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Medium-sized values for the prime number theorem for primes in arithmetic progressions

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Abstract. We give two improved explicit versions of the prime number theorem for primes in arithmetic progressions: the first isolating the contribution of the Siegel zero and the second completely explicit, where the improvement is for medium-sized values of x. This will give an improved explicit Bombieri–Vinogradov-like result for non-exceptional moduli.

Contents

1. Introduction

The prime number theorem for primes in arithmetic progressions (PNTPAP) states

$$\psi(x;q,a)=\frac{x}{\varphi(q)}+o_q(x),$$

and the strength of the result lies in the explicit value of $o(x)$ and how this depends on the range of q. The best known result is the one with the error term due to Siegel–Walfisz that is uniform for $q \leq (\log x)^A$, for any $A>0$, but this result can not be made explicit since the proof is ineffective. The classical explicit versions of the PNTPAP are the following. The first is by McCurley that in [13] obtained an explicit result for non-exceptional moduli and in [14] focused on the case where $q=3$. Improving [14], Ramaré and Rumley proved in [17]

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explicit results for $q \leq 72$ and other small moduli. A result for large moduli is obtained by Liu and Wang in [12]; they proved, for $q \leq \log^6 x$, a version of the PNTPAP with an explicit error term of size $\frac{x}{\log^{13} x}$. Dusart, in [8], obtained an explicit error term that is of size $o\left(\frac{x}{\log^A x}\right)$, for any $A > 0$. Note that while this result improves [12] for large x, it is worse for medium-sized values in the range $10^2 \leq \log x \leq 10^6$. Dusart also improved the result in [17] for $q = 3$. Yamada in [22] (unpublished) proved a generalized version of the result in [12], where for multiple small- to medium-sized A and for $q \leq \log^A x$, he isolated the contribution of the Siegel zero and obtains an error term of size $\frac{x}{\log^{A-2} x}$. Note that this result is better than the one in [8] for medium-sized values, aside for the non-explicit contribution of the Siegel zero. Yamada also used this result, joint with [1], to obtain an explicit version of a Bombieri–Vinogradov style theorem for non-exceptional moduli. The last explicit version of the PNTPAP is the one by Bennett et al. in [2]. Here they improved the previous results for $3 \leq q \leq 10^5$ for small x and for $q \geq 10^5$ for large x.

In this paper we will focus on a version of the PNTPAP for medium-sized x. In doing this we will draw inspiration from [22] and we will first obtain an improved explicit version isolating the contribution of the Siegel zero, see Theorem 1.2. To obtain this result it is fundamental to obtain, with χ a Dirichlet character modulo q, the best error term in

$$\psi(x,\chi) = \sum_{\rho \in z(\chi), |\gamma| \leq T} \frac{x^\rho}{\rho} + R(x)\frac{x \log x}{T}, \tag{1}$$

see Section 3 for the definition of $z(\chi)$. Setting $T = \log^A x$, with $0 < A \ll 1$, Yamada, drawing inspiration from [12], proved explicitly that $R(x) \ll \log x$. In Lemma 3.3 we improve this result by proving an explicit version of the result by Goldston in [9], to obtain $R(x) \ll \log\log x$. In doing this we reshape the proof of Goldston to obtain a better explicit upper bound. Note that all the results we obtain are as general as possible to make them useful in different ranges of x and for different choices of A. We then use Theorem 1.2, together with the explicit bound on the Siegel zeroes in [4] and [5], to obtain a completely explicit version of the PNTPAP, in Theorem 1.4, that improves the previous results for medium-sized values. We conclude the paper using Theorem 1.2 to improve upon Bombieri–Vinogradov style theorem for non-exceptional moduli in [22]. The result we prove is also more general.

We now introduce the three main results. We start with a result on zeroes on Dirichlet L-functions, namely Theorem 1.1 and 1.3 of [10].

Theorem 1.1. *Define* $\prod(s,q) = \prod_{\chi \pmod q} L(s,\chi)$, $R_0 = 6.3970$ *and* $R_1 = 2.0452$. *Then the function* $\prod(s,q)$ *has at most one zero* $\rho = \beta + i\gamma$, *in the region* $\beta \geq 1 - 1/R_0 \log\max\{q, q|\gamma|\}$. *Such zero is called a Siegel zero and if it exists, then it must be real, simple and correspond to a non-principal real character* χ

(mod q). *Moreover, for any given Q_1, among all the zeroes with $q \le Q_1$ there is at most one such a zero with $\beta \ge 1 - 1/2R_1 \log Q_1$.*

We are interested in an intermediate explicit result to the PNTPAP that isolates the possible contribution due to the Siegel zero. In this paper we aim to improve Theorem 1.1 in [22]; we do so in the following result.

Theorem 1.2. *Let α_1, $\alpha_2 \in \mathbb{R}^+$, $Y_0 = \log\log X_0$ and $C(\alpha_1, \alpha_2, Y_0)$ be the constants given in Table 6 in the Appendices. Let $q \le \log^{\alpha_1} x$. Let $E_0 = 1$ if β_0, the possible Siegel zero modulo q, exists and $E_0 = 0$ otherwise. If $\gcd(a, q) = 1$ and $x \ge X_0$, then*

$$\frac{\varphi(q)}{x}\left|\psi(x;q,a) - \frac{x}{\varphi(q)}\right| < \frac{C(\alpha_1,\alpha_2,Y_0)}{\log^{\alpha_2} x} + E_0\frac{x^{\beta_0-1}}{\beta_0} \tag{2}$$

and

$$-1 + x^{-1}\sum_{\chi \pmod q} |\psi(x,\chi)| < \frac{C(\alpha_1,\alpha_2,Y_0)}{\log^{\alpha_2} x} + E_0\frac{x^{\beta_0-1}}{\beta_0}. \tag{3}$$

See (33) for values of $C(\alpha_1, \alpha_2, Y_0)$ different from those in Table 6. Note again that, for medium-sized x, we obtain an improvement of size $\log\log x/\log x$ on Yamada's result in [22]. From Theorem 1.2, using the results in [4] and [5] to control the size of the Siegel zeroes, we obtain a completely explicit version of the PNTPAP.

Theorem 1.3. *Let α_1, $\alpha_2 \in \mathbb{R}^+$, $\alpha_1 < 2$, $q \le \log^{\alpha_1} x$, $Y_0 = \log\log X_0$, $Y_0 \ge \max\left\{\frac{11\log 10}{\alpha_1+\alpha_2+3}, 2\right\}$ and*

$$C_1(\alpha_1,\alpha_2,X_0) = 2\exp\left(-\frac{100(\log x)^{1-\alpha_1/2}}{(\alpha_1\log\log x)^2}\right)\log^{\alpha_2} x + C(\alpha_1,\alpha_2,Y_0).$$

We have

$$\left|\psi(x;q,a) - \frac{x}{\varphi(q)}\right| < \frac{C_1(\alpha_1,\alpha_2,X_0)x}{\varphi(q)\log^{\alpha_2} x}.$$

See Table 1 for some explicit upper bounds for $C_1(\alpha_1, \alpha_2, X_0)$.

Y_0	α_1	α_2	C_1
4.4	1	1	0.65
5	1	2	0.66
5.6	1	3	0.3
7.8	1	10	$8.4 \cdot 10^{-2}$
4.6	1.2	1	0.69
5.2	1.2	2	0.43

Y_0	α_1	α_2	C_1
5.7	1.2	3	0.63
14.2	1.2	7	0.46
4.7	1.3	1	0.68
10.9	1.3	2	0.9
14.4	1.3	3	0.84
11.6	1.4	1	0.99

Table 1: Upper bound for $C_1(x, \alpha_1, \alpha_2, X_0)$

Note that the bound $\alpha_1 < 2$ is needed to bound effectively the size of the Siegel zero. If we need an upper bound independent or $\varphi(q)$ we can obtain it using Theorem 15 in [18]. For medium-sized x this improves the results in [2], see for example Theorems 1.1, 1.2, 1.3 and Lemma 6.10.

Now, for a given Q_1, we call a modulus $q_0 \leq Q_1$ to be exceptional up to Q_1 if $\prod(s, q_0)$ has a zero with $\beta \geq 1 - 1/2R_1 \log Q_1$, with R_1 defined in Theorem 1.1. We then use Theorem 1.2 and Theorem 1.2 in [1] to improve Theorem 1.4 in [22], while taking care in giving a more general version of the result. This result is an explicit Bombieri–Vinogradov-like theorem where the sum is restricted to non-exceptional moduli.

