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Regularity and continuity of commutators of multilinear maximal operators

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ABSTRACT. This work is devoted to investigating the regularity and continuity properties for the commutators of multilinear maximal operators. More precisely, let $[\vec{b}, \mathfrak{M}]$ and $\mathfrak{M}_{\vec{b}}$ be the commutators and maximal commutators of the multilinear maximal operator $\mathfrak{M}$ with $\vec{b}$, respectively, where $\vec{b} = (b_1, \dots, b_m)$ with each b_i being a locally integrable function. It is proved that for $0 < s < 1$, $1 < p_1, \dots, p_{m+1}, p, q < \infty$, $1/p = 1/p_1 + \cdots + 1/p_{m+1}$, the operator $[\vec{b}, \mathfrak{M}]$ is bounded and continuous from $W^{1,p_1}(\mathbb{R}^n) \times \cdots \times W^{1,p_m}(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$ if each $b_i \in W^{1,p_{m+1}}(\mathbb{R}^n)$, from $W^{s,p_1}(\mathbb{R}^n) \times \cdots \times W^{s,p_m}(\mathbb{R}^n)$ to $W^{s,p}(\mathbb{R}^n)$ if each $b_i \in W^{s,p_{m+1}}(\mathbb{R}^n)$, from $F_s^{p_1,q}(\mathbb{R}^n) \times \cdots \times F_s^{p_m,q}(\mathbb{R}^n)$ to $F_s^{p,q}(\mathbb{R}^n)$ if each $b_i \in F_s^{p_{m+1},q}(\mathbb{R}^n)$ and from $B_s^{p_1,q}(\mathbb{R}^n) \times \cdots \times B_s^{p_m,q}(\mathbb{R}^n)$ to $B_s^{p,q}(\mathbb{R}^n)$ if each $b_i \in B_s^{p_{m+1},q}(\mathbb{R}^n)$. The corresponding results for $\mathfrak{M}_{\vec{b}}$ are also considered.

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1. Introduction and main results

The regularity theory of maximal operators has been the subject of many recent articles in harmonic analysis. The boundedness of multilinear operators is also always an active topic of current research. Based on the above topics, a natural question is that whether the multilinear maximal operator and its commutators have somewhat regularity properties. This is the main motivation of

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this work. To be more precise, we shall establish the boundedness and continuity for the above operators on the Sobolev spaces, fractional Sobolev spaces, Triebel–Lizorkin spaces and Besov spaces.

Let us start with a brief recollection of some recent developments on the regularity theory of maximal operators.

1.1. Regularity properties for maximal operators

For $f \in L^1_{\rm loc}(\mathbb{R}^n)$ with $n \geq 1$, the centered Hardy–Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\mathrm{d}y,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at x with radius r, and $|B(x,r)|$ denotes the volume of $B(x,r)$. Analogously, the uncentered maximal function $\widetilde{M}f$ at a point x is defined by taking the supremum of averages over open balls that contain the point. One famous result of harmonic analysis is the celebrated theorem of Hardy–Littlewood–Wiener that asserts that $M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded for $1 < p \leq \infty$. For $p = 1$ we have $M : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ bounded. The same bounds hold for $\widetilde{M}$.

Regularity properties of maximal operators have been studied extensively. The first work related to Sobolev regularity was due to Kinnunen [18] who established the boundedness of $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $1 < p \leq \infty$, where $W^{1,p}(\mathbb{R}^n)$ is the first order Sobolev space, i.e.

$$W^{1,p}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R} : \|f\|_{W^{1,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\nabla f = (D_1 f, \dots, D_n f)$ is the weak gradient of f. The same conclusion also holds for $\widetilde{M}$ by a simple modification of Kinnunen's arguments or [17, Theorem 1]. Since then, more and more works were devoted to extending the main result of [18] to various variants. For example, see [19] for the local case, [20] for the fractional case and [6, 25] for the multilinear case. Due to the lack of the sublinearity for the derivative of the maximal function, the continuity of $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$ is certainly a nontrivial issue. This question was addressed in the affirmative by Luiro [29] and was later extended to a local version in [30] and a multilinear version in [6, 24]. Another way to extend the regularity theory of maximal operators is to study its behaviour on different smooth function spaces. Korry [22] firstly showed that M is bounded on the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ defined by the Bessel potentials for $0 < s < 1$ and $1 < p < \infty$. The above result was extended by Korry [21] who proved that M is bounded on the inhomogeneous Triebel–Lizorkin spaces $F_s^{p,q}(\mathbb{R}^n)$ and inhomogeneous Besov spaces $B_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < p, q < \infty$. Later on, Luiro [30] established the continuity of $M : F_s^{p,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < p, q < \infty$. Recently, Liu and Wu [26] extended the above results to the maximal operators associated with polynomial mappings. In addition, the above authors established the continuity of $M : B_s^{p,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < p, q < \infty$. Other interesting works related to this topic are [1, 4, 5, 7, 16].

The study of multilinear operators has also been an active topic of current research, which originated in the works of Coifman and Meyer in the 70's (see [8, 9] for the background) and was later studied by many authors (see [15, 23] etc.). It is not motivated by a mere quest to generalize the theory of linear operators but rather by their natural appearance in analysis. It is well known that the multilinear maximal operator introduced originally by Lerner et al. [23] plays a key role in the theory of multilinear Calderón–Zygmund operator. In 2015, Liu and Wu [25] studied the Sobolev regularity for the multilinear maximal operator associated to balls

$$\mathfrak{M}(\vec{f})(x) = \sup_{B \ni x} \prod_{j=1}^{m} \frac{1}{|B|} \int_B |f_j(y_j)| \mathrm{d}y_j, \quad x \in \mathbb{R}^n,$$

where $m \geq 1$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The above supremum is taken over all the open balls B containing x. Liu and Wu [25] proved that for $1 < p_1, \dots, p_m < \infty$, $1 \leq p < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$,

$$\mathfrak{M} : W^{1,p_1}(\mathbb{R}^n) \times \cdots \times W^{1,p_m}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n) \text{ is bounded.}$$

Moreover, if $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in W^{1,p_j}(\mathbb{R}^n)$, then

$$\|\mathfrak{M}(\vec{f})\|_{W^{1,p}(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}. \tag{1.1}$$

The above results are based on the following bounds

$$\|\mathfrak{M}(\vec{f})\|_{L^p(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}, \tag{1.2}$$

where $1 < p_1, \dots, p_m \leq \infty$, $1 \leq p < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. One can easily check that

$$\mathfrak{M} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \text{ is continuous.} \tag{1.3}$$

Motivated by (1.1) and (1.3), Liu [24] showed that, among other things,

$$\mathfrak{M} : W^{1,p_1}(\mathbb{R}^n) \times \cdots \times W^{1,p_m}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n) \text{ is continuous,} \tag{1.4}$$

where $1 < p_1, \dots, p_m < \infty$, $1 \leq p < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$.

Based on the above, it is natural to ask that whether the multilinear maximal operator $\mathfrak{M}$ is bounded and continuous on the fractional Sobolev spaces, Triebel–Lizorkin spaces or Besov spaces. Thanks to the work in [27] in which the first two authors and Yabuta established the boundedness and continuity for the multilinear strong maximal operators on the Sobolev spaces, Triebel–Lizorkin spaces and Besov spaces, we have a good opportunity to obtain the following results by using similar methods. Here we only list these results without proofs, which are useful for our aim.

Theorem A. *Let $1 < p_1, \ldots, p_m, p, q < \infty$, $0 < s < 1$ and $1/p = 1/p_1 + \cdots + 1/p_m$.*

(i) *The map $\mathfrak{M} : F_s^{p_1,q}(\mathbb{R}^n) \times \cdots \times F_s^{p_m,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$ is bounded and continuous. Moreover, if $\vec{f} = (f_1, \ldots, f_m)$ with each $f_i \in F_s^{p_i,q}(\mathbb{R}^n)$, then*

$$\|\mathfrak{M}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}.$$

(ii) *The map $\mathfrak{M} : B_s^{p_1,q}(\mathbb{R}^n) \times \cdots \times B_s^{p_m,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$ is bounded and continuous. Moreover, if $\vec{f} = (f_1, \ldots, f_m)$ with each $f_i \in B_s^{p_i,q}(\mathbb{R}^n)$, then*

$$\|\mathfrak{M}(\vec{f})\|_{B_s^{p,q}(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}.$$

(iii) *The map $\mathfrak{M} : W^{s,p_1}(\mathbb{R}^n) \times \cdots \times W^{s,p_m}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n)$ is bounded and continuous. Moreover, if $\vec{f} = (f_1, \ldots, f_m)$ with each $f_i \in W^{s,p_i}(\mathbb{R}^n)$, then*

$$\|\mathfrak{M}(\vec{f})\|_{W^{s,p}(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} \|f_j\|_{W^{s,p_j}(\mathbb{R}^n)}.$$

1.2. Properties for commutators of maximal operators

It is well known that the commutator

$$[b, T](f)(x) = bTf(x) - T(bf)(x)$$

with suitable operator T and function b was initialized by Coifman et al. [10] who proved that the commutator $[b, T]$ with T being Riesz transform is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ under the condition that $b \in \mathrm{BMO}(\mathbb{R}^n)$. Later on, the study on commutator $[b, T]$ with various of operators T on a variety of function spaces have been studied by many authors. The commutator of Hardy–Littlewood maximal operator was first studied by Milman and Schonbek [31] who established the L^p $(1 < p < \infty)$ bounds for $[b, \widetilde{M}]$ if $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b \geq 0$. The above result was later improved by Bastero et al. [2] who stated that the operator $[b, \widetilde{M}]$ is of type (p, p) for $1 < p < \infty$ if $b \in \mathrm{BMO}(\mathbb{R}^n)$. In [3] Bonami et al. used $[b, \widetilde{M}]$ to study the product of a function in $H^1(\mathbb{R}^n)$ and a function in $\mathrm{BMO}(\mathbb{R}^n)$. Recall that the maximal commutator with b is defined by

$$\widetilde{M}_b f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |b(x) - b(y)||f(y)|dy,$$

where the supremum is taken over all the open balls B containing x. In 1991, García-Cuerva et al. [12] first proved that $\widetilde{M}_b$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. One can consult [12, 35] for the boundedness of $\widetilde{M}_b$.

