New York Journal of Mathematics
New York J. Math. **27** (2021) 1060–1084.

On curves with high multiplicity on $\mathbb{P}(a, b, c)$ for $\min(a, b, c) \leq 4$

David McKinnon, Rindra Razafy, Matthew Satriano and Yuxuan Sun

ABSTRACT. On a weighted projective surface $\mathbb{P}(a, b, c)$ with $\min(a, b, c) \leq 4$, we compute lower bounds for the *effective threshold* of an ample divisor, in other words, the highest multiplicity a section of the divisor can have at a specified point. We expect that these bounds are close to being sharp. This translates into finding divisor classes on the blowup of $\mathbb{P}(a, b, c)$ that generate a cone contained in, and probably close to, the effective cone.

CONTENTS

1. Introduction

Given a projective variety X and a point $Q \in X$, it is, in general, a notoriously difficult problem to calculate the pseudo-effective cone of the blow-up $\mathrm{Bl}_Q(X)$ in terms of the pseudo-effective cone of X. Even addressing the *a priori* easier question of when $\mathrm{Bl}_Q(X)$ is a Mori Dream Space, where $X = \mathbb{P}(a, b, c)$ is a weighted projective surface and Q is the identity of its torus, is already challenging and has a rich history [Hun82, Cut91, Sri91, GNW94, CT15, GK16, He17, GGK20]. To gain information about the pseudo-effective cone of $\mathrm{Bl}_Q(X)$, we consider the following quantity, cf. [Fuj92].

Received November 23, 2020.
2010 *Mathematics Subject Classification.* 14C15, 14F43.
Key words and phrases. weighted projective spaces, curves, Ehrhart polynomial, multiplicity.
The first author and third authors were partially supported by a Discovery Grant from the National Science and Engineering Board of Canada. The second and fourth authors were supported by an Undergraduate Student Research Award from the National Science and Engineering Board of Canada.

ISSN 1076-9803/2021

Definition 1.1. Let X be a projective variety defined over a field k, D a k-rational $\mathbb{Q}$-divisor, and Q a k-rational point of X. Let π be the blowup of X at Q and E the exceptional divisor of π. We say the *effective threshold* is

$$\gamma_Q(D) := \sup\{\gamma > 0 \mid \pi^*(D) - \gamma E \text{ is pseudo-effective}\}.$$

The quantity $\gamma_Q(D)$ can be reinterpreted concretely as follows:if there is a divisor in the class of D with multiplicity m at Q, then $\gamma_Q(D) \geq m$. Conversely, if $\gamma_Q(D) = m$, then for all $\epsilon > 0$, the class $\pi^* D - (m - \epsilon)E$ is pseudo-effective, so D contains divisors of multiplicity arbitrarily close to m, at least in a $\mathbb{Q}$-divisor sense. So, computing $\gamma_Q(D)$ essentially amounts to computing

$$\sup_{C,m} \left\{ \frac{1}{m} \operatorname{mult}_Q(C) \right\}$$

as m varies through positive integers and C varies through divisors in the divisor class mD.

In this paper, we give characteristic-free lower bounds for $\gamma_Q(D)$ in the case where X is the weighted projective surface $\mathbb{P}(a, b, c)$ and $\min(a, b, c) \leq 4$. In fact, we do more than this: we introduce a combinatorial quantity γ_{expected} which is a lower bound on γ, and compute γ_{expected} exactly. It is worth remarking here that although the motivation for studying γ_Q is geometric, our lower bounds on γ_Q also have consequences for Diophantine approximation problems related to generalizations of Roth's famous 1955 theorem [Ro55], see e.g., [MR16, Theorem 3.3] and [MS20, Section 8].

In [GGK20], the authors make a series of detailed calculations closely related to what we compute in this paper. In particular, they search the spaces of global sections of toric surfaces of Picard rank one for irreducible curves whose strict transforms have negative self-intersection upon blowing up a point. If there is such a curve, then the pseudo-effective cone of the blowup will be finitely generated by the exceptional divisor of the blowup and another curve of negative self-intersection.

In this paper, we compute not only curves, but also the corresponding value of the effective threshold. We do not prove that the curves we find are always generators of the pseudo-effective cone, but in most cases the value of γ we compute is expected to be equal or very close to the actual value. As the authors of [GGK20] also point out, our quantity γ_{expected} is expected to be very close to the actual value of γ.

Since $X = \mathbb{P}(a, b, c)$ is a toric surface, if the point Q does not lie in the main torus orbit T, then computing γ_Q is generally straightforward, so we may assume that Q lies in T. Furthermore, we can choose a, b, and c to be pairwise coprime, with $a \leq b \leq c$. These inequalities are always strict unless $a = b = 1$, in which case γ can be computed directly. Thus, we may assume that $a < b < c$. Finally, since X has Picard rank 1, it suffices to compute $\gamma_Q(H)$, where H is the generator of the Cartier class group.

Our first result concerns the case $a < 4$ and serves as a warm-up to our main result.

Proposition 1.2. *Let $a < b < c$ be pairwise coprime, so we may write $c = pa+qb$ with $p, q \in \mathbb{Z}$ and $0 \le q < a$. Let Q be in the torus of $\mathbb{P}(a,b,c)$ and H be the generator of the Cartier class group. Then*

$$\gamma_Q(H) \ge \begin{cases} (q+1)b, & p \ge 0 \\ (a-1)b, & p < 0 \text{ and } a \le 3. \end{cases}$$

We do not claim that this Proposition is new. Indeed, in [HKL18], Hausen, Keicher, and Laface prove a number of results along these lines, and moreover they obtain all the results of Proposition 1.2, as a consequence of their Theorems 1.1 and 1.2. Despite the lack of novelty in Proposition 1.2, we present a proof of it to illustrate our techniques in a simpler setting.

Proposition 1.2 yields lower bounds on γ_Q when $a \le 3$. Moving from $a \le 3$ to $a = 4$ is significantly more involved. In order to state our results, we first discuss our main technique of proof. Note that if $m \in \mathbb{Q}^+$ and mH is a Weil divisor such that $h^0(mH) > \binom{\nu+1}{2}$, then there is a global section g of mH that vanishes at Q to order ν. Writing $m = \frac{m_1}{m_2}$ with $m_1, m_2 \in \mathbb{Z}^+$, we see $g^{m_2} \in H^0(X, m_1 H)$ vanishes to order νm_2. By definition, it follows that $\gamma_Q(H) \ge \frac{\nu m_2}{m_1} = \frac{\nu}{m}$. This motivates the following definition.

Definition 1.3. For any Weil divisor D, let

$$\nu(D) := \max\left\{ d \in \mathbb{Z}^+ \mid h^0(D) > \binom{d+1}{2} \right\}.$$

If H denotes the generator of the Cartier class group of $\mathbb{P}(a,b,c)$ with $a < b < c$, then let

$$\gamma_{\text{expected}}(H) := \sup\left\{ \frac{\nu(mH)}{m} \mid m \in \frac{1}{b}\mathbb{Z}^+ \cup \frac{1}{c}\mathbb{Z}^+ \right\}.$$

Remark 1.4. Note that the definition of γ_{expected} considers only some of the Weil divisors. In particular, since the Weil class group of $\mathbb{P}(a,b,c)$ is generated by $\frac{1}{abc}H$, every Weil divisor that appears in the definition of γ_{expected} is a multiple of a in the Weil class group.

We can now state our main result. Recall that Proposition 1.2 already yields lower bounds on γ_Q when $p \ge 0$ in general, so we turn to the case $p < 0$.

Theorem 1.5. *With notation and hypotheses as in Proposition 1.2, assume $a = 4$ and $p < 0$. Then*

$$\gamma_Q(H) \ge \gamma_{\text{expected}}(H) = \frac{\nu(D_0)}{m_0},$$

where $D_0 \sim m_0 H$ and $\nu(D_0)$ are computed exactly as follows. Given our constraints, we have $2 < \frac{b}{-p} < \frac{16}{3}$. Divide the interval $[2, \frac{16}{3}]$ into a countably infinite sequence of intervals of the form

$$I_k := \left[\frac{16(k+1)^2}{8(k+1)^2 - 4(k+1) - 1}, \frac{16k^2}{8k^2 - 4k - 1} \right]$$

with $k \in \mathbb{Z}^+$. Then the class of D_0 is given as follows, depending on the value of $\frac{b}{-p} \in I_k$:

(1) *If $\frac{b}{-p} \in I'_{k,-} := \left[\frac{16(k+1)^2}{8(k+1)^2-4(k+1)-1}, \frac{2k+1}{k}\right]$, then $D_0 \sim \frac{2k+3}{c} H$ with $\nu(D_0) = 4(k+1)$.*
(2) *If $\frac{b}{-p} \in I'_{k,+} := \left[\frac{2k+1}{k}, \frac{4(2k+1)^2}{8k^2+4k-1}\right]$, then $D_0 \sim \frac{2k+1}{b} H$ with $\nu(D_0) = 4(k+1)$.*
(3) *If $\frac{b}{-p} \in I''_{k,-} := \left[\frac{4(2k+1)^2}{8k^2+4k-1}, \frac{4k}{2k-1}\right]$, then $D_0 \sim \frac{k+1}{c} H$ with $\nu(D_0) = 2k+1$.*
(4) *If $\frac{b}{-p} \in I''_{k,+} := \left[\frac{4k}{2k-1}, \frac{16k^2}{8k^2-4k-1}\right]$, then $D_0 \sim \frac{k}{b} H$ with $\nu(D_0) = 2k+1$.*

Remark 1.6. The quantity $\gamma_{\text{expected}}(H)$ is the lower bound for $\gamma(H)$ obtained by simple linear algebra: the vanishing to order n of a section of H is equivalent to the vanishing of $\binom{n+1}{2}$ linear forms on the space of sections of H. We therefore have $\gamma(H) \geq \gamma_{\text{expected}}(H)$ trivially.

However, one also expects that the two quantities are not so different. First of all, the divisibility restriction in $\gamma_{expected}$ does not exclude any Cartier divisors, and in the examples that we are aware of, does not change the value of γ at all. More significantly, if $\gamma(H) > \gamma_{\text{expected}}(H)$ at some point Q in the main torus orbit, then there is a section s of some multiple mH of H that has an order of vanishing that is greater than $\nu(mH)$ at Q. For any element σ of the torus, the section $\sigma(s)$ has unusually high order of vanishing at $\sigma(Q)$, so for *every* point of the main orbit, there is a section of mH that has unusually high order of vanishing there. This is unlikely – though not downright impossible – and so one expects the two quantities to be close.

Nevertheless, there are examples where γ and γ_{expected} do not agree. For example, if $(a,b,c) = (5,33,49)$ or $(8,15,43)$, Kurano and Matsuoka ([KM09]) showed that γ and γ_{expected} are not the same. Several other authors, including Gonzalez Anaya, Gonzalez, and Karu, have obtained other very interesting results along these lines, of which an excellent summary can be found in [GAGK21].

