C \to D$ be a 2-functor and construct an associated homology spectral sequence. This begins with a detailed description of the normal oplax nerve ND (Definition 2.5) in terms of Street's oriented simplices. Then in Section 3.2 we define a bisimplicial set associated to F. In Section 3.3

we analyze the spectral sequence associated to this bisimplicial set and give an identification of its E^2 page in the case that F is an opfibration (Lemma 3.22).
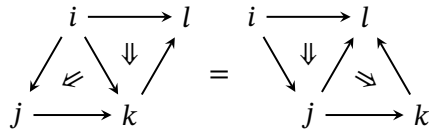
3.1. Oriented simplices. We give the 2-truncated version of Street's orientals from [Str87]. In *loc. cit.*, Street defines the nth oriental to be the "free strict n-category generated by an n-simplex," and thus our orientals are obtained from his by forcing all k-cells to be identities for $k > 2$.

Definition 3.1 ([Str87]). For $p \geq 0$, the *oriental* $\mathcal{O}(p)$ is the 2-category defined as follows.

- The objects of $\mathcal{O}(p)$ are the natural numbers $0, \ldots, p$.
- The nonidentity 1-cells of $\mathcal{O}(p)$ are generated by pairs $(i, j) \colon i \to j$ for each $i < j$.
- The nonidentity 2-cells of $\mathcal{O}(p)$ are generated by triples (i, j, k) for each $i < j < k$, with source and target as indicated below.

$$
\begin{array}{ccc}
 & (i,k) & \\
i & \xrightarrow{\hspace{2cm}} & k \\
 & \Downarrow (i,j,k) & \\
(i,j) & & (j,k) \\
 & j & \\
\end{array}
$$

The 2-cells are subject to the condition that each quadruple $i < j < k < l$ yields a commuting tetrahedron; that is, an equality of pastings as below.

$$
\begin{array}{ccc}
i \longrightarrow l & & i \longrightarrow l \\
\diagup \Downarrow \diagup & = & \diagdown \Downarrow \diagdown \\
j \longrightarrow k & & j \longrightarrow k \\
\end{array}
$$

Remark 3.2. Two observations may help the reader parse these tetrahedra: First, the boundary of the two halves consists of the "short spine" $i \to l$ along the top as the source, and the "long spine" $i \to j \to k \to l$ along the bottom 3 edges as the target. Second, each 2-cell has a single arrow as its source, and a composite of two arrows as its target. We use this oplax direction in virtually all of the discussion below.

The following two propositions can be read off from [Gur09] following [Str82, Str87].

Proposition 3.3. *For $p \geq 0$, normal oplax functors $[p] \to D$ are in bijective correspondence with strict 2-functors $\mathcal{O}(p) \to D$.*

Proposition 3.4. *The assignment $[p] \mapsto \mathcal{O}(p)$ is the function on objects of a functor $\mathcal{O}(-) \colon \Delta \to 2\mathcal{C}at$. This cosimplicial 2-category has an associated nerve functor $N \colon 2\mathcal{C}at \to s\mathcal{S}et$ which is the normal oplax nerve (Definition 2.5).*

Remark 3.5. Note that the oriental $\mathcal{O}(p)$ has oplax initial object 0 and oplax terminal object p (Definition 2.27).

Unpacking the definition of $\mathrm{Oplax}(\mathscr{O}(p), D)$, we have the following explicit description of its 1- and 2-cells. For $\sigma, \sigma' : \mathscr{O}(p) \to D$, a 1-cell $f : \sigma \to \sigma'$ is an oplax transformation. It consists of components $f_i : \sigma_i \to \sigma'_i$ for each i together with 2-cells f_{ij} for each $i < j$ as shown below.

$$
\begin{array}{ccc}
\sigma_i & \xrightarrow{\ \sigma_{(i,j)}\ } & \sigma_j \\
f_i \downarrow & \overset{\ \ }{\underset{f_{ij}}{\Leftarrow}} & \downarrow f_j \\
\sigma'_i & \xrightarrow[\ \sigma'_{(i,j)}\ ]{} & \sigma'_j
\end{array}
$$

These data are subject to the axiom that a triple $i < j < k$ yields the following equality of pasting diagrams, where the unlabeled 2-cells are given by $\sigma(i, j, k)$ and $\sigma'(i, j, k)$, respectively.

A 2-cell $m : f \Rightarrow f'$ between 1-cells $f, f' : \sigma \to \sigma'$ is a modification. This data consists of 2-cells $m_i : f_i \Rightarrow f'_i$ for each i, subject to the axiom that the following "soup-can" pastings are equal.

3.2. Constructing a bisimplicial set. Given a 2-functor $F : C \to D$, we construct a bisimplicial set and give two alternate descriptions of it in terms of nerves of certain (op)lax comma objects. This bisimplicial set is also considered (with the vertical and horizontal directions reversed) by Bullejos and Cegarra in their proof of Quillen's theorem A for 2-categories (see [BC03, §4.1]).

Definition 3.6. Let $F : C \to D$ be a 2-functor. Define a bisimplicial set B as follows. The set $B_{p,q}$ is the set of triples (ω, δ, σ) of strict 2-functors which make

the diagram below commute.

$$
\begin{array}{ccc}
\mathscr{O}(q) & \xrightarrow{\ \omega\ } & C \\
& \searrow & \downarrow F \\
& \mathscr{O}(q+1+p) & \\
& \nearrow \quad \searrow\ \delta & \\
\mathscr{O}(p) & \xrightarrow[\ \sigma\ ]{} & D
\end{array}
\tag{3.7}
$$

The map $\mathscr{O}(q) \to \mathscr{O}(q+1+p)$ is induced by the inclusion $[q] \to [q+1+p]$ in Δ that sends each $i \in [q]$ to itself, while the map $\mathscr{O}(p) \to \mathscr{O}(q+1+p)$ is induced by the inclusion $[p] \to [q+1+p]$ that sends each $j \in [p]$ to $q+1+j$. The faces and degeneracies are given below, where H denotes the horizontal direction (fix q, vary p) and V denotes the vertical direction (fix p, vary q).

$$
\begin{aligned}
d_i^H(\omega, \delta, \sigma) &= (\omega, d_{q+1+i}\delta, d_i\sigma) \\
s_i^H(\omega, \delta, \sigma) &= (\omega, s_{q+1+i}\delta, s_i\sigma) \\
d_i^V(\omega, \delta, \sigma) &= (d_i\omega, d_i\delta, \sigma) \\
s_i^V(\omega, \delta, \sigma) &= (s_i\omega, s_i\delta, \sigma)
\end{aligned}
$$

Unpacking Definition 3.6, the data of a strict 2-functor δ as above consists of the following.

- The values of δ on the sub-oriental with vertices $0 < \cdots < q$ are given by $F \circ \omega$.
- The values of δ on the sub-oriental with vertices $q+1 < \cdots < q+1+p$ are given by σ.
- For $i \in [q]$ and $j \in [p]$, $\delta_{(i,q+1+j)}$ is a 1-cell $F(\omega_i) \to \sigma_j$.
- For $i \in [q]$ and $j < j' \in [p]$, $\delta_{(i,q+1+j,q+1+j')}$ is a 2-cell as below.

$$
\begin{array}{ccc}
& F(\omega_i) & \\
{\scriptstyle\delta_{(i,q+1+j)}}\swarrow & \Swarrow & \searrow{\scriptstyle\delta_{(i,q+1+j')}} \\
\sigma_j & \xrightarrow[\ \sigma_{(j,j')}\ ]{} & \sigma_{j'}
\end{array}
$$

- For $i < i' \in [q]$ and $j \in [p]$, $\delta_{(i,i',q+1+j)}$ is a 2-cell as below.

$$
\begin{array}{ccc}
& F(\omega_i) & \\
{\scriptstyle F(\omega_{(i,i')})}\swarrow & \Swarrow & \searrow{\scriptstyle\delta_{(i,q+1+j)}} \\
F(\omega_{i'}) & \xrightarrow[\ \delta_{(i',q+1+j)}\ ]{} & \sigma_j
\end{array}
$$

- These 2-cells fit into commuting tetrahedra, as in the equalities of pasting diagrams below.

– For $i \in [q]$ and $j < j' < j'' \in [p]$,

$$
\begin{array}{ccc}
F(\omega_i) \longrightarrow \sigma_{j''} & & F(\omega_i) \longrightarrow \sigma_{j''} \\
\swarrow \Leftarrow \searrow \Downarrow \nearrow & = & \searrow \Downarrow \nearrow \searrow \\
\sigma_j \longrightarrow \sigma_{j'} & & \sigma_j \longrightarrow \sigma_{j'}
\end{array}
$$

– For $i < i' \in [q]$ and $j < j' \in [p]$,

$$
\begin{array}{ccc}
F(\omega_i) \longrightarrow \sigma_{j'} & & F(\omega_i) \longrightarrow \sigma_{j'} \\
\swarrow \Leftarrow \searrow \Downarrow \nearrow & = & \searrow \Downarrow \nearrow \searrow \\
F(\omega_{i'}) \longrightarrow \sigma_j & & F(\omega_{i'}) \longrightarrow \sigma_j
\end{array}
$$

– For $i < i' < i'' \in [q]$ and $j \in [p]$,

$$
\begin{array}{ccc}
F(\omega_i) \longrightarrow \sigma_j & & F(\omega_i) \longrightarrow \sigma_j \\
\swarrow \Leftarrow \searrow \Downarrow \nearrow & = & \searrow \Downarrow \nearrow \searrow \\
F(\omega_{i'}) \rightarrow F(\omega_{i''}) & & F(\omega_{i'}) \rightarrow F(\omega_{i''})
\end{array}
$$

The data of δ can be interpreted in two ways: as a lax transformation whose components are oplax transformations, or as an oplax transformation whose components are lax transformations, as the following two results explain. Recall from Notation 2.25 we let $\widehat{k} : \ast \to K$ denote the constant functor at an object k of a 2-category K.