Recently, Liu et al. [28] studied the regularity properties of $[b, \widetilde{M}]$ and $\widetilde{M}_b$. The main results of [28] can be listed as follows:

Theorem B ([28]). *Let $1 < p_1, p_2, p, q < \infty$ and $1/p = 1/p_1 + 1/p_2$.*

(i) *If $b \in W^{s,p_2}(\mathbb{R}^n)$ for some $s \in [0,1]$, then the map $[b, \widetilde{M}] : W^{s,p_1}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n)$ is bounded and continuous. Moreover, the map $\widetilde{M}_b : W^{s,p_1}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n)$ is bounded.*

(ii) *If $b \in F_s^{p_2,q}(\mathbb{R}^n)$ for some $s \in (0,1)$, then the map $[b, \widetilde{M}] : F_s^{p_1,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$ is bounded and continuous. The same result holds for $\widetilde{M}_b$.*

(iii) *If $b \in B_s^{p_2,q}(\mathbb{R}^n)$ for some $s \in (0,1)$, then the map $[b, \widetilde{M}] : B_s^{p_1,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$ is bounded and continuous. The same result holds for $\widetilde{M}_b$.*

1.3. Commutators of multilinear maximal operators

The primary aim of this work is to establish the bounds and continuity for commutators of multilinear maximal operators on the Sobolev spaces, Triebel–Lizorkin spaces and Besov spaces. We now introduce the following objectives of research.

Definition 1.1. (Commutators of multilinear maximal operator). Let $m \geq 1$ and $\vec{f} = (f_1, \dots, f_m)$ and $\vec{b} = (b_1, \dots, b_m)$ with each $f_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $b_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We define the commutator of $\mathfrak{M}$ and $\vec{b}$ by the formula

$$[\vec{b}, \mathfrak{M}](\vec{f})(x) = \sum_{i=1}^{m} [\vec{b}, \mathfrak{M}]_i(\vec{f})(x), \quad x \in \mathbb{R}^n,$$

where

$$[\vec{b}, \mathfrak{M}]_i(\vec{f})(x) = b_i(x)\mathfrak{M}(\vec{f})(x) - \mathfrak{M}(f_1, \dots, f_{i-1}, b_i f_i, f_{i+1}, \dots, f_m)(x).$$

The multilinear maximal commutator with $\vec{b}$ is defined by

$$\mathfrak{M}_{\vec{b}}(\vec{f})(x) = \sum_{i=1}^{m} \mathfrak{M}^i_{\vec{b}}(\vec{f})(x),$$

where

$$\mathfrak{M}^i_{\vec{b}}(\vec{f})(x) = \sup_{B \ni x} \frac{1}{|B|^m} \int_{B^m} |b_i(x) - b_i(y_i)| \prod_{j=1}^{m} |f_j(y_j)| \mathrm{d}\vec{y},$$

where the supremum is taken over all the open balls B containing x. Here $B^m = \overbrace{B \times B \times \cdots \times B}$ and $\mathrm{d}\vec{y} = \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_m$. When $m = 1$, the operator $[\vec{b}, \mathfrak{M}]$ reduces to $[b, \widetilde{M}]$. Respectively, the operator $\mathfrak{M}_{\vec{b}}$ reduces to $\widetilde{M}_b$.

The commutator in the multilinear setting was first studied by Pérez and Torres in [32] and was later developed by many authors (see [23] et al.). The commutators of multilinear maximal operators associated to cubes was first introduced by Zhang [36] who investigated the multiple weighted estimates for these commutators.

Before presenting our main results, let us point out the following comments, which are useful for our proofs of main results.

Remark 1.2. (i) For any fixed $1 \leq i \leq m$, the operator $[\vec{b}, \mathfrak{M}]_i$ is neither positive nor sublinear. However, the operator $\mathfrak{M}^i_{\vec{b}}$ is positive and sublinear.

(ii) Let $1 < p_1, \dots, p_{m+1}, p \leq \infty$ and $1/p = 1/p_1 + \cdots + 1/p_{m+1}$. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^{p_j}(\mathbb{R}^n)$ and $\vec{b} = (b_1, \dots, b_m)$ with each $b_j \in L^{p_{m+1}}(\mathbb{R})$. For any fixed $i \in \{1, \dots, m\}$, we get by (1.2) and Hölder's inequality that

$$\|[\vec{b}, \mathfrak{M}]_i(\vec{f})\|_{L^p(\mathbb{R}^n)} \leq C\|b_i\|_{L^{p_{m+1}}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f\|_{L^{p_j}(\mathbb{R}^n)}. \tag{1.5}$$

Combining (1.5) with (1.3) and Hölder's inequality implies that

$$[\vec{b}, \mathfrak{M}]_i : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \text{ is continuous.} \tag{1.6}$$

On the other hand, one can easily check that

$$\mathfrak{M}^i_{\vec{b}}(\vec{f})(x) \leq |b_i(x)|\mathfrak{M}(\vec{f})(x) + \mathfrak{M}(f_1, \dots, f_{i-1}, b_i f_i, f_{i+1}, \dots, f_m)(x). \tag{1.7}$$

By (1.2), (1.7) and Hölder's inequality, we obtain

$$\|\mathfrak{M}^i_{\vec{b}}(\vec{f})\|_{L^p(\mathbb{R}^n)} \leq C\|b_i\|_{L^{p_{m+1}}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f\|_{L^{p_j}(\mathbb{R}^n)}. \tag{1.8}$$

It is not difficult to see that

$$|\mathfrak{M}^i_{\vec{b}}(\vec{f}_j) - \mathfrak{M}^i_{\vec{b}}(\vec{f})| \leq \sum_{l=1}^{m} \mathfrak{M}^i_{\vec{b}}(\vec{F}_l),$$

where $\vec{f}_j = (f_{1,j}, \dots, f_{m,j})$ and $\vec{F}_l = (f_1, \dots, f_{l-1}, f_{l,j} - f_l, f_{l+1,j}, \dots, f_{m,j})$. This together with (1.8) implies that

$$\mathfrak{M}^i_{\vec{b}} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \text{ is continuous.} \tag{1.9}$$

(iii) For $y \in \mathbb{R}^n$, we define $f_y(x) = f(x - y)$. Let $\vec{f} = (f_1, \dots, f_m)$, $\vec{f}_y = ((f_1)_y, \dots, (f_m)_y)$, $\vec{b} = (b_1, \dots, b_m)$ and $\vec{b}_y = ((b_1)_y, \dots, (b_m)_y)$. Clearly, $(\mathfrak{M}(\vec{f}))_y = \mathfrak{M}(\vec{f}_y)$ and $(\mathfrak{M}^i_{\vec{b}}(\vec{f}))_y = \mathfrak{M}^i_{\vec{b}_y}(\vec{f}_y)$ for all $i = 1, \dots, m$.

Based on the above, some questions naturally arise as follows.

Question 1.3. Are the commutators of multilinear maximal operators bounded and continuous on the Sobolev spaces, fractional Sobolev spaces, Triebel–Lizorkin spaces or Besov spaces?

This is the main motivation of this work. We shall give a affirmative answer to Question 1.3 by the following results.

Theorem 1.4. *Let $1 < p_1, \dots, p_{m+1}, p < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_{m+1}$. Let $\vec{b} = (b_1, \dots, b_m)$ with each $b_i \in W^{1,p_{m+1}}(\mathbb{R}^n)$. Then*

$$[\vec{b}, \mathfrak{M}] : W^{1,p_1}(\mathbb{R}^n) \times \cdots \times W^{1,p_m}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, if $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in W^{1,p_i}(\mathbb{R}^n)$, we have

$$\|[\vec{b}, \mathfrak{M}](\vec{f})\|_{W^{1,p}(\mathbb{R}^n)} \le C\Big(\sum_{i=1}^m \|b_i\|_{W^{1,p_{m+1}}(\mathbb{R}^n)}\Big) \prod_{j=1}^m \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}. \tag{1.10}$$

The above boundedness result holds for $\mathfrak{M}_{\vec{b}}$.

Theorem 1.5. *Let $1 < p_1, \dots, p_{m+1}, p, q < \infty$, $0 < s < 1$ and $1/p = 1/p_1 + \cdots + 1/p_{m+1}$. Let $\vec{b} = (b_1, \dots, b_m)$ with each $b_i \in F_s^{p_{m+1},q}(\mathbb{R}^n)$. Then*

$$[\vec{b}, \mathfrak{M}] : F_s^{p_1,q}(\mathbb{R}^n) \times \cdots \times F_s^{p_m,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, if $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in F_s^{p_i,q}(\mathbb{R}^n)$, we have

$$\|[\vec{b}, \mathfrak{M}](\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \le C\Big(\sum_{i=1}^m \|b_i\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)}\Big) \prod_{j=1}^m \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \tag{1.11}$$

The same result holds for $\mathfrak{M}_{\vec{b}}$.

Theorem 1.6. *Let $1 < p_1, \dots, p_{m+1}, p, q < \infty$, $0 < s < 1$ and $1/p = 1/p_1 + \cdots + 1/p_{m+1}$. Let $\vec{b} = (b_1, \dots, b_m)$with each $b_i \in B_s^{p_{m+1},q}(\mathbb{R}^n)$. Then*

$$[\vec{b}, \mathfrak{M}] : B_s^{p_1,q}(\mathbb{R}^n) \times \cdots \times B_s^{p_m,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, if $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in B_s^{p_i,q}(\mathbb{R}^n)$, we have

$$\|[\vec{b}, \mathfrak{M}](\vec{f})\|_{B_s^{p,q}(\mathbb{R}^n)} \le C\Big(\sum_{i=1}^m \|b_i\|_{B_s^{p_{m+1},q}(\mathbb{R}^n)}\Big) \prod_{j=1}^m \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}. \tag{1.12}$$

The same result holds for $\mathfrak{M}_{\vec{b}}$.

By the facts $W^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n) = F_s^{p,2}(\mathbb{R}^n)$ for any $s > 0$ and $1 < p < \infty$ and Theorems 1.4 and 1.5, we can get the following result immediately.

Corollary 1.7. *Let $1 < p_1, \dots, p_{m+1}, p < \infty$, $0 \le s \le 1$ and $1/p = 1/p_1 + \cdots + 1/p_{m+1}$. Let $\vec{b} = (b_1, \dots, b_m)$ with each $b_i \in W^{s,p_{m+1}}(\mathbb{R}^n)$, then*

$$[\vec{b}, \mathfrak{M}] : W^{s,p_1}(\mathbb{R}^n) \times \cdots \times W^{s,p_m}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, if $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in W^{s,p_i}(\mathbb{R}^n)$, then

$$\|[\vec{b}, \mathfrak{M}](\vec{f})\|_{W^{s,p}(\mathbb{R}^n)} \le C\Big(\sum_{i=1}^{m} \|b_i\|_{W^{s,p_{m+1}}(\mathbb{R}^n)}\Big) \prod_{j=1}^{m} \|f_j\|_{W^{s,p_j}(\mathbb{R}^n)}.$$

The same result holds for $\mathfrak{M}_{\vec{b}}$.

Remark 1.8. Theorems 1.4-1.6 and Corollary 1.7 extend Theorem B to the multilinear version, which are of interest in their own right. On the other hand, the continuity of $\mathfrak{M}_{\vec{b}} : W^{1,p_1}(\mathbb{R}^n) \times \cdots \times W^{1,p_m}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ under the conditions in Theorem 1.4 is certainly an interesting issue, even in the special case $m = 1$.