Theorem 1.4. *Let $A \in \mathbb{R}^+$ be such that $A > 3$,*

$$c_0 = \frac{2^{\frac{13}{2}}(2 + \log(\log 2/\log(4/3)))}{9\pi(\log 2)^2}\left(\frac{1}{3} + \frac{3}{2\log 2}\right)\sqrt{\frac{\psi(113)}{113}}$$

and

$$c_1 = \prod_p \left(1 + \frac{1}{p(p-1)}\right).$$

Let $Q = \frac{x^{1/2}}{\log^A x}$ and $1 \leq Q_1 \leq \log^A x$. Let q_0 denote the exceptional modulus up to Q_1 if it exists. If $\log\log x \geq \max\{7, \frac{11\log 10}{2A}\}$, then we have the inequality

$$\sum_{q \leq Q, q_0 \nmid q} \max_{a \pmod q} \left|\psi(x; q, a) - \frac{\psi(x)}{\varphi(q)}\right| < \sqrt{x} + \frac{2c_1c_0 x}{\log^{A-\frac{9}{2}} x} + \frac{2c_1c_0 x \log^{\frac{9}{2}} x}{Q_1} + \frac{c_1^2 x C(A, A-3, X_0)(1 + A\log\log x)}{2\log^{A-4} x} + E(x, A),$$

with $C(A, A-3, X_0)$ from Theorem 1.2, and

$$E(x, A) = \frac{\sqrt{x}}{2\log 2\log^{A-1} x} + \frac{c_1^2(1 + A\log\log x)\log x}{2} \cdot \left(\frac{x^{1-\frac{1}{2AR_1\log\log x}}}{1 - \frac{1}{2AR_1\log\log x}} + 34x(\log x)^{1.52}\exp\left(-0.81\sqrt{\log x}\right) + \frac{A\log\log x}{\log 2}\right) + \frac{c_0c_1\log^{\frac{9}{2}} x}{2}\left(4\sqrt{x}\log^A x + 18\frac{x^{\frac{11}{12}}}{\log^{\frac{A}{2}} x} + 5x^{\frac{5}{6}} + \frac{5}{2}x^{\frac{5}{6}}\log x\right).$$

This paper is structured as follows. In Section 2 we introduce useful bounds on the zeta and the L-functions. In Section 3 we prove the explicit version of (1) and in Section 4 we conclude the proof of Theorem 1.2. In Section 5 we prove Theorem 1.4.

2. Some useful bounds

We start by introducing Corollary 2.1 of [3] that, improving [21], gives a bound on the value of $N(T,\chi)$, the number of zeroes up to T of $L(s,\chi)$.

Lemma 2.1 (Bennett et al.)**.** *Let χ be a character with conductor $q>1$. If $T \geq 5/7$, then*

$$\left| N(T,\chi) - \frac{T}{\pi}\log\frac{qT}{2\pi e}\right| \leq r_1(q,T),$$

with

$$r_1(q,T) = \min\{0.247\log qT + 6.894, 0.298\log qT + 4.358\}.$$

To bound the possible Siegel zeroes we will use the results in [4] and [5] that can be stated as follows.

Lemma 2.2. *Let β_0 be a Siegel zero and q the modulus of the corresponding character χ. Then we have*

$$\frac{100}{\sqrt{q}\log^2 q} \leq \beta_0 \leq 1 - \frac{100}{\sqrt{q}\log^2 q}.$$

Moreover, for $1 < b \leq 1.3$, by equation (1.17) in [19] we have

$$-\frac{\zeta'}{\zeta}(b) < \frac{1}{b-1} - C + 0.1877(b-1), \tag{4}$$

with $C = \lim_{n\to\infty}\left(-\log n + \sum_{k=1}^{n}\frac{1}{k}\right)$ the Euler–Mascheroni constant. Also, taking $b > 1$ and $t \geq 1126$, by partial summation and Theorems 4 and 14 in [18], we obtain the asymptotically sharp bound

$$\sum_{\frac{t}{2}<n<t-1/2} \Lambda(n) = \psi(t-1/2) - \psi\left(\frac{t}{2}\right) \leq r_2(t),$$

with

$$\begin{aligned} r_2(t) =& \frac{t}{2}\left(1 + \frac{1}{\log(t-1/2)} - \frac{1}{2\log t/2}\right) + \sqrt{t-1/2}\left(1+\frac{1}{\log(t-1/2)}\right) \\ &- 0.98\sqrt{\frac{t}{2}} + 3(t-1/2)^{1/3} - \frac{1}{2}\left(1 + \frac{1}{2\log(t-1/2)}\right). \end{aligned}$$

This gives

$$\sum_{\frac{t}{2}<n<t-1/2} \Lambda(n)n^{-b} \leq \left(\frac{2}{t}\right)^{b} r_2(t). \tag{5}$$

3. Preliminary results

Letting $\rho = \beta + i\gamma$, we define

$$z(\chi) = \{\rho : \rho \neq 0, \beta > 1/2, L(\rho,\chi) = 0\},$$

which is fundamental in estimating the error term in the PNTPAP. In this section we aim to prove the following fundamental result, that is an improved version of Theorem 8 in [12] and Lemma 2.1 in [22], obtained making Goldston's result in [9] explicit.

Lemma 3.1. *Let χ be a Dirichlet character modulo q and $T = \log^{\alpha} x$, with $\alpha = \alpha_1 + \alpha_2 + 3$, $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and $q \leq \log^{\alpha_1} x$. Assuming $x > \exp\exp C$, with $C \geq 2$, we have*

$$|\psi(x,\chi) - \delta(\chi)x| \leq \sum_{\rho \in z(\chi), |\gamma| \leq T} \frac{x^{\beta}}{|\rho|} + R(C,\alpha_2,\alpha_1)\frac{x \log x \log\log x}{T},$$

with $\delta(\chi) = 1$ if χ is principal and $\delta(\chi) = 0$ otherwise. Upper bounds for $R(C,\alpha_2,\alpha_1)$ are given in Table 2.

C	α_1	α_2	R
3.4	1	1	32.2
3.7	1	2	28.6
4.3	1	5	24.6
3.8	2	1	17.3
4.3	2	4	32.7

C	α_1	α_2	R
4	3	1	27.9
4.6	3	5	31.9
4.3	4	1	19.7
4.9	4	6	18.9
4.5	5	1	26.4

C	α_1	α_2	R
5.1	5	7	27.7
4.7	6	1	23.4
5.3	6	8	21.1
4.9	7	1	17.6
5.5	7	9	18.6

Table 2: Upper bound for $R(x,\alpha_2,\alpha_1)$

Note that using Theorem 3.4 we can bound R for different values of C, α_1 and α_2. Lemma 3.1 improves Lemma 2.1 in [22], in the error term, reducing a $\log x$ factor to a $\log\log x$ one. It is interesting to note that Littlewood, in [11], assuming the Riemann Hypothesis proved the above result for $\psi(x)$ with an error term of $\frac{x\log x}{T}$. This suggests that even if it should be possible to improve the error term in the above result, it will probably be highly complicated. Yamada's Lemma 2.1 is based on Lemma 1 in [6]. Splitting a sum in a similar way as done by Dudek in [7] and being more careful with the error terms, it is possible to obtain an upper bound for $R(x,\alpha_1,\alpha_2)$ that is half the size of Yamada's. We will not give more details on this as this result is superseded by the one obtained making Goldston's result in [9] explicit. Here, we will prove an explicit version of Lemma 2 in [9], with a partially different proof to better control the error term.

Lemma 3.2. *We have*

$$\sum_{\frac{x}{2} < n \leq x - 3/2} \frac{\Lambda(n)}{(xn^{-1} - 1)} \leq r_{3.1}(x) \text{ and } \sum_{x+3/2 \leq n < 2x} \frac{\Lambda(n)}{(1 - xn^{-1})} \leq r_{3.2},$$

where, for $x \geq 10$,

$$r_{3.1}(x) = x\log x\Big(2\log\log\left(\frac{x}{2}-1\right) + \frac{2}{3}\log\log x + 0.57 + \frac{2}{\log\left(\frac{x}{2}-1\right)} + \frac{1}{\sqrt{x}}\left(\frac{\pi^2}{6}-1\right)\Big(\log 2x^{\frac{3}{2}} + 1.76\Big)\Big).$$

and, for $x \geq 5$,

$$r_{3.2}(x) = x\log x\Big(2\log\log x + \frac{2}{3}\log\log 2x + 0.56 + \frac{2x}{(x-2)\log x} + \frac{1}{\sqrt{x}}\left(\frac{\pi^2}{6}-1\right)\Big(\log 2x^{\frac{3}{2}} + 1.76\Big)\Big) + 2.1x.$$

Proof. The proof is based on that given in [9]. We will prove the first of the two inequalities. We have

$$\sum_{\frac{x}{2}<n\leq x-3/2} \frac{\Lambda(n)}{(xn^{-1}-1)} \leq x \sum_{1\leq l\leq \frac{\log x}{\log 2}} \sum_{\frac{x}{2}<p^l\leq x-3/2} \frac{\log p}{(x-p^l)}. \tag{6}$$