Lemma 3.8. *Fix* $\sigma \in N_p D$ *and consider the lax comma object* $\Delta F \downarrow \widehat{\sigma}$ *in the square below.*

$$
\begin{array}{ccc}
\Delta F \downarrow \widehat{\sigma} & \longrightarrow & C \\
\downarrow & \Swarrow & \downarrow \Delta F \\
\ast & \xrightarrow{\widehat{\sigma}} & \mathrm{Oplax}(\mathscr{O}(p), D)
\end{array}
$$

Then the set of pairs (ω, δ) *such that* $(\omega, \delta, \sigma) \in B_{p,q}$ *is in bijective correspondence with* $N_q(\Delta F \downarrow \widehat{\sigma})$, *the set of q-simplices in the nerve of* $\Delta F \downarrow \widehat{\sigma}$. *Under this bijection, $d_i^V(-, -, \sigma)$ corresponds to d_i, and similarly for degeneracies.*

Proof. A q-simplex in $N(\Delta F \downarrow \widehat{\sigma})$ is given by a 2-functor

$$
\mathscr{O}(q) \to \Delta F \downarrow \widehat{\sigma}.
$$

By Proposition 2.19, such 2-functors are in bijection with pairs (ω, λ), where ω is a 2-functor and λ is a lax transformation λ as shown below.

$$
\begin{array}{ccc}
\mathcal{O}(q) & \xrightarrow{\ \omega\ } & C \\
\downarrow & \overset{\nwarrow\!\!\nwarrow}{\lambda} & \downarrow{\scriptstyle \Delta F} \\
* & \xrightarrow[\ \sigma\ ]{} & \mathrm{Oplax}(\mathcal{O}(p), D)
\end{array}
$$

The lax transformation λ consists of the following data:

- For each $i \in \mathcal{O}(q)$, an oplax transformation $\lambda_i : \Delta F(\omega_i) \to \sigma$. That is, for each $j \in \mathcal{O}(p)$, a 1-cell

$$\lambda_{i,j} : (\Delta F(\omega_i))(j) = F(\omega_i) \to \sigma(j) = \sigma_j$$

and for $j < j'$, a 2-cell $\lambda_{i,(j,j')}$ as below.

$$
\begin{array}{ccc}
 & F(\omega_i) & \\
{\scriptstyle \lambda_{i,j}}\swarrow & \overset{\nwarrow\!\!\nwarrow}{} & \searrow{\scriptstyle \lambda_{i,j'}} \\
 & \lambda_{i,(j,j')} & \\
\sigma_j & \xrightarrow[\ \sigma_{(j,j')}\ ]{} & \sigma_{j'}
\end{array}
$$

The condition on λ_i being an oplax transformation means for all $j < j' < j'' \in [p]$ the following equality of pasting diagrams holds.

$$
\begin{array}{ccc}
F(\omega_i) \longrightarrow \sigma_{j''} & & F(\omega_i) \longrightarrow \sigma_{j''} \\
\diagup \nwarrow\!\!\nwarrow \Big\downarrow\!\!\Downarrow \diagup & = & \diagdown \Big\downarrow\!\!\Downarrow \diagup \searrow\!\!\searrow \\
\sigma_j \longrightarrow \sigma_{j'} & & \sigma_j \longrightarrow \sigma_{j'}
\end{array}
$$

- For each $i < i'$ in $\mathcal{O}(q)$, a modification $\lambda_{(i,i')} : \lambda_i \Rightarrow \lambda_{i'} \circ \Delta F(\omega_{(i,i')})$. That is, for each $j \in \mathcal{O}(p)$, a 2-cell $\lambda_{(i,i'),j}$ as below.

$$
\begin{array}{ccc}
 & F(\omega_i) & \\
{\scriptstyle F(\omega_{(i,i')})}\swarrow & \overset{\nwarrow\!\!\nwarrow}{} & \searrow{\scriptstyle \lambda_{i,j}} \\
 & \lambda_{(i,i'),j} & \\
F(\omega_{i'}) & \xrightarrow[\ \lambda_{(i',j)}\ ]{} & \sigma_j
\end{array}
$$

To say that $\lambda_{(i,i')}$ is a modification means that for all $j, j' \in [p]$, we have the equality of pastings as below.

$$
\begin{array}{ccc}
F(\omega_i) \longrightarrow \sigma_{j'} & & F(\omega_i) \longrightarrow \sigma_{j'} \\
\diagup \nwarrow\!\!\nwarrow \Big\downarrow\!\!\Downarrow \diagup & = & \diagdown \Big\downarrow\!\!\Downarrow \diagup \searrow\!\!\searrow \\
F(\omega_{i'}) \longrightarrow \sigma_j & & F(\omega_{i'}) \longrightarrow \sigma_j
\end{array}
$$

The condition on λ being a lax transformation means there is a compatibility of the modifications $\lambda_{(i,i')}$ with composition in $\mathscr{O}(q)$: for all $i < i' < i'' \in [q]$ and $j \in [p]$, we have the following equality of pasting diagrams.

$$
\begin{array}{ccc}
F(\omega_i) \longrightarrow \sigma_j & & F(\omega_i) \longrightarrow \sigma_j \\
\swarrow \ \ {}_{\nwarrow}\searrow \ {}^{\Downarrow}\nearrow & = & \searrow \ {}^{\Downarrow}\nearrow \ {}_{\searrow} \\
F(\omega_{i'}) \rightarrow F(\omega_{i''}) & & F(\omega_{i'}) \rightarrow F(\omega_{i''})
\end{array}
$$

With this description, one can verify at once that defining

- $\delta_{(i,q+1+j)} = \lambda_{i,j}$,
- $\delta_{(i,q+1+j,q+1+j')} = \lambda_{i,(j,j')}$, and
- $\delta_{(i,i',q+1+j)} = \lambda_{(i,i'),j}$

gives precisely the data of a 2-functor $\delta \colon \mathscr{O}(q+p+1) \to D$ in Display (3.7) as described above. Verifying the formulas for faces and degeneracies is straightforward. $\qquad\square$

Lemma 3.9. *Fix $\omega \in N_q C$ and consider the oplax comma object in the square below.*

$$
\begin{array}{ccc}
(\widehat{F \circ \omega}) \!\downarrow^{\mathrm{op}}\! \Delta 1_D & \longrightarrow & * \\
\downarrow & {}_{\swarrow\!\!\!\!\swarrow} & \downarrow {}^{\widehat{F \circ \omega}} \\
D & \xrightarrow[\Delta 1_D]{} & \mathrm{Lax}(\mathscr{O}(q), D)
\end{array}
$$

Then the set of pairs (δ, σ) such that $(\omega, \delta, \sigma) \in B_{p,q}$ is in bijective correspondence with $N_p((\widehat{F \circ \omega})\!\downarrow^{\mathrm{op}}\!\Delta 1_D)$, the set of p-simplices in the nerve of $(\widehat{F \circ \omega})\!\downarrow^{\mathrm{op}}\!\Delta 1_D$. Under this bijection, $d_i^H(\omega, -, -)$ corresponds to d_i, and similarly for degeneracies.

3.3. Analysis of the homology spectral sequence. We now describe the spectral sequence in homology associated to a 2-functor $F \colon C \to D$, arising from the simplicial set B of Definition 3.6. In Theorem 3.25 we show that if F is an opfibration then the E^2 page is given by the homology of D with local coefficients in the homology of the fibers of F. We apply this machinery in Section 4.3 to prove Theorem 1.5. We begin with a discussion of local coefficients.

Definition 3.10. Let N be a simplicial set. Its *category of elements* $\int N$ has objects given by pairs $([p], x)$ where $x \in N_p$. A morphism $([p], x) \to ([q], y)$ consists of $\phi \colon [q] \to [p]$ in Δ such that $\phi^*(x) = y$.

Definition 3.11. A *local coefficient system* on a simplicial set N is a functor

$$ F \colon \int N \to \mathscr{A}b. $$

We say that F is *morphism-inverting* if it sends all morphisms to isomorphisms. A *local coefficient system* on a topological space X is a functor

$$ \Pi_1 X \to \mathscr{A}b $$

where $\Pi_1 X$ is the fundamental groupoid of X.

Remark 3.12. A local coefficient system on a simplicial set N does not necessarily induce a local coefficient system on its geometric realization $|N|$. However, since the fundamental groupoid of $|N|$ is equivalent to the free groupoid on $\int N$, local coefficient systems on $|N|$ correspond to morphism-inverting local coefficient systems on N. See [GJ10, Section VI.4]. This implies the following result.