Remark 1.9. There are some remarks on the proofs of Theorems 1.4-1.6:

(1) Theorem 1.4 for $[\vec{b}, \mathfrak{M}]$ follows easily from the known Sobolev bounds and continuity for $\mathfrak{M}$ (see (1.1) and (1.4)) and a characterization of product functions on Sobolev spaces (see Lemma 2.1). The main ingredients in the proof of the boundedness for $\mathfrak{M}_{\vec{b}}$ are some properties on Sobolev spaces (see (2.1) and (2.2)).
(2) Theorem 1.5 for $[\vec{b}, \mathfrak{M}]$ follows easily from Theorem A (i) and a characterization of product functions on Triebel-Lizorkin spaces (see Lemma 2.2). The main ingredients in the proof of Theorem 1.5 for $\mathfrak{M}_{\vec{b}}$ are the mixed vector-valued inequality for $\widetilde{M}$ (see Lemma 4.1) and some properties for Triebel–Lizorkin spaces (see (2.3)-(2.6)).
(3) Theorem 1.6 for $[\vec{b}, \mathfrak{M}]$ follows easily from Theorem A (ii) and a characterization of product functions on Besov spaces (see Lemma 2.3). The main ingredients in the proof of Theorem 1.6 for $\mathfrak{M}_{\vec{b}}$ are some properties for Besov spaces (see (2.10)-(2.13)).
(4) Our methods apply to the multilinear maximal operators associated to cubes and their commutators as well as the commutators of the multilinear strong maximal operators.

This paper will be organized as follows. Section 2 will be devoted to presenting some properties for Sobolev spaces, Triebel–Lizorkin spaces and Besov spaces, which are the main ingredients in the proofs of main theorems. Section 3 is devoted to proving Theorem 1.4. In Section 4, we shall prove Theorem 1.5. The proof of Theorem 1.6 will be given in Section 5. We would like to remark that the main ideas in the proofs of Theorems are motivated by [26, 27, 34].

Throughout this paper, the letter C will stand for positive constants,, not necessarily the same one at each occurrence, but is independent of the essential variables. In what follows, let $\mathfrak{R}_n = \{\zeta \in \mathbb{R}^n; 1/2 < |\zeta| \leq 1\}$ and we denote by $\Delta_\zeta(f)$ the difference of f for an arbitrary function f defined on $\mathbb{R}^n$ and $\zeta \in \mathfrak{R}_n$, i.e., $\Delta_\zeta f(x) = f(x+\zeta) - f(x)$.

2. Properties for Sobolev spaces, Triebel–Lizorkin spaces and Besov spaces

In this section we shall present some properties on Sobolev spaces, Triebel–Lizorkin spaces and Besov spaces, which are very useful in our proofs.

2.1. Properties on Besov spaces

Let $e_l = (0, \dots, 0, 1, 0, \dots, 0)$ be the canonical l-th base vector in $\mathbb{R}^n$ for $l = 1, 2, \dots, n$. For a fixed $f \in L^p(\mathbb{R}^n)$ with $p > 1$, all $h \in \mathbb{R}$ with $|h| > 0$, $y \in \mathbb{R}^n$ and $i = 1, 2, \dots, n$, we define the functions f_h^i and f_y by setting

$$f_h^i(x) = \frac{f(x+he_i) - f(x)}{|h|} \text{ and } f_y(x) = f(x-y).$$

It is well known that

$$\|f_h^i - D_i f\|_{L^p(\mathbb{R}^n)} \to 0 \text{ as } h \to 0 \tag{2.1}$$

if $f \in W^{1,p}(\mathbb{R}^n)$ for some $p > 1$. For convenience, we set

$$G(f;p) = \limsup_{|h|\to 0} \frac{\|f_h - f\|_{L^p(\mathbb{R}^n)}}{|h|}.$$

According to [13, Section 7.11], we have

$$u \in W^{1,q}(\mathbb{R}^n),\ 1 < q < \infty \Longleftrightarrow u \in L^q(\mathbb{R}^n) \text{ and } G(u;q) < \infty. \tag{2.2}$$

We now present the characterization of product functions on the Sobolev spaces, which followed from [28].

Lemma 2.1. ([28]). *Let $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$. If $f \in W^{1,p_1}(\mathbb{R}^n)$ and $g \in W^{1,p_2}(\mathbb{R}^n)$, then $fg \in W^{1,p}(\mathbb{R}^n)$. Moreover,*

$$\nabla(fg) = g\nabla f + f\nabla g,$$

almost everywhere in $\mathbb{R}^n$. In particular,

$$\|fg\|_{W^{1,p}(\mathbb{R}^n)} \leq \|f\|_{W^{1,p_1}(\mathbb{R}^n)}\|g\|_{W^{1,p_2}(\mathbb{R}^n)}.$$

2.2. Properties on Triebel–Lizorkin spaces

Denote by $\dot{F}_s^{p,q}(\mathbb{R}^n)$ the homogeneous Triebel–Lizorkin spaces. Let $s > 0$ and $1 < p < \infty$, $1 < q \leq \infty$, $1 \leq r < \infty$. We denote by $E_{p,q,r}^s$ the mixed norm of three variable functions $g(x,k,\zeta)$ by

$$\|g\|_{E_{p,q,r}^s} := \Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |g(x,k,\zeta)|^r d\zeta\Big)^{q/r}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

It was shown by Yabuta [34] that

$$\|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} \sim \|\Delta_{2^{-k}\zeta} f\|_{E^s_{p,q,r}} \tag{2.3}$$

for $0 < s < 1$, $1 < p < \infty$, $1 < q \le \infty$ and $1 \le r < \min\{p, q\}$. Moreover, it was pointed out in [11, 14, 33] that

$$\|f\|_{F_s^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}, \text{ for } s > 0,\ 1 < p, q < \infty, \tag{2.4}$$

$$\|f\|_{F_{s_1}^{p,q}(\mathbb{R}^n)} \le \|f\|_{F_{s_2}^{p,q}(\mathbb{R}^n)}, \text{ for } s_1 \le s_2,\ 1 < p,\ q < \infty, \tag{2.5}$$

$$\|f\|_{F_s^{p,q_2}(\mathbb{R}^n)} \le \|f\|_{F_s^{p,q_1}(\mathbb{R}^n)}, \quad \text{for } s \in \mathbb{R},\ 1 < p < \infty,\ 1 < q_1 \le q_2 < \infty. \tag{2.6}$$

The following presents a characterization of product functions on the Triebel–Lizorkin spaces.

Lemma 2.2. *Let $1 < p_1, p_2, p < \infty$, $1/p = 1/p_1 + 1/p_2$ and $0 < s < 1$. If $f \in F_s^{p_1,q}(\mathbb{R}^n)$ and $g \in F_s^{p_2,q}(\mathbb{R}^n)$, then $fg \in F_s^{p,q}(\mathbb{R}^n)$. Moreover,*

$$\|fg\|_{F_s^{p,q}(\mathbb{R}^n)} \le C\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{2.7}$$

Proof. It is clear that

$$\Delta_{2^{-k}\zeta}(fg)(x) = \Delta_{2^{-k}\zeta} f(x)\Delta_{2^{-k}\zeta} g(x) + f(x)\Delta_{2^{-k}\zeta} g(x) + g(x)\Delta_{2^{-k}\zeta} f(x), \tag{2.8}$$

for all $x \in \mathbb{R}^n$, $\zeta \in \mathfrak{R}_n$ and $k \in \mathbb{Z}$. In light of (2.3) and (2.8), we have

$$\begin{aligned}
\|fg\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} \le C&\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} f\Delta_{2^{-k}\zeta} g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&+C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |f\Delta_{2^{-k}\zeta} g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&+C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |g\Delta_{2^{-k}\zeta} f|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

By Minkowski's inequaltiy and Hölder's inequality, we get from (2.3)-(2.6) that

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} f\Delta_{2^{-k}\zeta} g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le \Big\|\Big(\sum_{k\in\mathbb{Z}} (2^{ksp/p_1}\|\Delta_{2^{-k}\zeta} f\|_{L^{p_1/p}(\mathfrak{R}_n)})^{p_1q/p}\Big)^{p/(p_1q)}\Big\|_{L^{p_1}(\mathbb{R}^n)}\\
&\quad\times\Big\|\Big(\sum_{k\in\mathbb{Z}} (2^{ksp/p_2}\|\Delta_{2^{-k}\zeta} g\|_{L^{p_2/p}(\mathfrak{R}_n)})^{p_2q/p}\Big)^{p/(p_2q)}\Big\|_{L^{p_2}(\mathbb{R}^n)}\\
&\le C\|f\|_{\dot{F}_{sp/p_1}^{p_1q/p,p_1}(\mathbb{R}^n)}\|g\|_{\dot{F}_{sp/p_2}^{p_2q/p,p_2}(\mathbb{R}^n)}\\
&\le C\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)},
\end{aligned}$$

$$
\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |f\Delta_{2^{-k}\zeta}g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leq \|f\|_{L^{p_1}(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ks}\|\Delta_{2^{-k}\zeta}g\|_{L^1(\mathfrak{R}_n)})^q\Big)^{1/q}\Big\|_{L^{p_2}(\mathbb{R}^n)}\\
&\leq C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{\dot{F}^{p_2q/p,p_2}_{sp/p_2}(\mathbb{R}^n)}\\
&\leq C\|f\|_{F^{p_1,q}_s(\mathbb{R}^n)}\|g\|_{F^{p_2,q}_s(\mathbb{R}^n)}.
\end{aligned}
$$

Similarly,

$$
\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |g\Delta_{2^{-k}\zeta}f|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{F^{p_1,q}_s(\mathbb{R}^n)}\|g\|_{F^{p_2,q}_s(\mathbb{R}^n)}.
$$

These estimates together with the estimate $\|fg\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$ and (2.4) give (2.7) and completes the proof. □

2.3. *Properties on Besov spaces*

We denote by $\dot{B}^{p,q}_s(\mathbb{R}^n)$ the homogeneous Besov spaces. It was proved by Yabuta [34] that if $0 < s < 1$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $1 \leq r \leq p$, then

$$
\|f\|_{\dot{B}^{p,q}_s(\mathbb{R}^n)} \sim \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big\|\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(f)|^r\mathrm{d}\zeta\Big)^{1/r}\Big\|^q_{L^p(\mathbb{R}^n)}\Big)^{1/q}. \tag{2.9}
$$

For a measurable function $g : \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n \to \mathbb{R}$, we define

$$
\|g\|_{p,q,s} := \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |g(x,k,\zeta)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}.
$$

Then, by (2.7) and Fubini's theorem, we have

$$
\|f\|_{\dot{B}^{p,q}_s(\mathbb{R}^n)} \sim \|\Delta_{2^{-k}\zeta}f\|_{p,q,s}. \tag{2.10}
$$

It is well known that (see [11, 14, 33])

$$
\|f\|_{B^{p,q}_s(\mathbb{R}^n)} \sim \|f\|_{\dot{B}^{p,q}_s(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}, \quad \text{for } s > 0,\ 1 < p,q < \infty, \tag{2.11}
$$

$$
\|f\|_{B^{p,q}_{s_1}(\mathbb{R}^n)} \leq \|f\|_{B^{p,q}_{s_2}(\mathbb{R}^n)}, \quad \text{for } s_1 \leq s_2,\ 1 < p,\ q < \infty, \tag{2.12}
$$

$$
\|f\|_{B^{p,q_2}_s(\mathbb{R}^n)} \leq \|f\|_{B^{p,q_1}_s(\mathbb{R}^n)}, \quad \text{for } s \in \mathbb{R},\ 1 < p < \infty,\ 1 < q_1 \leq q_2 < \infty. \tag{2.13}
$$

The following presents a characterization of product functions on the Besov spaces.