The rest of the paper is organized as follows. Section 2 proves Proposition 1.2 and describes some preliminary reductions for Theorem 1.5. Section 3 computes the main terms in the count of global sections of multiples of H. Section 4 then begins the process of bounding the error terms, and Section 5 finishes the proof of Theorem 1.5.

Acknowledgments

We are grateful to Kalle Karu for many enlightening email exchanges. We thank the anonymous referee for helpful suggestions. This paper is the outcome of an NSERC-USRA project; we thank NSERC for their support.

2. Proof of Proposition 1.2 and preliminary reductions

Throughout this paper, we let x, y, and z be the weighted projective coordinates on $\mathbb{P}(a,b,c)$ with weights a, b, and c, respectively. We let D_x, D_y, and D_z denote the Weil divisors defined by the vanishing of x, y, and z, respectively. We let H denote the generator of the Cartier class group, so we have $H \sim bcD_x \sim acD_y \sim abD_z$. Given any Weil divisor D, we let $P_D \subset \mathbb{R}^2$ be the associated polytope with the property that $h^0(D) = |P_D \cap \mathbb{Z}^2|$. We sometimes abusively denote $|P_D \cap \mathbb{Z}^2|$ by $|P_D|$.

It will frequently be useful to write our divisors as integer multiples of D_x. Note that if $\delta \in \{b,c\}$ and $m \in \frac{1}{\delta}\mathbb{Z}^+$, then $mH \sim n\delta D_x$ where $n = \frac{bcm}{\delta} \in \mathbb{Z}^+$.

After a preliminary lemma, we prove Proposition 2.2 which is a slightly more general version of Proposition 1.2.

Lemma 2.1. *With notation and hypotheses as in Proposition 1.2, if $p < 0$, then $q \neq 1$. In particular, if $p < 0$ and $a \leq 3$, then $a = 3$ and $q = 2$.*

Proof. If $q = 1$, then $c = pa + b < b$ which is a contradiction. If $p < 0$ and $a \leq 3$, then $q = 0$ or $q \geq 2$. The former case cannot occur as it implies $c = pa$ and hence $p > 0$. The latter case implies $2 \leq q \leq a-1$ so $q = 2$ and $a = 3$. □

Proposition 2.2. *With notation and hypotheses as in Proposition 1.2, we have*

$$\gamma_Q(H) \geq \begin{cases} (q+1)b, & p \geq 0 \\ (a-1)b, & p < 0,\ q = a-1, \text{ and } \frac{-pa}{b} \leq 1. \end{cases}$$

Furthermore, if $a \leq 3$ and $p < 0$, then $q = a-1$ and $\frac{-pa}{b} \leq 1$ automatically hold.

Proof. Note that since a, b, and c are pairwise coprime, $p \neq 0$. First suppose $p > 0$. Then the polytope P_{aD_z} is the convex hull of 0, $(q,-a)$, and $(\frac{-pa}{b},-a)$, so it contains the triangle T with vertices 0, $(q,-a)$, $(0,-a)$. By Pick's Theorem, $1 + \binom{q+2}{2} \leq |T \cap \mathbb{Z}^2| \leq |P_{aD_z} \cap \mathbb{Z}^2|$, which implies $\nu(aD_z) \geq q+1$. Since $aD_z \sim \frac{1}{b}H$, we find $\gamma_Q(H) \geq (q+1)b$.

Next, suppose $p < 0$, $q = a-1$, and $\frac{-pa}{b} \leq 1$. Notice that the polytope P_{aD_z} is given by the vertices as above, and it contains the triangle T with vertices 0, $(q,-a)$, and $(1,-a)$ by $p < 0$ and $\frac{-pa}{b} \leq 1$. From $q = a-1$ and Pick's Theorem, we have $1 + \binom{a}{2} = |T \cap \mathbb{Z}^2|$, which, as in the previous paragraph, implies $\gamma_Q(H) \geq (a-1)b$.

Finally, we note that if $a \leq 3$ and $p < 0$, then Lemma 2.1 tells us $a = 3$ and $q = 2 = a-1$. Then, $b < c = pa + 2b$ implies $\frac{-pa}{b} < 1$. □

The rest of the paper is concerned with the proof of Theorem 1.5. By Lemma 2.1, since $p < 0$ and a, b, c are pairwise coprime, we must have

$$q = 3 = a - 1.$$

We begin by analyzing $\nu(D_0)$.

Proposition 2.3. *With notation and hypotheses as in Theorem 1.5, if* $\frac{b}{-p} \in I'_k := I'_{k,+} \cup I'_{k,-}$, *resp.* $I''_k := I''_{k,+} \cup I''_{k,-}$, *and* D_0 *is as in the conclusion of the theorem, then* $\nu(D_0) \ge 4(k+1)$, *resp.* $2k+1$.

Proof. Let $n_0 \in \mathbb{Z}$ be such that $D_0 \sim \frac{n_0}{b}H \sim n_0 c D_x$ or $D_0 \sim \frac{n_0}{c}H \sim n_0 b D_x$. In the former (respectively latter) case, $h^0(D_0)$ is given by the number of integer lattice points lying in the polytope

$$P_{n_0cD_x} = \operatorname{Conv}\left((0,0),\ \left(-n_0\frac{c}{b},0\right),\ (-3n_0,4n_0)\right)$$

respectively

$$P_{n_0bD_x} = \operatorname{Conv}\left((0,0),\ (-n_0,0),\ \left(-3n_0\frac{b}{c},4n_0\frac{b}{c}\right)\right).$$

First consider the case $\frac{b}{-p} \in I'_k$. As in Theorem 1.5, $I'_{k,+} := [\frac{2k+1}{k}, \frac{4(2k+1)^2}{8k^2+4k-1}]$ and $I'_{k,-} := [\frac{16(k+1)^2}{8(k+1)^2-4(k+1)-1}, \frac{2k+1}{k}]$. If $\frac{b}{-p} \in I'_{k,+}$, then $D_0 \sim n_0 c D_x$ with $n_0 = 2k+1$ and if $\frac{b}{-p} \in I'_{k,-}$, then $D_0 \sim n_0 b D_x$ with $n_0 = 2k+3$. Let

$$P' = \operatorname{Conv}((0,0),\ (-(2k+3),0),\ (-3(2k+1),4(2k+1))).$$

We then have $P' \subset P$. Indeed, if $\frac{b}{-p} \in I'_{k,+}$, then the inclusion follows from $-(2k+1)\frac{c}{b} = -(2k+1)(4\frac{p}{b}+3) \le -(2k+1)(4\frac{-k}{2k+1}+3) = -(2k+3)$. If $\frac{b}{-p} \in I'_{k,-}$, then the inclusion follows from $-3(2k+3)\frac{b}{c} \le -3(2k+1)$ and $4(2k+3)\frac{b}{c} \ge 4(2k+1)$. So, in either case, we have

$$|P' \cap \mathbb{Z}^2| \le h^0(D_0).$$

Note that the area of P' is given by

$$A(P') = \frac{1}{2}(4(2k+1)(2k+3)) = 2(2k+1)(2k+3)$$

and the number of lattice points on its boundary is given by

$$B(P') = (2k+3) + (2k+1) + 4 = 4k+8.$$

Since P' is a lattice polygon, applying Pick's Theorem, we have

$$|P' \cap \mathbb{Z}^2| = A(P') + \frac{1}{2}B(P') + 1 = 8k^2 + 18k + 11 = \binom{4(k+1)+1}{2} + 1,$$

which shows $\nu(D_0) \ge \nu(P') = 4(k+1)$.

For $\frac{b}{-p} \in I''_k$, the same proof works when using

$$P'' = \operatorname{Conv}((0,0),\ (-(k+1),0),\ (-3k,4k))$$

in place of P'. □

Proposition 2.3 therefore gives the lower bound

$$\gamma_{\text{expected}}(H) \geq \frac{\nu(D_0)}{m_0}, \tag{2.4}$$

where $D_0 = m_0 H$ is the divisor class described in Theorem 1.5. To obtain upper bounds, we introduce the following quantities and make use of the subsequent lemma. Let

$$\gamma_{\text{expected},b}(H) := \sup\left\{\frac{1}{m}\nu(mH) \mid m \in \frac{1}{c}\mathbb{Z}^+\right\}$$

$$\gamma_{\text{expected},c}(H) := \sup\left\{\frac{1}{m}\nu(mH) \mid m \in \frac{1}{b}\mathbb{Z}^+\right\}.$$

Then, we may bound $\gamma_{\text{expected}}(H)$ from above by bounding $\gamma_{\text{expected},b}(H)$ and $\gamma_{\text{expected},c}(H)$ from above, given that

$$\gamma_{\text{expected}}(H) = \max\left\{\gamma_{\text{expected},b}(H), \gamma_{\text{expected},c}(H)\right\}. \tag{2.5}$$

Lemma 2.6. *Suppose $\mathbb{P}(4, b_0, c_0)$, $\mathbb{P}(4, b_L, c_L)$, and $\mathbb{P}(4, b_U, c_U)$ satisfy the hypotheses of Theorem 1.5, except we need not assume $p < 0$. Suppose $\frac{b_L}{-p_L} < \frac{b_0}{-p_0} < \frac{b_U}{-p_U}$ and let H_0, H_L, and H_U denote the generators of the respective Cartier class groups. Then*

$$\frac{1}{c_0}\gamma_{\text{expected},b_0}(H_0) \leq \frac{1}{c_L}\gamma_{\text{expected},b_L}(H_L)$$

and

$$\frac{1}{c_0}\gamma_{\text{expected},c_0}(H_0) \leq \frac{1}{c_U}\gamma_{\text{expected},c_U}(H_U).$$

Proof. Since $c = pa + qb = 4p + 3b$, we see $\frac{b}{c} = \frac{1}{4\frac{p}{b}+3}$. As a result, $\frac{b_U}{c_U} < \frac{b_0}{c_0} < \frac{b_L}{c_L}$. It follows that

$$\begin{aligned} P_0 &:= \operatorname{Conv}\left((0,0), (-1,0), \left(-3\frac{b_0}{c_0}, 4\frac{b_0}{c_0}\right)\right) \\ &\subset \operatorname{Conv}\left((0,0), (-1,0), \left(-3\frac{b_L}{c_L}, 4\frac{b_L}{c_L}\right)\right) \\ &=: P_L. \end{aligned}$$

Since P_0, resp. P_L, is the polytope of bD_x on $\mathbb{P}(4, b_0, c_0)$, resp. $\mathbb{P}(4, b_L, c_L)$, we have $\nu(nP_0) \leq \nu(nP_L)$ for all $n \geq 1$, and so

$$(1/c_0)\gamma_{\text{expected},b}(H_0) \leq (1/c_L)\gamma_{\text{expected},b_L}(H_L).$$

We obtain the inequality $(1/c_0)\gamma_{\text{expected},c}(H_0) \leq (1/c_U)\gamma_{\text{expected},c_U}(H_U)$ in a similar manner from the inclusion

$$\operatorname{Conv}\left((0,0), \left(-\frac{c_0}{b_0}, 0\right), (-3,4)\right) \subset \operatorname{Conv}\left((0,0), \left(-\frac{c_U}{b_U}, 0\right), (-3,4)\right),$$

the left-hand, resp. right-hand, side being the polytope of cD_x on $\mathbb{P}(4, b_0, c_0)$, resp. $\mathbb{P}(4, b_U, c_U)$. □

Remark 2.7. Lemma 2.6 may be used to reduce the proof of Theorem 1.5 to special classes of weighted projective spaces with desirable arithmetic properties. The key idea is to compare the values of $\gamma_{\text{expected},b}(H)$ and $\gamma_{\text{expected},c}(H)$ on different weighted projective spaces to generate upper bounds.