Proposition 3.13. *If $F : \int N \to \mathcal{A}b$ is morphism-inverting, then $H_*(N\,;\,F)$ is isomorphic to $H_*(|N|\,;\,F)$, the homology of the space $|N|$ with local coefficients given by the functor $\Pi_1(|N|) \to \mathcal{A}b$ induced by F.*

Definition 3.14. Let D be a 2-category, and $F : \int(ND) \to \mathcal{A}b$ a local coefficient system on ND. Then $H_*(D\,;\,F)$ denotes the homology of the simplicial set ND with local coefficients in F. If F is omitted, we implicitly take the constant coefficient system at the integers $\mathbb{Z}$.

Remark 3.15. The standard definition of a local coefficient system on a 1-category C is a functor $F : C \to \mathcal{A}b$. The theory of covering spaces and covering groupoids implies that there is a 1-1 correspondence between morphism-inverting local coefficient systems on C in this sense and local coefficient systems on the classifying space BC in the sense of Definition 3.11. This is discussed in [Wei13, Definition IV.3.5.1] and [Qui73, Section 1]. Proofs that the homology of the corresponding chain complexes are naturally isomorphic appear in [Qui73, Section 1] and [Whi78, Theorem VI.4.8]. Therefore, in the case that D is a 1-category, our Definition 3.14 agrees with the standard definition when the coefficients are morphism-inverting.

Proposition 3.16. *Let $F : C \to D$ be a 2-functor and $q \geq 0$. The assignment*

$$\sigma \mapsto H_q(\Delta F {\downarrow} \widehat{\sigma}\,)$$

defines a local coefficient system $\mathcal{H}_q(\Delta F {\downarrow}\, \frown)$ on the simplicial set ND.

Proof. Since $\Delta F {\downarrow} \widehat{\sigma}$ is a lax comma object, the assignment $\sigma \mapsto H_q(\Delta F {\downarrow} \widehat{\sigma})$ is a functor on the category of elements of ND (Definition 3.10). Indeed, a morphism $([p], \sigma) \to ([p'], \sigma')$ in $\int ND$, given by $\phi : [p'] \to [p]$ in Δ with $\sigma' = \phi^* \sigma$, yields a lax transformation

$$
\begin{array}{ccccc}
\Delta F {\downarrow} \widehat{\sigma} & \longrightarrow & C & =\!\!=\!\!= & C \\
\downarrow & \overset{\Swarrow}{} & \downarrow & & \downarrow \\
* & \underset{\widehat{\sigma}}{\longrightarrow} & \mathrm{Oplax}(\mathcal{O}(p), D) & \underset{\phi^*}{\longrightarrow} & \mathrm{Oplax}(\mathcal{O}(p'), D)
\end{array}
$$

and hence, by the universal property of $\Delta F {\downarrow} (\widehat{\phi^* \sigma})$, we get a 2-functor

$$\phi^* : \Delta F {\downarrow} \widehat{\sigma} \to \Delta F {\downarrow} (\widehat{\phi^* \sigma}).$$

This assignment is clearly functorial. $\qquad\square$

Given a bisimplicial set we have a double complex constructed by taking free abelian groups on simplices, and alternating sums of face maps. We then have two spectral sequences (vertical and horizontal) associated to that double complex. Let E^0 denote the double complex associated to the bisimplicial set B of Definition 3.6.

Theorem 3.17. *The vertical spectral sequence associated to the double complex E^0 has*

$$E^2_{p,q} = H_p\big(D \, ; \, \mathscr{H}_q(\Delta F \!\downarrow \frown)\big) \tag{3.18}$$

and abuts to $H_{p+q}(C)$.

Proof. First consider homology of E^0 in the horizontal direction (fix q, vary p). By Lemma 3.9, taking homology and summing over ω yields

$$\bigoplus_{\omega \in N_q(C)} H_p\big((\widehat{F \circ \omega})\!\downarrow^{\mathrm{op}} \Delta 1_D\big).$$

By Corollary 2.31, for fixed ω, $H_p\big((\widehat{F \circ \omega})\!\downarrow^{\mathrm{op}} \Delta 1_D\big) = 0$ for $p > 0$ and is isomorphic to $\mathbb{Z}$ when $p = 0$. Therefore, the horizontal spectral sequence of the double complex collapses, and the homology of the total complex $\mathrm{Tot}\, E^0$ is given by

$$H_q\big(\mathrm{Tot}\, E^0\big) \cong H_q(C).$$

Next consider homology of E^0 in the vertical direction (fix p, vary q). By Lemma 3.8, taking homology and summing over σ yields

$$E^1_{p,q} = \bigoplus_{\sigma \in N_p(D)} H_q(\Delta F \!\downarrow \widehat{\sigma}\,). \tag{3.19}$$

Taking homology of $E^1_{p,q}$ now yields

$$E^2_{p,q} = H_p\big(D \, ; \, \mathscr{H}_q(\Delta F \!\downarrow \frown)\big). \tag{3.20}$$

Hence we have a spectral sequence

$$E^2_{p,q} = H_p\big(D \, ; \, \mathscr{H}_q(\Delta F \!\downarrow \frown)\big) \Rightarrow H_{p+q}\big(\mathrm{Tot}\, E^0\big) \cong H_{p+q}(C). \tag{3.21}$$

$\square$

If F is an opfibration, the following result gives a further simplification of the E^2 page.

Lemma 3.22. *If $F : C \to D$ is an opfibration and all 2-cells of D are invertible, then*

$$H_q(\Delta F \!\downarrow \widehat{\sigma}\,) \cong H_q\Big(F^{-1}\big(\sigma(0)\big)\Big)$$

for all $\sigma \in N_p D$.

Proof. Since $\mathscr{O}(q)$ has oplax initial object 0, Lemma 2.28 gives the first isomorphism below. Applying Theorem 2.34 gives the second.

$$H_q(\Delta F \!\downarrow \widehat{\sigma}\,) \cong H_q\big(F \!\downarrow \sigma(0)\big) \cong H_q\Big(F^{-1}\big(\sigma(0)\big)\Big). \qquad \square$$

Notation 3.23. If $F : C \to D$ is an opfibration and all 2-cells of D are invertible, we denote by $\mathscr{H}_q F^{-1}$ the local coefficient system given on objects by

$$\sigma \mapsto H_q\Big(F^{-1}\big(\sigma(0)\big)\Big)$$

and given on morphisms $\phi : ([p], \sigma) \to ([p'], \phi^*\sigma)$ by composing the morphisms

$$H_q\Big(\Delta F \!\downarrow\! \widehat{\sigma}\Big) \to H_q\Big(\Delta F \!\downarrow\! \widehat{\phi^*\sigma}\Big)$$

of Proposition 3.16 with the isomorphisms of Lemma 3.22. Explicitly, this means that $\mathscr{H}_q F^{-1}(\phi)$ is given by taking nerves and applying H_q to the composite below, using the morphisms i and H of Theorem 2.34, and d and e from Lemma 2.28

$$
\begin{array}{ccccccc}
F^{-1}(\sigma(0)) & \xrightarrow{\ i\ } & F\!\downarrow\!\sigma(0) & \xrightarrow{\ d\ } & \Delta F\!\downarrow\!\widehat{\sigma} & \xrightarrow{\ \phi^*\ } & \Delta F\!\downarrow\!\widehat{\phi^*\sigma} \\
& & & \xrightarrow{\ e\ } & F\!\downarrow\!\phi^*\sigma(0) & \xrightarrow{\ H\ } & F^{-1}(\phi^*\sigma(0)).
\end{array}
$$

Remark 3.24. Tracing through the constructions, one can check that the composite

$$F\!\downarrow\!\sigma(0) \xrightarrow{\ d\ } \Delta F\!\downarrow\!\widehat{\sigma} \xrightarrow{\ \phi^*\ } \Delta F\!\downarrow\!\widehat{\phi^*\sigma} \xrightarrow{\ e\ } F\!\downarrow\!\phi^*\sigma(0)$$

is precisely the base change induced by the 1-cell $\sigma_{(0,\phi(0))} : \sigma(0) \to \phi^*\sigma(0)$.

Combining Theorem 3.17 with Lemma 3.22 we have the following.

Theorem 3.25. *If $F : C \to D$ is an opfibration and all 2-cells of D are invertible, then there is a spectral sequence*

$$E^2_{p,q} = H_p\big(D\,;\,\mathscr{H}_q F^{-1}\big) \tag{3.26}$$

that abuts to $H_{p+q}(C)$.

4. Construction of $S^{-1}S$

In this section we construct a 2-categorical model for group completion that generalizes Quillen's $S^{-1}S$ [Gray76]. Here S is assumed to be both a 2-groupoid and a permutative Gray monoid with faithful translations; these terms are defined in Section 4.1. In Section 4.2 we define $S^{-1}X$ when S acts on a 2-category X. In Section 4.3 we prove, under the additional hypothesis that the 2-cells of X are invertible, that the map

$$\rho : S^{-1}X \to S^{-1} *$$

is an opfibration (Proposition 4.35). We then use the homology spectral sequence for ρ (Theorem 3.25) to show that it induces localization by $\pi_0 S$ on homology (Theorem 4.48). In the case $X = S$, this proves Theorem 1.5 (*ii*): $S^{-1}S$ models the group-completion of S.