Lemma 2.3. *Let $1 < p_1, p_2, p < \infty$, $1/p = 1/p_1 + 1/p_2$ and $0 < s < 1$. If $f \in B^{p_1,q}_s(\mathbb{R}^n)$ and $g \in B^{p_2,q}_s(\mathbb{R}^n)$, then $fg \in B^{p,q}_s(\mathbb{R}^n)$. Moreover,*

$$
\|fg\|_{B^{p,q}_s(\mathbb{R}^n)} \leq C\|f\|_{B^{p_1,q}_s(\mathbb{R}^n)}\|g\|_{B^{p_2,q}_s(\mathbb{R}^n)}. \tag{2.14}
$$

Proof. By (2.11) and the trivial estimate $\|fg\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$, to prove (2.14), it suffices to show that

$$\|fg\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} \leq C\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}. \tag{2.15}$$

By (2.10), (2.12) and Minkowski's inequality, we have

$$\begin{aligned}\|fg\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} \leq\ & C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}f(x)\Delta_{2^{-k}\zeta}g(x)|^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &+C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |f(x)\Delta_{2^{-k}\zeta}g(x)|^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &+C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |g(x)\Delta_{2^{-k}\zeta}f(x)|^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}.\end{aligned}$$

By (2.10)-(2.13) and Hölder's inequality, we conclude that

$$\begin{aligned}&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}f(x)\Delta_{2^{-k}\zeta}g(x)|^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\leq \Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_1}\|\Delta_{2^{-k}\zeta}f\|_{L^{p_1}(\mathbb{R}^n\times\mathfrak{R}_n)})^{qp_1/p}\Big)^{p/(qp_1)}\\ &\quad\times\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_2}\|\Delta_{2^{-k}\zeta}g\|_{L^{p_2}(\mathbb{R}^n\times\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\\ &\leq C\|f\|_{\dot{B}_{sp/p_1}^{p_1,qp_1/p}(\mathbb{R}^n)}\|g\|_{\dot{B}_{sp/p_2}^{p_2,qp_2/p}(\mathbb{R}^n)}\\ &\leq C\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)},\end{aligned}$$

$$\begin{aligned}&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |f(x)\Delta_{2^{-k}\zeta}g(x)|^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\leq C\|f\|_{L^{p_1}(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}}(2^{ks}\|\Delta_{2^{-k}\zeta}g\|_{L^{p_2}(\mathbb{R}^n\times\mathfrak{R}_n)})^q\Big)^{1/q}\\ &\leq C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{\dot{B}_s^{p_2,q}(\mathbb{R}^n)}\\ &\leq C\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}.\end{aligned}$$

Similarly one has

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |g(x)\Delta_{2^{-k}\zeta}f(x)|^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q} \leq C\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}.$$

Then (2.15) follows from the above estimates. □

3. Proof of Theorem 1.4

Throughout this section, let us fix $1 < p_1, \dots, p_{m+1}, p < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_{m+1}$. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in W^{1,p_i}(\mathbb{R}^n)$. For convenience, let s, t be such that $1/s = 1/p_1 + \cdots + 1/p_m$ and $1/t = 1/p_1 + 1/p_{m+1}$. It is clear that $p < s < p_1$, $p < t < p_1$, $1/p = 1/p_{m+1} + 1/s$ and $1/p = 1/t + 1/p_2 + \cdots + 1/p_m$.

3.1. Proof of Theorem 1.4 for $[\vec{b},\mathfrak{M}]$

To prove (1.10), it suffices to show that

$$\|[\vec{b},\mathfrak{M}]_i(\vec{f})\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|b_i\|_{W^{1,p_{m+1}}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)} \tag{3.1}$$

for each $i = 1, \dots, m$. By (1.5), inequality (3.1) reduces to the following

$$\|\nabla[\vec{b},\mathfrak{M}]_i(\vec{f})\|_{L^p(\mathbb{R}^n)} \leq C\|b_i\|_{W^{1,p_{m+1}}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)} \tag{3.2}$$

for each $i = 1, \dots, m$.

We now work with (3.2) for $i = 1$ and other cases are analogous. By (1.1) and Lemma 2.1, we have

$$\begin{aligned} \|b_1\mathfrak{M}(\vec{f})\|_{W^{1,p}(\mathbb{R}^n)} &\leq \|b_1\|_{W^{1,p_{m+1}}(\mathbb{R}^n)}\|\mathfrak{M}(\vec{f})\|_{W^{1,s}(\mathbb{R}^n)} \\ &\leq C\|b_1\|_{W^{1,p_{m+1}}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}. \end{aligned} \tag{3.3}$$

For convenience, we set $\vec{f}_{1,b_1} = (b_1 f_1, f_2, \dots, f_m)$. Invoking Lemma 2.1,

$$\|b_1 f_1\|_{W^{1,t}(\mathbb{R}^n)} \leq C\|b_1\|_{W^{1,p_{m+1}}(\mathbb{R}^n)}\|f_1\|_{W^{1,p_1}(\mathbb{R}^n)},$$

which combine with (1.1) leads to

$$\|\mathfrak{M}(\vec{f}_{1,b_1})\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|b_1\|_{W^{1,p_{m+1}}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}. \tag{3.4}$$

Then (3.2) with $i = 1$ follows from (3.3) and (3.4).

Next we prove the continuity result. Let $\vec{f}_j = (f_{1,j}, \dots, f_{m,j})$ with $f_{i,j} \to f_i$ in $W^{1,p_i}(\mathbb{R}^n)$ as $j \to \infty$ for all $i = 1, \dots, m$. We want to show that

$$\|[\vec{b},\mathfrak{M}]_i(\vec{f}_j) - [\vec{b},\mathfrak{M}]_i(\vec{f})\|_{W^{1,p}(\mathbb{R}^n)} \to 0 \ \text{ as } \ j \to \infty \tag{3.5}$$

for all $i = 1, \dots, m$. Without loss of generality we only work with the case $i = 1$ and other cases are analogous. By Lemma 2.1, we have that, $b_1 f_1 \in W^{1,t}(\mathbb{R}^n)$, $b_1 f_{1,j} \in W^{1,t}(\mathbb{R}^n)$ and

$$\begin{aligned} &\|b_1 f_{1,j} - b_1 f_1\|_{W^{1,t}(\mathbb{R}^n)} \\ &= \|b_1(f_{1,j} - f_1)\|_{W^{1,t}(\mathbb{R}^n)} \\ &\leq \|b_1\|_{W^{1,p_{m+1}}(\mathbb{R}^n)}\|f_{1,j} - f_1\|_{W^{1,p_1}(\mathbb{R}^n)} \to 0 \ \text{ as } j \to \infty, \end{aligned}$$

which together with (1.4) implies

$$\|\mathfrak{M}(b_1 f_{1,j}, f_{2,j}, \dots, f_{m,j}) - \mathfrak{M}(b_1 f_1, f_2, \dots, f_m)\|_{W^{1,p}(\mathbb{R}^n)} \to 0 \ \text{ as } j \to \infty. \tag{3.6}$$

Observe from (1.1) that $\mathfrak{M}(\vec{f}_j) \in W^{1,s}(\mathbb{R}^n)$ and $\mathfrak{M}(\vec{f}) \in W^{1,s}(\mathbb{R}^n)$. By (1.4) and Lemma 2.1, we have

$$\begin{aligned} &\|b_1\mathfrak{M}(\vec{f}_j) - b_1\mathfrak{M}(\vec{f})\|_{W^{1,p}(\mathbb{R}^n)} \\ &\leq \|b_1\|_{W^{1,p_{m+1}}(\mathbb{R}^n)}\|\mathfrak{M}(\vec{f}_j) - \mathfrak{M}(\vec{f})\|_{W^{1,s}(\mathbb{R}^n)} \to 0 \ \text{ as} j \to \infty. \end{aligned}$$

This together with (3.6) yields (3.5) for $i = 1$.

3.2. Proof of Theorem 1.4 for $\mathfrak{M}_{\vec{b}}$

We want to show that

$$\|\mathfrak{M}^i_{\vec{b}}(\vec{f})\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|b_i\|_{W^{1,p_{m+1}}(\mathbb{R}^n)} \prod_{j=1}^m \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)} \tag{3.7}$$

for each $i = 1, \dots, m$. By Remark 1.2 (ii), to prove (3.7), it suffices to show that

$$\|\nabla\mathfrak{M}^i_{\vec{b}}(\vec{f})\|_{L^p(\mathbb{R}^n)} \leq C\|b_i\|_{W^{1,p_{m+1}}(\mathbb{R}^n)} \prod_{j=1}^m \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)} \tag{3.8}$$

for each $i = 1, \dots, m$. We only prove (3.8) for $i = 1$ and other cases are analogous. Fix $y \in \mathbb{R}^n$, we get by Remark 1.2 (iii) that

$$\begin{aligned}
&|(\mathfrak{M}^1_{\vec{b}}(\vec{f}))_y(x) - \mathfrak{M}^1_{\vec{b}}(\vec{f})(x)| \\
&= |\mathfrak{M}^1_{\vec{b}_y}(\vec{f}_y)(x) - \mathfrak{M}^1_{\vec{b}}(\vec{f})(x)| \\
&\leq \sup_{B\ni x} \frac{1}{|B|^m} \int_{B^m} \Big| |(b_1)_y(x) - (b_1)_y(z_1)| \Big| \prod_{j=1}^m (f_j)_y(z_j) \Big| \\
&\quad - |b_1(x) - b_1(z_1)| \Big| \prod_{j=1}^m f_j(z_j) \Big| \Big| \mathrm{d}z_1 \mathrm{d}z_2 \cdots \mathrm{d}z_m \\
&\leq \sup_{B\ni x} \frac{1}{|B|^m} \int_{B^m} |(b_1)_y(x) - (b_1)_y(z_1) - b_1(x) + b_1(z_1)| \\
&\quad \times \Big| \prod_{j=1}^m (f_j)_y(z_j) \Big| \mathrm{d}z_1 \mathrm{d}z_2 \cdots \mathrm{d}z_m \\
&\quad + \sup_{B\ni x} \frac{1}{|B|^m} \int_{B^m} |b_1(x) - b_1(z_1)| \\
&\quad \times \Big| \prod_{j=1}^m (f_j)_y(z_j) - \prod_{j=1}^m f_j(z_j) \Big| \mathrm{d}z_1 \mathrm{d}z_2 \cdots \mathrm{d}z_m.
\end{aligned} \tag{3.9}$$

