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Duplications in the k-generalized Fibonacci sequences

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ABSTRACT. Let $k \geq 3$ be an odd integer. Consider the k-generalized Fibonacci sequence backward. The characteristic polynomial of this sequence has no dominating zero, therefore the application of Baker's method becomes more difficult. In this paper, we investigate the coincidence of the absolute values of two terms. The principal theorem gives a lower bound for the difference of two terms (in absolute value) if the larger subscript of the two terms is large enough. A corollary of this theorem makes possible to bound the coincidences in the sequence. The proof essentially depends on the structure of the zeros of the characteristic polynomial, and on the application of linear forms in the logarithms of algebraic numbers. Then we reduced the theoretical bound in practice for $3 \leq k \leq 99$, and determined all the coincidences in the corresponding sequences. Finally, we explain certain patterns of pairwise occurrences in each sequence depending on k if k exceeds a suitable entry value associated to the pair.

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1. Introduction

Let $k \geq 2$ be an integer. The k-generalized Fibonacci sequence $(F_n^{(k)})_{n\in\mathbb{Z}}$ has initial values

$$F_{-k+2}^{(k)} = \cdots = F_0^{(k)} = 0,\ F_1^{(k)} = 1, \tag{1}$$

and satisfies the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \qquad \text{for all } n \in \mathbb{Z}. \tag{2}$$

The case $k = 2$ gives the Fibonacci sequence. There exist several results in the literature related to Diophantine equations with members of the sequences $(F_n^{(k)})$ with positive indices n but not many results deal with problems in which negative subscripts are considered.

In this paper, we look at repeated values of $F_n^{(k)}$ for $n \leq 0$. For our convenience, we introduce the sequence $(H_n^{(k)})$ by $H_n^{(k)} := F_{-n}^{(k)}$ for $n \geq 0$. It means the reverse-direction interpretation of k-generalized Fibonacci sequences, such that $H_n^{(k)} = 0$ holds for $n = 0, \ldots, k-2$, further $H_{k-1}^{(k)} = 1$, and if $n \geq k$, then

$$H_n^{(k)} = -H_{n-1}^{(k)} - \cdots - H_{n-k+1}^{(k)} + H_{n-k}^{(k)}. \tag{3}$$

The characteristic polynomial of this sequence has no dominating zero if k is odd. Therefore, as we will see, the application of Baker's method becomes more difficult. Since we provide now a short survey on the related literature here in the introduction, we will use the notation $H_n^{(k)}$, and analyze the properties later when it is really favourable.

In fact, we look at the slightly more general Diophantine equation

$$|F_n^{(k)}| = |F_m^{(k)}|, \qquad \text{where} \qquad (m,n) \in \mathbb{Z}^2, \quad n \neq m, \quad |n| \geq |m|. \tag{4}$$

For k even, Pethő and Szalay [16] gave an explicit upper bound on $|n|$ in terms of k provided both m and n are negative. Their method uses classical algebraic number theory but does not use transcendental methods (i.e., Baker's theory of linear forms in logarithms). The case $k = 3$ has been handled by Bravo et al. [2]. Their paper [2], together with the earlier paper [1], determined the "total multiplicity of Tribonacci sequence"; namely, all the integer solutions (m,n) of the Diophantine equation $F_n^{(3)} = F_m^{(3)}$ with $n \neq m$. They did not study the more general equation $|F_n^{(3)}| = |F_m^{(3)}|$ (i.e., they did not include the situation $F_n^{(3)} = -F_m^{(3)}$), although their methods based on Baker's theory clearly allow for the study of this similar equation as well. In this paper, we also fill in this gap. Thus, we assume that $k \geq 3$. By Theorem 4.2 of [17], equation (4) has only finitely many effectively computable solutions. However, that theorem does not give an explicit upper bound on $|n|$ in terms of k. Our main result gives an explicit lower bound on $||F_n^{(k)}| - |F_m^{(k)}||$ for $n < m \leq 0$, when $|n| \geq C(k)$, where $C(k)$ is an explicit constant depending on k. In particular, if (4) holds then the above expression is zero, so $|n| < C(k)$.

Bravo and Luca [4] found all the solutions of the equation $F_n^{(k)} = F_m^{(\ell)}$ when $(n,k) \neq (m,\ell)$, $k \geq \ell$ and $n,\ m$ are both non–negative. There are parametric trivial solutions arising from the fact that $F_1^{(k)} = 1$ and $F_n^{(k)} = 2^{n-2}$ for all $n \in [2, k+1]$. In particular, every power of 2, say 2^a, is a term of $(F_n^{(k)})_{n\in\mathbb{N}}$ for all $k \geq a+2$. There is a "nontrivial" power of 2 sitting in the Fibonacci sequence, namely $F_6^{(2)} = 8$, which is nontrivial in the sense that it is not part of the initial string of powers of 2 as described above. Aside from these trivial solutions and the nontrivial power of 2 mentioned above, the only other solutions of the equation are $(m,n,k,\ell) = (7,6,3,2),\ (12,11,7,3)$. The particular case $(k,\ell) = (3,2)$ was worked out earlier by Marques in [12]. When (m,n) are allowed to vary in the set of all integers (so, one or both of them are allowed to be negative), Pethő [15] proved that if $k,\ \ell$ are fixed then the Diophantine equation

$$F_n^{(k)} = F_m^{(\ell)}$$

possesses only finitely many solutions $(n,m) \in \mathbb{Z}^2$. This result is ineffective and the proof is based on the theory of S-unit equations. An effective finiteness result from [15] states that if $k,\ \ell$ are given positive even integers and the integers n and m satisfy

$$|F_n^{(k)}| = |F_m^{(\ell)}|$$

then $\max\{|m|,|n|\} < C(k,\ell)$, where $C(k,\ell)$ is a constant which is effectively computable and depends only on k and ℓ.

Our main result is the following. Recall that $H_n^{(k)} = F_{-n}^{(k)}$ with non-negative integers n.