4.1. Background on permutative Gray monoids. In this section we recall from [GJO17, GJO19] the notion of permutative Gray monoid, a semi-strict type of symmetric monoidal bicategory. We refer the reader to [Gra74, Gur13] for further background on the Gray tensor product and to [GJO17, GJOS17] for further background on permutative Gray monoids. Just as permutative categories provide a strict model for symmetric monoidal categories, permutative Gray monoids provide a strict model for symmetric monoidal bicategories. We sketch the relevant definitions and then state the strictification result.

Definition 4.1. Let X and Y be 2-categories. The *Gray tensor product* of X and Y, written $X \otimes Y$, is the 2-category given by

- 0-cells consisting of pairs $x \otimes y$ with x an object of X and y an object of Y;
- 1-cells generated under composition by two kinds of basic 1-cells denoted $f \otimes 1 : x \otimes y \to x' \otimes y$ for $f : x \to x'$ in X and $1 \otimes g : x \otimes y \to x \otimes y'$ for $g : y \to y'$ in Y; and
- 2-cells generated by basic 2-cells of the form $\alpha \otimes 1$ for 2-cells α in X; $1 \otimes \delta$ for 2-cells δ in Y; and new interchange isomorphism 2-cells $\Sigma_{f,g} : (f \otimes 1)(1 \otimes g) \cong (1 \otimes g)(f \otimes 1)$.

These cells satisfy axioms related to composition, naturality and bilinearity; for a complete list, see [Gur13, Section 3.1] or [GJO17, Definition 3.16].

Remark 4.2. The definition given above is sometimes called the *pseudo* Gray tensor product. The definition given by Gray [Gra74] does not require that the 2-cells $\Sigma_{f,g}$ be isomorphisms. Our definition follows that of [GPS95], where the pseudo version, defined here, is shown to be a monoidal product for *2Cat* and is essential to the coherence theory of tricategories. See [GPS95, Gur13] for further details.

Theorem 4.3 ([GPS95, Section 4.8], [Gur13, Theorem 3.16]). *The assignment*

$$(X, Y) \mapsto X \otimes Y$$

extends to a functor of categories

$$\textit{2Cat} \times \textit{2Cat} \longrightarrow \textit{2Cat}$$

which defines a symmetric monoidal structure on 2Cat. The unit for this monoidal structure is the terminal 2-category.

Definition 4.4. A *Gray monoid* is a monoid object in $(\textit{2Cat}, \otimes)$. This consists of a 2-category S, a 2-functor

$$\oplus : S \otimes S \to S,$$

and an object e of S satisfying associativity and unit axioms.

Definition 4.5. A *permutative Gray monoid* S consists of a Gray monoid $(S, \oplus, e)$ together with a 2-natural isomorphism,

$$
\begin{array}{ccc}
S \otimes S & \xrightarrow{\ \tau\ } & S \otimes S \\
& \nearrow \beta & \\
\oplus \searrow & S & \swarrow \oplus
\end{array}
$$

where $\tau : S \otimes S \to S \otimes S$ is the symmetry isomorphism in *2Cat* for the Gray tensor product, such that the following axioms hold.

- The following pasting diagram is equal to the identity 2-natural transformation for the 2-functor $\oplus$.

$$
\begin{array}{ccccc}
& & 1 & & \\
& \nearrow & & \searrow & \\
S \otimes S & \xrightarrow{\ \tau\ } & S \otimes S & \xrightarrow{\ \tau\ } & S \otimes S \\
& \nearrow \beta & \downarrow & \Rightarrow \beta & \\
\oplus \searrow & & \oplus & & \swarrow \oplus \\
& & S & &
\end{array}
$$

- The following equality of pasting diagrams holds where we have abbreviated the tensor product to concatenation when labeling 1- or 2-cells.

Remark 4.6. If S is a permutative Gray monoid, we abuse notation and let $\Sigma_{f,g}$ denote the image under $\oplus$ of the Gray structure 2-cell $\Sigma_{f,g}$. The hexagon axiom and the 2-naturality of β together imply that $\Sigma_{f,g}$ is an identity 2-cell in S whenever f or g is a component of β [GJO17, Proposition 3.42].

Theorem 4.7 ([SP11, Theorem 2.97], [GJO17, Theorem 3.14]). *Every symmetric monoidal bicategory is equivalent, via a symmetric monoidal pseudofunctor, to a permutative Gray monoid.*

Definition 4.8. Let $(S, \oplus, e, \beta)$ and $(S', \oplus', e', \beta')$ be permutative Gray monoids. A *strict functor* is a 2-functor $F : S \to S'$ of the underlying 2-categories satisfying the following conditions.

- $F(e) = e'$, so that F strictly preserves the unit object.

- The diagram

$$\begin{array}{ccc} S \otimes S & \xrightarrow{\ F \otimes F\ } & S' \otimes S' \\ {\scriptstyle\oplus}\Big\downarrow & & \Big\downarrow{\scriptstyle\oplus'} \\ S & \xrightarrow{\ F\ } & S' \end{array}$$

    commutes, so that F strictly preserves the sum.
- The equation

$$\beta' * (F \otimes F) = F * \beta$$

    holds, so that F strictly preserves the symmetry. This equation is equivalent to requiring that

$$\beta'_{Fx,Fy} = F(\beta_{x,y})$$

    as 1-cells from $Fx \oplus' Fy = F(x \oplus y)$ to $Fy \oplus' Fx = F(y \oplus x)$.

Notation 4.9. The category of permutative Gray monoids, $\mathcal{PGM}$, has objects permutative Gray monoids and morphisms the strict functors between them.

Definition 4.10. Let $(S, \oplus, e, \beta)$ be a permutative Gray monoid, and let X be a 2-category. An *action* of S on X consists of a 2-functor $\mu : S \otimes X \to X$ such that

    i. $\mu(e, -)$ is the identity 2-functor on X,

    ii. $\mu \circ (\oplus \otimes 1_X) = \mu \circ (1_S \otimes \mu)$ as 2-functors $S \otimes S \otimes X \to X$, and

    iii. for any 1-cell f in X, and $\beta : s \oplus t \to t \oplus s$ in S, the image under μ of the 2-cell $\Sigma_{\beta,f} \in S \otimes X$ is an identity 2-cell in X.

Remark 4.11. Let $\mu : S \otimes X \to X$ be a 2-functor. Given objects in s, t in S and x in X, there is a 1-cell $\mu(\beta_{s,t} \otimes 1) : \mu((s \oplus t) \otimes x) \to \mu((t \oplus s) \otimes x)$. As x varies over the objects of X, these are the components of a pseudonatural isomorphism $\mu((s \oplus t) \otimes -) \Rightarrow \mu((t \oplus s) \otimes -)$. Condition ((*iii*)) of Definition 4.10 is equivalent to requiring that these components form a 2-natural transformation.

Notation 4.12. For an action $\mu : S \otimes X \to X$, we often write the image $\mu(s \otimes x)$ as merely sx, and similarly for higher cells. We also use juxtaposition for the action of S on itself via $\oplus$. Generalizing Remark 4.6, we denote by Σ the image under μ of any 2-cell Σ in $S \otimes X$.

We now turn to the notion of invertible cells in a permutative Gray monoid.

Definition 4.13. Let $(S, \oplus, e)$ be a Gray monoid.

    i. A 2-cell of S is invertible if it has an inverse in the usual sense.

    ii. A 1-cell $f : x \to y$ is invertible if there exists a 1-cell $g : y \to x$ together with invertible 2-cells $g \circ f \cong 1_x$, $f \circ g \cong 1_y$. In other words, f is invertible if it is an internal equivalence in S.

    iii. An object x of S is invertible if there exists another object y together with invertible 1-cells $x \oplus y \to e$, $y \oplus x \to e$.

A 2-category satisfying the first and second condition for all 1- and 2-cells is a *2-groupoid*, and a Gray monoid (or more generally, a monoidal bicategory) satisfying the third condition for all objects is called *grouplike*.

Definition 4.14. A *Picard 2-category* is a grouplike symmetric monoidal 2-groupoid. We say that a Picard 2-category is *strict* if it is a permutative Gray monoid.

We also have a notion of group-completion for monoid-like structures on spaces.

Definition 4.15. Let X be a homotopy commutative, homotopy associative H-space. A *group completion* of X is an H-space Y, together with an H-space map $f : X \to Y$, such that

- $\pi_0(f)$ exhibits $\pi_0(Y)$ as the group completion of the abelian monoid $\pi_0(X)$; and
- for all commutative rings k, the induced map on homology
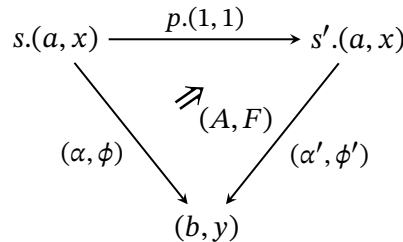
$$H_* f : H_*(X; k) \to H_*(Y; k)$$

   exhibits $H_*(Y; k)$ as the localization $\pi_0(X)^{-1} H_*(X; k)$.