Noting that

$$\prod_{j=1}^m (f_j)_y(z_j) - \prod_{j=1}^m f_j(z_j) = \sum_{l=1}^m ((f_l)_y(z_l) - f_l(z_l)) \Big(\prod_{\mu=1}^{l-1} f_\mu(z_\mu) \Big) \Big(\prod_{\nu=l+1}^m (f_\nu)_y(z_\nu) \Big),$$

which together with (3.9) implies that

$$\begin{aligned}
&|(\mathfrak{M}^1_{\vec{b}}(\vec{f}))_y(x) - \mathfrak{M}^1_{\vec{b}}(\vec{f})(x)| \\
&\leq |(b_1)_y(x) - b_1(x)|\mathfrak{M}(\vec{f})(x) + \mathfrak{M}(\vec{f}_{1,b_1,y})(x) + \sum_{l=1}^m \mathfrak{M}^1_{\vec{b}}(\vec{F}_{l,y})(x),
\end{aligned} \tag{3.10}$$

where

$$\vec{f}_{1,b_1,y} = (((b_1)_y - b_1)f_1, f_2, \dots, f_m),$$

$$\vec{F}_{l,y} = (f_1, \dots, f_{l-1}, (f_l)_y - f_l, (f_{l+1})_y, \dots, (f_m)_y).$$

By Hölder's inequality and Minkowski's inequality, we get from (1.2), (1.8) and (3.10) that

$$\begin{aligned}
&\|(\mathfrak{M}^1_{\vec{b}}(\vec{f}))_y - \mathfrak{M}^1_{\vec{b}}(\vec{f})\|_{L^p(\mathbb{R}^n)} \\
&\le \|((b_1)_y - b_1)\mathfrak{M}(\vec{f})\|_{L^p(\mathbb{R}^n)} + \|\mathfrak{M}(\vec{f}_{1,b_1,y})\|_{L^p(\mathbb{R}^n)} \\
&\quad + \sum_{l=1}^m \|\mathfrak{M}^1_{\vec{b}}(\vec{F}_{l,y})\|_{L^p(\mathbb{R}^n)} \\
&\le C\|(b_1)_y - b_1\|_{L^{p_{m+1}}(\mathbb{R}^n)} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \\
&\quad + C\sum_{l=1}^m \|b_1\|_{L^{p_{m+1}}(\mathbb{R}^n)} \|(f_l)_y - f_l\|_{L^{p_l}(\mathbb{R}^n)} \\
&\qquad \times \prod_{\mu=1}^{l-1} \|f_\mu\|_{L^{p_\mu}(\mathbb{R}^n)} \prod_{\nu=l+1}^m \|(f_\nu)_y\|_{L^{p_\nu}(\mathbb{R}^n)}.
\end{aligned} \tag{3.11}$$

We note that $\|(f_j)_y\|_{L^{p_j}(\mathbb{R}^n)} \le \|(f_j)_y - f_j\|_{L^{p_j}(\mathbb{R}^n)} + \|f_j\|_{L^{p_j}(\mathbb{R}^n)}$ and $G(b_1, p_{m+1}) < \infty$, $G(f_j, p_j) < \infty$. These facts, together with (3.11), imply

$$G(\mathfrak{M}^1_{\vec{b}}(\vec{f}); p) < \infty. \tag{3.12}$$

Combining (3.12) with (2.2) and (1.8) implies $\mathfrak{M}^1_{\vec{b}}(\vec{f}) \in W^{1,p}(\mathbb{R}^n)$. Fix $1 \le i \le n$. Given $l \in \{1, \dots, m\}$, we get from (2.1) that $(f_l)^i_h \to D_i f_l$ in $L^{p_l}(\mathbb{R}^n)$ and $(b_1)^i_h \to D_i b_1$ in $L^{p_{m+1}}(\mathbb{R}^n)$ when $h \to 0$. Moreover, $(\mathfrak{M}^1_{\vec{b}}(\vec{f}))^i_h \to D_i \mathfrak{M}^1_{\vec{b}}(\vec{f})$ in $L^p(\mathbb{R}^n)$ as $h \to 0$. We also know that $(f_l)_{-he_i} \to f_l$ in $L^{p_l}(\mathbb{R}^n)$ and $(b_1)_{-he_i} \to b_1$ in $L^{p_{m+1}}(\mathbb{R}^n)$ when $|y| \to 0$. Therefore, we can find a sequence of numbers $\{h_k\}_{k\ge1}$ with $\lim_{k\to\infty} h_k = 0$ and a measurable set E with $|\mathbb{R}^n \setminus E| = 0$ such that

(i) $(f_l)_{h_k e_i}(x) \to f_l(x)$, $(b_1)_{h_k e_i}(x) \to b_1(x)$, $(f_l)^i_{h_k}(x) \to D_i f_l(x)$, $(b_1)^i_{h_k} \to D_i b_1(x)$ as $k \to \infty$ for all $x \in E$;

(ii) $(\mathfrak{M}^1_{\vec{b}}(\vec{f}))^i_{h_k}(x) \to D_i \mathfrak{M}^1_{\vec{b}}(\vec{f})(x)$ as $k \to \infty$ for all $x \in E$.

By (3.11) and Fatou's lemma, we have

$$
\begin{aligned}
&\|D_i\mathfrak{M}^1_{\vec{b}}(\vec{f})\|_{L^p(\mathbb{R}^n)}\\
&\le \Big\|\liminf_{k\to\infty}(\mathfrak{M}^1_{\vec{b}}(\vec{f}))^i_{h_k}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le C\Big\|\liminf_{k\to\infty}(b_1)^i_{h_k}\Big\|_{L^{p_{m+1}}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)}\\
&\quad+C\sum_{l=1}^m\|b_1\|_{L^{p_{m+1}}(\mathbb{R}^n)}\Big\|\liminf_{k\to\infty}(f_l)^i_{h_k}\Big\|_{L^{p_l}(\mathbb{R}^n)}\prod_{\mu=1}^{l-1}\|f_\mu\|_{L^{p_\mu}(\mathbb{R}^n)}\\
&\quad\quad\times\prod_{\nu=l+1}^m\Big\|\liminf_{k\to\infty}(f_\nu)_{-h_ke_i}\Big\|_{L^{p_\nu}(\mathbb{R}^n)}\\
&\le C\Big\|D_ib_1\|_{L^{p_{m+1}}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)}\\
&\quad+C\sum_{l=1}^m\|b_1\|_{L^{p_{m+1}}(\mathbb{R}^n)}\|D_if_l\|_{L^{p_l}(\mathbb{R}^n)}\prod_{\substack{1\le j\le m\\ j\ne l}}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}\\
&\le C\|b_1\|_{W^{1,p_{m+1}}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}.
\end{aligned}
$$

This gives (3.8) for $i=1$ and completes the proof of Theorem 1.4. □

4. Proof of Theorem 1.5

In order to prove Theorem 1.5, we need the following lemma.

Lemma 4.1. ([34]) *For any* $1<p,\ q,\ r<\infty$, *it holds that*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\|\widetilde{M}f_{k,\zeta}\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\le C_{p,q,r}\Big\|\Big(\sum_{k\in\mathbb{Z}}\|f_{k,\zeta}\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

We now turn to the proof of Theorem 1.5. Throughout this section we fix $1<p_1,\dots,p_{m+1},p,q<\infty$, $0<s<1$ and $1/p=1/p_1+\cdots+1/p_{m+1}$. Let $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in F^{p_j,q}_s(\mathbb{R}^n)$. Let α,α_m be such that $1/\alpha=1/p_1+\cdots+1/p_m$ and $1/\alpha_m=1/p_1+1/p_{m+1}$. Clearly, $p<\alpha,\alpha_m<p_1$ and $1/p=1/\alpha+1/p_{m+1}$. The proof of Theorem 1.5 will be divided into two subsections:

4.1. Proof of Theorem 1.5 for $[\vec{b},\mathfrak{M}]$

By Minkowski's inequality, inequality (1.11) reduces to the following

$$\|[\vec{b},\mathfrak{M}]_i(\vec{f})\|_{F^{p,q}_s(\mathbb{R}^n)}\le C\|b_i\|_{F^{p_{m+1},q}_s(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{F^{p_j,q}_s(\mathbb{R}^n)}\tag{4.1}$$

for each $i = 1, \dots, m$. We only work with (4.1) for the case $i = 1$ and other cases are analogous. By Theorem A (i) and invoking Lemma 2.2, we have

$$\begin{aligned}\|b_1\mathfrak{M}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} &\leq C\|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)}\|\mathfrak{M}(\vec{f})\|_{F_s^{\alpha,q}(\mathbb{R}^n)} \\ &\leq C\|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)}\prod_{j=1}^{m}\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)},\end{aligned} \tag{4.2}$$

$$\begin{aligned}\|\mathfrak{M}(b_1f_1, f_2, \dots, f_m)\|_{F_s^{p,q}(\mathbb{R}^n)} &\leq C\|b_1f_1\|_{F_s^{\alpha_m,q}(\mathbb{R}^n)}\prod_{j=2}^{m}\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)} \\ &\leq C\|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)}\prod_{j=1}^{m}\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}.\end{aligned} \tag{4.3}$$

Combining (4.3) with (4.2) leads to (4.1) for the case $i = 1$.

Let $\vec{f}_j = (f_{1,j}, \dots, f_{m,j})$ with each $f_{i,j} \to f_i$ in $F_s^{p_i,q}(\mathbb{R}^n)$ as $j \to \infty$ for all $i \in \{1, \dots, m\}$. It suffices to show that

$$\|[\vec{b}, \mathfrak{M}]_i(\vec{f}_j) - [\vec{b}, \mathfrak{M}]_i(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j \to \infty \tag{4.4}$$

for all $i = 1, \dots, m$. We only prove (4.4) for $i = 1$ since other cases can be proved similarly. Invoking Lemma 2.2, one has

$$\|b_1f_{1,j} - b_1f_1\|_{F_s^{\alpha_m,q}(\mathbb{R}^n)} \leq C\|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)}\|f_{1,j} - f_1\|_{F_s^{p_1,q}(\mathbb{R}^n)},$$

which together with the continuity result in Theorem A (i) yields that

$$\|\mathfrak{M}(b_1f_{1,j}, f_{2,j}, \dots, f_{m,j}) - \mathfrak{M}(b_1f_1, f_2, \dots, f_m)\|_{F_s^{p,q}(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j \to \infty. \tag{4.5}$$

On the other hand, by invoking Lemma 2.2 and Theorem A (i) again,

$$\|b_1\mathfrak{M}(\vec{f}_j) - b_1\mathfrak{M}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j \to \infty,$$

which together with (4.5) leads to (4.4) with $i = 1$.