Fix a weighted projective space $\mathbb{P}(4,b,c)$ satisfying the hypotheses of Theorem 1.5 and let $D_0 \sim m_0 H$ and $\nu_0 := \nu(D_0)$ be as predicted by theorem. We will suppose that $m_0 = \frac{n_0}{c}$ with $n_0 \in \mathbb{Z}^+$ (the case where $m_0 \in \frac{1}{b}\mathbb{Z}^+$ is handled similarly). We must prove $\gamma_{\text{expected}}(H) = \frac{\nu_0}{m_0} = \frac{c\nu_0}{n_0}$. Let $I = [\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}]$ be an interval of the form $I'_{k,\pm}$ or $I''_{k,\pm}$ as in Proposition 2.3, where $\frac{b}{-p} \in I$.

Assume $\frac{b}{-p}$ is in the interior of I. We must show $\gamma_{\text{expected}}(H) = \frac{c\nu_0}{n_0}$. By (2.5), this is equivalent to proving

$$\gamma_{\text{expected},b}(H) \leq \frac{c\nu_0}{n_0} \quad \text{and} \quad \gamma_{\text{expected},c}(H) \leq \frac{c\nu_0}{n_0}.$$

This may be done as follows: fix increasing sequences of positive integers $\{b_i\}_i$ and $\{-p_i\}_i$ for which $\alpha_1 b_i - \beta_1(-p_i) = 1$ and $c_i := 4p_i + 3b_i > b_i$ is such that 4, b_i, c_i are pairwise coprime. We may always find such sequences, since $\alpha_1, \beta_1 > 0$ are coprime in all cases listed in Theorem 1.5. Let H_i denote the generator of the Cartier class group of $\mathbb{P}(4, b_i, c_i)$. Then for i sufficiently large, $\frac{b_i}{-p_i} \in I$ is monotonically decreasing with $\frac{b_i}{-p_i} \to \frac{\beta_1}{\alpha_1}$. Given $\frac{b}{-p} \in I$, there exists an N large enough such that $\frac{b_N}{-p_N} < \frac{b}{-p}$, so by Lemma 2.6,

$$\frac{1}{c}\gamma_{\text{expected},b}(H) \leq \frac{1}{c_N}\gamma_{\text{expected},b_N}(H_N) \leq \frac{1}{c_{N+1}}\gamma_{\text{expected},b_{N+1}}(H_{N+1}) \leq \dots \tag{2.8}$$

Similarly, fix increasing sequences of positive integers $\{b'_i\}_i$ and $\{-p'_i\}_i$ for which $\alpha_2 b'_i - \beta_2(-p'_i) = -1$ and $c'_i := 4p'_i + 3b'_i > b'_i$ is such that 4, b'_i, c'_i are pairwise coprime. As above, such sequences always exist. Let H'_i denote the generator of the Cartier class group of $\mathbb{P}(4, b'_i, c'_i)$. Then for i sufficiently large, $\frac{b'_i}{-p'_i} \in I$ is monotonically increasing with $\frac{b'_i}{-p'_i} \to \frac{\beta_2}{\alpha_2}$. Choosing an N large enough such that $\frac{b}{-p} < \frac{b'_n}{-p'_n}$, we have by Lemma 2.6

$$\frac{1}{c}\gamma_{\text{expected},c}(H) \leq \frac{1}{c'_N}\gamma_{\text{expected},c'_N}(H'_N) \leq \frac{1}{c'_{N+1}}\gamma_{\text{expected},c'_{N+1}}(H'_{N+1}) \leq \dots \tag{2.9}$$

We claim that to prove Theorem 1.5 for $\mathbb{P}(4,b,c)$, it is enough to show

$$\frac{\nu_0}{n_0} \geq \frac{1}{n}\nu\left(\frac{n}{c_i}H_i\right) \tag{2.10}$$

and

$$\frac{c'_i\nu_0}{n_0} > \frac{b'_i}{n}\nu\left(\frac{n}{b_i}H'_i\right) \tag{2.11}$$

for all i sufficiently large and all $n \in \mathbb{Z}^+$. Indeed, (2.10) implies

$$\begin{aligned} c_i \frac{\nu_0}{n_0} &\geq c_i \sup\left\{\frac{1}{n}\nu\left(\frac{n}{c_i}H_i\right) \mid n \in \mathbb{Z}^+\right\} \\ &= \sup\left\{\frac{1}{m}\nu\left(mH_i\right) \mid m \in \frac{1}{c_i}\mathbb{Z}^+\right\} \\ &= \gamma_{\text{expected},b_i}(H_i) \end{aligned}$$

for all i sufficiently large, which when combined with (2.8) shows

$$\gamma_{\text{expected},b}(H) \leq \frac{c\nu_0}{n_0}.$$

Similarly, (2.11) implies

$$c_i' \frac{\nu_0}{n_0} > \gamma_{\text{expected},c_i'}(H_i'),$$

which when combined with (2.9) shows

$$\gamma_{\text{expected},c}(H) < \frac{c\nu_0}{n_0}.$$

which is exactly our goal. Further, the strict inequality in this latter case will allow us to conclude that no D of the form $\frac{n}{c}H$ (other than the case where n is divisible by n_0) gives the required upper bound for $\gamma_{\text{expected}}(H)$. Together with the lower bound (2.4), this computes $\gamma_{\text{expected}}(H)$.

Note that the above computes $\gamma_{\text{expected}}(H)$ assuming $\frac{b}{-p}$ is in the interior of I. If $\frac{b}{-p}$ is an endpoint of I, then it is straightforward to check that $b = 2k+1$ and $-p = k$, in which case $c = 2k+3$. In that case, one may verify Theorem 1.5 directly using that the Erhart quasi-polynomial computed in Section 3 is an actual polynomial.

3. Ehrhart quasi-polynomials for bD_x and cD_x

Our first goal in this section is to give an expression for the number of lattice points in the polytopes P_{nbD_x} and P_{ncD_x}.

Proposition 3.1. *Keep the notation and hypotheses of Theorem 1.5, let $\delta \in \{b, c\}$, and set $s = \frac{b}{c}$. Then*

$$|P_{n\delta D_x} \cap \mathbb{Z}^2| = c_2(\delta D_x, n)n^2 + c_1(\delta D_x, n)n + c_0(\delta D_x, n),$$

where the c_i's are given as follows.

(1) *For* $\delta = b$, *we have* $c_2(bD_x, n) = 2s$, $c_1(bD_x, n) = \frac{1}{2}(1 + s + \frac{4}{c})$, *and*

$$c_0(bD_x, n) = 1 - \frac{1}{8s}\left(\{4sn\}^2 - \{4sn\}\right) - \frac{5}{2}\{sn\} + \sum_{j=0}^{4\left\{\frac{\lfloor 4sn\rfloor}{4}\right\}} \left\{\frac{3}{4}j\right\}$$

$$+ \frac{b-1}{2}\left\{\frac{4n}{c}\right\} - \sum_{j=0}^{b\left\{\frac{\lfloor 4sn\rfloor}{b}\right\}} \left\{\frac{-p}{b}j\right\}.$$

(2) *For* $\delta = c$, *we have* $c_2(cD_x, n) = \frac{2}{s}$, $c_1(cD_x, n) = \frac{1}{2}(1 + \frac{1}{s} + \frac{4}{b})$, *and*

$$c_0(cD_x, n) = 1 - \left\{\frac{4n}{b}\right\} - \frac{b-1}{2}\left\{\frac{4n}{b}\right\} + \sum_{j=0}^{b\left\{\frac{4n}{b}\right\}} \left\{\frac{-p}{b}j\right\}.$$

Proof. First consider $\delta = b$.

$$|P_{nbD_x} \cap \mathbb{Z}^2| := \left|\mathrm{Conv}(A, B, C)\right| := \left|\mathrm{Conv}\left((0,0), (-n, 0), \left(-3n\frac{b}{c}, 4n\frac{b}{c}\right)\right)\right|$$

where BC is given by $y = \frac{b}{p}x + \frac{b}{p}n$ and AC is given by $y = -\frac{4}{3}x$. We will compute $|P_{nbD_x} \cap \mathbb{Z}^2|$ by counting the number of lattice points lying on each line segment A_jB_j, where $A_j = (-\frac{3}{4}j, j)$ lies on AC and $B_j = (\frac{p}{b}j - n, j)$ lies on BC, for $j = 0, 1, \ldots, \lfloor 4sn\rfloor$. Here, our approach is similar to that of [L11, Theorem 3.1]. Denote $M = \lfloor 4sn\rfloor = 4sn - \{4sn\}$. Then,

$$\begin{aligned}
&|P_{nbD_x} \cap \mathbb{Z}^2| \\
&= \sum_{j=0}^{M}\left(\left\lfloor -\frac{3}{4}j\right\rfloor - \left\lceil \frac{p}{b}j - n\right\rceil + 1\right) \\
&= (n+1)(M+1) + \sum_{j=0}^{M}\left(-\left\lceil \frac{3}{4}j\right\rceil + \left\lfloor \frac{-p}{b}j\right\rfloor\right) \\
&= (n+1)(M+1) + \sum_{j=0}^{M}\left(-\left(\frac{3}{4}j + 1 - \left\{\frac{3}{4}j\right\}\right) + \frac{-p}{b}j - \left\{\frac{-p}{b}j\right\}\right) + \left\lfloor \frac{M}{4}\right\rfloor + 1 \\
&= n(M+1) + \frac{M}{4} - \left\{\frac{M}{4}\right\} + 1 + \sum_{j=0}^{M}\left(\frac{-p}{b} - \frac{3}{4}\right)j + \sum_{j=0}^{M}\left\{\frac{3}{4}j\right\} - \sum_{j=0}^{M}\left\{\frac{-p}{b}j\right\}.
\end{aligned}$$