Definition 4.16. We say that a functor of symmetric monoidal categories or 2-categories is a *homotopy group completion* if it is a group completion, as in Definition 4.15, on classifying spaces.

4.2. Definition of $S^{-1}X$. Let S be a permutative Gray monoid, and suppose that S acts on a 2-category X, with action denoted by juxtaposition. There is an induced diagonal action of S on $S \times X$, and we denote this with a lower dot as in $s.(a, x) = (sa, sx)$ for $s \in S$ and $(a, x) \in S \times X$.

Definition 4.17. We describe the 0-, 1-, 2-cells of $S^{-1}X$ as follows.

- An object of $S^{-1}X$ consists of a pair $(a, x) \in S \times X$.
- A 1-cell $(a, x) \to (b, y)$ is given by a triple $(s, (\alpha, \phi))$ where $s \in S$ and (α, ϕ) is a morphism in $S \times X$ from $s.(a, x)$ to (b, y).
- A 2-cell from $(s, (\alpha, \phi))$ to $(s', (\alpha', \phi'))$ is given by an equivalence class $\langle p, (A, F) \rangle$ where $p : s \to s'$ is a 1-cell in S and (A, F) is a 2-cell in $S \times X$ as below.

$$
\begin{array}{ccc}
s.(a, x) & \xrightarrow{\quad p.(1, 1) \quad} & s'.(a, x) \\
& \searrow^{\nearrow (A, F)} \swarrow & \\
(\alpha, \phi) & & (\alpha', \phi') \\
& (b, y) &
\end{array}
$$

Two equivalence classes $\langle p, (A, F) \rangle$ and $\langle q, (B, G) \rangle$ are equal if there is a 2-cell isomorphism $\Theta : p \cong q$ in S such that we have the following equality of pastings in $S \times X$; the unmarked 2-cells are given by (A, F)

and (B, G), respectively.

$$
\begin{array}{ccc}
& q.(1,1) & \\
s.(a,x) \quad \Uparrow\Theta.(1,1) \quad s'.(a,x) & & s.(a,x) \qquad\qquad s'.(a,x) \\
p.(1,1) & = & q.1 \\
(\alpha,\phi) \quad \nearrow \quad (\alpha',\phi') & & (\alpha,\phi) \quad \nearrow \quad (\alpha',\phi') \\
(b,y) & & (b,y)
\end{array}
\tag{4.18}
$$

Proposition 4.19. *The data of $S^{-1}X$ given in Definition 4.17 forms a 2-category. This construction is functorial with respect to strict functors of permutative Gray monoids on the first variable and maps that preserve the action strictly on the second variable. There is a 2-functor $i: X \to S^{-1}X$ given by $i(x) = (e, x)$ on objects; it is natural on both variables.*

Proof. We define composition of 1-cells

$$(s, (\alpha, \phi)): (a, x) \to (b, y) \quad \text{and} \quad (t, (\gamma, \psi)): (b, y) \to (c, z)$$

by the formula

$$(t, (\gamma, \psi)) \circ (s, (\alpha, \phi)) = (ts, (\gamma \circ t\alpha, \psi \circ t\phi)).$$

Vertical composition of 2-cells is given by pasting their defining triangles in $S \times X$. Horizontal composition of 2-cells

$$
\begin{array}{ccc}
(s,(\alpha,\phi)) & & (t,(\gamma,\psi)) \\
(a,x) \quad \Downarrow \langle p, (A,F)\rangle \quad (b,y) & & \Downarrow \langle q, (B,G)\rangle \quad (c,z) \\
(s',(\alpha',\phi')) & & (t',(\gamma',\psi'))
\end{array}
$$

is given by $\langle r, (C, H) \rangle = \langle r', (C', H') \rangle$ where

$$
\begin{aligned}
r &= qs' \circ tp \\
C &= (\gamma' * \Sigma_{q,\alpha'} * tpa) \circ (B * tA) \\
H &= (\psi' * \Sigma_{q,\phi'} * tpx) \circ (G * tF) \\
r' &= t'p \circ qs \\
C' &= (\gamma' * t'A * qsa) \circ (\gamma' * \Sigma_{q,\alpha}) \circ (B * t\alpha) \\
H' &= (\psi' * t'F * qsx) \circ (\psi' * \Sigma_{q,\phi}) \circ (G * t\phi).
\end{aligned}
$$

Verification of the axioms is routine. The statements about functoriality and naturality of the constructions follow directly from the definitions. $\square$

The special case where $X = *$, the terminal 2-category with the unique action, will be useful in later sections.

Lemma 4.20. *If the 2-cells of S are invertible, then the topological space $|NS^{-1} * |$ is contractible.*

Proof. We will produce a lax transformation from the constant 2-functor $S^{-1} *$ $\to S^{-1} *$ at e to the identity functor on $S^{-1} *$. The result then follows by Theorem 2.6.

The objects of $S^{-1} *$ can be identified with the objects of S and we omit the coordinate for the cells appearing in $*$. The 1-cells of $S^{-1} *$ are given by pairs $(s, \alpha) : a \to b$ where $s \in S$ and $\alpha : sa \to b$. The 2-cells of $S^{-1} *$ are given by equivalence classes $\langle p, A \rangle$ where $A : \alpha \Rightarrow \alpha' \circ (p \oplus 1_a)$. Two equivalence classes $\langle p, A \rangle$ and $\langle q, B \rangle$ are equal if there is a 2-cell $\Theta : p \cong q$ in S such that $(1_{\alpha'} * (\Theta \oplus 1)) \circ A = B$.

For each $a \in S$ there is a canonical 1-cell $(a, 1_a) : e \to a$ in $S^{-1} *$ and for each other 1-cell $(s, \alpha) : e \to a$ in $S^{-1} *$ there is a canonical 2-cell $\langle \alpha, 1_\alpha \rangle : (s, \alpha) \Rightarrow (a, 1)$. If the 2-cells in S are invertible, then this 2-cell is unique since, for any other such 2-cell $\langle q, B \rangle$, we have $B : \alpha \Rightarrow q$ invertible by hypothesis, and therefore taking $\Theta = B$ gives $\langle \alpha, 1_\alpha \rangle = \langle q, B \rangle$. Thus $(a, 1)$ is terminal in each hom-category $(S^{-1} *)(e, a)$. The 1-cells $(a, 1)$ therefore assemble to a lax transformation from the constant functor $S^{-1} * \to S^{-1} *$ at e to the identity functor on $S^{-1} *$. $\qquad\square$

Next we discuss the action of S on $S^{-1} X$.

Lemma 4.21. *The action $S \otimes X \to X$ induces an action of S on $S^{-1} X$ which we write as*

$$\xi : S \otimes S^{-1} X \to S^{-1} X.$$

Proof. We define $\xi : S \otimes S^{-1} X \to S^{-1} X$ on cells as follows; we remind the reader that the symmetry for S is written $\beta_{p,q} : pq \cong qp$.

$$
\begin{array}{rrcl}
\text{0-cells:} & s \otimes (a, x) & \mapsto & (a, sx) \\
\text{1-cells:} & s \otimes (t, (\alpha, \phi)) & \mapsto & (t, (\alpha, s\phi \circ \beta_{t,s} x)) \\
 & f \otimes (a, x) & \mapsto & (e, (1_a, fx)) \\
\text{2-cells:} & s \otimes \langle p, (A, F) \rangle & \mapsto & \langle p, (A, sF * \beta_{t,s} x) \rangle \\
 & \gamma \otimes (a, x) & \mapsto & \langle 1_e, (1, \gamma x) \rangle \\
 & \Sigma_{f, (t(\alpha, \phi))} & \mapsto & \langle 1_e, (1, \beta_{t,s} x * \Sigma_{f, \phi}) \rangle
\end{array}
$$

All the axioms (2-functoriality of ξ plus those in Definition 4.10) are all straightforward consequences of the Gray tensor product, permutative Gray monoid, and action axioms for X. Note in particular that condition (*iii*) of Definition 4.10 is essential for the 2-functoriality of ξ. $\qquad\square$

Definition 4.22. We say that S *acts homotopy invertibly* on X or that *the action is homotopy invertible* if each 2-functor $\mu(s, -) : X \to X$ induces a homotopy equivalence on the classifying space, $|NX|$ (see Definition 2.5).

Remark 4.23. Since classifying spaces are CW-complexes, an action is homotopy invertible if and only each $\mu(s, -)$ induces a weak homotopy equivalence.

Proposition 4.24. *The action in Lemma 4.21 is homotopy invertible.*

Proof. We construct a 2-functor $s^{-1} : S^{-1}X \to S^{-1}X$ which serves as a homotopy inverse to $\xi(s, -)$. The 2-functor s^{-1} is defined on cells below.

$$
\begin{array}{lrcl}
\text{0-cells:} & (a, x) & \mapsto & (sa, x) \\
\text{1-cells:} & (t, (\alpha, \phi)) & \mapsto & (t, (s\alpha \circ \beta_{t,s} a, \phi)) \\
\text{2-cells:} & \langle p, (A, F) \rangle & \mapsto & \langle p, (sA * \beta_{t,s} a, F) \rangle
\end{array}
$$

As in Lemma 4.21, the 2-functor axioms are simple to check.