We will first bound the right-hand side sum with $l = 1$. Let $\pi(x)$ count the number of primes less than x and define $P(x,y) = \pi(x) - \pi(x-y)$. In [15] it is proved that

$$P(x,y) \leq \frac{2y}{\log y} \quad 1 < y \leq x. \tag{7}$$

Now

$$\begin{aligned}\sum_{\frac{x}{2}<p\leq x-3/2} \frac{1}{(x-p)} &\leq \sum_{1\leq n<\frac{x}{2}} \frac{1}{n+1/2}\Big(P(x-\frac{3}{2},n) - P(x-\frac{3}{2},n-1)\Big)\\ &\leq \sum_{1\leq n\leq \frac{x}{2}-1} P(x-\frac{3}{2},n)\left(\frac{1}{n+1/2} - \frac{1}{n+3/2}\right)\\ &\quad + \frac{2}{x-1}P(x-\frac{3}{2},\frac{x}{2}-1) - \frac{2}{3}P(x-\frac{3}{2},0).\end{aligned}$$

Seen that we assumed $x \geq 10$ and using (7), this can be bounded with

$$\begin{aligned}&2\sum_{4\leq n\leq \frac{x}{2}-1} \frac{1}{(n+3/2)\log n} + \frac{4}{15} + \frac{4}{35} + \frac{8}{63} + \frac{2}{\log\left(\frac{x}{2}-1\right)}\\ &\leq 2\log\log\left(\frac{x}{2}-1\right) - 2\log\log 3 + \frac{4}{15} + \frac{4}{35} + \frac{8}{63} + \frac{2}{\log\left(\frac{x}{2}-1\right)}.\end{aligned}$$

We now bound the right-hand side sum in (6) for $l \geq 2$. We have

$$\sum_{2\leq l\leq \frac{\log x}{\log 2}} \sum_{\frac{x}{2}<p^l\leq x-3/2} \frac{\log p}{(x-p^l)} \leq \log x \sum_{2\leq l\leq \frac{\log x}{\log 2}} \frac{1}{l} \sum_{\frac{x}{2}<p^l\leq x-3/2} \frac{1}{(x-p^l)}.$$

For $l \geq 2$, using Euler–Maclaurin summation formula, we have

$$\sum_{\frac{x}{2}<p^l\leq x-3/2} \frac{1}{(x-p^l)} \leq \int_{(\frac{x}{2})^{1/l}}^{(x-3/2)^{1/l}} \frac{1}{(x-t^l)}dt + \frac{2}{3}$$

$$\leq \frac{2}{l}\int_{\sqrt{\frac{x}{2}}}^{\sqrt{x-3/2}} \frac{1}{(x-y^2)}dy + \frac{2}{3},$$

where in the last step we used the change of variables $t = y^{2/l}$. Using

$$\int \frac{1}{(x-y^2)}dy = -\frac{\log \frac{|y-\sqrt{x}|}{|y+\sqrt{x}|}}{2\sqrt{x}} + C, \tag{8}$$

we have

$$\sum_{2\leq l\leq \frac{\log x}{\log 2}} \frac{1}{l} \sum_{\frac{x}{2}<p^l\leq x-3/2} \frac{1}{(x-p^l)}$$

$$\leq \frac{2}{3}\log\left(\frac{\log x}{\log 2}\right) + \left(\frac{\pi^2}{6}-1\right)\left(-\frac{\log \frac{|y-\sqrt{x}|}{|y+\sqrt{x}|}}{\sqrt{x}}\right)\Bigg|_{\sqrt{\frac{x}{2}}}^{\sqrt{x-3/2}}.$$

We finish the proof of the first upper bound observing, for $x \geq 3$, that

$$\frac{\sqrt{x}-\sqrt{x-3/2}}{\sqrt{x}+\sqrt{x-3/2}} \geq \frac{1}{2x^{\frac{3}{2}}},$$

which gives

$$\left(-\log\frac{|y-\sqrt{x}|}{|y+\sqrt{x}|}\right)\Bigg|_{\sqrt{\frac{x}{2}}}^{\sqrt{x-3/2}} \leq \log 2x^{\frac{3}{2}} + \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right).$$

We now focus on the second of the two inequalities. We can see that

$$\sum_{x+3/2\leq n<2x} \frac{\Lambda(n)}{(1-xn^{-1})} = \sum_{x+3/2\leq n<2x} \Lambda(n) + x\sum_{x+3/2\leq n<2x} \frac{\Lambda(n)}{(n-x)}.$$

By Theorem 12 in [18], it follows

$$\sum_{x+3/2\leq n<2x} \Lambda(n) \leq 2.1x.$$

While this bound is far from optimal, the sum is asymptotic to x, it is enough for our purposes as it accounts for a negligible term. We now have

$$\sum_{x+3/2<n\leq 2x} \frac{\Lambda(n)}{(n-x)} \leq \sum_{1\leq l\leq \frac{\log 2x}{\log 2}} \sum_{x+3/2<p^l\leq 2x} \frac{\log p}{(p^l-x)}. \tag{9}$$

We will first bound the right-hand side sum with $l = 1$. Define $P^*(x, y) = \pi(x + y) - \pi(x)$, with $1 < y \leq x$. Now

$$\begin{aligned}\sum_{x+3/2<p\leq 2x} \frac{1}{(p-x)} &\leq \sum_{1\leq n<x-1} \frac{1}{\max\{n, 3/2\}} (P^*(x, n+1) - P^*(x, n)) \\ &\leq \sum_{2\leq n\leq x-1} P^*(x, n+1) \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &\quad + \frac{P^*(x, x)}{x-2} + \frac{P^*(x, 2)}{6} - \frac{2}{3} P^*(x, 1).\end{aligned}$$

Seen that we assumed $x \geq 5$ and using (7), this can be bounded with

$$\begin{aligned}&2 \sum_{4\leq n\leq x-1} \frac{1}{n \log(n+1)} + \frac{1}{2} + \frac{2x}{(x-2)\log x} \\ &\leq 2 \log\log(x-1) - 2\log\log 3 + \frac{1}{2} + \frac{2x}{(x-2)\log x}.\end{aligned}$$

We now bound the right-hand side sum in (9) for $l \geq 2$. We have

$$\sum_{2\leq l\leq \frac{\log 2x}{\log 2}} \sum_{x+3/2<p^l\leq 2x} \frac{\log p}{(p^l - x)} \leq \log x \sum_{2\leq l\leq \frac{\log x}{\log 2}} \frac{1}{l} \sum_{x+3/2<p^l\leq 2x} \frac{1}{(p^l - x)}.$$

For $l \geq 2$, using the Euler–Maclaurin summation formula, we have

$$\begin{aligned}\sum_{x+3/2<p^l\leq 2x} \frac{1}{(p^l - x)} &\leq \int_{(x+3/2)^{1/l}}^{(2x)^{1/l}} \frac{1}{(t^l - x)} dt + \frac{2}{3} \\ &\leq \frac{2}{l} \int_{\sqrt{x+3/2}}^{\sqrt{2x}} \frac{1}{(y^2 - x)} dy + \frac{2}{3},\end{aligned}$$

where in the last step we used the change of variables $t = y^{2/l}$. Remembering (8), we have

$$\begin{aligned}&\sum_{2\leq l\leq \frac{\log 2x}{\log 2}} \frac{1}{l} \sum_{x+3/2<p^l\leq 2x} \frac{1}{(p^l - x)} \\ &\qquad \leq \frac{2}{3} \log\left(\frac{\log 2x}{\log 2}\right) + \left(\frac{\pi^2}{6} - 1\right) \left(\frac{\log \frac{|y-\sqrt{x}|}{|y+\sqrt{x}|}}{\sqrt{x}}\right)\Bigg|_{\sqrt{x+3/2}}^{\sqrt{2x}}.\end{aligned}$$

We conclude the proof of the first upper bound by observing that, for $x \geq 3$,

$$\frac{\sqrt{x+3/2} - \sqrt{x}}{\sqrt{x+3/2} + \sqrt{x}} \geq \frac{1}{2x^{\frac{3}{2}}},$$

which gives

$$\left(\log\frac{|y-\sqrt{x}|}{|y+\sqrt{x}|}\right)\Bigg|_{\sqrt{x+3/2}}^{\sqrt{2x}} \leq \log 2x^{\frac{3}{2}} + \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right).$$

□

Note that the term $2/3$ that appears in the proof above can probably be improved using Euler–Maclaurin summation formula to a higher order. An improvement should also be possible in the ≈ 0.5 constant. We now introduce a variation of Chen and Wang Lemma 1 in [6], this is obtained using Lemma 3.2. Being a bit more careful than Chen and Wang we also obtain a $\log 2$ saving in the remainder term, not counting the use of Lemma 3.2.