4.2. Proof of Theorem 1.5 for $\mathfrak{M}_{\vec{b}}$

We want to show that

$$\|\mathfrak{M}_{\vec{b}}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \leq C\Big(\sum_{i=1}^{m}\|b_i\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)}\Big)\prod_{j=1}^{m}\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \tag{4.6}$$

To prove (4.6), it suffices to show that

$$\|\mathfrak{M}_{\vec{b}}^{i}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \leq C\|b_i\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)}\prod_{j=1}^{m}\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)} \tag{4.7}$$

for each $i = 1, \dots, m$. Without loss of generality, we only work with (4.7) for the case $i = 1$ and other cases are analogous. By (3.10) and (1.7), we have that, for any $(x,k,\zeta)\in\mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n$,

$$\begin{aligned}
&|\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)|\\
&\le|\Delta_{2^{-k}\zeta}b_1(x)|\mathfrak{M}(\vec{f})(x)+\mathfrak{M}(\vec{f}_{1,b_1,-2^{-k}\zeta})(x)\\
&\quad+|b_1(x)|\sum_{l=1}^m\mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta})(x)+\sum_{l=1}^m\mathfrak{M}(\vec{G}_{b_1,l,-2^{-k}\zeta})(x)\\
&=:\ \Gamma(x,k,\zeta),
\end{aligned}\tag{4.8}$$

where

$$\vec{f}_{1,b_1,-2^{-k}\zeta}=(\Delta_{2^{-k}\zeta}b_1f_1,f_2,\dots,f_m),$$

$$\vec{G}_{b_1,1,-2^{-k}\zeta}=(b_1\Delta_{2^{-k}\zeta}f_1,(f_2)_{-2^{-k}\zeta},\dots,(f_m)_{-2^{-k}\zeta}),$$

$$\vec{G}_{l,-2^{-k}\zeta}=(f_1,\dots,f_{l-1},\Delta_{2^{-k}\zeta}f_l,(f_{l+1})_{-2^{-k}\zeta},\dots,(f_m)_{-2^{-k}\zeta}),$$

and for $l=2,\dots,m$,

$$\vec{G}_{b_1,l,-2^{-k}\zeta}=(b_1f_1,f_2,\dots,f_{l-1},\Delta_{2^{-k}\zeta}f_l,(f_{l+1})_{-2^{-k}\zeta},\dots,(f_m)_{-2^{-k}\zeta}).$$

In light of (4.8) and (2.3), we have

$$\begin{aligned}
&\|\mathfrak{M}^1_{\vec{b}}(\vec{f})\|_{\dot{F}^{p,q}_s(\mathbb{R}^n)}\\
&\le C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}b_1|\mathfrak{M}(\vec{f})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad+C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\vec{f}_{1,b_1,-2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad+C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\sum_{l=1}^m|b_1|\mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad+C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\sum_{l=1}^m\mathfrak{M}(\vec{G}_{b_1,l,-2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&=:\ A_1+A_2+A_3+A_4.
\end{aligned}\tag{4.9}$$

By Hölder's inequality, (1.2), (2.3) and (2.4), we have

$$\begin{aligned}
A_1&\le C\|\mathfrak{M}(\vec{f})\|_{L^\alpha(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}b_1|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^{p_{m+1}}(\mathbb{R}^n)}\\
&\le C\|b_1\|_{\dot{F}^{p_{m+1},q}_s(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)}\\
&\le C\|b_1\|_{F^{p_{m+1},q}_s(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{F^{p_j,q}_s(\mathbb{R}^n)}.
\end{aligned}\tag{4.10}$$

Fix $r \in (1, \min\{q, p_{m+1}\})$, by Hölder's inequality, the bounds for $\widetilde{M}$, Lemma 4.1, (2.3) and (2.4), we have

$$
\begin{aligned}
A_2 &\le \Big\| \prod_{j=2}^{m} \widetilde{M} f_j \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \widetilde{M}(\Delta_{2^{-k}\zeta} b_1 f_1) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\
&\le C \prod_{j=2}^{m} \|\widetilde{M} f_j\|_{L^{p_j}(\mathbb{R}^n)} \\
&\quad \times \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \widetilde{M}(\Delta_{2^{-k}\zeta} b_1 f_1) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^{\alpha_m}(\mathbb{R}^n)} \\
&\le C \prod_{j=2}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \\
&\quad \times \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \|\widetilde{M}(\Delta_{2^{-k}\zeta} b_1 f_1)\|^q_{L^r(\mathfrak{R}_n)} \Big)^{1/q} \Big\|_{L^{\alpha_m}(\mathbb{R}^n)} \\
&\le C \prod_{j=2}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \|\Delta_{2^{-k}\zeta} b_1 f_1\|^q_{L^r(\mathfrak{R}_n)} \Big)^{1/q} \Big\|_{L^{\alpha_m}(\mathbb{R}^n)} \\
&\le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \|\Delta_{2^{-k}\zeta} b_1\|^q_{L^r(\mathfrak{R}_n)} \Big)^{1/q} \Big\|_{L^{p_{m+1}}(\mathbb{R}^n)} \\
&\le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \|b_1\|_{\dot{F}_s^{p_{m+1},q}(\mathbb{R}^n)} \\
&\le C \|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}.
\end{aligned} \tag{4.11}
$$

For A_3, by Hölder's inequality and Minkowski's inequality

$$
\begin{aligned}
A_3 &\le C \sum_{l=1}^{m} \|b_1\|_{L^{p_{m+1}}(\mathbb{R}^n)} \\
&\quad \times \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta}) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^{\alpha}(\mathbb{R}^n)}.
\end{aligned} \tag{4.12}
$$

Let us fix $l \in \{1, \dots, m\}$. Noting that

$$
\mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta}) \le \sum_{\tau \subset E_l} \prod_{\mu \in \tau \cup \{l\}} \widetilde{M}(\Delta_{2^{-k}\zeta} f_\mu) \prod_{\nu \in \tau'} \widetilde{M} f_\nu, \tag{4.13}
$$

where $E_l = \{l+1, \dots, m\}$ and $\tau' = \{1, \dots, m\} \setminus (\tau \cup \{l\})$. Let α_τ be such that $1/\alpha_\tau = \sum_{\ell\in\tau} 1/p_\ell + 1/p_l$. It is clear that $p < \alpha_\tau < p_l$ and $1/p = 1/\alpha_\tau + \sum_{\kappa\in\tau'} 1/p_\kappa$. By (4.13), Minkowski's inequality, Hölder's inequality and the bounds for $\widetilde{M}$, we have

$$
\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^{\alpha}(\mathbb{R}^n)}\\
&\le\sum_{\tau\subset E_m}\Big\|\prod_{\nu\in\tau'}\widetilde{M}f_\nu\\
&\quad\times\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\prod_{\mu\in\tau\cup\{l\}}\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le C\sum_{\tau\subset E_m}\prod_{\nu\in\tau'}\|\widetilde{M}f_\nu\|_{L^{p_\nu}(\mathbb{R}^n)}\\
&\quad\times\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\prod_{\mu\in\tau\cup\{l\}}\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}\\
&\le C\sum_{\tau\subset E_m}\prod_{\nu\in\tau'}\|f_\nu\|_{L^{p_\nu}(\mathbb{R}^n)}\\
&\quad\times\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\prod_{\mu\in\tau\cup\{l\}}\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}.
\end{aligned}
\tag{4.14}
$$

Given $\tau\in E_m$. By Hölder's inequality and Lemma 4.1, one has

$$
\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\prod_{\mu\in\tau\cup\{l\}}\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}\\
&\le\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\prod_{\mu\in\tau\cup\{l\}}\|\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\|^q_{L^{p_\mu/\alpha_\tau}(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}\\
&\le\Big\|\prod_{\mu\in\tau\cup\{l\}}\Big(\sum_{k\in\mathbb{Z}}\Big(2^{ks\alpha_\tau/p_\mu}\|\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\|_{L^{p_\mu/\alpha_\tau}(\mathfrak{R}_n)}\Big)^{p_\mu q/\alpha_\tau}\Big)^{\alpha_\tau/(qp_\mu)}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}\\
&\le\prod_{\mu\in\tau\cup\{l\}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\Big(2^{ks\alpha_\tau/p_\mu}\|\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\|_{L^{p_\mu/\alpha_\tau}(\mathfrak{R}_n)}\Big)^{p_\mu q/\alpha_\tau}\Big)^{\alpha_\tau/(qp_\mu)}\Big\|_{L^{p_\mu}(\mathbb{R}^n)}\\
&\le C\prod_{\mu\in\tau\cup\{l\}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\Big(2^{ks\alpha_\tau/p_\mu}\|\Delta_{2^{-k}\zeta}f_\mu\|_{L^{p_\mu/\alpha_\tau}(\mathfrak{R}_n)}\Big)^{p_\mu q/\alpha_\tau}\Big)^{\alpha_\tau/(qp_\mu)}\Big\|_{L^{p_\mu}(\mathbb{R}^n)}.
\end{aligned}
$$

This together with (2.3)-(2.6) leads to

$$
\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\prod_{\mu\in\tau\cup\{l\}}\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}\\
&\le C\prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{\dot{F}^{p_\mu,p_\mu q/\alpha_\tau}_{\alpha_\tau s/p_\mu}(\mathbb{R}^n)}\\
&\le C\prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{F^{p_\mu,p_\mu q/\alpha_\tau}_{\alpha_\tau s/p_\mu}(\mathbb{R}^n)}\\
&\le C\prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{F^{p_\mu,q}_{s}(\mathbb{R}^n)}.
\end{aligned}
\tag{4.15}
$$

It follows from (4.14), (4.15) and (2.4) that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^\alpha(\mathbb{R}^n)}\leq C\prod_{j=1}^m\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \tag{4.16}$$

Combining (4.16) with (4.12) and (2.4) yields

$$A_3\leq C\|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \tag{4.17}$$

Similar arguments to those used to derive (4.16) may give that

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\vec{G}_{b_1,l,-2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\leq C\|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)},\end{aligned}$$

for each $l=1,\dots,m$, which together with Minkowski's inequality implies

$$A_4\leq C\|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \tag{4.18}$$

Then (4.7) with $i=1$ follows from (4.9)-(4.11), (4.17) and (4.18).