Now we show that s^{-1} and $\xi(s, -)$ are homotopy inverses to each other. First note that it is a simple matter of applying the definitions on cells in Lemma 4.21 and above to show that $\xi(s, -) \circ s^{-1} = s^{-1} \circ \xi(s, -)$ as 2-functors. In particular, we only need to define a single transformation between this composite and the identity to produce a homotopy exhibiting these 2-functors as homotopy inverse to each other (see Theorem 2.6). A pseudonatural transformation $T : 1 \Rightarrow \xi(s, -) \circ s^{-1}$ is given by the data below.

Component for 0-cells:

$$
T_{(a,x)} = (s, (1, 1)) : (a, x) \to (sa, sx)
$$

Pseudonaturality for 1-cells:

$$
T_{(t,(\alpha,\phi))} = \langle \beta_{t,s}, (1, 1) \rangle : (ts, (s\alpha \circ \beta_{t,s} a, s\phi \circ \beta_{t,s} x)) \Rightarrow (st, (s\alpha, s\phi)) \qquad \square
$$

Remark 4.25. The composite $\xi(s, -) \circ s^{-1}$ in the proof of Proposition 4.24 is the action of S on $S^{-1}X$ induced by the diagonal action of S on $S \times X$.

There are two conditions we can study which are weaker than all objects of a permutative Gray monoid being invertible.

Definition 4.26. Let S be a permutative Gray monoid.

  i. We say an object s is *homotopy invertible* if the 2-functor $s \oplus - : S \to S$ induces a homotopy equivalence on the classifying space.
  ii. If every object is homotopy invertible, we say that S is *homotopy grouplike*. This is equivalent to the action of S on itself being homotopy invertible.

Definition 4.27. Let S be a permutative Gray monoid. We say that S has *faithful translations* if, for each $s, x, y \in S$, the functor

$$
s \oplus - : S(x, y) \to S(sx, sy)
$$

is faithful. Explicitly, this means that if $\alpha, \beta : f \Rightarrow g$ are parallel 2-cells such that $s\alpha = s\beta$, then $\alpha = \beta$.

Theorem 4.28. *Let S be a permutative Gray monoid.*

  i. *$S^{-1}S$ is a permutative Gray monoid.*
  ii. *The 2-functor $i : S \to S^{-1}S$ of Proposition 4.19 is a strict functor of permutative Gray monoids (Definition 4.8).*
  iii. *$S^{-1}S$ is homotopy grouplike.*

Proof. We begin by defining a 2-functor $S^{-1}S \otimes S^{-1}S \to S^{-1}S$ which we also write as $\oplus$ using infix notation. An object of $S^{-1}S \otimes S^{-1}S$ is a pair $((a, x), (a', x'))$, and we define

$$(a, x) \oplus (a', x') = (aa', xx').$$

Given $(s, (\alpha, \phi)) : (a, x) \to (b, y)$, we define

$$(s, (\alpha, \phi)) \oplus (a', x') : (a, x) \oplus (a', x') \to (b, y) \oplus (a', x')$$

to be the 1-cell $(s, (\alpha a', \phi x'))$. The 1-cell $(a', x') \oplus (s, (\alpha, \phi))$ is defined to be

$$(s, (a'\alpha \circ \beta_{s,a'} a, x'\phi \circ \beta_{s,x'} x)).$$

The 2-cells in $S^{-1}S \otimes S^{-1}S$ come in three varieties: $\Gamma \otimes 1$ and $1 \otimes \Gamma$ for a 2-cell Γ in $S^{-1}S$, and the invertible 2-cells Σ indexed by pairs of 1-cells in $S^{-1}S$. For a 2-cell $\langle p, (A, F) \rangle$, we define

$$\langle p, (A, F) \rangle \oplus (a', x') = \langle p, (Aa', Fx') \rangle.$$

We define

$$(a', x') \oplus \langle p, (A, F) \rangle = \langle p, (a'A * \beta_{s,a'} a, x'F * \beta_{s,x'} x) \rangle.$$

Let $(s, (\alpha, \phi)) : (a, x) \to (b, y)$ and $(s', (\alpha', \phi')) : (a', x') \to (b', y')$ be a pair of 1-cells in $S^{-1}S$. The image of Σ indexed by this pair is

$$\langle \beta_{s,s'}, (\Sigma_{\alpha,\alpha'}^{-1} * s\beta a', \Sigma_{\phi,\phi'}^{-1} * s\beta x') \rangle.$$

With the definitions in place, we leave the routine verifications to the reader, noting only that all the axioms follow from the corresponding axioms for S and the Gray tensor product axioms. The symmetry isomorphism

$$\beta : (a, x) \oplus (a', x') \cong (a', x') \oplus (a, x)$$

is defined to be $(e, (\beta_{a,a'}, \beta_{x,x'}))$.

The second claim follows by straightforward application of the definitions of the 2-functor i and the symmetric monoidal structure on $S^{-1}S$.

For the third part, to show that $(a, x) \oplus -$ is a homotopy equivalence upon taking nerves, we need only note that $(x, a) \oplus -$ is an inverse up to homotopy. Indeed, there is a pseudonatural transformation

$$\theta : 1 \Rightarrow (x, a) \oplus (a, x) \oplus -$$

with component

$$\theta_{(b,y)} = (xa, (1, \beta_{x,a} y)) : (b, y) \to (xab, axy)$$

for an object (b, y) and pseudofunctoriality constraint

$$\theta_{(t, (\alpha, \phi))} = \langle \beta_{t, xa}, (1, 1) \rangle$$

for a 1-cell $(t, (\alpha, \phi)) : (b, y) \to (b', y')$. $\square$

Remark 4.29. In the argument for Theorem 4.28 (*iii*), the component of θ at (e, e) is a morphism

$$\theta_{e,e} = (xa, (1, \beta_{x,a})) \colon (e, e) \to (xa, ax)$$

that we might call $\eta_{a,x}$. In the case that S is a permutative 1-category, Thomason [Tho80] has noted that the $\eta_{a,x}$ are not natural with respect to morphisms $(a, x) \to (a', x')$ unless the symmetry of S is trivial. A similar statement is true here, with the added caveat that the assignment $(a, x) \mapsto (xa, ax)$ is only a pseudofunctor, and the maps η cannot be made the components of any kind of transformation.

We end this section by comparing our construction with Quillen's classical construction for 1-categories [Gray76]. For a category C, we write dC for the locally discrete 2-category obtained by adding identity 2-cells. We note that for a permutative category C, dC is a permutative Gray monoid.

Proposition 4.30. *Let S be a permutative category and X be a category equipped with a strict action of S. Then there is a bijective on objects biequivalence*

$$(dS)^{-1}(dX) \to d(S^{-1}X)$$

where the source is the 2-categorical construction of Definition 4.17 applied to dS and dX and the target is the locally discrete 2-category obtained from Quillen's 1-categorical $S^{-1}X$. This map is compatible with the action of dS.

Proof. The 2-cells of $(dS)^{-1}(dX)$ are not necessarily all identities, but for any pair of 1-cells in $(dS)^{-1}(dX)$ there is at most one 2-cell between them, and one exists if and only if the corresponding 1-cells in Quillen's $S^{-1}X$ are in the same equivalence class. Taking the quotient of the 2-category $(dS)^{-1}(dX)$ by identifying isomorphic 1-cells yields a category which is readily seen to be Quillen's $S^{-1}X$. By the uniqueness of 2-cells between a given pair of parallel 1-cells, this quotienting process is then the bijective on objects biequivalence we desire. $\square$

4.3. The canonical projection is an opfibration. In this section we prove that the canonical projection

$$\rho \colon S^{-1}X \to S^{-1} * .$$

induced by the unique 2-functor $X \to *$ is an opfibration. As in the proof of Lemma 4.20 we omit the final coordinate (that of the terminal category) when describing cells in $S^{-1} *$. The following result will be useful for our applications in Theorem 4.48.

Lemma 4.31. *A 2-cell $\langle p, (A, F) \rangle$ is an isomorphism in $S^{-1}X$ if and only if the 1-cell p is an equivalence in S, the 2-cell A is an isomorphism in S, and the 2-cell F is an isomorphism in X.*

Proof. The identity 2-cell in $S^{-1}X$ is $\langle 1, (1,1)\rangle$, and $\langle p, (A, F)\rangle = \langle 1, (1,1)\rangle$ if there exists a 2-cell isomorphism $\Theta\colon p \cong 1$ satisfying the equality in Display (4.18). This implies that A, F are invertible 2-cells. The claim in the statement of the lemma follows by considering when a composite

$$\langle p', (A', F')\rangle \circ \langle p, (A, F)\rangle$$

is equal to the identity.   □

We provide the following lemma to help the reader identify the data of a lift (in the sense of Definition 2.9 (i)) for the special case of the 2-functor

$$\rho\colon S^{-1}X \to S^{-1} * .$$

This will aid in the proof of Proposition 4.35.