Lemma 3.3. *Take $f(s)=\sum_{n=1}^{\infty}\Lambda(n)/n^s$, that is absolutely convergent for $\Re(s)>1$. Then for any $b>1$, $T\geq 1$ and $x=N+1/2\geq 6$, with N a positive integer, we have*

$$\left|\sum_{n\leq x}\Lambda(n)-\frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)s^{-1}x^s ds\right| \leq R_1(x,T,b),$$

with

$$R_1(x,T,b)=\frac{x^b}{\pi T\log 2}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^b}+\frac{2^{-1}x^b}{\pi T}\sum_{\frac{x}{2}<n<x-1/2}\frac{\Lambda(n)}{n^b}$$
$$+\frac{1}{\pi T}\Bigg(2^b\Bigg(\log(x-1/2)(x-1/2)\left(2+\frac{1}{x-1/2}\right)\left(\frac{x}{x-1/2}\right)^b$$
$$+\log(x+1/2)2(x+1/2)\left(\frac{x}{x+1/2}\right)^b+r_{3.1}(x)\Bigg)+r_{3.2}(x)\Bigg).$$

Proof. Following the proof of Lemma 1 in [6] we obtain

$$\left|\sum_{n\leq x}\Lambda(n)-\frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)s^{-1}x^s ds\right|$$
$$\leq \frac{x^b}{\pi T\log 2}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^b}+\frac{1}{\pi T}\sum_{\frac{x}{2}<n<2x}\Lambda(n)\frac{(xn^{-1})^b}{\log(xn^{-1})}.$$

We are now left with obtaining an upper bound for the right-hand side sum. We start splitting the sum in two at $n=N$, obtaining

$$\sum_{\frac{x}{2}<n<2x}\Lambda(n)\frac{(xn^{-1})^b}{\log(xn^{-1})}= \sum_{\frac{x}{2}<n<N}\Lambda(n)\frac{(xn^{-1})^b}{\log(xn^{-1})}+\sum_{N\leq n<2x}\Lambda(n)\frac{(xn^{-1})^b}{\log(xn^{-1})}. \tag{10}$$

The first term of (10), remembering that $x = N + 1/2$, is

$$\sum_{\frac{x}{2}<n<x-1/2} \Lambda(n)\frac{(xn^{-1})^b}{\log(xn^{-1})}. \tag{11}$$

For $x \geq 1$ the Taylor expansion for $\log x$ gives

$$\log x \geq \frac{2(x-1)}{(x+1)},$$

which allows to bound (11) with

$$\begin{aligned}
&2^{-1} \sum_{\frac{x}{2}<n<x-1/2} \Lambda(n)\frac{(xn^{-1})^b(xn^{-1}+1)}{(xn^{-1}-1)} \\
&= 2^{-1}x^b \sum_{\frac{x}{2}<n<x-1/2} \Lambda(n)n^{-b} + 2^b \sum_{\frac{x}{2}<n<x-1/2} \frac{\Lambda(n)}{(xn^{-1}-1)}.
\end{aligned}$$

We can now bound the right-hand side sum using Lemma 3.2. The second term of (10), remembering that $x = N + 1/2$, is

$$\begin{aligned}
\sum_{x-1/2\leq n<2x} \Lambda(n)\frac{(xn^{-1})^b}{|\log(xn^{-1})|} &\leq \log(x-1/2)\frac{(x/(x-1/2))^b}{|\log(x/(x-1/2))|} \\
&+ \log(x+1/2)\frac{(x/(x+1/2))^b}{|\log(x/(x+1/2))|} + \sum_{x+3/2\leq n<2x} \frac{\Lambda(n)}{|\log(xn^{-1})|}.
\end{aligned}$$

We can bound the right-hand side sum using $|\log(1-x)| \geq x$, which holds for $0 \leq x \leq 1$, and Lemma 3.2. This concludes the proof. □

It is interesting to note that in Lemma 3.3 it would be possible to obtain a slightly better result splitting (10) at a point different than 2, but this would require proving a customized variation of Lemma 3.2.

We can now prove Lemma 3.1, proceeding similarly to the proof of Theorem 8 in [12] and Lemma 2.1 in [22].

Theorem 3.4. *Let χ be a Dirichlet character modulo q, and let $T > 1$ and $x \geq 1126$. Then*

$$|\psi(x,\chi) - \delta(\chi)x| \leq \sum_{\rho\in z(\chi),|\gamma|\leq T} \frac{x^\beta}{|\rho|} + R^*(x,T,q),$$

with

$$\begin{aligned}
R^*(x,T,q) =& \frac{\log q}{\log 2} + R_2(T,x) + R_3(T,x) + \log 2 + R_5(T,x) + R_7(T,q,x) \\
&+ R_8(T,q,x) + \log x + \frac{x}{T-1}r_4(T,q) + R_{11}(T,q,x),
\end{aligned}$$

$$r_4(T,q) = \frac{T+1}{\pi}\log\frac{q(T+1)}{2\pi e} - \frac{T-1}{\pi}\log\frac{q(T-1)}{2\pi e} + r_1(T+1,q) + r_1(T-1,q),$$

$$R_2(x,T) = R_1(x,T,1+\log^{-1}x),$$

where we use (4) *and* (5) *to make this last term completely explicit,*

$$R_3(x,T) = \frac{1}{\pi T\log 2}\Big(\log x - C + \frac{0.1877}{\log x} + 2^{1+\frac{1}{\log x}} \cdot \Big(x\log x + \frac{3}{2}x - \frac{1}{2}\Big)\log x + x(\log x + \log 2 + 2)\log 2x\Big),$$

$$R_5(T,x) = \frac{1}{2\pi}\left(\log x - C + \frac{0.1877}{\log x}\right)(T+1)\exp\left(\frac{\log(x+1/2)}{\log x}\right),$$

$$r_5(x,q) = 2.5\Big(1.75\log(qx) + \frac{1}{2.5+x^2} + \frac{1}{x(2.25+x^2)^{1/2}} + \frac{\pi}{4x} + 3.31\Big) + 0.62,$$

$$R_7(T,q,x) = \frac{(3/2+\frac{1}{\log x})(ex+2.5^{1+\frac{1}{\log x}})}{2\pi(T-1)}\Big(r_4(T+1,q)\,(r_4(T+1,q)+1) + \max_{T-1\le x\le T+1} r_5(x,q)\Big),$$

$$r_6(x) = \Big(1.75\log(2+|x|) + \frac{1}{2.5+x^2} + 2.43 + \frac{1}{((4+x^2)(0.25+x^2))^{1/2}}\Big) + 0.62,$$

$$R_8(T,q,x) = \frac{\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{2.5}}\right)(\frac{3}{2}+\frac{1}{\log x})}{2\pi(T-1)} \cdot \left(2r_4(T+1,q) + \max_{-(T+1)\le x\le T+1} r_6(x)\right),$$

$$R_{11}(T,q,x) = (\sqrt{x}+2)\frac{\sqrt{q}\log^2 q}{100} + (\sqrt{x}+1)\Big(R_0 r_1(q,1)\log q + \frac{r_1(q,T+1)}{T+1} + \frac{\log T}{\pi}\log\frac{q\sqrt{T}}{2\pi e} + r_1(q,T)\Big).$$

Proof. Aside for the result in Lemma 3.3 the proof is a more general version of Theorem 8 in [12]. If $\chi \pmod q$ is induced by the primitive character $\chi_1 \pmod{q_1}$ we have,

$$|\psi(\chi,x)-\psi(\chi_1,x)| \le \frac{\log q}{\log 2}. \tag{12}$$

Thus, taking note of the above error term, we can focus on primitive characters. Note that if $\chi=\chi_0$, then the following argument holds with $\psi(\chi,x)$ replaced by $\psi(\chi,x)-x$. By Corollary 2.1 in [3], we have

$$\sum_{|\gamma-T|\le 1} 1 \le r_4(T,q), \tag{13}$$

thus we can see that there exists a real T_0, such that $|T-T_0|\le 1$ and

$$\frac{1}{|\gamma-T_0|} \le r_4(T,q)+1, \tag{14}$$

for any non-trivial zero $\rho=\beta+i\gamma$ of $L(s,\chi)$. Defining $x_0=\lfloor x\rfloor+1/2$, we have

$$\psi(x,\chi)=\psi(x_0,\chi).$$

Now,

$$\left|\psi(x,\chi)-\frac{1}{2\pi i}\int_{b-iT_0}^{b+iT_0}\left(-\frac{L'}{L}(s,\chi)\right)\frac{x_0^s}{s}ds\right| \le R_2(x,T). \tag{15}$$

By Lemma 1 in [6] and using (4), we obtain

$$\left|\psi(2.5,\chi)-\frac{1}{2\pi i}\int_{b-iT_0}^{b+iT_0}\left(-\frac{L'}{L}(s,\chi)\right)\frac{2.5^s}{s}ds\right| \le R_3(x,T). \tag{16}$$

Considering the difference between $\psi(x,\chi)$ and $\psi(2.5,\chi)$, we obtain

$$\psi(x,\chi)=\frac{1}{2\pi i}\int_{b-iT_0}^{b+iT_0}\left(-\frac{L'}{L}(s,\chi)\right)\frac{x_0^s-2.5^s}{s}ds+R_4(x,T), \tag{17}$$

where, observing that $|\psi(2.5,\chi)|\le\log 2$, $|R_4(x,T)|\le R_2(x,T)+R_3(x,T)+\log 2$. The difference between the main term of the right-hand side of (17) and its analogue with x_0 replaced by x is at most

$$\frac{1}{2\pi}\int_{-T_0}^{T_0}\left|\frac{L'}{L}(b+iu,\chi)\right|\left|\int_{x_0}^{2.5}x^{b-1+iu}dx\right|du \le R_5(T,x).$$