Rewrite the sums involving fractional parts as sums of a linear term in n and a c−periodic term in n:

$$\sum_{j=0}^{M}\left\{\frac{3}{4}j\right\}=\left\lfloor\frac{M}{4}\right\rfloor\sum_{j=0}^{3}\left\{\frac{3}{4}j\right\}+\sum_{j=0}^{4\left\{\frac{M}{4}\right\}}\left\{\frac{3}{4}j\right\}=\frac{3}{2}\left(sn-\{sn\}\right)+\sum_{j=0}^{4\left\{\frac{\lfloor 4sn\rfloor}{4}\right\}}\left\{\frac{3}{4}j\right\},$$

$$\sum_{j=0}^{M}\left\{\frac{-p}{b}j\right\}=\left\lfloor\frac{M}{b}\right\rfloor\sum_{j=0}^{b-1}\left\{\frac{-p}{b}j\right\}+\sum_{j=0}^{b\left\{\frac{M}{b}\right\}}\left\{\frac{-p}{b}j\right\}=\frac{b-1}{2}\left(\frac{4n}{c}-\left\{\frac{4n}{c}\right\}\right)+\sum_{j=0}^{b\left\{\frac{\lfloor 4sn\rfloor}{b}\right\}}\left\{\frac{-p}{b}j\right\},$$

where we have used the identity

$$\sum_{j=0}^{b-1}\left\{\frac{-p}{b}j\right\}=\frac{b-1}{2}$$

given that b and p are coprime. Moreover, by a direct computation,

$$\sum_{j=0}^{M}\left(\frac{-p}{b}-\frac{3}{4}\right)j=-\binom{M+1}{2}\frac{c}{4b}=-2sn^2+\left(\{4sn\}-\frac{1}{2}\right)n-\frac{1}{8s}\left(\{4sn\}^2-\{4sn\}\right).$$

Thus, we can write $|P_{nbD_x}\cap\mathbb{Z}^2|=c_2(bD_x,n)n^2+c_1(bD_x,n)n+c_0(bD_x,n)$, where

$$\begin{aligned}
c_2(bD_x,n)&=2s,\\
c_1(bD_x,n)&=\frac{1}{2}\left(1+s+\frac{4}{c}\right),\\
c_0(bD_x,n)&=1-\frac{1}{8s}\left(\{4sn\}^2-\{4sn\}\right)-\frac{5}{2}\{sn\}\\
&\qquad+\sum_{j=0}^{4\left\{\frac{\lfloor 4sn\rfloor}{4}\right\}}\left\{\frac{3}{4}j\right\}+\frac{b-1}{2}\left\{\frac{4n}{c}\right\}-\sum_{j=0}^{b\left\{\frac{\lfloor 4sn\rfloor}{b}\right\}}\left\{\frac{-p}{b}j\right\}.
\end{aligned}$$

Likewise, we may find the Ehrhart quasi-polynomial for $|P_{ncD_x}\cap\mathbb{Z}^2|$. To simplify our calculations, we may consider $naD_z=4nD_z\sim ncD_x$. By linear equivalence, $|P_{naD_z}\cap\mathbb{Z}^2|=|P_{ncD_x}\cap\mathbb{Z}^2|$. The polytope of naD_z is given by

$$P_{naD_z}=\operatorname{Conv}(A',B',C'):=\operatorname{Conv}\left((0,0),\left(-\frac{4p}{b}n,-4n\right),(3n,-4n)\right),$$

with $A'C'$ contained in the line $y = -\frac{4}{3}x$ and $A'B'$ contained in the line $y = \frac{b}{p}x$. Similarly as before:

$$
\begin{aligned}
|P_{naD_z} \cap \mathbb{Z}^2| &= \sum_{j=0}^{4n} \left(\left\lfloor \frac{3}{4} j \right\rfloor - \left\lceil \frac{-p}{b} j \right\rceil + 1 \right) \\
&= \sum_{j=0}^{4n} \left(\frac{3}{4} j - \left\{ \frac{3}{4} j \right\} \right) - \sum_{j=0}^{4n} \left(\frac{-p}{b} j + 1 - \left\{ \frac{-p}{b} j \right\} \right) + \left\lfloor \frac{4n}{b} \right\rfloor + 1 + 4n + 1 \\
&= \frac{c}{4b} \binom{4n+1}{2} - \frac{3}{2} n + \frac{2(b-1)}{b} n - \frac{b-1}{2} \left\{ \frac{4n}{b} \right\} \\
&\qquad + \sum_{j=0}^{b\{\frac{4n}{b}\}} \left\{ \frac{-p}{b} j \right\} + \frac{4n}{b} + 1 - \left\{ \frac{4n}{b} \right\},
\end{aligned}
$$

using expressions that we obtained previously. Thus, we can write

$$
|P_{ncD_x} \cap \mathbb{Z}^2| = |P_{naD_z} \cap \mathbb{Z}^2| = c_2(cD_x, n)n^2 + c_1(cD_x, n)n + c_0(cD_x, n),
$$

where

$$
\begin{aligned}
c_2(cD_x, n) &= \frac{2}{s}, \\
c_1(cD_x, n) &= \frac{1}{2} \left(1 + \frac{1}{s} + \frac{4}{b} \right) n, \\
c_0(cD_x, n) &= 1 - \left\{ \frac{4n}{b} \right\} - \frac{b-1}{2} \left\{ \frac{4n}{b} \right\} + \sum_{j=0}^{b\{\frac{4n}{b}\}} \left\{ \frac{-p}{b} j \right\}. \qquad \square
\end{aligned}
$$

Our next goal is to give upper bounds on the constant terms of the Ehrhart quasi-polynomials of $|P_{n\delta D_x} \cap \mathbb{Z}^2|$, $\delta = b, c$. In Proposition 3.1, notice that the expressions of the last two terms of $c_0(bD_x, n)$ and $c_0(cD_x, n)$ are of the same form, which we will analyze in depth in Section 4. In the following, we give a uniform upper bound on $c_0(bD_x, n)$ minus its last two terms.

Lemma 3.2. *In the expression of $c_0(bD_x, n)$, we have*

$$
-\frac{5}{2} \{sn\} + \sum_{j=0}^{4\left\{\frac{\lfloor 4sn \rfloor}{4}\right\}} \left\{ \frac{3}{4} j \right\} \leq \frac{1}{8}
$$

for all $n \geq 0$, where $s = \frac{b}{c}$. Furthermore:

(1) The above expression is positive if and only if $\frac{1}{4} < \{sn\} < \frac{3}{10}$.

(2) The above expression is greater than $\frac{-1}{32s}$ only if $\{sn\} < \frac{1}{2} + \frac{1}{80s}$.

Proof. Let $bn = mc + r$ with $0 \leq r \leq c-1$, so that $\{sn\} = \frac{r}{c}$. Let $\ell = 0, 1, 2, 3$ be the integer such that $\frac{\ell c}{4} \leq r < \frac{(\ell+1)c}{4}$. Then $\lfloor 4sn \rfloor = 4m + \ell$ and so $4\left\{\frac{\lfloor 4sn \rfloor}{4}\right\} = \ell$.

Now, we will bound the given expression from the above for each $\ell = 0, 1, 2, 3$. If $\ell = 0$, then the given expression in the lemma is $-\frac{5r}{2c} \leq 0$. If $\ell = 1$, we have $-\frac{5r}{2c} + \frac{3}{4} \leq -\frac{5}{2} \cdot \frac{1}{4} + \frac{3}{4} = \frac{1}{8}$. If $\ell = 2$, we have $-\frac{5r}{2c} + \frac{3}{4} + \frac{2}{4} \leq -\frac{5}{2} \cdot \frac{1}{2} + \frac{5}{4} = 0$. Lastly, if $\ell = 3$, we have $-\frac{5r}{2c} + \frac{3}{4} + \frac{2}{4} + \frac{1}{4} \leq -\frac{5}{2} \cdot \frac{3}{4} + \frac{3}{2} = -\frac{3}{8}$.

For the final statements of the lemma, we see the expression is non-positive if $\ell \neq 1$, and so we must have $\frac{1}{4} < \{sn\}$. When $\ell = 1$, we computed the expression is equal to $-\frac{5r}{2c} + \frac{3}{4}$, which is positive if and only if $\{sn\} = \frac{r}{c} < \frac{3}{10}$.

Similarly, the expression could be greater than $\frac{-1}{32s}$ in cases $\ell = 0, 1, 2$. (Note that $s < 1$ because $b < c$.) Working case by case with the expressions obtained, we obtain that $\{sn\} = \frac{r}{c} < \frac{1}{2} + \frac{1}{80s}$ in order for the expression to be greater than $\frac{-1}{32s}$. □

Note that $-\frac{1}{8s}(\{4sn\}^2 - \{4sn\}) \leq \frac{1}{32s}$ since the function $x - x^2$ is maximized at $x = \frac{1}{2}$. Combining this observation with Lemma 3.2, we obtain the following corollary.

Corollary 3.3. *We have*

$$c_0(bD_x, n) \leq \frac{9}{8} + \frac{1}{32s} + \frac{b-1}{2}\left\{\frac{4n}{c}\right\} - \sum_{j=0}^{b\left\{\frac{\lfloor 4sn \rfloor}{b}\right\}} \left\{\frac{-p}{b} j\right\}.$$

Moreover, if $\{sn\} \geq \frac{1}{2} + \frac{1}{80s}$, *we may improve the above bound as follows:*

$$c_0(bD_x, n) \leq 1 + \frac{b-1}{2}\left\{\frac{4n}{c}\right\} - \sum_{j=0}^{b\left\{\frac{\lfloor 4sn \rfloor}{b}\right\}} \left\{\frac{-p}{b} j\right\}.$$

4. Bounding $c_0(bD_x, n)$ and $c_0(cD_x, n)$

In this section, we prove the key results needed to bound $c_0(bD_x, n)$ and $c_0(cD_x, n)$. This amounts to obtaining bounds for the expression $\frac{b-1}{2}\left\{\frac{4n}{c}\right\} - \sum_{j=0}^{r}\{\frac{-p}{b} j\}$, where $r = b\{\frac{\lfloor 4sn \rfloor}{b}\}$. We begin by recording the following lemma.

Lemma 4.1. *Let* $n, b, c \in \mathbb{Z}^+$ *with* $4 < b < c$ *and* $\gcd(4, b, c) = 1$. *Let* $p < 0$ *be an integer satisfying* $4p + 3b = c$ *and* $s = \frac{b}{c}$. *If* $r = b\left\{\frac{\lfloor 4sn \rfloor}{b}\right\}$, *then*

$$\frac{b-1}{2}\left\{\frac{4n}{c}\right\} - \sum_{j=0}^{r}\left\{\frac{-p}{b} j\right\} \leq \frac{(b-1)(r+1)}{2b} - \sum_{j=0}^{r}\left\{\frac{-p}{b} j\right\}.$$

Proof. Since r is the reminder when b is divided into $\lfloor 4sn \rfloor$, we have $4sn = \lfloor 4sn \rfloor + \{4sn\} = b\lfloor \frac{\lfloor 4sn \rfloor}{b} \rfloor + r + \{4sn\}$. So,

$$\left\{\frac{4n}{c}\right\} = \left\{\frac{4sn}{b}\right\} = \left\{\left\lfloor \frac{\lfloor 4sn \rfloor}{b} \right\rfloor + \frac{r}{b} + \frac{\{4sn\}}{b}\right\} = \frac{r + \{4sn\}}{b},$$

where the last equality uses $0 \leq \frac{r}{b} \leq \frac{b-1}{b}$ and $0 \leq \frac{\{4sn\}}{b} < \frac{1}{b}$. □

By the above lemma, it suffices to bound the expression on the righthand side. In §4.1, we give a general algorithm to obtain bounds on expressions of the form $\frac{(\beta-1)(u+1)}{2\beta} - \sum_{j=0}^{u}\{\frac{\alpha}{\beta}j\}$ when α and β satisfy particular Diophantine equations, see Corollary 4.3.