Lemma 4.32. *Let $(s, \alpha, \varphi)\colon (a, x) \to (d, z)$ and $(u, \gamma, \chi)\colon (a, x) \to (b, y)$ be 1-cells in $S^{-1}X$ and let $\langle p, A\rangle\colon (t, \beta)\circ(s, \alpha) \cong (u, \gamma)$ be a 2-cell isomorphism in $S^{-1} *$. A 1-cell $(v, \delta, \lambda) \in S^{-1}X$ together with 2-cells $\langle p_1, A_1\rangle \in S^{-1} *$ and $\langle p_2, A_2, F_2\rangle \in S^{-1}X$ give a lift of the triple $\big((u, \gamma, \chi), (t, \beta), \langle p, A\rangle\big)$ if and only if there is $\Theta\colon p \cong p_2 \circ (p_1 s)$ in S such that the following equality of pasting diagrams holds in S.*

$$(4.33)$$

Definition 4.34. In the context of Lemma 4.32, we say that Θ is the *witness* of this lift.

Proposition 4.35. *Let S be a permutative Gray monoid acting on a 2-category X. Assume that*

- *S has faithful translations,*
- *S has invertible 1- and 2-cells, and*
- *X has invertible 2-cells.*

Then the 2-functor $\rho\colon S^{-1}X \to S^{-1} *$ induced by the unique 2-functor $X \to *$ is an opfibration.

Proof. We will show that any 1-cell in $S^{-1}X$ which is of the form

$$(s, \alpha, 1)\colon (a, x) \to (d, sx)$$

is opcartesian with respect to ρ; this immediately shows that any 1-cell (s, α) in $S^{-1} *$ has an opcartesian lift. Given $(u, \gamma, \varphi)\colon (a, x) \to (b, y)$ in $S^{-1}X$ and

$\langle p, A \rangle \colon (t, \beta) \circ (s, \alpha) \cong (u, \gamma)$ in $S^{-1} *$, we can choose the lift in the sense of Definition 2.9 (i) as follows.

- We require a 1-cell $(v, \delta, \lambda) \colon (d, sx) \to (b, y)$, and choose it to have $v = t, \delta = \beta, \lambda = \varphi \circ px$.
- We require a 2-cell $\langle p_1, A_1 \rangle \colon (t, \beta) \cong (v, \delta)$, and choose it to be the identity 2-cell.
- We require a 2-cell $\langle p_2, A_2, F_2 \rangle \colon (v, \delta, \lambda) \circ (s, \alpha, 1) \cong (u, \gamma, \varphi)$, and choose it to have $p_2 = p$, $A_2 = A$, $F_2 = 1$.

This verifies Definition 2.9 (i). To verify Definition 2.9 (ii), suppose we are given the following data:

- a 2-cell $\langle q, B \rangle \colon (t, \beta) \Rightarrow (t', \beta')$ in $S^{-1} *$;
- a pair of 1-cells $(u, \gamma, \varphi), (u', \gamma', \varphi') \colon (a, x) \to (b, y)$ in $S^{-1}X$;
- a 2-cell $\langle r, C, G \rangle \colon (u, \gamma, \varphi) \Rightarrow (u', \gamma', \varphi')$ in $S^{-1}X$;
- a pair of 2-cell isomorphisms

$$\langle p, A \rangle \colon (t, \beta) \circ (s, \alpha) \cong (u, \gamma), \quad \langle p', A' \rangle \colon (t', \beta') \circ (s, \alpha) \cong (u', \gamma')$$

  in $S^{-1} *$;
- a lift $(v, \delta, \lambda), \langle p_1, A_1 \rangle, \langle p_2, A_2, F_2 \rangle$ of $(u, \gamma, \varphi), (t, \beta), \langle p, A \rangle$ with witness (see Definition 4.34) Θ; and
- a lift $(v', \delta', \lambda'), \langle p'_1, A'_1 \rangle, \langle p'_2, A'_2, F'_2 \rangle$ of $(u', \gamma', \varphi'), (t', \beta'), \langle p', A' \rangle$ with witness Θ',

satisfying equation (2.10).

We require a unique isomorphism 2-cell $\langle \tilde{q}, \tilde{B}, \tilde{H} \rangle \colon (v, \delta, \lambda) \Rightarrow (v', \delta', \lambda')$ in $S^{-1}X$ satisfying the two equations (2.11) and (2.12).

Equation (2.10) yields the existence of a 2-cell isomorphism $\Phi \colon r \circ p \cong p' \circ qs$ in S such that the equality

$$(4.36)$$

holds. To check the two required equations of Definition 2.9 (ii), we need the following:

- a 2-cell isomorphism $\Psi\colon p_1'\circ q \cong \tilde{q}\circ p_1$ such that the equality

$$(4.37)$$

holds, and

- a 2-cell isomorphism $\Lambda\colon r\circ p_2 \cong p_2'\circ\tilde{q}s$ such that

$$(4.38)$$

$$(4.39)$$

both hold.

Since the 1-cell $p_1\colon t\to v$ is an equivalence, we fix a pseudo-inverse $m\colon v\to t$ to p_1 together with $\eta\colon 1\cong mp_1$, $\varepsilon\colon p_1m\cong 1$ satisfying the triangle identities.

Let $\tilde{q} = p_1' \circ q \circ m$, and let $\tilde{B}$ be given by the pasting diagram below.

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
\tag{4.40}
$$

The 2-cell isomorphism Ψ is the whiskering

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
$$
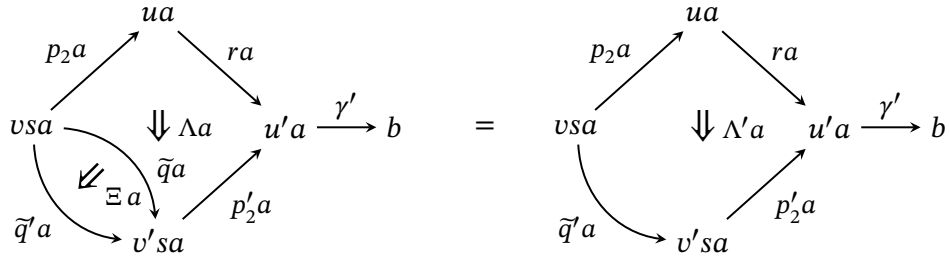
and equation (4.37) is satisfied by construction. The 2-cell isomorphism Λ is defined to be the following pasting.

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
\tag{4.41}
$$

Now we verify equation (4.38) as follows. The two diagrams below are equal by the definition of Θ' (see 4.34).

$$
\begin{array}{c}
\text{(left diagram)} \quad = \quad \text{(right diagram)}
\end{array}
$$

(4.42)

Next, the right hand diagram above and left hand diagram below are equal by definition of Φ (Display (4.36)) and the naturality of Σ with respect to 2-cells. Lastly, the two diagrams below are equal by the definition of Θ.
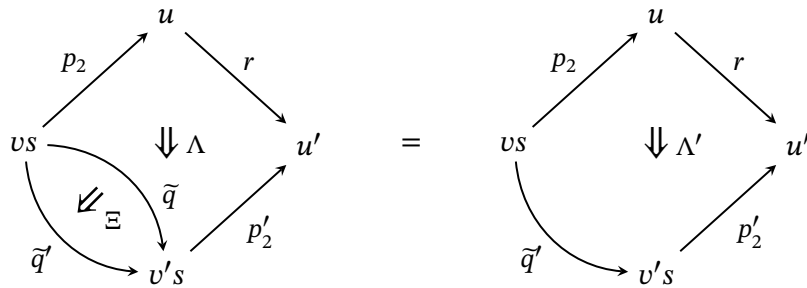
(4.43)

Using this definition of Λ and the fact that F_2' is invertible, we get a unique choice of $\tilde{H}$, concluding the construction of the 2-cell $\langle \tilde{q}, \tilde{B}, \tilde{H} \rangle$. To show uniqueness, suppose that $\langle \tilde{q}, \tilde{B}, \tilde{H} \rangle$ and $\langle \tilde{q}', \tilde{B}', \tilde{H}' \rangle$ are two lifts, with their corresponding 2-isomorphisms Ψ, Λ, Ψ' and Λ'. The 2-cell

defines a 2-isomorphism $\Xi : \tilde{q} \cong \tilde{q}'$, which by (4.37) for Ψ and Ψ' satisfies (4.18) for $\tilde{B}$ and $\tilde{B}'$. Using Displays (4.37) and (4.38), one can show the following equality of 2-cells in S.

From the assumption of faithful translations and invertibility of 1-cells in S, we have the equality below, which together with (4.39) implies that Ξ satisfies (4.18) for $\tilde{H}$ and $\widetilde{H'}$.

This shows that $\langle \tilde{q}, \tilde{B}, \tilde{H} \rangle = \langle \tilde{q}', \tilde{B}', \tilde{H}' \rangle$. Thus we have proved that 1-cells of the form $(s, \alpha, 1)$ are opcartesian with respect to $\rho : S^{-1}X \to S^{-1} * $ and further that every 1-cell of $S^{-1} *$ has an opcartesian lift.