Theorem 1.1. *Assume that $k \geq 3$ is an odd integer. If $n > m \geq 0$ then*

$$\left|\left|H_n^{(k)}\right| - \left|H_m^{(k)}\right|\right| > \frac{\left|H_n^{(k)}\right|}{\exp(7\cdot 10^{30}\cdot k^{16}(\log k)^5(\log n)^2)} \tag{5}$$

provided

$$n \geq C(k) := 10^{32}\cdot 1.454^{k^3}k^{22}(\log k)^5.$$

Our theorem immediately implies

Corollary 1.2. *Assume that $k \geq 3$ is an odd integer. Then there is no integer solution $0 < m < n$ to the equation*

$$\left|H_n^{(k)}\right| = \left|H_m^{(k)}\right|$$

with $n > C(k)$.

2. Preliminaries

The main problem with Diophantine equations with members of $(H_n^{(k)})_{n\in\mathbb{N}}$ with fixed k is that while the characteristic polynomial

$$T_k(x) = x^k - x^{k-1} - \cdots - x - 1$$

of $(F_n^{(k)})_{n\in\mathbb{N}}$ has a positive real dominating zero, the characteristic polynomial

$$\tilde{T}_k(x) := -x^k T_k\left(\frac{1}{x}\right) = x^k + x^{k-1} + \cdots + x - 1$$

of $(H_n^{(k)})$ has no dominating root when k is odd. When k is even, $\tilde{T}_k(x)$ possesses a dominating zero which is a negative real number but its dominance over the remaining roots is not strong. So, in this section we collect some estimates pertaining to the roots of $T_k(x)$ as well as estimates concerning the values of $F_n^{(k)}$ in terms of these roots.

It is known that the polynomial $T_k(x)$ has simple zeros and the largest one in absolute value is a positive real number denoted by α_1 and is greater than 1. Furthermore, $T_k(x)$ is a Pisot polynomial, i.e. all zeros but α_1 lie inside the unit circle. The other zeros are complex non-real numbers when k is odd. When k is even, $T_k(x)$ has an additional real zero which is in the interval $(-1, 0)$. If two zeros have common absolute value then they form a complex conjugate pair. This was proved in [15] but it also follows rather easily from a result of Mignotte [14] which states that there are no nontrivial multiplicative relations among the conjugates of a Pisot number. Recalling that $k \geq 3$ is odd, the zeros of the characteristic polynomial $T_k(x)$ can be ordered by

$$|\alpha_k| = |\alpha_{k-1}| < |\alpha_{k-2}| = |\alpha_{k-3}| < \cdots < |\alpha_3| = |\alpha_2| < \alpha_1,$$

where $\alpha_{k-1} = \overline{\alpha_k}, \alpha_{k-3} = \overline{\alpha_{k-2}}, \ldots$, etc. For brevity, put $\varrho := |\alpha_k|$, and $\varrho_2 := |\alpha_{k-2}|$. The explicit Binet formula

$$F_n^{(k)} = \sum_{j=1}^{k} g_k(\alpha_j)\alpha_j^{n-1} \quad \text{for all} \quad n \geq 0, \tag{6}$$

where

$$g_k(x) = \frac{x-1}{2 + (k+1)(x-2)}$$

was given by Dresden and Du in [5]. It remains true when negative indices n are allowed. For simplicity, we put

$$a_j := g_k(\alpha_j)\alpha_j^{-1} \quad \text{for all} \quad j = 1, \ldots, k.$$

Thus, $F_n^{(k)} = \sum_{j=1}^{k} a_j \alpha_j^n$ is a simpler than but equivalent form to (6).

In the sequel, we list a few estimates which are used later. The next three lemmata do not depend on the parity of k.

Lemma 2.1. *For $k \geq 2$, the following inequalities hold.*

$$2 - \frac{1}{2^{k-1}} < \alpha_1 < 2 - \frac{1}{2^k}.$$

Proof. This is Lemma 3.6, and a consequence of Theorem 3.9 in [20]. □

Lemma 2.2. *If $j \neq 1$, then*

$$\frac{1}{3^{1/k}} < |\alpha_j| < 1 - \frac{1}{2^8 k^3}.$$

Proof. See Lemma 2.1 in [9] for the left-hand side. The right-hand side can be found in Theorem 2 in [10]. □

The next statement is Corollary 3 in [6].

Lemma 2.3. *If* $|\alpha_j| > |\alpha_i|$, *then*

$$\frac{|\alpha_j|}{|\alpha_i|} > c_k := 1 + \frac{1}{1.454^{k^3}}.$$

An essential part of the proof of the main theorem depends on Baker's method. Here we describe the principal tool due to Matveev. Let $\mathbb{K}$ be an algebraic number field of degree $d_{\mathbb{K}}$ and let $\eta_1, \eta_2, \ldots, \eta_t \in \mathbb{K}$ not 0 or 1, and $b_1, \ldots, b_t$ be nonzero integers. Put

$$B := \max\{|b_1|, \ldots, |b_t|, 3\} \qquad \text{and} \qquad \Gamma := \prod_{i=1}^{t} \eta_i^{b_i} - 1.$$

Let $A_1, \ldots, A_t$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{K}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for} \quad j = 1, \ldots t,$$

where for an algebraic number η with minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(u)}) \in \mathbb{Z}[X]$$

with positive a_0 we write $h(\eta)$ for its Weil height given by

$$h(\eta) := \frac{1}{u}\left(\log a_0 + \sum_{j=1}^{u} \max\{0, \log |\eta^{(j)}|\}\right).$$

Under these circumstances, Matveev [13] proved

Lemma 2.4. *If* $\Gamma \neq 0$, *then*

$$\log |\Gamma| > -3 \cdot 30^{t+4}(t+1)^{5.5} d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log tB) A_1 A_2 \cdots A_t.$$

We next list some well known properties of the logarithmic height function. For the proof, see e.g. [19] Ch. 3.2.

Lemma 2.5. *The properties*

(i) $h(\mu + \nu) \leq h(\mu) + h(\nu) + \log 2$,
(ii) $h(\mu \nu^{\pm 1}) \leq h(\mu) + h(\nu)$,
(iii) $h(\mu^{\ell}) \leq |\ell| h(\mu)$

are valid for all algebraic numbers μ, ν, *and integers* ℓ.

We also refer the Baker-Davenport reduction method of Dujella and Pethő (see [7, Lemma 5a]). Let $\| c \|$ denote the distance of c from the nearest integer.