4.1. An algorithm to bound expressions of the form $\frac{(\beta-1)(u+1)}{2\beta} - \sum_{j=0}^{u}\{\frac{\alpha}{\beta}j\}$.

Our goal in this subsection is to prove:

Proposition 4.2. *Suppose that $\alpha_0 > \alpha_1$, $\beta_0 > \beta_1$, and*

$$\alpha_1\beta_0 - \beta_1\alpha_0 = \sigma = \pm 1$$

where $\alpha_i, \beta_i \in \mathbb{Z}^+$. Let $u_0 = \beta_1 t_1 + u_1$, where $u_0, u_1, t_1 \in \mathbb{Z}^{\geq 0}$ and $0 \leq u_i < \beta_i$ for all i. Then

$$\frac{(u_0+1)(\beta_0-1)}{2\beta_0} - \sum_{j=0}^{u_0}\left\{\frac{\alpha_0}{\beta_0}j\right\} = \frac{(u_1+1)(\beta_1-1)}{2\beta_1} - \sum_{j=0}^{u_1}\left\{\frac{\alpha_1 j}{\beta_1}\right\} + \epsilon(\sigma, t_1, u_1, \beta_0, \beta_1),$$

where

$$\epsilon(\sigma, t, u, \beta', \beta) = \frac{(u+1)(\sigma u + \beta' - \beta)}{2\beta'\beta} + \frac{\sigma t(\beta(t-\sigma) + 2u + 1 - \beta')}{2\beta'} + \Delta(\sigma, t, u, \beta', \beta)$$

and $\Delta(\sigma, t, u, \beta', \beta) = 1$ if $\beta' - \beta \leq \beta t + u$ and $\sigma = -1$, and $\Delta(\sigma, u, \beta', \beta) = 0$ otherwise.

When applied iteratively, we arrive at the following algorithm.

Corollary 4.3. *Suppose we have sequences of positive integers $\alpha_0 > \alpha_1 > \cdots > \alpha_N$ and $\beta_0 > \beta_1 > \cdots > \beta_N$ such that for all i,*

$$\alpha_i\beta_{i-1} - \beta_i\alpha_{i-1} = \sigma_i = \pm 1.$$

Let $u_0, \ldots, u_N$ and $t_1, \ldots, t_N$ be non-negative integers satisfying $u_{i-1} = \beta_i t_i + u_i$ and $0 \leq u_i < \beta_i$ for all i. Then

$$\frac{(u_0+1)(\beta_0-1)}{2\beta_0} - \sum_{j=0}^{u_0}\left\{\frac{\alpha_0}{\beta_0}j\right\} = \frac{(u_N+1)(\beta_N-1)}{2\beta_N} - \sum_{j=0}^{u_N}\left\{\frac{\alpha_N j}{\beta_N}\right\} + \sum_{i=1}^{N}\epsilon(\sigma_i, t_i, u_i, \beta_{i-1}, \beta_i),$$

where ϵ is as in Proposition 4.2.

We begin with the following preliminary lemmas.

Lemma 4.4. *Let $\alpha, \beta \in \mathbb{Z}^+$ be relatively prime. Then*

$$\sum_{k=0}^{\beta-1}\left\lfloor \frac{\alpha}{\beta}k \right\rfloor = \frac{1}{2}(\alpha-1)(\beta-1) \qquad \text{and} \qquad \sum_{k=0}^{\beta-1}\left\lceil \frac{\alpha}{\beta}k \right\rceil = \frac{1}{2}(\alpha+1)(\beta-1).$$

Proof. Notice that $\sum_{k=0}^{\beta}\lfloor \frac{\alpha}{\beta}k\rfloor + \beta + 1$ is the number of lattice points in the triangle with vertices 0, $(\beta, 0)$, and (β, α). So, by Pick's Theorem,

$$\sum_{k=0}^{\beta}\left\lfloor \frac{\alpha}{\beta}k \right\rfloor + \beta + 1 = \frac{1}{2}(\alpha\beta + \alpha + \beta + 1) + 1.$$

Since $\sum_{k=0}^{\beta}\lfloor \frac{\alpha}{\beta}k\rfloor = \sum_{k=0}^{\beta-1}\lfloor \frac{\alpha}{\beta}k\rfloor + \alpha$, the first result follows. The second result follows from the first and the fact that $\sum_{k=0}^{\beta-1}\lceil \frac{\alpha}{\beta}k\rceil = (\beta-1) + \sum_{k=0}^{\beta-1}\lfloor \frac{\alpha}{\beta}k\rfloor$. □

Lemma 4.5. *Suppose that $\alpha_0 > \alpha_1$, $\beta_0 > \beta_1$, and*

$$\alpha_1\beta_0 - \beta_1\alpha_0 = \sigma = \pm 1$$

where $\alpha_i, \beta_i \in \mathbb{Z}^+$. Then

(1) $\{\frac{\alpha_0 j}{\beta_0}\} = \{\frac{\alpha_1 j}{\beta_1}\} - \sigma\frac{j}{\beta_1\beta_0}$ for all $0 \le j < \beta_1$, and
(2) for any integer u satisfying $0 \le u < \beta_1$,

$$\sum_{j=0}^{u}\left\{\frac{\alpha_0 j}{\beta_0}\right\} = \sum_{j=0}^{u}\left\{\frac{\alpha_1 j}{\beta_1}\right\} - \frac{\sigma}{\beta_0\beta_1}\binom{u+1}{2}$$

(3)

$$\sum_{j=0}^{\beta_1}\left\{\frac{\alpha_0 j}{\beta_0}\right\} = \frac{1-\sigma+(\beta_1+\sigma)(\beta_0-\sigma)}{2\beta_0}.$$

Proof. We begin with the proof of (1). The case $j = 0$ is clear, so we assume $1 \le j \le \beta_1 - 1$. Since $\frac{\alpha_0 j}{\beta_0} = \frac{\alpha_1 j}{\beta_1} - \sigma\frac{j}{\beta_1\beta_0}$, it suffices to show $0 \le \{\frac{\alpha_1 j}{\beta_1}\} - \sigma\frac{j}{\beta_1\beta_0} < 1$. Since α_1 and β_1 are relatively prime, we see $\frac{1}{\beta_1} \le \{\frac{\alpha_1 j}{\beta_1}\} \le 1 - \frac{1}{\beta_1}$. The result then follows from the fact that $|\sigma\frac{j}{\beta_1\beta_0}| < \frac{1}{\beta_0} < \frac{1}{\beta_1}$.

Part (2) follows from (1) by noting

$$\sum_{j=0}^{u}\left\{\frac{\alpha_0 j}{\beta_0}\right\} = \sum_{j=0}^{u}\left(\left\{\frac{\alpha_1 j}{\beta_1}\right\} - \sigma\frac{j}{\beta_0\beta_1}\right) = \sum_{j=0}^{u}\left\{\frac{\alpha_1 j}{\beta_1}\right\} - \frac{\sigma}{\beta_0\beta_1}\binom{u+1}{2}.$$

To prove (3), we use $\sum_{j=0}^{\beta_1}\{\frac{\alpha_0 j}{\beta_0}\} = \frac{\sigma+1}{2} - \frac{\sigma}{\beta_0} + \sum_{j=0}^{\beta_1-1}\{\frac{\alpha_0 j}{\beta_0}\}$ and part (2) to see

$$\sum_{j=0}^{\beta_1}\left\{\frac{\alpha_0 j}{\beta_0}\right\} = \frac{\sigma+1}{2} - \frac{\sigma}{\beta_0} + \sum_{j=0}^{\beta_1-1}\left(\frac{\alpha_1 j}{\beta_1} - \left\lfloor \frac{\alpha_1 j}{\beta_1}\right\rfloor\right) - \frac{\sigma}{\beta_0\beta_1}\binom{\beta_1}{2}$$
$$= \frac{\sigma+1}{2} - \frac{\sigma}{\beta_0} + \frac{\alpha_1}{\beta_1}\binom{\beta_1}{2} - \frac{1}{2}(\alpha_1-1)(\beta_1-1) - \sigma\frac{1}{\beta_0\beta_1}\binom{\beta_1}{2},$$

where the second equality uses Lemma 4.4. The result follows by algebraic manipulation. □

The following result is the first step in proving Proposition 4.2.

Corollary 4.6. *With hypotheses as in Proposition 4.2, we have*

$$\frac{(u_1+1)(\beta_0-1)}{2\beta_0} - \sum_{j=0}^{u_1}\left\{\frac{\alpha_0 j}{\beta_0}\right\} = \frac{(u_1+1)(\beta_1-1)}{2\beta_1} - \sum_{j=0}^{u_1}\left\{\frac{\alpha_1 j}{\beta_1}\right\} + \epsilon'(\sigma, t_1, u_1, \beta_0, \beta_1),$$

where

$$\epsilon'(\sigma, t, u, \beta', \beta) = \frac{(u+1)(\sigma u + \beta' - \beta)}{2\beta'\beta}.$$

Proof. By Lemma 4.5 (2),

$$\frac{(u_1+1)(\beta_0-1)}{2\beta_0} - \sum_{j=0}^{u_1}\left\{\frac{\alpha_0 j}{\beta_0}\right\} = \frac{(u_1+1)(\beta_0-1)}{2\beta_0} - \sum_{j=0}^{u_1}\left\{\frac{\alpha_1 j}{\beta_1}\right\} + \frac{\sigma}{\beta_0\beta_1}\binom{u_1+1}{2}.$$

Since

$$\frac{(u_1+1)(\beta_0-1)}{2\beta_0} + \frac{\sigma}{\beta_0\beta_1}\binom{u_1+1}{2} = \frac{(u_1+1)(\beta_1-1)}{2\beta_1} + \epsilon'(\sigma, t_1, u_1, \beta_0, \beta_1),$$

the result follows. □

The next step in proving Proposition 4.2 is provided by:

Corollary 4.7. *With hypotheses as in Proposition 4.2, we have*

$$\frac{(u_0+1)(\beta_0-1)}{2\beta_0} - \sum_{j=0}^{u_0}\left\{\frac{\alpha_0}{\beta_0}j\right\} = \frac{(u_1+1)(\beta_0-1)}{2\beta_0} - \sum_{j=0}^{u_1}\left\{\frac{\alpha_0 j}{\beta_0}\right\} + \epsilon''(\sigma, t_1, u_1, \beta_0, \beta_1),$$

where

$$\epsilon''(\sigma, t, u, \beta', \beta) = \frac{\sigma t(\beta(t-\sigma) + 2u + 1 - \beta')}{2\beta'} + \Delta(\sigma, t, u, \beta', \beta)$$

and Δ is as in Proposition 4.2.