In the last step we used (4), $b=1+\log^{-1}x$ and $|T_0-T|\le 1$. Thus (17) becomes

$$\psi(x,\chi)=\frac{1}{2\pi i}\int_{b-iT_0}^{b+iT_0}\left(-\frac{L'}{L}(s,\chi)\right)\frac{x^s-2.5^s}{s}ds+R_6(x,T), \tag{18}$$

with $|R_6(x,T)|\le|R_4(x,T)|+R_5(x,T)$. We now take Ω to be a rectangle with vertices $b\pm iT_0$ and $1/2\pm iT_0$. By Cauchy's residue theorem we obtain

$$\frac{1}{2\pi i}\int_{\Omega}\left(-\frac{L'}{L}(s,\chi)\right)\frac{x^s-2.5^s}{s}ds=-\sum_{\rho\in z(\chi),|\gamma|\le T_0}\frac{x^\rho-2.5^\rho}{\rho}+\theta\log x, \tag{19}$$

with $|\theta| \le 1$. By Lemma 9 in [6], (13) and (14), we obtain

$$\left|\frac{L'}{L}(\sigma + iT_0, \chi)\right| \le r_4(T+1, q)\,(r_4(T+1, q) + 1) + r_5(T_0, q).$$

Also by (9') in [6] and (13)

$$\left|\frac{L'}{L}(-1/2 + ix, \chi)\right| \le 2r_4(T, q) + r_6(x, T).$$

Hence

$$\left|\frac{1}{2\pi i}\int_{-1/2\pm iT_0}^{b\pm iT_0}\left(-\frac{L'}{L}(s,\chi)\right)\frac{x^s - 2.5^s}{s}ds\right| \le R_7(T, q, x)$$

and

$$\left|\frac{1}{2\pi i}\int_{-1/2-iT_0}^{-1/2+iT_0}\left(-\frac{L'}{L}(s,\chi)\right)\frac{x^s - 2.5^s}{s}ds\right| \le R_8(T, q, x).$$

Thus by (18) and (19)

$$\psi(t,\chi) = -\sum_{\rho\in z(\chi), |\gamma|\le T_0}\frac{x^\rho - 2.5^\rho}{\rho} + R_9(T, q, x) \tag{20}$$

with $|R_9(T, q, x)| \le |R_6(x, T)| + R_7(T, q, x) + R_8(T, q, x) + \log x$. By (13)

$$\left|\sum_{\rho\in z(\chi), |\gamma|\le T}\frac{x^\rho}{\rho} - \sum_{\rho\in z(\chi), |\gamma|\le T_0}\frac{x^\rho}{\rho}\right| \le \frac{x}{T-1}r_4(T, q).$$

Thus (20) can be rewritten as

$$\psi(t,\chi) = -\sum_{\rho\in z(\chi), |\gamma|\le T}\frac{x^\rho}{\rho} + \sum_{\rho\in z(\chi), |\gamma|\le T_0}\frac{2.5^\rho}{\rho} + R_{10}(T, q, x), \tag{21}$$

with $|R_{10}(T, q, x)| \le |R_9(T, q, x)| + \frac{x}{T-1}r_4(T, q)$. We can see that

$$\begin{aligned}\left|\sum_{\rho\in z(\chi), |\gamma|\le T_0}\frac{2.5^\rho}{\rho}\right| &\le \sum_{\rho\in z(\chi), |\gamma|\le T+1}\frac{2.5}{|\rho|}\\ &\le \sum_{\rho\in z(\chi), |\gamma|\le 1}\frac{2.5}{\beta} + \sum_{\rho\in z(\chi), 1<|\gamma|\le T+1}\frac{2.5}{|\rho|}.\end{aligned} \tag{22}$$

Now using Lemma 2.2 to bound the possible two Siegel zeroes, Theorem 1.1 to bound the other zeroes and Lemma 2.1, we obtain

$$\sum_{\rho\in z(\chi), |\gamma|\le 1}\frac{1}{\beta(\rho)} \le \frac{\sqrt{q}\log^2 q}{50} + R_0 r_1(q, 1)\log q. \tag{23}$$

Furthermore, by Lemma 2.1

$$\sum_{\rho\in z(\chi), 1<|\gamma|\le T+1}\frac{1}{|\rho|} \le \frac{r_1(q, T+1)}{T+1} + \int_1^{T+1}\frac{r_1(q, y)}{y^2}dy. \tag{24}$$

We can also see that

$$\sum_{\substack{\rho\in z(\chi)\\|\gamma|\le T,\beta<1/2}}\frac{x^\rho}{\rho}=\sum_{\substack{\rho\in z(\chi)\\|\gamma|\le 1,\beta<1/2}}\frac{x^\rho}{\rho}+\sum_{\substack{\rho\in z(\chi)\\1<|\gamma|\le T+1,\beta<1/2}}\frac{x^\rho}{\rho}. \tag{25}$$

Similarly to (23), as there is only one possible Siegel zero in this range,

$$\sum_{\substack{\rho\in z(\chi)\\|\gamma|\le 1,\beta<1/2}}\frac{x^\sigma}{|\rho|}\le\sqrt{x}\left(\frac{\sqrt{q}\log^2 q}{100}+R_0r_1(q,1)\log q\right) \tag{26}$$

and similarly to (24)

$$\sum_{\substack{\rho\in z(\chi)\\1<|\gamma|\le T+1,\beta<1/2}}\frac{x^\sigma}{|\rho|}\le\sqrt{x}\left(\frac{r_1(q,T+1)}{T+1}+\int_1^{T+1}\frac{r_1(q,y)}{y^2}dy\right). \tag{27}$$

Here we can note that

$$\int_1^{T+1}\frac{r_1(q,y)}{y^2}dy\le\frac{\log T}{\pi}\log\frac{q\sqrt{T}}{2\pi e}+r_1(T,q).$$

Thus by (21)–(27), we obtain

$$\left|\psi(t,\chi)+\sum_{\substack{\rho\in z(\chi)\\|\gamma|\le T,\beta\ge 1/2}}\frac{x^\rho}{\rho}\right|\le|R_{10}(T,q,x)|+R_{11}(T,q,x).$$

This concludes the proof. □

We can now prove Lemma 3.1.

Proof. (Lemma 3.1) The result follows from Theorem 3.4, the choices for q,T,x done in Lemma 3.1 and simple computations. We also used that

$$\max_{T-1\le x\le T+1}r_5(x,q)\le 2.5\Big(1.75\log(q(T+1))+\frac{1}{2.5+(T-1)^2}+\frac{1}{(T-1)(2.25+(T-1)^2)^{1/2}}+\frac{\pi}{4(T-1)}+3.31\Big)+0.62,$$

$$\max_{-(T+1)\le x\le T+1}r_6(x)\le 2.5\left(1.75\log(2T+3)+\frac{1}{2.5}+3.43\right)+0.62.$$

□

4. Proof of Theorem 1.2

Since (3) implies (2), we will focus on proving (3). We set

$$\Sigma = \sum_{\chi} \sum_{\rho \in z(\chi), |\lambda| \le T} \frac{x^{\beta-1}}{|\rho|},$$

where the external sum is over all Dirichlet characters modulo q. Now we need to bound Σ and to do this successfully we split the sum as follows. For $H > 1$ and $R > 0$, we define

$$z_0(\chi, H, R) = \{\rho : \frac{1}{2} \le \beta \le 1 - \frac{1}{R \log qH}, |\gamma| < H, L(\rho, \chi) = 0\}$$

and

$$z_1(\chi, H, R) = \{\rho : \frac{1}{R_0} \le (1-\beta) \log qH \le \frac{1}{R}, |\gamma| < H, L(\rho, \chi) = 0\}.$$

We define

$$\Sigma_0 = \sum_{\chi} \sum_{\rho \in z_0(\chi, T, R)} \frac{x^{\beta-1}}{|\rho|} \text{ and } \Sigma_1 = \sum_{\chi} \sum_{\rho \in z_1(\chi, T, R)} \frac{x^{\beta-1}}{|\rho|}.$$

This gives us

$$\Sigma = \Sigma_0 + \Sigma_1 + E_0 \frac{x^{\beta_0 - 1}}{\beta_0}. \tag{28}$$

We can see that

$$\sum_{\rho \in z_0(\chi, T, R)} \frac{x^{\beta-1}}{|\rho|} \le \frac{1}{2} x^{-1/R \log qT} \left(2N(\chi, 1) + \int_1^T \frac{dN(\chi, t)}{t} \right)$$

and thus, by Lemma 2.1,

$$\Sigma_0 \le \frac{qS(T, q)}{2} x^{-1/R \log qT}, \tag{29}$$

with

$$\begin{aligned} S(T, q) = &\frac{2}{\pi} \log \frac{q}{2\pi e} + 2r_1(q, 1) \\ &+ \frac{1}{2\pi} \log T \left(\log T + 2 \left(\log \frac{q}{2\pi e} + 1 \right) \right) + 0.298. \end{aligned}$$

We are now left with estimating Σ_1. We start with the following estimate, that is Lemma 2.2 in [22] with the corrected upper bound. In Lemma 2.2 the condition $\exp \sqrt{\frac{\log x}{R_0}} \le (qT)^{1/R_0 \lambda}$ is misstated as $\exp \sqrt{\frac{x}{R_0}} \le (qT)^{1/R_0 \lambda}$.