Lemma 2.6. *Let* $\kappa \neq 0$ *and* μ *be real numbers. Assume that* M *is a positive integer. Let* P/Q *be the convergent of the continued fraction expansion of* κ *such that* $Q > 6M$, *and put*

$$\xi := \|\mu Q\| - M \cdot \|\kappa Q\|.$$

If $\xi > 0$, then there is no solution of the inequality

$$0 < |m\kappa - n + \mu| < AB^{-k}$$

for positive integers m, n, and k with

$$\frac{\log(AQ/\xi)}{\log B} \le k \qquad \text{and} \qquad m \le M.$$

The final result of this section is Lemma 7 in [11].

Lemma 2.7. *If $s \ge 1$, $T \ge (4s^2)^s$, and $x/(\log x)^s < T$, then*

$$x < 2^s T(\log T)^s.$$

3. Preparation

The proof of the main theorem requires a result concerning the size of $|H_n^{(k)}|$. This is Lemma 3.2 for which we need the following preparation.

Lemma 3.1. *If $n > d_k := 2 \cdot 10^{15} \cdot 1.454^{k^3} k^{11} (\log k)^3$, then*

(i) $|H_n^{(k)}| > \frac{1}{2}\left|a_k \alpha_k^{-n} + \overline{a_k}(\overline{\alpha_k})^{-n}\right|$;

(ii) $|H_n^{(k)}| > 3\varrho_2^{-n}$.

Proof. First we prove (i). It is sufficient to show that for n large enough we have

$$\frac{1}{2}\left|a_k \alpha_k^{-n} + \overline{a_k}(\overline{\alpha_k})^{-n}\right| \ge \left|\sum_{j=1}^{k-2} a_j \alpha_j^{-n}\right|. \tag{7}$$

Indeed, then

$$|H_n^{(k)}| = \left|\sum_{j=1}^{k} a_j \alpha_j^{-n}\right| \ge \left|a_k \alpha_k^{-n} + \overline{a_k}(\overline{\alpha_k})^{-n}\right| - \left|\sum_{j=1}^{k-2} a_j \alpha_j^{-n}\right|,$$

and now we conclude the statement (i) of the lemma from (7).

Assume $n > 2k^2 \log(4k)$. We first bound the left-hand side (in short LHS) of (7) from below as follows:

$$\begin{aligned} LHS &:= \frac{1}{2}|a_k|\varrho^{-n}\left|1 - \left(-\frac{\overline{a_k}}{a_k}\right)\left(\frac{\alpha_k}{\overline{\alpha_k}}\right)^n\right| \\ &> \frac{1}{2}|a_k|\varrho^{-n}\exp(-4.74 \cdot 10^{14} k^8 (\log k)^3 \log n) \\ &> \frac{1}{2^{11}k^4}\varrho^{-n}\exp(-4.74 \cdot 10^{14} k^8 (\log k)^3 \log n). \end{aligned} \tag{8}$$

Here we used the following two observations. The lower bound on

$$\left|1 - \left(-\frac{\overline{a_k}}{a_k}\right)\left(\frac{\alpha_k}{\overline{\alpha_k}}\right)^n\right|$$

comes from inequality (4.4) in [9]. It assumes that $n > 2k^2 \log(4k)$, which we are also assuming. Furthermore,

$$\begin{aligned} |a_k| &= \left|\frac{g(\alpha_k)}{\alpha_k}\right| = \frac{1}{|2+(k+1)(\alpha_k-2)|}\left|\frac{1}{\alpha_k}-1\right| \\ &\geq \frac{1}{2+(k+1)(|\alpha_k|+2)}\left(\frac{1}{|\alpha_k|}-1\right) > \frac{1}{(3k+5)(2^8k^3-1)} \\ &> \frac{1}{2^{10}k^4}. \end{aligned}$$

For $k \geq 5$, the above inequality follows from Lemma 2.2 and the fact that $1/(3k+5) \geq 1/(4k)$ which holds when $k \geq 5$. For $k = 3$, one checks directly that $|a_3| > 0.35 > 1/(2^{10} \cdot 3^4)$.

For the right-hand side (in short RHS) of (7), we see that for $k \geq 5$ we have

$$\begin{aligned} \left|\sum_{j=1}^{k-2} a_j \alpha_j^{-n}\right| &\leq \sum_{j=1}^{k-2} |a_j||\alpha_j|^{-n} \\ &= |\alpha_{k-2}|^{-n} \sum_{j=1}^{k-2} |a_j| \left|\frac{\alpha_j}{\alpha_{k-2}}\right|^{-n} \leq \varrho_2^{-n} \sum_{j=1}^{k-2} |a_j| \\ &< \varrho_2^{-n}(|a_1| + (k-3) \max_{2\leq j\leq k-2} |a_j|) < 3\varrho_2^{-n}. \end{aligned} \tag{9}$$

The above inequality also holds for $k = 3$ since in that case the left-hand side only has one term which is real and positive, namely $a_1\alpha_1^{-n}$ and $a_1 \in (0.18, 0.19)$, so $a_1 < 3$. We need to justify upper bounds for $|a_j|$ for $j = 1, \ldots, k-1$. For $k \geq 5$, $j \in \{2, \ldots, k\}$ we have

$$\begin{aligned} |a_j| &= \left|\frac{g_k(\alpha_j)}{\alpha_j}\right| \leq \frac{1}{|2+(k+1)(\alpha_j-2)|}\left|\frac{1}{\alpha_j}-1\right| \\ &< \frac{1}{(k+1)(2-|\alpha_j|)-2}\left(1+\frac{1}{|\alpha_j|}\right) < \frac{1+3^{1/k}}{k-1} < \frac{2.5}{k-1}, \end{aligned} \tag{10}$$

where we used Lemma 2.2, and for $j = 1$ we have

$$\begin{aligned} |a_1| &= \frac{1}{2+(k+1)(\alpha_1-2)}\left(\frac{\alpha_1-1}{\alpha_1}\right) \\ &< \frac{1}{(2-(k+1)/2^{k-1})(2-1/2^{k-1})} < 0.5 \end{aligned} \tag{11}$$

since $k \geq 5$, where we used Lemma 2.1. By inspection, as we have done already, these bounds also hold for $k = 3$. Hence, (i) of the lemma follows for $n > 2k^2 \log(4k)$ such that

$$\frac{1}{2^{11}k^4} \exp(-4.74 \cdot 10^{14} k^8 (\log k)^3 \log n)\varrho^{-n} > 3\varrho_2^{-n}, \tag{12}$$

holds. The above inequality is implied by

$$\left(\frac{\varrho_2}{\varrho}\right)^n > 2^{13}k^4 \exp(4.74 \cdot 10^{14}k^8(\log k)^3 \log n). \tag{13}$$