Proof. We first claim that if $j \in \mathbb{Z}^+$ and $1 \leq j < \beta_0$, then

$$\left\{\frac{\alpha_0(j+\beta_1)}{\beta_0}\right\} = \left\{\frac{\alpha_0 j}{\beta_0}\right\} - \frac{\sigma}{\beta_0} - \Delta'(\sigma, j), \tag{4.8}$$

where $\Delta'(\sigma, j) = 1$ if $j = \beta_0 - \beta_1$ and $\sigma = -1$, and $\Delta'(\sigma, j) = 0$ otherwise. Since $\alpha_0\beta_1 \equiv -\sigma \bmod \beta_0$, to prove our claim, it is enough to show the righthand side of (4.8) lies in the interval $[0,1)$. Note that $\{\frac{\alpha_0 j}{\beta_0}\} - \frac{\sigma}{\beta_0} \in [0,1)$ unless either: (i) $\sigma = 1$ and $\{\frac{\alpha_0 j}{\beta_0}\} = 0$, or (ii) $\sigma = -1$ and $\{\frac{\alpha_0 j}{\beta_0}\} = \frac{\beta_0 - 1}{\beta_0}$. Case (i) never occurs since $\gcd(\alpha_0, \beta_0) = 1$ and $1 \le j < \beta_0$, so $\alpha_0 j$ is not divisible by β_0. Case (ii) occurs exactly when $\alpha_0 j \equiv -1 \mod \beta_0$, i.e. when $j = \beta_0 - \beta_1$. This establishes our claim.

Recalling that $u_0 = \beta_1 t_1 + u_1$, we see from equation (4.8) and Lemma 4.5 (3) that

$$\begin{aligned}
\sum_{j=0}^{u_0} \left\{\frac{\alpha_0 j}{\beta_0}\right\} &= \sum_{j=\beta_1 t_1+1}^{u_0} \left\{\frac{\alpha_0 j}{\beta_0}\right\} + \sum_{j=1}^{\beta_1 t_1} \left\{\frac{\alpha_0 j}{\beta_0}\right\} \\
&= \sum_{j=1}^{u_1} \left\{\frac{\alpha_0 j}{\beta_0}\right\} - \sigma \frac{u_1 t_1}{\beta_0} + t_1 \sum_{j=1}^{\beta_1} \left\{\frac{\alpha_0 j}{\beta_0}\right\} - \sigma \frac{\beta_1}{\beta_0}\binom{t_1}{2} - \Delta(\sigma, t_1, u_1, \beta_0, \beta_1) \\
&= \sum_{j=0}^{u_1} \left\{\frac{\alpha_0 j}{\beta_0}\right\} - \sigma \frac{u_1 t_1}{\beta_0} + t_1 \frac{1 - \sigma + (\beta_1 + \sigma)(\beta_0 - \sigma)}{2\beta_0} \\
&\qquad - \sigma \frac{\beta_1}{\beta_0}\binom{t_1}{2} - \Delta(\sigma, t_1, u_1, \beta_0, \beta_1) \\
&= \sum_{j=0}^{u_1} \left\{\frac{\alpha_0 j}{\beta_0}\right\} + \frac{(u_0+1)(\beta_0-1)}{2\beta_0} \\
&\qquad - \frac{(u_1+1)(\beta_0-1)}{2\beta_0} - \epsilon''(\sigma, t_1, u_1, \beta_0, \beta_1). \qquad \square
\end{aligned}$$

We now turn to the main result of this subsection.

Proof of Proposition 4.2. Noting that $\epsilon(\sigma, t_1, u_1, \beta_0, \beta_1) = \epsilon'(\sigma, t_1, u_1, \beta_0, \beta_1) + \epsilon''(\sigma, t_1, u_1, \beta_0, \beta_1)$, we see

$$\begin{aligned}
\frac{(u_0+1)(\beta_0-1)}{2\beta_0} - \sum_{j=0}^{u_0} \left\{\frac{\alpha_0}{\beta_0} j\right\} &= \frac{(u_1+1)(\beta_0-1)}{2\beta_0} - \sum_{j=0}^{u_1} \left\{\frac{\alpha_0 j}{\beta_0}\right\} + \epsilon''(\sigma, t_1, u_1, \beta_0, \beta_1) \\
&= \frac{(u_1+1)(\beta_1-1)}{2\beta_1} - \sum_{j=0}^{u_1} \left\{\frac{\alpha_1 j}{\beta_1}\right\} + \epsilon(\sigma, t_1, u_1, \beta_0, \beta_1)
\end{aligned}$$

where the first equality is by Corollary 4.7 and the second equality is by Corollary 4.6. □

We end the subsection by giving a bound on ϵ.

Lemma 4.9. *For $t, u, \beta', \beta \in \mathbb{Z}$ such that $0 \le u < \beta < \beta'$ and $0 \le t \le \lfloor \frac{\beta' - 1 - u}{\beta} \rfloor$, we have*

$$\frac{-(\beta' + \beta - 1)^2 + 4(\beta' - \beta)}{8\beta'\beta} \le \epsilon(1, t, u, \beta', \beta) \le \frac{\beta' - 1}{2\beta'}$$

and

$$\frac{\beta'-\beta}{2\beta'\beta} \le \epsilon(-1,t,u,\beta',\beta) \le \begin{cases} \frac{1}{8\beta\beta'}(\beta'-\beta+3)(\beta'-\beta-1), & \text{if } u+\beta t < \beta'-\beta \\ 1, & \text{otherwise.} \end{cases}$$

Furthermore, letting $v = u + \beta t$,

$$\epsilon(-1,t,u,\beta',\beta) = -\frac{1}{2\beta'\beta}(v+1)(v+\beta-\beta') + \Delta$$

and

$$\epsilon(1,t,u,\beta',\beta) = \frac{1}{2\beta'\beta}(v+1)(v-\beta'-\beta) + \frac{1}{\beta}(u+1).$$

As functions of v, the former (resp. latter) is increasing (resp. decreasing) if and only if $v \le \frac{1}{2}(\sigma\beta + \beta' - 1)$, where u is viewed as a constant in the latter.

Proof. Throughout the proof, we treat β and β' as fixed constants. Letting $v = u + \beta t$, we find

$$\eta := 2\beta'\beta\epsilon = \sigma v^2 + (\sigma(1-\beta') - \beta)v + \beta'(1+\sigma)u + \beta' - \beta + 2\beta'\beta\Delta$$

Then, the expressions $\epsilon(\pm 1, t, u, \beta', \beta)$ are obtained by substituting $\sigma = \pm 1$ into η. It suffices to bound ϵ on the larger region $0 \le t \le \frac{\beta'-1-u}{\beta}$, where our constraints become $0 \le u \le \beta - 1$ and $u \le v \le \beta' - 1$.

We first consider the case $\sigma = -1$ and bound ϵ from the above. Recall that $\Delta = 0$ if $v < \beta' - \beta$ and $\Delta = 1$ otherwise. Then $\eta = -(v+1)(v+\beta-\beta') + 2\beta'\beta\Delta$ has a global maximum at $v_{\max} := \frac{\beta'-\beta+1}{2}$. Since $0 \le v_{\max} \le \beta' - \beta$, we see that if $v < \beta' - \beta$, then $\eta(u,v) \le \eta(0, v_{\max}) = \frac{1}{4}(\beta'-\beta+3)(\beta'-\beta-1)$. If, on the other hand, $\beta' - \beta \le v$, then $\eta(u,v) \le \eta(0, \beta'-\beta) = 2\beta'\beta$. The lower bound of ϵ is then obtained by calculating ϵ when $v = 0, \beta' - 1$ and taking the minimum of the two.

We next consider the case $\sigma = 1$. For fixed u, the function $\eta(u,v)$ has a global minimum at $v_{\min} := \frac{\beta+\beta'-1}{2}$. Since $u \le \beta - 1 < v_{\min} \le \beta' - 1$ and $|v_{\min} - (\beta'-1)| < |v_{\min} - (\beta - 1)|$, we see $\eta(u,v) \le \eta(u,u)$. As $\eta(u,u)$ is a quadratic in u with global minimum at $\frac{\beta-\beta'-1}{2} < 0$, we find $\eta(u,u) \le \eta(\beta - 1, \beta - 1) = \beta(\beta'-1)$. This gives the upper bound on ϵ when $\sigma = 1$, and the lower bound is obtained by substituting $v = v_{\min} = \frac{\beta+\beta'-1}{2}$ and $u = 0$ into the expression $\epsilon(1,t,u,\beta',\beta)$.

The final statement concerning where $\epsilon(\sigma,t,u,\beta',\beta)$ is increasing is clear from the expression of η. ☐

5. Proof of Theorem 1.5

We prove Theorem 1.5 using the procedure outlined in Remark 2.7. Throughout this section, we fix the following notation. Let $I = (\frac{\beta^{(1)}}{\alpha^{(1)}}, \frac{\beta^{(2)}}{\alpha^{(2)}})$ be one of the four types of intervals listed in Theorem 1.5 and let $D_0 \sim m_0 H \sim n_0 \delta D_x$ be as listed in Theorem 1.5, with $\delta \in \{b, c\}$, $n_0 = \frac{bcm_0}{\delta} \in \mathbb{Z}^+$, and ν_0 be the proposed $\nu(D_0)$ from Theorem 1.5; for example, if $\beta^{(1)} = 16k^2$ and $\alpha^{(1)} = 8k^2 - 4k - 1$, then $D_0 \sim (2k+1)bD_x$ and $\nu_0 = 4k$. Throughout this section, for $\delta' \in \{b, c\}$, we let $|n\delta' D_x| := |P_{n\delta' D_x} \cap \mathbb{Z}^2|$. To prove Theorem 1.5, we must show

$$|n\delta' D_x| < \binom{\lceil \frac{\delta'}{\delta} \frac{\nu_0}{n_0} n \rceil + 1}{2} + 1 \tag{5.1}$$

for each n whenever $\delta' \neq \delta$, and for each n not a multiple of n_0 whenever $\delta' = \delta$. Moreover, for each n a multiple of n_0 with $\delta' = \delta$, we need to show

$$|n\delta D_x| < \binom{\frac{\nu_0}{n_0} n + 2}{2} \tag{5.2}$$

Note that inequality (5.2) implies that $\nu(D_0) \leq \nu_0$ for each D_0 as listed in Theorem 1.5, by Definition 1.3. Combining this with the result $\nu(D_0) \geq \nu_0$ established by Proposition 2.3, we may prove the claim $\nu(D_0) = \nu_0$ in Theorem 1.5.