It remains to prove the continuity result for $\mathfrak{M}_{\vec{b}}$. Let $\vec{f}_j=(f_{1,j},\dots,f_{m,j})$ with each $f_{i,j}\to f_i$ in $F_s^{p_i,q}(\mathbb{R}^n)$ as $j\to\infty$ for all $i\in\{1,\dots,m\}$. It suffices to show that

$$\|\mathfrak{M}_{\vec{b}}^i(\vec{f}_j)-\mathfrak{M}_{\vec{b}}^i(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)}\to 0\quad\text{as}\quad j\to\infty \tag{4.19}$$

for all $i=1,\dots,m$. We only prove (4.19) for $i=1$ since other cases are analogous. By (2.4) we have that $f_{i,j}\to f_i$ in $\dot{F}_s^{p_i,q}(\mathbb{R}^n)$ and in $L^{p_i}(\mathbb{R}^n)$ as $j\to\infty$ for all $i\in\{1,\dots,m\}$. By (1.9) and (2.4), to prove (4.19) with $i=1$, it is enough to prove that

$$\|\mathfrak{M}_{\vec{b}}^1(\vec{f}_j)-\mathfrak{M}_{\vec{b}}^1(\vec{f})\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)}\to 0\ \text{ as } j\to\infty. \tag{4.20}$$

Now we prove (4.20) by contradiction. Assume that (4.20) doesn't hold. We may assume without loss of generality that there exists a constant $c>0$ such that

$$\|\mathfrak{M}_{\vec{b}}^1(\vec{f}_j)-\mathfrak{M}_{\vec{b}}^1(\vec{f})\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)}>c,\ \text{ for all } j\geq 1. \tag{4.21}$$

Thanks to (1.9), we may assume, without loss of generality, by extracting a subsequence that $\mathfrak{M}_{\vec{b}}^1(\vec{f}_j)(x)-\mathfrak{M}_{\vec{b}}^1(\vec{f})(x)\to 0$ as $j\to\infty$ for almost every $x\in\mathbb{R}^n$. Hence,

$$\Delta_{2^{-k}\zeta}(\mathfrak{M}_{\vec{b}}^1(\vec{f}_j)-\mathfrak{M}_{\vec{b}}^1(\vec{f}))(x)\to 0\ \text{ as }\ j\to\infty \tag{4.22}$$

for every $(k,\zeta)\in\mathbb{Z}\times\mathfrak{R}_n$ and almost every $x\in\mathbb{R}^n$. From (4.8), we have that, for $(x,k,\zeta)\in\mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n$,

$$|\Delta_{2^{-k}\zeta}(\mathfrak{M}_{\vec{b}}^1(\vec{f}_j)-\mathfrak{M}_{\vec{b}}^1(\vec{f}))(x)|\leq\Gamma_j(x,k,\zeta)+\Gamma(x,k,\zeta), \tag{4.23}$$

where Γ is given as in (4.8) and

$$\begin{aligned}\Gamma_j(x,k,\zeta) = \Big| &|\Delta_{2^{-k}\zeta} b_1(x)|\mathfrak{M}(\vec{f}_j)(x) + \mathfrak{M}(\vec{f}_{1,j,b_1,-2^{-k}\zeta})(x) \\ &+ |b_1(x)| \sum_{l=1}^{m} \mathfrak{M}(\vec{G}_{l,j,-2^{-k}\zeta})(x) \\ &+ \sum_{l=1}^{m} \mathfrak{M}(\vec{G}_{b_1,l,j,-2^{-k}\zeta})(x) - \Psi(x,k,\zeta)\Big| \\ \le& \sum_{i=1}^{4} \varphi_{1,j}(x,k,\zeta),\end{aligned} \tag{4.24}$$

where

$$\varphi_{1,j}(x,k,\zeta) := |\Delta_{2^{-k}\zeta} b_1(x)||\mathfrak{M}(\vec{f}_j)(x) - \mathfrak{M}(\vec{f})(x)|,$$

$$\varphi_{2,j}(x,k,\zeta) := |\mathfrak{M}(\vec{f}_{1,j,b_1,-2^{-k}\zeta})(x) - \mathfrak{M}(\vec{f}_{1,b_1,-2^{-k}\zeta})(x)|,$$

$$\varphi_{3,j}(x,k,\zeta) := |b_1(x)| \sum_{l=1}^{m} |\mathfrak{M}(\vec{G}_{l,j,-2^{-k}\zeta})(x) - \mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta})(x)|,$$

$$\varphi_{4,j}(x,k,\zeta) = \sum_{l=1}^{m} |\mathfrak{M}(\vec{G}_{b_1,l,j,-2^{-k}\zeta})(x) - \mathfrak{M}(\vec{G}_{b_1,l,-2^{-k}\zeta})(x)|,$$

$$\vec{f}_{1,j,b_1,-2^{-k}\zeta} := (\Delta_{2^{-k}\zeta} b_1 f_{1,j}, f_{2,j}, \dots, f_{m,j}),$$

$$\vec{G}_{b_1,1,j,-2^{-k}\zeta} := (b_1 \Delta_{2^{-k}\zeta} f_{1,j}, (f_{2,j})_{-2^{-k}\zeta}, \dots, (f_{m,j})_{-2^{-k}\zeta}),$$

$$\vec{G}_{l,j,-2^{-k}\zeta} := (f_{1,j}, \dots, f_{l-1,j}, \Delta_{2^{-k}\zeta} f_{l,j}, (f_{l+1,j})_{-2^{-k}\zeta}, \dots, (f_{m,j})_{-2^{-k}\zeta}),$$

and for $l = 2, \dots, m$,

$$\vec{G}_{b_1,l,j,-2^{-k}\zeta} := (b_1 f_{1,j}, f_{2,j}, \dots, f_{l-1,j}, \Delta_{2^{-k}\zeta} f_{l,j}, (f_{l+1,j})_{-2^{-k}\zeta}, \dots, (f_{m,j})_{-2^{-k}\zeta}).$$

Similar arguments as in deriving (4.16) may imply that

$$\begin{aligned}&\|\varphi_{i,j}\|_{E^s_{p,q,1}} \\ &\le C\|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)} \sum_{l=1}^{m} \|f_{l,j} - f_l\|_{F_s^{p_l,q}(\mathbb{R}^n)} \\ &\quad \times \prod_{\substack{1\le\mu\le m \\ \mu\ne l}} (\|f_{\mu,j} - f_\mu\|_{F_s^{p_\mu,q}(\mathbb{R}^n)} + \|f_\mu\|_{F_s^{p_\mu,q}(\mathbb{R}^n)}), \quad i = 1,2,3,4.\end{aligned} \tag{4.25}$$

Combining (4.25) with (4.24) implies that

$$\begin{aligned}\|\Gamma_j\|_{E^s_{p,q,1}} \le C\|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)} &\sum_{l=1}^{m} \|f_{l,j} - f_l\|_{F_s^{p_l,q}(\mathbb{R}^n)} \\ \times &\prod_{\substack{1\le\mu\le m \\ \mu\ne l}} (\|f_{\mu,j} - f_\mu\|_{F_s^{p_\mu,q}(\mathbb{R}^n)} + \|f_\mu\|_{F_s^{p_\mu,q}(\mathbb{R}^n)}).\end{aligned} \tag{4.26}$$

Also, we get from (4.9)-(4.11), (4.17) and (4.18) that

$$\|\Gamma\|_{E^s_{p,q,1}} \leq C\|b_1\|_{F_s^{p_{m+1},q}(\mathbb{R}^n)} \prod_{j=1}^m \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \tag{4.27}$$

By (4.26), we have that $\|\Gamma_j\|_{E^s_{p,q,1}} \to 0$ as $j \to \infty$, which yields that, there exists a subsequence $\{j_\ell\}_{\ell=1}^\infty \subset \{j\}_{j=1}^\infty$ such that

$$\sum_{\ell=1}^\infty \|\Gamma_{j_\ell}\|_{E^s_{p,q,1}} < \infty. \tag{4.28}$$

From (4.23), we see that

$$|\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)| \leq \sum_{\ell=1}^\infty \Gamma_{j_\ell}(x,k,\zeta) + \Gamma(x,k,\zeta) =: \Phi(x,k,\zeta), \tag{4.29}$$

for all $(x,k,\zeta) \in \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n$. Combining (4.28) with (4.29) and Minkowski's inequality implies $\|\Phi\|_{E^s_{p,q,1}} < \infty$. It follows that $\int_{\mathfrak{R}_n} \Phi(x,k,\zeta)\mathrm{d}\zeta < \infty$ for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$. This, together with (4.22), (4.29) and the dominated convergence theorem, implies

$$\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)|\mathrm{d}\zeta \to 0 \ \text{ as } \ell \to \infty \tag{4.30}$$

for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$. Using the fact $\|\Phi\|_{E^s_{p,q,1}} < \infty$, we deduce that

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \Phi(x,k,\zeta)\mathrm{d}\zeta\Big)^q\Big)^{1/q} < \infty \tag{4.31}$$

for almost every $x \in \mathbb{R}^n$. We get from (4.29) that

$$\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)|\mathrm{d}\zeta \leq \int_{\mathfrak{R}_n} \Phi(x,k,\zeta)\mathrm{d}\zeta, \tag{4.32}$$

for all $(x,k,\zeta) \in \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n$ and $\ell \geq 1$. By the dominated convergence theorem, we get from (4.30)-(4.32) that

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)|\mathrm{d}\zeta\Big)^q\Big)^{1/q} \to 0 \ \text{ as } \ell \to \infty. \tag{4.33}$$

By (4.29) and the fact that $\|\Phi\|_{E^s_{p,q,1}} < \infty$,

$$\begin{aligned}&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\\ &\leq \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Phi(x,k,\zeta)|\mathrm{d}\zeta\Big)^q\Big)^{1/q} < \infty,\end{aligned} \tag{4.34}$$

for almost every $x \in \mathbb{R}^n$. It follows from (4.33), (4.34) and the dominated convergence theorem that

$$\|\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))\|_{E^s_{p,q,1}} \to 0 \text{ as } \ell \to \infty,$$

which together with (2.3) leads to $\|\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f})\|_{\dot{F}^{p,q}_s(\mathbb{R}^n)} \to 0$ as $\ell \to \infty$. This is in contradiction with (4.21). □

5. Proof of Theorem 1.6

This section is devoted to presenting the proof of Theorem 1.6. By Lemma 2.3, Theorem A (ii) and the arguments similar to those used in deriving the corresponding result for $[\vec{b}, \mathfrak{M}]$, we obtain (1.12) and the continuity for $[\vec{b}, \mathfrak{M}]$: $B^{p_1,q}_s(\mathbb{R}^n) \times \cdots \times B^{p_m,q}_s(\mathbb{R}^n) \to B^{p,q}_s(\mathbb{R}^n)$.

It remains to prove Theorem 1.6 for $\mathfrak{M}_{\vec{b}}$. In what follows, we fix $0 < s < 1$, $1 < p_1, \dots, p_{m+1}, p, q < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_{m+1}$. Let α, α_m be such that $1/\alpha = 1/p_1 + \cdots + 1/p_m$ and $1/\alpha_m = 1/p_1 + 1/p_{m+1}$. It is clear that $p < \alpha$, $\alpha_m < p_1$ and $1/p = 1/\alpha + 1/p_{m+1}$.