Lemma 4.1. *Let $\rho = \beta + i\gamma$ be a zero of $L(s,\chi)$, with $\beta < 1 - 1/R_0 \log q|\gamma|$, $|\gamma| \le T$ and $\rho \ne 0$. Let λ be such that $\beta = 1 - \lambda/\log qT$, then*

$$\left|\frac{x^{\rho-1}}{\rho}\right| \le \mu(\lambda) = \begin{cases} \frac{x^{-1/R_0 \log q}}{1-1/R_0 \log q} & \text{if } |\gamma| \le 1, \\ q e^{-2\sqrt{\frac{\log x}{R_0}}} & \text{if } |\gamma| > 1 \text{ and } e^{\sqrt{\frac{\log x}{R_0}}} < (qT)^{1/R_0\lambda}, \\ q\frac{x^{-\lambda/\log qT}}{(qT)^{1/R_0\lambda}} & \text{otherwise.} \end{cases}$$

Proof. When $|\gamma| \le 1$ we obtain

$$\left|\frac{x^{\rho-1}}{\rho}\right| \le \frac{x^{\beta-1}}{\beta} \le \frac{x^{-1/R_0 \log q}}{1 - 1/R_0 \log q},$$

using in the last step that $\beta \ge \frac{1}{R_0 \log q} \ge \frac{1}{\log x}$, which follows from Theorem 1.1. We may now assume $\gamma \ge 0$. If $\gamma \ge 1$, observing that

$$\frac{1}{R_0 \log q\gamma} + \frac{\log \gamma}{\log x} \ge 2\sqrt{\frac{1}{R_0 \log x}} - \frac{\log q}{\log x},$$

we obtain

$$\left|\frac{x^{\rho-1}}{\rho}\right| \le \frac{x^{\beta-1}}{\gamma} \le q \exp -2\sqrt{\frac{\log x}{R_0}}.$$

Let $\gamma_0 = \frac{1}{q} \exp\sqrt{\frac{\log x}{R_0}}$ and $\gamma_1 = q^{1/R_0\lambda-1} T^{1/R_0\lambda}$. We may assume $\gamma_0 \ge \gamma_1$. In the case $\gamma \le \gamma_1$ we have

$$\left|\frac{x^{\rho-1}}{\rho}\right| \le \frac{x^{1/R_0 \log q\gamma}}{\gamma} \le \frac{x^{1/R_0 \log q\gamma_1}}{\gamma_1} = q\frac{x^{-\lambda/\log qT}}{(qT)^{1/R_0\lambda}},$$

since $\frac{x^{1/R_0 \log q\gamma}}{\gamma}$ is increasing below γ_0. In the other case we have

$$\left|\frac{x^{\rho-1}}{\rho}\right| \le \frac{x^{\lambda/\log qT}}{\gamma} \le \frac{x^{\lambda/\log qT}}{\gamma_1} = q\frac{x^{-\lambda/\log qT}}{(qT)^{1/R_0\lambda}}.$$

□

We now split Σ_1 as follows

$$\Sigma_1 = \sum_{\chi} \sum_{\rho \in z_1(\chi,T,R), |\gamma| \le 1} \frac{x^{\beta-1}}{|\rho|} + \sum_{\chi} \sum_{\rho \in z_1(\chi,T,R), |\gamma| \ge 1} \frac{x^{\beta-1}}{|\rho|}. \tag{30}$$

By Lemma 2.1, we obtain

$$\sum_{\chi} \sum_{\rho \in z_1(\chi,T,R), |\gamma| \le 1} \frac{x^{\beta-1}}{|\rho|} \le 2q\left(\frac{1}{\pi} \log \frac{q}{2\pi e} + r_1(q,1)\right) \frac{x^{-1/R_0 \log q}}{1 - 1/R_0 \log q}. \tag{31}$$

Now let $p_j = \beta_j + i\gamma_j$ be all zeroes of $\prod(s,q)$ with $\beta_j = 1 - \lambda_j/\log qT$, $1 \le \gamma_j \le T$ and $1/R_0 < \lambda_1 \le \lambda_2 \cdots$.

We now summarize some of the results in [12] about zeroes of $\prod(s,q)$.

Lemma 4.2. *Assuming $qT \geq 8\cdot 10^9$, then $\lambda_1 \leq \eta_i$ and $\lambda_2 > \xi_i$ for each $i = 1,\cdots,12$ and η_i, ξ_i from Table 3 and $\lambda_3 \geq 0.26213$. Moreover, if $qT \geq 10^{11}$ then $\lambda_n \geq \varrho_n$, with ϱ_n in Table 4.*

Proof. See Theorems 1-2 and Tables 1, 3-5 in [12]. □

i	η_i	ξ_i
1	0.16	0.2605
2	0.17	0.2477
3	0.18	0.2356
4	0.19	0.2242
5	0.20	0.2135
6	0.206	0.2074
7	0.2067	0.2067

Table 3: Bounds for λ_1 and λ_2

n	4	5	6	7	10	18
ϱ_n	0.28	0.31	0.32	0.33	0.36	0.39
n	45	91	146	332	834	7000
ϱ_n	0.42	0.45	0.46	0.47	0.475	0.478

Table 4: Bounds for λ_n, for $n \geq 4$

Now using Lemmas 4.1 and 4.2, we can obtain an upper bound for Σ_1, with the sum restricted to $|\gamma| \geq 1$. We are thus summing $x^{\beta-1}/|\rho|$ over the zeroes ρ of $L(s,\chi)$, with $1 - 1/R\log qT \leq \beta_j \leq 1 - 1/R_0 \log qT$ and $1 \leq |\gamma_j| \leq T$.

Corollary 4.3. *Assume $qT \geq 10^{11}$, then, for some $i = 1,\cdots,6$,*

$$\begin{aligned}&\sum_{\chi}\sum_{\rho\in z_1(\chi,T,R),|\gamma|\geq 1}\frac{x^{\beta-1}}{|\rho|}\\ &\leq 2q\min_{0\leq J\leq 12}\left(\mu(\eta_i)+\mu(\xi_{i+1})+\sum_{j=1}^{J}M_j\mu(\max\{\xi_{i+1},\nu_j\})\right),\end{aligned} \tag{32}$$

with $R = R_{i,J} = 1/\max\{\xi_{i+1},\nu_{J+1}\}$ and M_j and ν_j defined in Table 5.

j	1	2	3	4	5	6	7
ν_j	0.26213	0.27	0.30	0.32	0.33	0.36	0.39
M_j	1	1	1	1	4	7	47
j	8	9	10	11	12	13	
ν_j	0.42	0.45	0.46	0.47	0.475	0.478	
M_j	57	55	186	502	6166	-	

Table 5: Sizes of ν_j and M_j

By Theorem 1.1, Theorem 3.4, (28)–(32) we prove Theorem 1.2, with

$$\begin{aligned} C(\alpha_1,\alpha_2,Y_0) = \max_{x\geq Y_0}\max_{q\leq \log^{\alpha_1}} \Big(& \frac{R^*(x,T,q)\log\log x}{\log^2 x} + q\log^{\alpha_2} x \\ & \cdot \left(2\left(\frac{1}{\pi}\log\frac{q}{2\pi e} + r_1(q,1)\right)\frac{x^{-1/R_0\log q}}{1-1/R_0\log q}\right. \\ & + \max_{1\leq i\leq 6}\min_{0\leq J\leq 12}\Big(\frac{1}{2}S(T,q)x^{-1/R_{i,J}\log qT} \\ & + 2\Big(\mu(\eta_i)+\mu(\xi_{i+1})+\sum_{j=1}^{J} M_j\mu(\max\{\xi_{i+1},\nu_j\})\Big)\Big)\Big)\Big). \end{aligned} \tag{33}$$

Note that with our choice of T we have $qT \geq 10^{11}$ for $\log\log x \geq \frac{11\log 10}{\alpha_1+\alpha_2+3}$. Table 6 follows by simple computations.