By Remark 2.7, it suffices to prove (5.1) and (5.2) for weighted projective surfaces satisfying either $\alpha^{(1)} b + \beta^{(1)} p = 1$ or $\alpha^{(2)} b + \beta^{(2)} p = -1$ for $\frac{b}{-p} \in I$. It also suffices to consider b (thus $-p$ and c) sufficiently large. We begin with the proof of (5.1), the more challenging of the above two equations:

Theorem 5.3. *Inequality (5.1) holds for each n whenever $\delta' \neq \delta$, and for each n not a multiple of n_0 whenever $\delta' = \delta$.*

Proof. First, suppose that $\delta' = \delta$. Thus, we need to consider weighted projective surfaces satisfying $\alpha_1 b + \beta_1 p = \pm 1$ with $\frac{b}{-p} \in I'_{k,\mp}$ or $\frac{b}{-p} \in I''_{k,\mp}$ as listed in Theorem 1.5. Notice that α_1, β_1 are the corresponding ones listed in Entries 1 and 2 of Table 1. For the sake of brevity, we prove the result when $\delta' = \delta = b$, $\beta_1 := \beta^{(1)} = 16k^2$ and $\alpha_1 := \alpha^{(1)} = 8k^2 - 4k - 1$, over the interval $I'_{k,-} = [\frac{\beta_1}{\alpha_1}, \frac{2k-1}{k-1}]$ for $k \geq 2$. The other cases are similar and in fact easier.[1] By Remark 2.7, it suffices to prove the result for b sufficiently large, where

$$\alpha_1 b - \beta_1(-p) = 1.$$

Let $\alpha_0 := -p$, $\beta_0 := b$, and

$$|n\delta' D_x| = c_2 n^2 + c_1 n + c_0$$

where the c_i are given as in Section 3.

[1] The reason the case considered in this proof is the most difficult is because the upper bounds on ϵ are weakest when $\sigma = -1$; the case considered here corresponds to Entry 1 of Table 1 which has the most number of $\sigma_i = -1$. In addition, among the entries of the table, Entry 1 has the most number of steps.

We begin by giving an upper bound for c_0. Letting $r = b\left\{\frac{\lfloor 4sn \rfloor}{b}\right\}$, we see from Corollary 3.3 and Lemma 4.1 that

$$c_0 \le \frac{9}{8} + \frac{1}{32s} + \varkappa$$

where $\varkappa$ is an upper bound on $\frac{b-1}{2}\{\frac{4n}{c}\} - \sum_{j=0}^{r}\{\frac{-p}{b}j\}$. To obtain such a bound, we apply Corollary 4.3 with $(\alpha_i, \beta_i, \sigma_i)$ given as in Entry 1 of Table 1 below (the other entries listed in the table are used to address the remaining cases whose proof we omit). Note that for each $i \ge 1$, we have $\alpha_i\beta_{i-1} - \beta_i\alpha_{i-1} = \sigma_i$. Therefore, if we let $u_0 := r$ and $u_1, \dots, u_5$ and $t_1, \dots, t_5$ be as in Corollary 4.3, and

$$\epsilon_i := \epsilon(\sigma_i, t_i, u_i, \beta_{i-1}, \beta_i),$$

we have

$$c_0 \le \frac{9}{8} + \frac{1}{32s} + \varkappa' + \sum_{i=1}^{5} \epsilon_i,$$

where

$$\varkappa' := \frac{u_5 + 1}{4} - \sum_{j=0}^{u_5} \left\{\frac{j}{2}\right\} \le \frac{1}{4}.$$

Entry	α_i	β_i	σ_i
1	$\alpha_1 = 8k^2 - 4k - 1$ $\alpha_2 = 4k^2 - 4k + 1$ $\alpha_3 = 4k - 3$ $\alpha_4 = k - 1$ $\alpha_5 = 1$	$\beta_1 = 16k^2$ $\beta_2 = 8k^2 - 4k + 1$ $\beta_3 = 8k - 2$ $\beta_4 = 2k - 1$ $\beta_5 = 2$	$\sigma_1 = 1$ $\sigma_2 = 1$ $\sigma_3 = -1$ $\sigma_4 = -1$ $\sigma_5 = 1$
2	$\alpha_1 = 8k^2 + 4k - 1$ $\alpha_2 = 4k^2$ $\alpha_3 = 4k - 1$ $\alpha_4 = k$ $\alpha_5 = 1$	$\beta_1 = 4(2k+1)^2$ $\beta_2 = 8k^2 + 4k + 1$ $\beta_3 = 8k + 2$ $\beta_4 = 2k + 1$ $\beta_5 = 2$	$\sigma_1 = 1$ $\sigma_2 = 1$ $\sigma_3 = -1$ $\sigma_4 = 1$ $\sigma_5 = 1$
3	$\alpha_1 = 2k - 1$ $\alpha_2 = k$ $\alpha_3 = 1$	$\beta_1 = 4k$ $\beta_2 = 2k + 1$ $\beta_5 = 2$	$\sigma_1 = 1$ $\sigma_2 = 1$ $\sigma_3 = 1$
4	$\alpha_1 = k$ $\alpha_2 = 1$	$\beta_1 = 2k + 1$ $\beta_2 = 2$	$\sigma_1 = 1$ $\sigma_2 = 1$

TABLE 1. α_i and β_i used to bound $\frac{b-1}{2}\{\frac{4n}{c}\} - \sum_{j=0}^{r}\{\frac{-p}{b}j\}$ via Corollary 4.3, when considering $D \sim nbD_x$. We remark that when considering $D \sim ncD_x$, i.e., bounding $-\frac{b-1}{2}\{\frac{4n}{b}\} + \sum_{j=0}^{r}\{\frac{-p}{b}j\}$ in the constant term of $|P_{ncD_x} \cap \mathbb{Z}^2|$, $\sigma_1 = 1$ needs to be replaced by $\sigma_1 = -1$.

We begin by taking crude upper bounds on the ϵ_i. For small n, we will need to replace these crude bounds with more refined ones. By Lemma 4.9, we have

$$\epsilon_i \leq \frac{\beta_{i-1}-1}{2\beta_{i-1}} =: \epsilon_i^+$$

for $i \in \{1,2,5\}$ and

$$\epsilon_4 \leq 1 =: \epsilon_4^+.$$

Furthermore, one checks that for $k \geq 11$,

$$\epsilon_3 \leq \frac{1}{8\beta_2\beta_3}(\beta_2-\beta_3+3)(\beta_2-\beta_3-1) =: \epsilon_3^+$$

as $\epsilon_3^+ \geq 1$. It is enough to prove Theorem 5.3 for $k \geq 11$, leaving the remaining finitely many cases $2 \leq k \leq 10$ to be checked by hand.

Next, solving for p in terms of b, we have $p = \frac{1-\alpha_1 b}{\beta_1}$ from which we find $c = 4p+3b = \frac{1+b(1+2k)^2}{4k^2}$. It follows that $1 = \frac{1}{4k^2}\frac{1}{c} + \frac{(1+2k)^2}{4k^2}s$, and so

$$\frac{1}{c} = 4k^2 - (1+2k)^2 s.$$

Note also that from our expression for c in terms of b, we have

$$s = \frac{4k^2}{(1+2k)^2+\frac{1}{b}}.$$

Combining this with our results from Section 3, we see

$$c_1 = \frac{1}{2} + \frac{\beta_1}{\frac{8}{b}+2(3\beta_1-4\alpha_1)} + 2(4k^2-(1+2k)^2 s).$$

Recall also that

$$c_2 = \frac{2\beta_1}{\frac{4}{b}+(3\beta_1-4\alpha_1)}.$$

We have therefore expressed c_2, c_1, s, p, and c all in terms of k and b.

Let $n = (2k+1)t+u$ for $t \geq 0$, $1 \leq u \leq 2k$, where $u \neq 0$ since $n_0 = 2k+1$ does not divide n. We can then express

$$\left\lceil \frac{\nu_0}{n_0} n \right\rceil = \left\lceil \frac{4k}{2k+1} n \right\rceil = 4kt+2u+\epsilon := \begin{cases} 4kt+2u & \text{if } 1 \leq u \leq k, \\ 4kt+2u-1 & \text{if } k+1 \leq u \leq 2k. \end{cases}$$

We are now ready to show

$$f := c_2 n^2 + c_1 n + c_0 - \binom{\left\lceil \frac{\nu_0}{n_0} n \right\rceil + 1}{2} < 1.$$

Replacing c_0 by quantity $\frac{9}{8}+\frac{1}{32s}+\frac{1}{4}+\sum_{i=1}^5 \epsilon_i^+$, we obtain a larger function $g = \frac{g_1}{g_2}$ where the g_i are polynomials in t, u, b, k and $g_2 > 0$. One checks that

$g_2 - g_1$ is decreasing in t and that it is a quadratic in b with positive b^2-coefficient for $t > \frac{k}{16}$. Thus, for $t > \frac{k}{16}$ and b sufficiently large, we have shown $f < 1$.

We next turn to the case where $t \le \frac{k}{16}$. Then $n = (2k+1)t + u \le \frac{k^2}{8} + \frac{33k}{16}$ and so

$$r \le 4sn < \frac{\beta_2 - \beta_3 - 1}{2} < \beta_2.$$

As a result, $r = u_1 = u_2$ and $t_1 = t_2 = 0$. We may therefore plug in directly to the definition of ϵ_1 and ϵ_2 to obtain better bounds than ϵ_1^+ and ϵ_2^+; using the final statement of Lemma 4.9 and the fact that $\frac{\beta_3 - \beta_2 - 1}{2} < 0 \le r$, we see

$$\epsilon_i \le \frac{1}{2\beta_{i-1}\beta_i}(4sn+1)(4sn+\beta_{i-1}-\beta_i) =: \epsilon_i^{++}$$

for $i \in \{1,2\}$. Similarly, we find

$$\epsilon_3 \le -\frac{1}{2\beta_2\beta_3}(4sn+1)(4sn+\beta_3-\beta_2) =: \epsilon_3^{++}.$$

Using the same argument as in the previous paragraph, replacing the use of ϵ_i^+ with ϵ_i^{++} for $i \in \{1,2,3\}$, we now find that $g_1 - g_2$ is a cubic in b with positive b^3-coefficient whenever $n \ge \sqrt{k}$ and $n \ne k+1$.

It therefore remains to handle the cases $n = k+1$ and $n < \sqrt{k}$. We consider $n < \sqrt{k}$ first. Here, $t_1 = t_2 = t_3 = t_4 = 0$ and $r = u_1 = u_2 = u_3 = u_4$. Furthermore, $r \le 4sn < \frac{\beta_3 - \beta_4 - 1}{2}$, so we have

$$\epsilon_4 \le -\frac{1}{2\beta_3\beta_4}(4sn+1)(4sn+\beta_4-\beta_3) =: \epsilon_4^{++}.$$

Now, since $\beta_5 = 2$, we know $u_5 = 0$ or $u_5 = 1$. Plugging back into the definition of κ' and ϵ_5 and using that ϵ_5 is increasing on the range from r to $4sn$, we find

$$\kappa' + \epsilon_5 \le \epsilon_5^{++} := \begin{cases} \frac{1}{4} + \frac{1}{2\beta_4}((4sn)^2 - (1+\beta_4)(4sn) + \beta_4 - 2), & r \text{ is even} \\ \frac{1}{2\beta_4}((4sn)^2 - (1+\beta_4)(4sn) + 3\beta_4 - 2), & r \text{ is odd.} \end{cases}$$

Treating these cases separately and replacing our use of c_0 with $\frac{9}{8} + \frac{1}{32s} + \sum_{i=1}^{5} \epsilon_i^{++}$ yields $f < 1$ for all $n < \sqrt{k}$.