Note that to prove the second and third condition for ρ to be an opfibration, it is sufficient to prove that any 2-cell in $S^{-1}X$ is cartesian. Indeed, the

second condition of Definition 2.14 states that ρ is locally a fibration, or equivalently that every 2-cell with *target* in the image of ρ has a cartesian lift. If $(t, \beta, \psi) \colon (a, x) \to (b, y)$ is a 1-cell in $S^{-1}X$ and $\langle p, A \rangle \colon (s, \alpha) \Rightarrow \rho(t, \beta, \psi)$ is any 2-cell in $S^{-1} *$, then $\langle p, A, 1_\varphi \rangle \colon (s, \alpha, \varphi) \Rightarrow (t, \beta, \psi)$ is a cartesian lift where $\varphi = \psi \circ px$. The third condition of Definition 2.14 states that the horizontal composite of cartesian 2-cells is again cartesian, and this follows if all 2-cells are cartesian.

Fix the data of a pair of 2-cells in $S^{-1}X$

$$\langle p, A, F \rangle \colon (s, \alpha, \varphi) \Rightarrow (t, \beta, \psi), \quad \langle q, B, G \rangle \colon (u, \gamma, \chi) \Rightarrow (t, \beta, \psi),$$

and let $\langle r, C \rangle \colon (u, \gamma) \Rightarrow (s, \alpha)$ be a 2-cell in $S^{-1} *$ such that

$$\rho \langle q, B, G \rangle = \rho \langle p, A, F \rangle \circ \langle r, C \rangle.$$

We must show there is a unique lift $\langle r', C', H' \rangle$ of $\langle r, C \rangle$, such that the equality $\langle q, B, G \rangle = \langle p, A, F \rangle \circ \langle r', C', H' \rangle$ holds. To prove existence, we construct H such that $\langle r, C, H \rangle$ satisfies the condition. By assumption, we have a 2-cell isomorphism $\Theta \colon p \circ r \cong q$ such that the following pastings are equal.

$$(4.44)$$

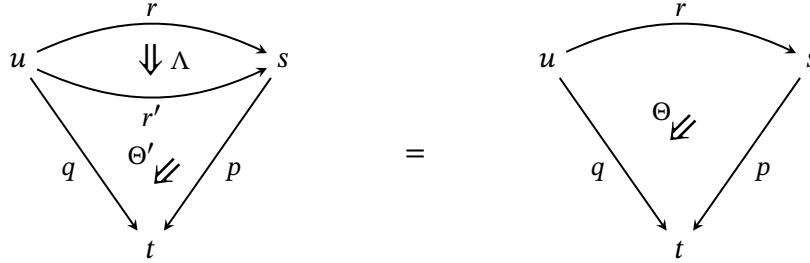It suffices to construct a 2-cell H in X such that the equality

$$(4.45)$$

holds, but since 2-cells are invertible in X there is always such an H.

To show uniqueness, we assume there is another lift of the form $\langle r', C', H' \rangle$ and we prove that it is equal to the lift we just constructed. By assumption, there exists a 2-cell isomorphism $\Theta' \colon p \circ r' \cong q$ satisfying the appropriate versions of

Displays (4.44) and (4.45). Since

$$\langle r, C \rangle = \rho\langle r', C', H' \rangle = \langle r', C' \rangle,$$

there exists a 2-isomorphism $\Lambda : r \cong r'$ that satisfies Display (4.18) with respect to C and C'. It remains to show that Λ satisfies Display (4.18) with respect to H and H', which follows from the equality below.

This equality in turn follows from Displays (4.44) and (4.45) using invertibility of 1- and 2-cells in S and faithful translations. Therefore $\langle p, A, F \rangle$ is cartesian. □

Corollary 4.46. *Let S be a permutative Gray monoid acting on a 2-category X. Assume that*

- *S has faithful translations,*
- *S has invertible 1- and 2-cells, and*
- *X has invertible 2-cells.*

Then for any 2-functor $F : Y \to S^{-1} *$, we have a homotopy equivalence

$$\mathrm{pb}(\rho, F) \to \rho{\downarrow}F$$

on classifying spaces. In particular this map induces an isomorphism on homotopy and homology groups.

Proof. This follows by combining Theorem 2.34 and Proposition 4.35. □

Before stating the main theorem, we note an explicit description of the fibers of the projection ρ.

Lemma 4.47. *For any $a \in S^{-1} *$, there is an isomorphism $\rho^{-1}(a) \cong X$ commuting with the action of S.*

Proof. Objects of $\rho^{-1}(a)$ are (a, x) for $x \in X$. 1-cells $(a, x) \to (a, x')$ are $(e, (1_a, \phi))$. 2-cells are $\langle 1_e, (1_{1_a}, F) \rangle$. □

Theorem 1.5 is a special case the following, with $X = S$.

Theorem 4.48. *Let S be a permutative Gray monoid acting on a 2-category X. Assume that*

- *S has faithful translations,*
- *S has invertible 1- and 2-cells, and*
- *X has invertible 2-cells.*

Then the inclusion $i: X \to S^{-1}X$ induces an isomorphism

$$[\pi_0 S]^{-1} H_q(X) \xrightarrow{\;\cong\;} H_q(S^{-1}X).$$

In particular, when $X = S$, we obtain that $i: S \to S^{-1}S$ is a group completion on classifying spaces.

Proof. The projection

$$\rho: S^{-1}X \to S^{-1} *$$

is an opfibration by Proposition 4.35. Since all the 1- and 2-cells of S are invertible, Lemma 4.31 implies that $S^{-1} *$ has invertible 2-cells. Thus, we have the (simplicial) local coefficient system $\mathscr{H}_q \rho^{-1}$ on $S^{-1} *$ of Notation 3.23, and we consider the spectral sequence

$$E^2_{p,q} = H_p\big(S^{-1} *; \mathscr{H}_q \rho^{-1}\big) \;\Rightarrow\; H_{p+q}(S^{-1}X)$$

of Theorem 3.25 for $F = \rho$.

Let E^0 be the double complex associated to the bisimplicial set B constructed in Definition 3.6. Since S is a permutative Gray monoid, the set of objects ob$S$ becomes a monoid by restriction. The actions of S on $S^{-1}X$ and $S^{-1} *$ of Lemma 4.21, the latter trivial, induce an action of ob$S$ on B; explicitly, $s \cdot (\omega, \delta, \sigma) = (s\omega, s\delta, s\sigma)$. A 1-cell $s \to t$ gives rise to a triple of pseudonatural transformations $(s\omega, s\delta, s\sigma) \to (t\omega, t\delta, t\sigma)$ and hence to a chain homotopy between the corresponding chain maps $s \cdot -$ and $t \cdot -$ on Tot. Therefore $\pi_0 S$ acts on the homology of the total complex.

By Lemma 4.47 we have an isomorphism of $\pi_0 S$-modules

$$\mathscr{H}_q \rho^{-1}(\sigma) = H_q\big(\rho^{-1}\big(\sigma(0)\big)\big) \cong H_q(X)$$

for all σ and all q. We can thus define a local coefficient system M_q on $S^{-1} *$ by setting $M_q(\sigma) = H_q(X)$ and using the isomorphism above to define the action on morphisms. It is important to remark that as a coefficient system, M_q is generally not constant. Recall that σ consists of objects σ_i and 1-cells $\sigma_{(i,j)} = (s_{(i,j)}, \alpha_{(i,j)})$ in $S^{-1} *$, for $i < j \in [p]$. Let $\phi: [p'] \to [p]$ in Δ. Using Remark 3.24 and tracing through the isomorphism in Lemma 4.47, one can check that the map

$$M_q(\phi): M_q(\sigma) \to M_q(\phi^*\sigma)$$

is given by the action of $s_{0,\phi(0)}$ on X.

Because S acts trivially on $S^{-1} *$, the action of ob$S$ on E^0 descends to an action of $\pi_0 S$ on E^2 and can be identified with the action on $M_q = H_q(X)$ induced by the action of S on X. Localizing with respect to the action of $\pi_0 S$ is exact, and thus we have

$$[\pi_0 S]^{-1} E^2_{p,q} \cong H_p\big(S^{-1} *; [\pi_0 S]^{-1} M_q\big) \;\Rightarrow\; [\pi_0 S]^{-1} H_{p+q}(S^{-1}X).$$

The edge homomorphism of this localized spectral sequence is the localization of the map induced on homology by $i: X \to S^{-1}X$ in Proposition 4.19.

Consider the coefficient system L_q given by

$$L_q(\sigma) = [\pi_0 S]^{-1} H_q(X)$$

for all σ. Since it is morphism-inverting, by Proposition 3.13 it induces a topological local coefficient system on $|NS^{-1} * |$, also denoted L_q. Thus we have

$$H_p(S^{-1} * ; L_q) \xrightarrow{\cong} H_p(|NS^{-1} * | ; L_q)$$

for all p and q. Next we recall that $|NS^{-1} * |$ is contractible by Lemma 4.20, and therefore $H_p(|NS^{-1} * | ; L_q)$ is 0 for all $p > 0$. Thus the localized spectral sequence collapses and the edge map induces an isomorphism

$$[\pi_0 S]^{-1} H_q(X) \xrightarrow{\cong} H_q(S^{-1}X). \qquad \square$$

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