5. Proof of Theorem 1.4

Let χ^* be the primitive character modulo q^*, that induces χ modulo q. Then by (12) and noting that $\psi(x,\chi_0)=\psi(x)$, we have

$$\left|\psi(x,q,a)-\frac{\psi(x)}{\varphi(q)}\right| \leq \frac{1}{\varphi(q)}\left|\sum_{\chi \pmod q,\chi\neq\chi_0}\psi(x,\chi^*)\right| + \frac{\log q}{\log 2}.$$

Hence we obtain

$$\begin{aligned} & \sum_{q\leq Q, q_0\nmid q}\left|\psi(x,q,a)-\frac{\psi(x)}{\varphi(q)}\right| \\ & \leq \frac{Q\log Q}{\log 2} + \sum_{q\leq Q,q_0\nmid q}\frac{1}{\varphi(q)}\left|\sum_{\chi \pmod q,\chi\neq\chi_0}\psi(x,\chi^*)\right| \\ & \leq \frac{Q\log Q}{\log 2} + \left(\sum_{1\leq m\leq Q}\frac{1}{\varphi(m)}\right)\sum_{1<q\leq Q,q_0\nmid q}\frac{1}{\varphi(q)}\left|\sideset{}{^*}\sum_{\chi \pmod q}\psi(x,\chi)\right|, \end{aligned}$$

where $\sum^*_{\chi \pmod q}$ denotes the sum over all primitive characters $\chi \pmod q$. By Theorem A.17 in [16], we can bound the last equation with

$$\frac{Q\log Q}{\log 2}+\frac{c_1}{2}\log x\sum_{1<q\le Q, q_0\nmid q}\frac{1}{\varphi(q)}\left|\sum^*_{\chi \pmod q}\psi(x,\chi)\right|. \tag{34}$$

We now split the sum in (34) at $Q_1=\log^A x$. We start bounding the sum up to Q_1, by Theorem A.17 in [16], with

$$\begin{aligned}\sum_{1<q\le Q_1, q_0\nmid q}\frac{1}{\varphi(q)}&\left|\sum^*_{\chi \pmod q}\psi(x,\chi)\right|\\ &\le c_1(1+A\log\log x)\max_{1<q\le Q_1, q_0\nmid q}\left|\sum^*_{\chi \pmod q}\psi(x,\chi)\right|.\end{aligned}$$

We can now see that

$$\begin{aligned}\sum^*_{\chi \pmod q}|\psi(x,\chi)| &\le x\left(-1+\frac{1}{x}\sum_{\chi \pmod q}|\psi(x,\chi)|\right)+x\left|\frac{\psi(x,\chi_0)}{x}-1\right|\\ &\le x\left(-1+\frac{1}{x}\sum_{\chi \pmod q}|\psi(x,\chi)|\right)+x\left|\frac{\psi(x)}{x}-1\right|+\frac{\log q}{\log 2}\end{aligned}$$

and, by Theorem 1.2, Theorem 1 from [20] and $\log\log x\ge\max\left\{7,\frac{11\log 10}{2A}\right\}$, we can bound this with

$$\begin{aligned}&\frac{C(A,A-3,X_0)x}{\log^{A-3}x}+\frac{x^{\beta_0}}{\beta_0}\\ &+34x(\log x)^{1.52}\exp\left(-0.81\sqrt{\log x}\right)+\frac{A\log\log x}{\log 2}.\end{aligned} \tag{35}$$

Since $q_0\nmid q$ implies $q\le Q_1$ is not exceptional, by Theorem 1.1, we have

$$\frac{x^{\beta_0}}{\beta_0}\le\frac{x^{1-\frac{1}{2AR_1\log\log x}}}{1-\frac{1}{2AR_1\log\log x}} \tag{36}$$

We now want to bound the part of the sum in (34) from Q_1 to Q. By Theorem 1.2 in [1] and partial summation, we obtain

$$\begin{aligned}\sum_{Q_1\le q\le Q}\frac{1}{\varphi(q)}\left|\sum^*_{\chi \pmod q}\psi(x,\chi)\right| &\le\frac{4c_0x\log^{\frac{7}{2}}x}{Q_1}+\\ +c_0(\log x)^{\frac{7}{2}}\Big(4\sqrt{x}\log^A x+4\frac{x}{\log^A x}&+18\frac{x^{\frac{11}{12}}}{\log^{\frac{A}{2}}x}+5x^{\frac{5}{6}}+\frac{5}{2}x^{\frac{5}{6}}\log x\Big).\end{aligned} \tag{37}$$

Now Theorem 1.4 follows from (34)–(37).

Appendix

Y_0	α_1	α_2	C
4.3	1	1	1.67
4.4	1	1	0.65
7.4	1	1	$9.3 \cdot 10^{-6}$
4.9	1	2	2.25
5	1	2	0.66
7.5	1	2	$7.8 \cdot 10^{-6}$
5.5	1	3	1.81
5.6	1	3	0.3
7.6	1	3	$6.6 \cdot 10^{-6}$
5.9	1	4	9.19
6	1	4	0.93
7.6	1	4	$6.7 \cdot 10^{-6}$
6.3	1	5	11.67
6.4	1	5	0.6
7.7	1	5	$5.6 \cdot 10^{-6}$
6.7	1	6	1.78
6.8	1	6	$3.6 \cdot 10^{-2}$
8	1	6	$3.2 \cdot 10^{-6}$
7	1	7	1.33
7.1	1	7	$1.2 \cdot 10^{-2}$
8.3	1	7	$1.8 \cdot 10^{-6}$
7.2	1	8	28.3
7.3	1	8	0.16
8.5	1	8	$1.2 \cdot 10^{-6}$
7.5	1	9	0.96
7.6	1	9	$1.7 \cdot 10^{-3}$
8.7	1	9	$8.1 \cdot 10^{-7}$
7.7	1	10	2.28
7.8	1	10	$1.9 \cdot 10^{-3}$
8.9	1	10	$5.5 \cdot 10^{-7}$
5.2	2	1	3.61
5.3	2	1	0.82
6.9	2	1	$2.7 \cdot 10^{-5}$
5.7	2	2	1.91
5.8	2	2	0.27
7.3	2	2	$1.3 \cdot 10^{-5}$
6.1	2	3	4.21
6.2	2	3	0.35
7.6	2	3	$6.8 \cdot 10^{-6}$
6.5	2	4	2.17
6.6	2	4	$8.2 \cdot 10^{-2}$
7.8	2	4	$4.7 \cdot 10^{-6}$
6.8	2	5	4.79
6.9	2	5	$9.4 \cdot 10^{-2}$
8.1	2	5	$2.6 \cdot 10^{-6}$
7.1	2	6	2.48
7.2	2	6	$2.1 \cdot 10^{-62}$
8.4	2	6	$1.5 \cdot 10^{-6}$
7.3	2	7	36.1
7.4	2	7	0.19

Y_0	α_1	α_2	C
8.6	2	7	$9.9 \cdot 10^{-7}$
7.5	2	8	255.04
7.6	2	8	0.71
8.8	2	8	$6.7 \cdot 10^{-7}$
7.7	2	9	751.4
7.7	2	9	1
9	2	9	$4.6 \cdot 10^{-7}$
7.9	2	10	765.84
8	2	10	0.43
9.2	2	10	$3.1 \cdot 10^{-7}$
5.9	3	1	7.58
6	3	1	0.87
7.3	3	1	$1.3 \cdot 10^{-5}$
6.3	3	2	1.78
6.4	3	2	0.11
7.6	3	2	$7 \cdot 10^{-6}$
6.6	3	3	5.52
6.7	3	3	0.21
7.9	3	3	$3.9 \cdot 10^{-6}$
6.9	3	4	8.66
7	3	4	0.17
8.2	3	4	$2.2 \cdot 10^{-6}$
7.2	3	5	3.08
7.3	3	5	$2.4 \cdot 10^{-2}$
8.5	3	5	$1.2 \cdot 10^{-6}$
7.4	3	6	30.87
7.5	3	6	0.14
8.7	3	6	$8.2 \cdot 10^{-7}$
7.6	3	7	144.06
7.7	3	7	0.34
8.9	3	7	$5.6 \cdot 10^{-7}$
7.8	3	8	263.6
7.9	3	8	0.29
9.1	3	8	$3.8 \cdot 10^{-7}$
8	3	9	154.44
8.1	3	9	$6.5 \cdot 10^{-2}$
9.3	3	9	$2.6 \cdot 10^{-7}$
8.2	3	10	22.78
8.3	3	10	$3.2 \cdot 10^{-3}$
9.4	3	10	$2.2 \cdot 10^{-7}$
6.5	4	1	5.9
6.6	4	1	0.28
7.8	4	1	$4.8 \cdot 10^{-6}$
6.8	4	2	1.43
6.9	4	2	$3.5 \cdot 10^{-2}$
8.1	4	2	$2.7 \cdot 10^{-6}$
7	4	3	13.81
7.1	4	3	0.25
8.3	4	3	$1.8 \cdot 10^{-6}$
7.3	4	4	2.69