Next, we turn to the case $n = k+1$. Here $r = 4k-1$, so $r = u_1 = u_2 = u_3$, $u_4 = u_5 = 1$, $t_1 = t_2 = t_3 = t_5 = 0$, and $t_4 = 2$. Since $\{sn\} = sn - (k-1) \ge \frac{1}{2} + \frac{1}{80s}$, Corollary 3.3 tells us $c_0 \le 1 + \kappa' + \sum_{i=1}^{5} \epsilon_i$. Directly using the definition of the ϵ_i functions, we find $g_1 - g_2$ is a quadratic in b with positive b^2-coefficient. This concludes our proof for $\delta' = \delta = b$, $\beta_1 = \beta^{(1)} = 16k^2$ and $\alpha_1 = \alpha^{(1)} = 8k^2 - 4k - 1$.

Finally, if we suppose $\delta' \ne \delta$ instead, then the weighted projective spaces considered are the ones satisfying $\alpha_1 b + \beta_1 p = \pm 1$ with $\frac{b}{-p} \in I'_{k,\pm}$ or $\frac{b}{-p} \in I''_{k,\pm}$.

Notice that α_1 and β_1 are the corresponding ones listed in Entries 3 and 4 of Table 1, which have considerably fewer steps than the ones for $\delta' = \delta$. The proof is almost exactly the same as the above, with the only difference being the technique used to rewrite $\lceil \frac{\delta'}{\delta}\frac{\nu_0}{n_0}n \rceil$ as a piecewise linear function in t, u such that $n_0 t + u = n$, $t \geq 0$, $0 \leq u \leq n_0 - 1$. We illustrate this with the case $\beta^{(2)} = \beta_1 = 2k-1, \alpha^{(2)} = \alpha_1 = k-1, \sigma_1 = -1$ over the interval $I'_{k,-} = [\frac{\beta^{(1)}}{\alpha^{(1)}}, \frac{\beta^{(2)}}{\alpha^{(2)}}]$ for $k \geq 2$, $\alpha^{(1)}$ and $\beta^{(1)}$ as in the previous paragraph. Here, $\delta = b$, $\delta' = c$, and $D_0 \sim (2k+1)bD_x$ with $\nu_0 = 4k$ as before. Notice that

$$\frac{c}{b} = 4\frac{p}{b} + 3 = \frac{4}{\beta^{(2)}}\left(-\alpha^{(2)} - \frac{1}{b}\right) + 3 = \frac{2k+1}{2k-1} - \frac{4}{(2k-1)b},$$

so that

$$\left\lceil \frac{c}{b}\frac{4k}{2k+1}n \right\rceil = \left\lceil \frac{4k}{2k-1}n \right\rceil$$

for all $\frac{16k}{(2k+1)(2k-1)b}n < \frac{1}{2k+1} \iff n < \frac{(2k-1)b}{16k}$. Thus, we may use our previous technique to rewrite the above ceiling function as a polynomial for all $n < \frac{(2k-1)b}{16k}$. Replacing the function f with an upper bound obtained by the same process as before, we may also conclude for $n > \frac{(2k-1)b}{16k}$ by examining the asymptotic behaviour of f, similarly as in the previous case.

We remark that for all other cases where $\delta' \neq \delta$, the ceiling function may be simplified in such a manner for all $n < Cb$ with $C > 0$ a constant. □

To finish the proof of Theorem 1.5, we must now handle the case where n_0 divides n with $\delta' = \delta$. This is substantially easier than Theorem 5.3.

Proposition 5.4. *Inequality (5.2) holds when n_0 divides n and $\delta' = \delta$.*

Proof. As in the proof of Theorem 5.3, we handle the case where $\beta_1 := \beta^{(1)} = 16k^2$, $\alpha_1 := \alpha^{(1)} = 8k^2 - 4k - 1$, $\delta = b$, $n_0 = 2k+1$, and $\nu_0 = 4k$. The other cases are similar and easier. By Remark 2.7, it suffices to prove the result for b sufficiently large, where

$$\alpha_1 b - \beta_1(-p) = 1.$$

Let $\alpha_0 := -p$, $\beta_0 := b$, and $(\alpha_i, \beta_i, \sigma_i)$ be as in Table 1. It is enough to show

$$f := c_2(n_0 t)^2 + c_1 n_0 t + c_0 - \binom{\nu_0 t + 2}{2} < 0$$

for all $t \geq 1$. Indeed, if $n = n_0 t$ and $|nbD_x| < \binom{\nu_0 n + 2}{2}$, then $\nu(nbD_x) < \nu_0 n + 1$ which implies $\nu(nD_0) \leq \nu_0 n$, as required. Replacing c_0 by the crude upper bound

$$c_0 \leq \frac{9}{8} + \frac{1}{32s} + \frac{1}{4} + \sum_{i=1}^{5} \epsilon_i^+$$

as in the proof of Theorem 5.3, we obtain a larger function $g \geq f$. One computes $\frac{\partial g}{\partial t} < 0$, so it is enough show $g|_{t=1} < 0$. After clearing denominators, one is left with a quadratic in b whose b^2-coefficient is negative. Thus, for b sufficiently large, $f \leq g < 0$. □

References

[CT15] Castravet, Ana-Maria; Tevelev, Jenia. $\overline{M}_{0,n}$ is not a Mori dream space. *Duke Math. J.* **164** (2015), no. 8, 1641–1667. MR3352043, Zbl 1343.14013, arXiv:1311.7673, doi: 10.1215/00127094-3119846. 1060

[Cut91] Cutkosky, Steven D. Symbolic algebras of monomial primes. *J. Reine Angew. Math* **416** (1991) 71–89. MR1099946 (92b:13009), Zbl 0719.14033, doi: 10.1515/crll.1991.416.71. 1060

[Fuj92] Fujita, Takao. On Kodaira energy and adjoint reduction of polarized manifolds. *Manuscripta Math.* **76** (1992), no. 1, 59–84. MR1171156 (93i:14032), Zbl 0766.14027, doi: 10.1007/BF02567747. 1060

[GK16] González, José L.; Karu, Kalle. Some non-finitely generated Cox rings. *Compos. Mathematica* **152** (2016), no. 5, 984–996. MR3505645, Zbl 1383.14015, arXiv:1407.6344, doi: 10.1112/S0010437X15007745. 1060

[GGK20] González-Anaya, Javier; González, José L.; Karu, Kalle. Curves generating extremal rays in blowups of weighted projective planes. Preprint, 2020. arXiv:2002.07123. 1060, 1061

[GAGK21] González-Anaya, Javier; González, José L.; Karu, Kalle. The geography of negative curves. Preprint, 2021. arXiv:2104.03950v1 1063

[HKL18] Hausen, Jürgen; Keicher, Simon; Laface, Antonio. On blowing up the weighted projective plane. *Math. Z.* **290** (2018), no. 3-4, 1339—1358. MR3856856, Zbl 1401.14143, doi: 10.1007/s00209-018-2065-6. 1062

[GNW94] Goto, Shiro; Nishida, Koji; Watanabe, Kei-Ichi. Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question. *Proc. Amer. Math. Soc.* **120** (1994), no. 2, 383–392, 1994. MR1163334 (94d:13005), Zbl 0796.13005, doi: 10.2307/2159873. 1060

[He17] He, Z. New examples and non-examples of Mori dream spaces when blowing up toric surfaces. Preprint, 2017. arXiv:1703.008196v2. 1060

[Hun82] Huneke, Craig. On the finite generation of symbolic blow-ups. *Math. Z.* **179** (1982), no. 4, 465–472. MR0652854 (83h:13029), Zbl 0488.13013, doi: 10.1007/BF01215060. 1060

[KM09] Kurano, Kazuhiko; Matsuoka, Naoyuki. On finite generation of symbolic Rees rings of space monomial curves and existence of negative curves. *J. Algebra* **322** (2009), no. 9, 3268–3290. MR2567420 (2010k:13007), Zbl 1188.14021, arXiv:0801.3896. doi: 10.1016/j.jalgebra.2008.08.015. 1063

[L11] Linke, Eva. Rational Ehrhart quasi-polynomials. *J. Combin. Theory Ser. A* **118** (2011), no. 7, 1966–1978. MR2802181, Zbl 1234.52010, arXiv:1006.5612, doi: 10.1016/j.jcta.2011.03.007. 1069

[MR15] McKinnon, David; Roth, Mike. Seshadri constants, diophantine approximation, and Roth's theorem for arbitrary varieties. *Invent. Math.* **200** (2015), no. 2, 513–583. MR3338009, Zbl 1337.14023, arXiv:1306.2976, doi: 10.1007/s00222-014-0540-1.

[MR16] McKinnon, David; Roth, Mike. An analogue of Liouville's theorem and an application to cubic surfaces. *Eur. J. Math.* **2** (2016), no. 4, 929–959. MR3572552, Zbl 1412.14016, doi: 10.1007/s40879-016-0113-5. 1061

[MS20] McKinnon, David; Satriano, Matthew. Approximating rational points on toric varieties. *Trans. Amer. Math. Soc.* **374** (2021), no. 5, 3557-3577. MR4237956, Zbl 07331130, doi: 10.1090/tran/8318. 1061

[Ro55] Roth, Klaus F. Rational approximations to algebraic numbers. *Mathematika* **2** (1955), 1–20; corrigendum, 168. MR0072182 (17,242d), Zbl 0064.28501, doi: 10.1112/S0025579300000644. 1061

[Sri91] Srinivasan, Hema. On finite generation of symbolic algebras of monomial primes. *Comm. Algebra* **19** (1991), no. 9, 2557–2564. MR1125189 (92m:13003), Zbl 0734.13001, doi: 10.1080/00927879108824279. 1060

(David McKinnon) University of Waterloo, Department of Pure Mathematics, Waterloo, Ontario N2L 3G1, Canada
dmckinnon@uwaterloo.ca

(Rindra Razafy) University of Waterloo, Department of Pure Mathematics, Waterloo, Ontario N2L 3G1, Canada
rrazafy@uwaterloo.ca

(Matthew Satriano) University of Waterloo, Department of Pure Mathematics, Waterloo, Ontario N2L 3G1, Canada
msatrian@uwaterloo.ca

(Yuxuan Sun) University of Waterloo, Department of Pure Mathematics, Waterloo, Ontario N2L 3G1, Canada
y376sun@uwaterloo.ca

This paper is available via http://nyjm.albany.edu/j/2021/27-41.html.