We have

$$\frac{\varrho_2}{\varrho} > 1 + \frac{1}{1.454^{k^3}}$$

by Lemma 2.3 and

$$\log\left(1 + \frac{1}{1.454^{k^3}}\right) > \frac{1}{2 \cdot 1.454^{k^3}}.$$

Thus, in order for (13) to hold it is enough for n to satisfy

$$\frac{n}{2 \cdot 1.454^{k^3}} > \log(2^{13}k^4) + 4.74 \cdot 10^{14}k^8(\log k)^3 \log n.$$

For $k = 3$, $\rho_2/\rho > \alpha_1 > 1.8$, so $\log(\rho_2/\rho) > \log(1.8) > 1/2$, so we can ignore the factor 1.454^{k^3} from the denominator on the left-hand side. The first member on the right-hand side above is small. That is, $\log(2^{13}k^4) < 0.26 \cdot 10^{14}k^8(\log k)^3 \log n$ for all $k \geq 3$ and $n > 2k^2 \log(4k)$. Hence, it suffices that

$$n > \delta_k \cdot 10^{15}k^8(\log k)^3 \log n, \quad \text{where} \quad \delta_k := \begin{cases} 1.454^{k^3} & \text{if} \quad k \geq 5; \\ 1 & \text{if} \quad k = 3. \end{cases} \tag{14}$$

Thus, $n > n_k$, where n_k is the largest solution of the inequality

$$\frac{n}{\log n} \leq 10^{15}\delta_k k^8(\log k)^3.$$

Assume $k \geq 5$. To bound n_k, we use Lemma 2.7 with $s = 1$. We take

$$T := 10^{15} \cdot 1.454^{k^3}k^8(\log k)^3.$$

Then

$$\log T = k^3\left(\log 1.454 + \frac{15 \log 10 + 8 \log k + 3 \log\log k}{k^3}\right) < k^3$$

since $k \geq 5$. Hence,

$$n_k < 2T \log T < 2 \cdot 1.454^{k^3} \cdot 10^{15}k^{11}(\log k)^3 = d_k,$$

subsequently (i) holds if $n > d_k$. Note that d_k exceeds $2k^2 \log(4k)$ so such n also satisfy that $n > 2k^2 \log(4k)$ and this last inequality holds for $k = 3$ as well. Finally, for $k = 3$, a computation shows that

$$n_3 < 5 \cdot 10^{20} < 10^{25} < d_3,$$

and the inequality $n_k < d_k$ fulfils for $k = 3$ as well.

Now we turn to the proof of (ii). Using (i), we get that (ii) is true provided

$$\frac{1}{2}\left|a_k\alpha_k^{-n} + \overline{a_k}(\overline{\alpha_k})^{-n}\right| > 3\varrho_2^{-n}.$$

Our previous computation (8) shows that the left-hand side of this inequality is larger than

$$\frac{1}{2^{11}k^4}\varrho^{-n}\exp(-4.74\cdot 10^{14}k^8(\log k)^3\log n),$$

while inequality (12) shows that the above expression exceeds $3\varrho_2^{-n}$ provided $n > 2\cdot 1.454^{k^3}\cdot 10^{15}k^{11}(\log k)^3$, which implies the desired conclusion. □

Now we are able to bound $|H_n^{(k)}|$ as follows.

Lemma 3.2. *Let $k \geq 3$. The inequality*

$$|H_n^{(k)}| < 3\varrho^{-n}$$

holds for all $n \geq 0$. Furthermore,

$$\varrho^{-n+1.3\cdot 10^{17}k^{11}(\log k)^3\log n} < |H_n^{(k)}|$$

is valid for all $n > d_k$.

Proof. The lower bound follows from (8), the observation that $2^{11} < k^9$ holds for $k \geq 3$ together with the fact that $\varrho < 1 - 1/(2^8k^3)$.

For the upper bound, we go back to (9). The only difference that the sum is up to k instead of $k-2$ and we factor out $\varrho = |\alpha_k|$ instead of $\varrho_2 = |\alpha_{k-2}|$. Thus,

$$\left|\sum_{j=1}^{k} a_j\alpha_j^{-n}\right| \leq \varrho^{-n}(|a_1| + (k-1)\max_{1\leq j\leq k-1}\{|a_j|\}) < 3\varrho^{-n},$$

where we used (10) and (11). □

4. Proof of Theorem 1.1

Set

$$A_{n,m} := \left|\left|H_n^{(k)}\right| - \left|H_m^{(k)}\right|\right|. \tag{15}$$

We assume $n > m$ and $n > d_k$. Suppose first that

$$6\varrho^{-m} < \varrho^{-n+4.4\cdot 10^{14}k^9(\log k)^3\log n}. \tag{16}$$

It then follows by Lemma 3.2 that

$$|H_m^{(k)}| < 3\varrho^{-m} < \frac{1}{2}\varrho^{-n+4.4\cdot 10^{14}k^9(\log k)^3\log n} < \frac{1}{2}|H_n^{(k)}|,$$

so that

$$A_{n,m} = \left|\left|H_n^{(k)}| - |H_m^{(k)}\right|\right| > 0.5|H_n^{(k)}|,$$

which is a better inequality than (5). Thus, let us assume that (16) does not hold. Then

$$m - n > -4.4\cdot 10^{14}k^9(\log k)^3\log n - (\log 6)/\log\left(\frac{1}{\varrho}\right) > -G_k\log n,$$

where $G_k := 4.45\cdot 10^{14}k^9(\log k)^3$.

Next, equation (15) can be rewritten as

$$H_n^{(k)} = \pm H_m^{(k)} \pm A, \qquad \text{where} \qquad A := A_{m,n},$$

and yields

$$a_k\alpha_k^{-n}(1\mp\alpha_k^{-(m-n)})+\overline{a_k}(\overline{\alpha_k})^{-n}(1\mp(\overline{\alpha_k})^{-(m-n)}) = \pm A-\sum_{j=1}^{k-2} a_j(\alpha_j^{-n}\mp\alpha_j^{-m}). \quad (17)$$

The absolute value of the second term of the right-hand side of (17) satisfies

$$\begin{aligned}\left|-\sum_{j=1}^{k-2} a_j(\alpha_j^{-n}\mp\alpha_j^{-m})\right| &\le \sum_{j=1}^{k-2}|a_j|(|\alpha_j|^{-n}+|\alpha_j|^{-m})\\ &\le 3\varrho_2^{-n}+3\varrho_2^{-m}<6\varrho_2^{-n}, \quad (18)\end{aligned}$$

by a previous argument.