We want to show that

$$\|\mathfrak{M}_{\vec{b}}(\vec{f})\|_{B^{p,q}_s(\mathbb{R}^n)} \le C\Big(\sum_{i=1}^m \|b_i\|_{B^{p_{m+1},q}_s(\mathbb{R}^n)}\Big) \prod_{j=1}^m \|f_j\|_{B^{p_j,q}_s(\mathbb{R}^n)}. \tag{5.1}$$

By Minkowski's inequality, to prove (5.1), it suffices to show that

$$\|\mathfrak{M}^i_{\vec{b}}(\vec{f})\|_{B^{p,q}_s(\mathbb{R}^n)} \le C\|b_i\|_{B^{p_{m+1},q}_s(\mathbb{R}^n)} \prod_{j=1}^m \|f_j\|_{B^{p_j,q}_s(\mathbb{R}^n)} \tag{5.2}$$

for each $i = 1, \dots, m$. Without loss of generality, we only prove (5.2) for the case $i = 1$ since other cases are analogous. From (2.10) and (4.8), we have

$$\begin{aligned} &\|\mathfrak{M}^1_{\vec{b}}(\vec{f})\|_{\dot{B}^{p,q}_s(\mathbb{R}^n)} \\ &\le C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(|\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)|)^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q} \\ &\le C(B_1 + B_2 + B_3 + B_4). \end{aligned} \tag{5.3}$$

where

$$B_1 := \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_d}\int_{\mathbb{R}^n}(|\Delta_{2^{-k}\zeta} b_1(x)|\mathfrak{M}(\vec{f})(x))^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q},$$

$$B_2 := \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\vec{f}_{1,b_1,-2^{-k}\zeta})(x))^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q},$$

$$B_3 := \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\sum_{l=1}^m |b_1(x)|\mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta})(x)\Big)^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q},$$

$$B_4 := \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\sum_{l=1}^m \mathfrak{M}(\vec{G}_{b_1,l,-2^{-k}\zeta})(x)\Big)^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}.$$

By (1.2), (2.10), (2.11) and Hölder's inequality, we have

$$\begin{aligned} B_1 &\le C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\vec{f})(x)|\Delta_{2^{-k}\zeta}b_1(x)|)^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q} \\ &\le C\|\mathfrak{M}(\vec{f})\|_{L^\alpha(\mathbb{R}^n\times\mathfrak{R}_n)} \\ &\quad\times\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|\Delta_{2^{-k}\zeta}b_1(x)|^{p_{m+1}}\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p_{m+1}}\Big)^{1/q} \\ &\le C\prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}\|b_1\|_{\dot{B}_s^{p_{m+1},q}(\mathbb{R}^n)} \\ &\le C\|b_1\|_{B_s^{p_{m+1},q}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} B_2 &\le C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\|\Delta_{2^{-k}\zeta}b_1 f_1\|_{L^{\alpha_m}(\mathbb{R}^n\times\mathfrak{R}_n)}\prod_{j=2}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}\Big)^q\Big)^{1/q} \\ &\le C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\|\Delta_{2^{-k}\zeta}b_1\|_{L^{p_{m+1}}(\mathbb{R}^n\times\mathfrak{R}_n)}\prod_{j=1}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}\Big)^q\Big)^{1/q} \\ &\le C\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)}\|b_1\|_{\dot{B}_s^{p_{m+1},q}(\mathbb{R}^n)} \\ &\le C\|b_1\|_{B_s^{p_{m+1},q}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}. \end{aligned} \tag{5.5}$$

By Minkowski's inequality and Hölder's inequality, we conclude that

$$\begin{aligned} B_3 &\le C\sum_{l=1}^m\|b_1\|_{L^{p_{m+1}}(\mathbb{R}^n)} \\ &\quad\times\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta})(x)\Big)^\alpha \mathrm{d}x\mathrm{d}\zeta\Big)^{q/\alpha}\Big)^{1/q}. \end{aligned} \tag{5.6}$$

Fix $l \in \{1, \dots, m\}$. By the bounds for $\widetilde{M}$ and (2.10)-(2.13), we have

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big\|\prod_{\mu\in\tau\cup\{l\}}\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\Big\|^q_{L^{\alpha}_{\tau}(\mathbb{R}^n\times\mathfrak{R}_n)}\Big)^{1/q}\\
&\le \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\prod_{\mu\in\tau\cup\{l\}}\|\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\|_{L^{p_\mu}(\mathbb{R}^n\times\mathfrak{R}_n)}\Big)^q\Big)^{1/q}\\
&\le \prod_{\mu\in\tau\cup\{l\}}\Big(\sum_{k\in\mathbb{Z}}(2^{ks\alpha_\tau/p_\mu}\|\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\|_{L^{p_\mu}(\mathbb{R}^n\times\mathfrak{R}_n)})^{p_\mu q/\alpha_\tau}\Big)^{\alpha_\tau/(p_\mu q)}\\
&\le \prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{\dot{B}^{p_\mu,p_\mu q/\alpha_\tau}_{s\alpha_\tau/p_\mu}(\mathbb{R}^n)}\\
&\le \prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{B^{p_\mu,p_\mu q/\alpha_\tau}_{s\alpha_\tau/p_\mu}(\mathbb{R}^n)} \le \prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{B^{p_\mu,q}_s(\mathbb{R}^n)}.
\end{aligned}\tag{5.7}$$

Using Minkowski's inequality, Hölder's inequality, (4.13) and the bounds for $\widetilde{M}$, one finds that

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta})(x))^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\le \sum_{\tau\subset E_l}\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\prod_{\mu\in\tau\cup\{l\}}\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\\
&\quad\times\prod_{\nu\in\tau'}\widetilde{M}f_\nu\Big)^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\le C\sum_{\tau\subset E_l}\prod_{\nu\in\tau'}\|\widetilde{M}f_\nu\|_{L^{p_\nu}(\mathbb{R}^n)}\\
&\quad\times\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big\|\prod_{\mu\in\tau\cup\{l\}}\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\Big\|^q_{L^{\alpha_\tau}(\mathbb{R}^n\times\mathfrak{R}_n)}\Big)^{1/q}\\
&\le C\sum_{\tau\subset E_l}\prod_{\nu\in\tau'}\|f_\nu\|_{L^{p_\nu}(\mathbb{R}^n)}\\
&\quad\times\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big\|\prod_{\mu\in\tau\cup\{l\}}\widetilde{M}(\Delta_{2^{-k}\zeta}f_\mu)\Big\|^q_{L^{\alpha_\tau}(\mathbb{R}^n\times\mathfrak{R}_n)}\Big)^{1/q}.
\end{aligned}\tag{5.8}$$

If follows from (2.11), (5.7) and (5.8) that

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\vec{G}_{l,-2^{-k}\zeta})(x))^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\le C\prod_{j=1}^m\|f_j\|_{B^{p_j,q}_s(\mathbb{R}^n)}.
\end{aligned}\tag{5.9}$$

Combining (5.9) with (2.9) and (5.6) implies that

$$B_3 \le C\|b_1\|_{B^{p_{m+1},q}_s(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{B^{p_j,q}_s(\mathbb{R}^n)}.\tag{5.10}$$

Similar arguments to those used in deriving (5.9) may imply

$$\begin{aligned}&\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\mathfrak{M}(\vec{G}_{b_1,l,-2^{-k}\zeta})(x)\Big)^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\&\leq C\|b_1\|_{B_s^{p_{m+1},q}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)},\end{aligned}$$

for each $l=1,\dots,m$, which together with Minkowski's inequality gives that

$$B_4\leq C\|b_1\|_{B_s^{p_{m+1},q}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}. \tag{5.11}$$

Combining (5.11) with (5.3)-(5.5) and (5.10) implies (5.2) for $i=1$.

Next we shall prove the continuity result for $\mathfrak{M}_{\vec{b}}$. The proof is similar as in the proof of the continuity part for $\mathfrak{M}_{\vec{b}}$ in Theorem 1.5. Let $\vec{f}_j=(f_{1,j},\dots,f_{m,j})$ with each $f_{i,j}\to f_i$ in $B_s^{p_i,q}(\mathbb{R}^n)$ as $j\to\infty$ for all $i\in\{1,\dots,m\}$. It suffices to show that

$$\|\mathfrak{M}^i_{\vec{b}}(\vec{f}_j)-\mathfrak{M}^i_{\vec{b}}(\vec{f})\|_{B_s^{p,q}(\mathbb{R}^n)}\to 0\quad\text{as}\quad j\to\infty \tag{5.12}$$

for all $i=1,\dots,m$. We only prove (5.12) for $i=1$ since other cases are analogous. By (2.11), we have that, $f_{i,j}\to f_i$ in $\dot{B}_s^{p_i,q}(\mathbb{R}^n)$ and in $L^{p_i}(\mathbb{R}^n)$ as $j\to\infty$ for all $i\in\{1,\dots,m\}$. By (1.9), to conclude (5.12) with $i=1$, it suffices to prove that

$$\|\mathfrak{M}^1_{\vec{b}}(\vec{f}_j)-\mathfrak{M}^1_{\vec{b}}(\vec{f})\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}\to 0\ \text{ as } j\to\infty. \tag{5.13}$$

Now we prove (5.13) by contradiction. Assume that (5.13) doesn't hold. We may assume, without loss of generality that, there exists a constant $c>0$ such that

$$\|\mathfrak{M}^1_{\vec{b}}(\vec{f}_j)-\mathfrak{M}^1_{\vec{b}}(\vec{f})\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}>c,\ \text{ for all } j\geq 1.$$

Let $\{\varphi_{i,j}\}_{i=1}^4$, and Γ_j, Γ, Ψ be given as in the proof of Theorem 1.5. Similar arguments as in deriving (5.9) may give that

$$\begin{aligned}\|\varphi_{i,j}\|_{p,q,s}\leq C\sum_{l=1}^m&\|f_{l,j}-f_l\|_{B_s^{p_l,q}(\mathbb{R}^n)}\\&\times\prod_{\substack{1\leq\mu\leq m\\ \mu\neq l}}(\|f_{\mu,j}-f_\mu\|_{B_s^{p_\mu,q}(\mathbb{R}^n)}+\|f_\mu\|_{B_s^{p_\mu,q}(\mathbb{R}^n)}),\ \ i=1,2,3,4.\end{aligned}$$

It follows that

$$\begin{aligned}\|\Gamma_j\|_{p,q,s}\leq C\|b_1\|_{B_s^{p_{m+1},q}(\mathbb{R}^n)}&\sum_{l=1}^m\|f_{l,j}-f_l\|_{B_s^{p_l,q}(\mathbb{R}^n)}\\&\times\prod_{\substack{1\leq\mu\leq m\\ \mu\neq l}}(\|f_{\mu,j}-f_\mu\|_{B_s^{p_\mu,q}(\mathbb{R}^n)}+\|f_\mu\|_{B_s^{p_\mu,q}(\mathbb{R}^n)}).\end{aligned} \tag{5.14}$$

By Minkowski's inequality, (4.8), (5.4), (5.5), (5.10) and (5.11), we deduce that

$$\|\Psi\|_{p,q,s} \le C\|b_1\|_{B_s^{p_{m+1},q}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}. \tag{5.15}$$

The rest of proof follows from (5.14), (5.15) and the arguments similar to the proof of Theorem 1.5. □

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