Y_0	α_1	α_2	C
7.4	4	4	$1.9 \cdot 10^{-2}$
8.6	4	4	$1 \cdot 10^{-6}$
7.5	4	5	18.33
7.6	4	5	$7.1 \cdot 10^{-2}$
8.8	4	5	$6.8 \cdot 10^{-6}$
7.7	4	6	55.75
7.8	4	6	0.11
9	4	6	$4.6 \cdot 10^{-7}$
7.9	4	7	62.04
8	4	7	$5.1 \cdot 10^{-2}$
9.2	4	7	$3.2 \cdot 10^{-7}$
8.1	4	8	20.26
8.2	4	8	$6.2 \cdot 10^{-3}$
9.3	4	8	$2.6 \cdot 10^{-7}$
8.3	4	9	1.5
8.4	4	9	$1.5 \cdot 10^{-}$
9.5	4	9	$1.8 \cdot 10^{-7}$
8.4	4	10	276.47
8.5	4	10	$1.9 \cdot 10^{-2}$
9.6	4	10	$1.5 \cdot 10^{-7}$
7	5	1	2.83
7.1	5	1	$4.7 \cdot 10^{-2}$
8.2	5	1	$2.2 \cdot 10^{-6}$
7.2	5	2	4.14
7.3	5	2	$4.3 \cdot 10^{-2}$
8.4	5	2	$1.5 \cdot 10^{-6}$
7.4	5	3	4.44
7.5	5	3	$2.7 \cdot 10^{-2}$
8.7	5	3	$8.3 \cdot 10^{-7}$
7.6	5	4	8.57
7.7	5	4	$2.8 \cdot 10^{-2}$
8.9	5	4	$5.6 \cdot 10^{-7}$
7.8	5	5	15.08
7.9	5	5	$2.3 \cdot 10^{-2}$
9.1	5	5	$3.8 \cdot 10^{-7}$
8	5	6	9.86
8.1	5	6	$6.1 \cdot 10^{-3}$
9.2	5	6	$3.2 \cdot 10^{-7}$
8.2	5	7	1.73
8.3	5	7	$3.7 \cdot 10^{-4}$
9.4	5	7	$2.2 \cdot 10^{-7}$
8.3	5	8	391.48
8.4	5	8	$6.2 \cdot 10^{-2}$
9.5	5	8	$1.8 \cdot 10^{-7}$
8.5	5	9	7.64
8.6	5	9	$3.2 \cdot 10^{-4}$
9.7	5	9	$1.2 \cdot 10^{-7}$
8.6	5	10	701.51
8.7	5	10	$1.9 \cdot 10^{-2}$
9.8	5	10	$9.9 \cdot 10^{-8}$

Y_0	α_1	α_2	C
7.4	6	1	3.1
7.5	6	1	$1.9 \cdot 10^{-2}$
8.5	6	1	$1.3 \cdot 10^{-6}$
7.5	6	2	170.94
7.6	6	2	0.92
8.7	6	2	$8.4 \cdot 10^{-7}$
7.7	6	3	41.41
7.8	6	3	0.12
8.9	6	3	$5.7 \cdot 10^{-7}$
7.9	6	4	6.34
8	6	4	$1.1 \cdot 10^{-2}$
9.1	6	4	$3.9 \cdot 10^{-7}$
8.1	6	5	1.18
8.2	6	5	$8.4 \cdot 10^{-4}$
9.3	6	5	$2.6 \cdot 10^{-7}$
8.2	6	6	285.77
8.3	6	6	$9.9 \cdot 10^{-2}$
9.4	6	6	$2.2 \cdot 10^{-7}$
8.4	6	7	15.34
8.5	6	7	$1.7 \cdot 10^{-3}$
9.6	6	7	$1.5 \cdot 10^{-7}$
8.5	6	8	1872.72
8.6	6	8	0.13
9.7	6	8	$1.2 \cdot 10^{-7}$
8.7	6	9	7.58
8.8	6	9	$1.2 \cdot 10^{-4}$
9.9	6	9	$8.2 \cdot 10^{-7}$
8.8	6	10	315.52
8.9	6	10	$2.8 \cdot 10^{-3}$
10	6	10	$6.7 \cdot 10^{-8}$
7.7	7	1	45.31
7.8	7	1	0.16
8.8	7	1	$7 \cdot 10^{-7}$
7.9	7	2	3.98
8	7	2	$8.2 \cdot 10^{-3}$
9	7	2	$4.7 \cdot 10^{-7}$
8	7	3	86.56
8.1	7	3	0.13
9.2	7	3	$3.2 \cdot 10^{-7}$
8.2	7	4	2.28
8.3	7	4	$3.7 \cdot 10^{-3}$
9.4	7	4	$2.2 \cdot 10^{-7}$
8.3	7	5	76.36
8.4	7	5	0.11
9.5	7	5	$1.8 \cdot 10^{-7}$
8.5	7	6	1.59
8.6	7	6	$1.2 \cdot 10^{-3}$
9.7	7	6	$1.2 \cdot 10^{-7}$
8.6	7	7	39.92
8.7	7	7	$6.9 \cdot 10^{-3}$
9.8	7	7	$9.9 \cdot 10^{-8}$
8.7	7	8	2071.16
8.8	7	8	0.12
9.9	7	8	$8.2 \cdot 10^{-8}$
8.9	7	9	1.34
9	7	9	$3 \cdot 10^{-5}$
10.1	7	9	$5.6 \cdot 10^{-8}$
9	7	10	20.07
9.1	7	10	$6.6 \cdot 10^{-5}$
10.2	7	10	$4.6 \cdot 10^{-8}$
8	8	1	121.48
8.1	8	1	0.25
9.1	8	1	$3.9 \cdot 10^{-7}$
8.2	8	2	2.95
8.3	8	2	$4.4 \cdot 10^{-3}$
9.3	8	2	$2.7 \cdot 10^{-7}$
8.3	8	3	64.3
8.4	8	3	0.11
9.4	8	3	$2.2 \cdot 10^{-7}$
8.5	8	4	1.22
8.6	8	4	$1.2 \cdot 10^{-3}$
9.6	8	4	$1.5 \cdot 10^{-7}$
8.6	8	5	8.83
8.7	8	5	$5.7 \cdot 10^{-3}$
9.7	8	5	$1.3 \cdot 10^{-7}$
8.7	8	6	166.66
8.8	8	6	$7.3 \cdot 10^{-2}$
9.9	8	6	$8.2 \cdot 10^{-8}$
8.8	8	7	495.45
8.9	8	7	0.15
10	8	7	$6.8 \cdot 10^{-8}$
8.9	8	8	1363.62
9	8	8	0.2
10.1	8	8	$5.6 \cdot 10^{-8}$
9	8	9	7602.09
9.1	8	9	0.21
10.2	8	9	$4.6 \cdot 10^{-8}$
9.1	8	10	71285.9
9.2	8	10	0.26
10.3	8	10	$3.8 \cdot 10^{-8}$
8.3	9	1	90.88
8.4	9	1	0.14
9.3	9	1	$2.7 \cdot 10^{-7}$
8.5	9	2	1.32
8.6	9	2	$1.2 \cdot 10^{-3}$
9.5	9	2	$1.8 \cdot 10^{-7}$
8.6	9	3	8.98
8.7	9	3	$5.7 \cdot 10^{-3}$
9.6	9	3	$1.5 \cdot 10^{-7}$
8.7	9	4	166.47
8.8	9	4	$7.3 \cdot 10^{-3}$
9.8	9	4	$1 \cdot 10^{-7}$
8.8	9	5	483.25
8.9	9	5	0.15
9.9	9	5	$8.3 \cdot 10^{-8}$
8.9	9	6	1030.28
9	9	6	0.0.2
10	9	6	$6.8 \cdot 10^{-8}$
9	9	7	1571.49
9.1	9	7	0.19
10.2	9	7	$4.6 \cdot 10^{-8}$
9.1	9	8	1673.84
9.2	9	8	0.13
10.3	9	8	$3.8 \cdot 10^{-8}$
9.2	9	9	1243.31
9.3	9	9	$5.2 \cdot 10^{-2}$
10.4	9	9	$3.2 \cdot 10^{-8}$
9.3	9	10	741.93
9.4	9	10	$1.4 \cdot 10^{-2}$
10.5	9	10	$2.6 \cdot 10^{-8}$
8.6	10	1	10.61
8.7	10	1	$6.3 \cdot 10^{-3}$
9.5	10	1	$1.9 \cdot 10^{-7}$
8.7	10	2	169.88
8.8	10	2	$7.4 \cdot 10^{-2}$
9.7	10	2	$1.3 \cdot 10^{-7}$
8.8	10	3	486.8
8.9	10	3	0.15
9.8	10	3	$1.1 \cdot 10^{-7}$
8.9	10	4	1031.98
9	10	4	0.2
10	10	4	$6.8 \cdot 10^{-8}$
9	10	5	1571.13
9.1	10	5	0.19
10.1	10	5	$5.6 \cdot 10^{-8}$
9.1	10	6	1666.25
9.2	10	6	0.13
10.2	10	6	$4.6 \cdot 10^{-8}$
9.2	10	7	1194.71
9.3	10	7	$5.2 \cdot 10^{-2}$
10.3	10	7	$3.8 \cdot 10^{-8}$
9.3	10	8	561.72
9.4	10	8	$1.4 \cdot 10^{-2}$
10.4	10	8	$3.2 \cdot 10^{-8}$
9.4	10	9	167.76
9.5	10	9	$2.4 \cdot 10^{-3}$
10.5	10	9	$2.6 \cdot 10^{-8}$
9.5	10	10	30.8
9.6	10	10	$2.3 \cdot 10^{-4}$
10.6	10	10	$2.2 \cdot 10^{-8}$

Table 6: Values for $C(\alpha_1, \alpha_2, Y_0)$ (Theorem 1.2)

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