We now turn our attention to the left-hand side of (17). Put $\alpha_k := \varrho z$ with $z := e^{i\vartheta}$, where $|z| = 1$ and $\vartheta := \arg\alpha_k$. Obviously, $\overline{\alpha_k} = \varrho z^{-1}$. Using this notation, the absolute value of the left-hand side of (17) equals

$$\begin{aligned}&\varrho^{-n}\left|a_k z^{-n}(1\mp\varrho^{n-m}z^{n-m})+\overline{a_k}z^n(1\mp\varrho^{n-m}z^{-(n-m)})\right|\\ &\quad=\varrho^{-n}\left|\overline{a_k}z^n(1\mp\varrho^{n-m}z^{-(n-m)})\right|\left|\frac{a_k}{\overline{a_k}}z^{-2n}\frac{1\mp\varrho^{n-m}z^{n-m}}{1\mp\varrho^{n-m}z^{-(n-m)}}-1\right|. \quad (19)\end{aligned}$$

Now we provide lower bounds for two factors of the product in the inequality above. The first bound is analytical, the second one is coming from the theorem of Matveev with $t = 3$. Hence,

$$\begin{aligned}\left|\overline{a_k}z^n(1\mp\varrho^{n-m}z^{-(n-m)})\right| &= |\overline{a_k}||z|^n|1\mp\varrho^{n-m}z^{-(n-m)}|\\ &\ge \frac{1}{2^{10}k^4}\left(1-\varrho^{n-m}|z|^{-(n-m)}\right)\\ &\ge \frac{1}{2^{10}k^4}(1-\varrho)\ge\frac{1}{2^{18}k^7}, \quad (20)\end{aligned}$$

by Lemma 2.2. In order to prepare the application of Lemma 2.4, let

$$\eta_1 := -\frac{a_k}{\overline{a_k}}, \qquad \eta_2 := z^{-2}, \qquad \eta_3 := \frac{1\mp\varrho^{n-m}z^{n-m}}{1\mp\varrho^{n-m}z^{-(n-m)}}.$$

Thus, $b_1 = 1$, $b_2 = -n$, $b_3 = 1$, so $B = n$. Moreover, all three numbers η_1, η_2, η_3 are in $\mathbb{K} := \mathbb{Q}(\alpha_k, \overline{\alpha_k})$, therefore $D = d_{\mathbb{K}} \le k^2$. In the forthcoming calculations, we use the properties of the heights of algebraic numbers (Lemma 2.5). Clearly, $h(\eta_1) \le 2h(a_k)$, and then

$$h(a_k) \le 3h(\alpha_k) + 5\log 2 + \log(k+1) < 8\log 2 + \log(k+1).$$

In the above, we used that $3h(\alpha_k) < 3\log\alpha_1)/k < 3\log 2/k < 1$. So, $h(\eta_1) \le 2\log(2^8(k+1))$, and then we take $A_1 = 2k^2\log(2^8(k+1))$. Secondly,

$$h(\eta_2) = h(z^2) = h\left(\frac{\alpha_k}{\overline{\alpha_k}}\right) \le 2h(\alpha_k) \le \frac{2\log 2}{k},$$

so we can take $A_2 = 2(\log 2)k$. Furthermore,

$$\begin{aligned} h(\eta_3) &\leq (h(\varrho^{n-m}z^{n-m}) + \log 2) + (h(\varrho^{n-m}z^{-(n-m)}) + \log 2) \\ &= 2\log 2 + h(\alpha_k^{n-m}) + h((\overline{\alpha_k})^{n-m}) \\ &\leq 2\log 2 + 2(n-m)\frac{\log 2}{k}. \end{aligned}$$

So, we can take $A_3 = 2k(k+n-m)\log 2$. With the above ingredients, Matveev's theorem provides

$$\begin{aligned} \log|\Gamma| &> -3\cdot 30^7\cdot 4^{5.5}(k^2)^2(1+\log(k^2))(1+\log(3n)) \\ &\quad \cdot 2k^2\log(2^8(k+1))\cdot 2(\log 2)k\cdot 2k(k+n-m)\log 2 \\ &> -7.5\cdot 10^{14}k^8\cdot 3\log k\cdot 6.4\log k\cdot 1.04\log n \\ &\quad \cdot (4.5\cdot 10^{14}k^8(\log k)^3\log n) \\ &> -6.9\cdot 10^{30}k^{16}(\log k)^5(\log n)^2. \end{aligned} \tag{21}$$

In the above calculations, we used that $1+\log(3n) < 1.04\log n$ provided $n > 10^{23}$, together with $1+\log(k^2) < 3\log k$ and $\log(2^8(k+1)) < 6.4\log k$ both valid for $k \geq 3$. Moreover,

$$k+n-m < k+G_k\log n < 4.5\cdot 10^{14}k^9(\log k)^3\log n.$$

At this point, we return to (17) which, together with the estimates (18), (19), (20) and (21) above, provides

$$\begin{aligned} A &\geq \frac{\varrho^{-n}}{2^{18}k^7}\exp(-6.9\cdot 10^{30}k^{16}(\log k)^5(\log n)^2) - 6\varrho_2^{-n} \\ &\geq \frac{\varrho^{-n}}{2^{18}k^7}\exp(-6.9\cdot 10^{30}k^{16}(\log k)^5(\log n)^2) \\ &\quad \cdot\left(1-\frac{6\cdot 2^{18}k^7\exp(6.9\cdot 10^{30}k^{16}(\log k)^5(\log n)^2)}{(\varrho_2/\varrho)^n}\right). \end{aligned} \tag{22}$$

To finish, using Lemma 3.2, we want that the last factor on the right-hand side above is greater than $1/2$, and

$$12\cdot 2^{18}k^7 < \exp(10^{29}k^{16}(\log k)^5(\log n)^2). \tag{23}$$

Taking logarithms (23) is obvious for all $k \geq 3$ and $n > \max\{d_k, 10^{23}\}$. So, it remains to deal with the condition that the last factor on the right-hand side of (22) exceeds $1/2$. This is equivalent to

$$12\cdot 2^{18}k^7\exp(6.9\cdot 10^{30}k^{16}(\log k)^5(\log n)^2) < \left(\frac{\varrho_2}{\varrho}\right)^n. \tag{24}$$

By Lemma 2.3, the last inequality holds provided

$$7\cdot 10^{30}k^{16}(\log k)^5(\log n)^2 < n\log\left(1+\frac{1}{1.454^{k^3}}\right),$$

for $k \geq 5$. As in the proof of Lemma 3.1, the right-hand side above can be replaced by $1/2$ when $k = 3$. The last inequality above is satisfied provided

$$n > 1.4\delta_k \cdot 10^{31} k^{16} (\log k)^5 (\log n)^2,$$

where δ_k has the same meaning as in (14). Thus, we want $n > C(k)$, where now $C(k)$ is the largest solution of

$$\frac{n}{(\log n)^2} < 1.4\delta_k \cdot 10^{31} k^{16} (\log k)^5. \tag{25}$$

Let $k \geq 5$ and let T be the right-hand side above. By Lemma 2.7 with $s = 2$, we get

$$C(k) < 4T(\log T)^2.$$

Now

$$\begin{aligned} \log T &< k^3 \left(\log(1.454) + \frac{\log(1.4 \cdot 10^{31}) + 16 \log k + 5 \log\log k}{k^3} \right) \\ &< 1.2k^3. \end{aligned}$$

Thus, we can take

$$4 \cdot 1.4 \cdot 1.454^{k^3} \cdot 10^{31} k^{16} (\log k)^5 (1.2k^3)^2 < 10^{32} \cdot 1.454^{k^3} k^{22} (\log k)^5 := C(k),$$

which is what we wanted. When $k = 3$, the largest solution of (25) is smaller than $10^{43} < 10^{47} < C(3)$. Finally, let us note that at some point we did make the assumption that $n > 10^{23}$, which now is justified in light of the fact that $C(k) > 10^{23}$ holds for all $k \geq 3$. □

5. Computations

First, we computed the approximate values of α_k, α_{k-2}, ϱ, ϱ_2, ϱ_2/ϱ and $|a_k|$ in the range $k = 5, 7, \ldots, 99$ with 200 digits precision. We found that

$$0.8187 < \varrho < 0.9891, \qquad 0.8710 < \varrho_2 < 0.9891,$$

$$1.000008 < \frac{\varrho_2}{\varrho} < 1.0639, \qquad 0.0067 < |a_k| < 0.1483.$$

Now follow Lemma 3.1, supposing $n \geq \lfloor 2 \cdot 99^2 \log(4 \cdot 99) \rfloor = 50921$, and in this case for (8) we have

$$LHS > \frac{1}{300} \cdot \exp(-6 \cdot 10^{32} \log n) \varrho^{-n}.$$

Comparing this with $RHS < 3\varrho_2^{-n}$, we finally obtain that statement (i) of Lemma 3.1 is true if $n > 6 \cdot 10^{39}$. In the next step, we return to (24), and using the numerical estimates we conclude $n < 4.2 \cdot 10^{75}$. This upper bound makes it possible to jump back to the left-hand side of (7), and apply Dujella-Pethő reduction for each odd k in $[5, 99]$. These procedures provide, in summary,

$$\frac{1}{2} \left| a_k \alpha_k^{-n} + \overline{a_k} (\overline{\alpha_k})^{-n} \right| > \frac{1}{2} \left| a_k \alpha_k^{-n} \right| \cdot 10^{-82}.$$

Suppose now that $|H_n^{(k)}| = |H_m^{(k)}|$, which leads to

$$\frac{1}{2}\left|a_k\alpha_k^{-n}\right| \cdot 10^{-82} < 3\varrho^{-m} < \varrho^{-m-100},$$

and then we get $n - m < 17300$.

Consider now again (19). In the third term, we have finitely many positive integer values for k and $n-m$, and an upper bound on n. We target to reduce this bound by the application of Dujella-Pethő reduction. It means approximately $2 \cdot 48 \cdot 17298$ reductions as follows. Put $\delta := n - m$ and

$$e^{i\nu_{k,\delta}} := \frac{a_k}{\overline{a_k}} \cdot \frac{1 \mp \varrho^{n-m} z^{n-m}}{1 \mp \varrho^{n-m} z^{-(n-m)}}, \tag{26}$$

where $-\pi < \nu_{k,\delta} < \pi$. Note that in (26), we used the fact that the right-hand side has absolute value 1. Recall that $z = e^{i\vartheta}$. Then

$$\left|\frac{a_k}{\overline{a_k}} z^{-2n} \frac{1 \mp \varrho^{n-m} z^{n-m}}{1 \mp \varrho^{n-m} z^{-(n-m)}} - 1\right| = \left|e^{i(-2n\vartheta+\nu_{k,\delta})} - 1\right| > |\sin(-2n\vartheta + \nu_{k,\delta})|.$$

Put

$$\ell_{k,n,\delta} := \left\lfloor \frac{-2n\vartheta + \nu_{k,\delta}}{\pi} \right\rceil,$$

where $\lfloor c \rceil$ means the nearest integer to c. Obviously, we have that $-\pi/2 \le -2n\vartheta + \nu_{k,\delta} - \ell_{k,n,\delta} \le \pi/2$, and

$$\begin{aligned} |\sin(-2n\vartheta + \nu_{k,\delta})| &= |\sin(-2n\vartheta + \nu_{k,\delta} - \ell_{k,n,\delta}\pi)| \\ &\ge 2\left|\left(-\frac{2\vartheta}{\pi}\right)n - \ell_{k,n,\delta} + \frac{\nu_{k,\delta}}{\pi}\right|. \end{aligned}$$

Now we are ready to apply Lemma 2.6 together with

$$\left|\left(-\frac{2\vartheta}{\pi}\right)n - \ell_{k,n,\delta} + \frac{\nu_{k,\delta}}{\pi}\right| < \frac{3}{b}\left(\frac{\varrho_2}{\varrho}\right)^{-n},$$

via (18) and (19), where

$$b = \left|\overline{a_k} z^n (1 \mp \varrho^{n-m} z^{-(n-m)})\right| = |a_k| \left|(1 \mp \varrho^{n-m} z^{-(n-m)})\right| \ge |a_k| \cdot |1 - \varrho|.$$

Now the brief summary on the application of the reduction method is presented. First, we mention that the description here refers the two cases $\pm$ together. The upper bounds on n we obtained by the first reduction were not sufficiently small for larger values k. Thus, we applied Lemma 2.6 as many times as it essentially reduced the bound, and this resulted a quasi-optimal range for n.

Suppose that the final bound on n is denoted by $b_n(k)$. The experimental formula $b_n(k) \approx 4.72k^3$ shows the approximate behavior of $b_n(k)$. We note that the inequality $b_n(k) < 4.72k^3$ holds for all $k \le 75$. The largest value appears when $k = 99$, namely $b_n(99) = 4597520$. In comparison, in the middle of the range, $b_n(51) = 3144305$. On the other hand, a brute force search indicated that there is no repetition (in absolute value) in the sequences if $n > 12000$.

The algorithm which verified the possible cases of n (for fixed k) can be split into two parts. The first part is a direct verification of the equality between the terms (in absolute value) of the sequence for $n \leq 13000$. For $k = 5, \ldots, 15$ this was sufficient. From $k = 17$, after the threshold 13000 the terms of the sequence were generated modulo M, a suitable modulus larger then the first 13000 terms (in absolute value) of the sequence. M is constructed as a product of an initial interval of primes. Then, the checking of the coincidence happened modulo M. We expected no coincidences by this way. If it might have occurred, then the procedure chose a new modulus M, and started again the verification from $n = 13000$. The check of the largest value $k = 99$ took approximately 8 and half days on an average desk computer.

In the sequel, we give a survey of the results provided by the algorithm.

• *Occurrence of* 0*, and* ± 1. The large number of coincidences of 0, and ± 1, respectively makes it not possible to list them up. Thus, we restrict ourselves to give the number of occurrences $o_k(0)$, and $o_k(\pm 1)$. It is very interesting that they can be given by polynomial functions of k if $5 \leq k \leq 99$. The last occurrence $l_k(.)$ can also be described by quadratic functions. The exact expressions are $o_k(0) = k(k-1)/2$, $o_k(\pm 1) = k$, $l_k(0) = (k-2)(k+1)$, $l_k(\pm 1) = k^2 - 2$. We remark that a very recent paper [8] has proved $o_k(0) = k(k-1)/2$ for $k \leq 500$. We think it would be a challenging problem to prove the correctness of these formulae for arbitrary $k \geq 5$. In the case $k = 3$, we found $o_3(0) = 4, o_3(\pm 1) = 3, l_3(0) = 17, l_3(\pm 1) = 7$.

• *Occurrence of pairs.* It is also interesting that if an integer not equal to $0, \pm 1$ appears twice (in absolute value) for some $k = k_0$, then it appears twice for all $k_0 < k \leq 99$ if the appearance of the first pair is fast enough. In addition, the subscripts of the terms of such pairs can be given by linear functions of k. This phenomenon is summarized briefly in Table 1. Let e_k stand for the entry value k such that a pair appears in $(H_n^{(k)})$, moreover put $V_0 := 84480$, $V_1 := 131072$, $V_2 := 17179869184$, $V_3 := 147573952589676412928$, $V_4 := 1119260188007982330192621320750271712696717855948 80$. Note that only -1568 is the integer which occurs twice, in the other cases the coincidence is valid for only the absolute values. Legend of Table 1: for instance, the row of ∓ 8 indicates that first -8 occurs at $H_{3k}^{(k)}$, and then 8 at $H_{4k+1}^{(k)}$, moreover it is true for $k \geq 5$.

In the next section, we will show that these formulae of subscripts hold for all $k \geq e_k$.

• *Exceptional occurrences.* There are two cases when a matching appears, but it does not appear later. For $k = 3$, $H_{16}^{(3)} = 56$, and $H_{20}^{(3)} = -56$. We note that if $k = 3$ this is the only coincidence which differs from 0 and ± 1. For $k = 5$, $H_{26}^{(5)} = H_{39}^{(5)} = 56$.

value	e_k	subscripts	value	e_k	subscripts
∓ 8	5	$(3k, 4k+1)$	± 32	7	$(5k+2, 6k-1)$
± 128	9	$(8k-1, 9k+6)$	∓ 256	9	$(7k, 9k-1)$
± 512	17	$(10k-1, 17k+14)$	-1568	29	$(9k+4, 29k+26)$
± 2048	33	$(12k-1, 33k+30)$	± 2816	9	$(9k+1, 10k)$
± 8192	65	$(14k-1, 65k+62)$	$\pm V_0$	9	$(12k+3, 13k+6)$
$\mp V_1$	19	$(15k, 18k-1)$	$\mp V_2$	35	$(31k, 35k-1)$
$\mp V_3$	69	$(63k, 68k-1)$	$\mp V_4$	97	$(114k+19, 115k+18)$

TABLE 1. Repetition formulae.

6. Regularities in the sequence $(H_n^{(k)})$

During the computation of multiple values in the sequence $(H_n^{(k)})$, we observed certain regularities. For example, we mentioned above that *if an integer not equal to* $0, \pm 1$ *appears twice (in absolute value) for some* $k = k_0$*, then it appears twice for all* $k_0 < k \le 99$ *if the appearance of the first pair is fast enough. In addition, the subscripts of the terms of such pairs can be given by linear functions of* k. In this part, we prove that this is not an accidental coincidence, but follows from the fact that the beginning of $(H_n^{(k_0)})$ is repeated with minor modification in $(H_n^{(k)})$ for all $k \ge k_0$.

The main tool is to split the sequence $(H_n^{(k)})$ into consecutive blocks with length $k+1$, and write the blocks in a top-down list. Assume that $k \ge 2$, and

$$n = j(k+1) + i$$

holds with the condition $0 \le i \le k$. This division with remainders admits that the term $H_n^{(k)}$ is located on the place i in the jth block. Thus, the arrangement of the blocks yields a rectangular table with width $k+1$, where one row is one block, and a column is belonging to a given value i. The principal result of this section is

Theorem 6.1. *Assume* $j = 0, \dots, k-2$ *and* $i = 0, \dots, k-2-j$*. Then* $H^{(k)}_{j(k+1)+i} = 0$*. Furthermore if either* $j = 0, \dots, k-2$*,* $i = k-1-j, \dots, k$ *or* $j = k-1, \dots, 2k-2$*,* $i = 0, \dots, 2k-2-j$*, then*

$$H^{(k)}_{j(k+1)+i} = (-1)^{j+i+1-k} \cdot 2^{k-1-i} \left[\binom{j+1}{j+i+1-k} + \binom{j}{j+i-k} \right]. \tag{27}$$

A direct application of this theorem shows a connection between the first few terms of the two sequences $(H_n^{(k)})$ and $(H_n^{(k+1)})$.

Corollary 6.2. *If either* $j = 0, \dots, k-2$ *and* $i = 0, \dots, k$ *or* $j = k-1, \dots, 2k-2$ *and* $i = 0, \dots, 2k-2-j$*, then*

$$H^{(k)}_{j(k+1)+i} = H^{(k+1)}_{j(k+2)+i+1}.$$

This corollary proves that if $|H_{n_1}^{(k)}| = |H_{n_2}^{(k)}| \notin \{0,1\}$ such that the locations (j_1, i_1) and (j_2, i_2) are in the range of Corollary 6.2, then the coincidence appears for all larger k values, of course with other subscripts. This also explains the so called exceptional solutions in the previous section, for instance why $56 = H_{26}^{(5)} = H_{39}^{(5)}$ is not repeated later. Indeed, $26 = 4{\cdot}6{+}2$ is possible, but $39 = 6{\cdot}6{+}3$ is out of the range ($k = 5$, $j = 6$, but $i = 3 > 2k - 2 - j$). Similarly, there is no guaranteed repetition associated to $56 = H_{16}^{(3)} = |H_{20}^{(3)}|$.

It is well known that the k-generalized Fibonacci sequences start in the positive direction with powers of 2. Moreover, Bravo and Luca [3] established all powers of 2 in these sequences. Our final statement shows that many powers of 2 appear regularly in the negative direction, too.

Corollary 6.3. *If $k \geq 2$ and $j = 0, \ldots, k-1$, then $H_{(j+1)k-1}^{(k)} = 2^j$.*

Proof of Theorem 6.1. The combination of two consecutive terms in (3), together with the new notation provides

$$H_n^{(k)} = 2H_{n-k}^{(k)} - H_{n-k-1}^{(k)}.$$[1]

The table arrangement of the blocks shows that an entry of the table located not in the right-most column is the double of the upper right neighbor element minus the upper neighbor element. The last entry of a row can be given as the double of the first entry of the row minus the upper neighbor. We can unify the two cases if we construct a virtual $(k+1)$th column as a copy of the 0th column lifted by one unit (see Figure 1).

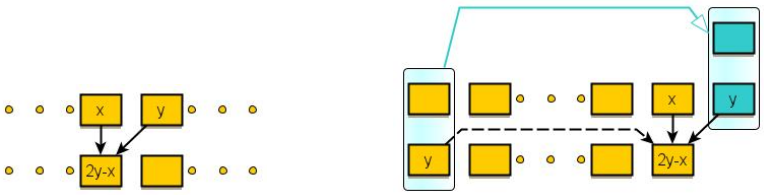

FIGURE 1. Construction rule of the table.

By this rule, we can easily fill the table for a given value k. But, this approach works also in case of a general k for $0 \leq j \leq k-2$, and partially for $k-1 \leq j \leq 2k-2$.

First, deal with the cases $0 \leq j \leq k-2$. It is illustrated by Table 2.

The 0th block is

$$\overbrace{0, 0, \ldots, 0}^{k-1}, 1, -1,$$

[1] This relation appeared in Garcia, Gómez, and Luca [8], equation (19).

j \ i	**0**	**1**	**2**	...	**k − 3**	**k − 2**	**k − 1**	**k**
0	0	0	0	...	0	0	1	−1
1	0	0	0	...	0	2	−3	1
2	0	0	0	...	4	−8	5	−1
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮
k − 2	0	2^{k-2}		...				$(-1)^{k-1}$

TABLE 2. The block scheme of $(H_n^{(k)})$, rows $0, \dots, k-2$.

and the zeros ensure that there are $k-2$ zeros at the beginning of the first block. Clearly, the number of the zeros are decreasing block by block. Hence

$$H^{(k)}_{j(k+1)+i} = 0 \quad \text{if} \quad j = 0, \dots, k-2; \;\; i = 0, \dots, k-2-j.$$

Recall the construction rule sketched in Figure 1. The non-zero parts of the blocks are gradually widening in a truncated triangular shape: $1, -1$ in row 0, and $2, -3, 1$ in row 1, etc. While the virtual column (the $(k+1)$th) contains 0 values then the non-zero triangle in the table coincides the triangle A118800 of OEIS [18]. No wonder, since A118800 possesses the same construction rule. Thanks to this coincidence, we see that (27) holds if $0 \le j \le k-2$. In particular, the left leg of the triangle contains increasing powers of 2, more precisely if $i = k-1-j$, then $n = j(k+1)+i = (j+1)k-1$ and $H_n^{(k)} = 2^j$. This proves Corollary 6.3. We explain why formula (27) is descending from row by row. This will be useful if we study the cases $k-1 \le j \le 2k-2$. Put $C_{a,b} := \binom{a}{b} + \binom{a-1}{b-1}$ (if the lower subscript is negative, then the binomial coefficient takes value 0). First, observe that in row 0 we have

$$1 = (-1)^0 \cdot 2^0 \cdot C_{1,0}, \quad -1 = (-1)^1 \cdot 2^{-1} \cdot C_{1,1}.$$

Then, introducing $\tau_{j,r} := (-1)^s \cdot 2^t \cdot C_{r,j}$ for some integers s and $-1 \le t$, one can easily verify $\tau_{j+1,r} = 2\tau_{j,r+1} - \tau_{j,r}$.

Examine now the rows $k-1 \le j \le 2k-2$. The main difference is that in these blocks, starting with block $k-1$, the left-most elements are non-zero. Thus, the table is perturbed by the virtual column, and the influence is growing from right by one additional entry, row by row. This is the reason that (27) is conditioned by $0 \le i \le 2k-2-j$ when $k-1 \le j \le 2k-2$. □

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