New York Journal of Mathematics

New York J. Math. **27** (2021) 1580–1596.

Necessary density conditions for d-approximate interpolation sequences in the Bargmann-Fock space

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ABSTRACT. Inspired by Olevskii and Ulanovskii [12], we introduce the concept of d-approximate interpolation in weighted Bargmann-Fock spaces as a natural extension of the classical concept of interpolation. We then show that d-approximate interpolation sets satisfy a density condition, similar to the one that classical interpolation sets satisfy. More precisely, we show that the upper Beurling density of any d-approximate interpolation set must be bounded from above by $1/(1-d^2)$.

CONTENTS

1. Introduction

Let $\phi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic function such that for all $z \in \mathbb{C}^n$

$$i\partial\bar{\partial}\phi \simeq i\partial\bar{\partial}|z|^2, \tag{1}$$

in the sense of positive currents. Here, and throughout the paper, we use the standard notation $A \lesssim B$ to denote that there exists a constant $C > 0$ such that $A \leq CB$, and $A \simeq B$ which means that $A \lesssim B$ and $B \lesssim A$. The implied constants may change from line to line.

Received July 27, 2020.

2010 *Mathematics Subject Classification.* 30D15, 30E05, 46E22.

Key words and phrases. Bargmann-Fock space, approximate interpolation set, reproducing kernel, concentration operator.

† Research supported in part by National Science Foundation DMS grant # 1600874.

ISSN 1076-9803/2021

The weighted Bargmann-Fock space $\mathcal{F}_\phi(\mathbb{C}^n)$ is the space of all entire functions $f : \mathbb{C}^n \to \mathbb{C}$ satisfying the integrability condition

$$\|f\|_\phi^2 := \int_{\mathbb{C}^n} |f(z)|^2 e^{-2\phi(z)} dm(z) < \infty,$$

where dm denotes the Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Equipped with the norm $\|\cdot\|_\phi$, the weighted Bargmann-Fock space $\mathcal{F}_\phi(\mathbb{C}^n)$ is a reproducing kernel Hilbert space (RKHS) (for more information about this space see e.g. [15]). We will denote its reproducing kernel at $\lambda \in \mathbb{C}^n$ by $K_\lambda^\phi(z)$, and its normalized reproducing kernel $K_\lambda^\phi(z)/\|K_\lambda^\phi\|_\phi$ by $k_\lambda^\phi(z)$. The classical Bargmann-Fock space $\mathcal{F}(\mathbb{C}^n)$ is an important special case obtained when $\phi(z) = \frac{\pi}{2}|z|^2$. We denote its norm simply by $\|\cdot\|$, i.e.,

$$\|f\|^2 := \int_{\mathbb{C}^n} |f(z)|^2 e^{-\pi|z|^2} dm(z).$$

In this case, explicit formulas for the reproducing kernels are known: $K_\lambda(z) = e^{\pi\langle z,\lambda\rangle}$, $k_\lambda(z) = e^{\pi\langle z,\lambda\rangle - \frac{\pi}{2}|\lambda|^2}$ (see e.g. [20]).

A countable set $\Lambda = \{\lambda\} \subseteq \mathbb{C}^n$ is said to be an *interpolation set* for $\mathcal{F}_\phi(\mathbb{C}^n)$, if for every square summable sequence of complex numbers $(c_\lambda) \in l^2(\Lambda)$ there exists $f \in \mathcal{F}_\phi(\mathbb{C}^n)$ such that $\langle f, k_\lambda^\phi\rangle_\phi = c_\lambda$ for all $\lambda \in \Lambda$. The following equivalent definition of interpolation sets is most relevant for our purposes. Namely, Λ is an interpolation set if and only if the following two conditions hold [19]: (i) for every $\lambda \in \Lambda$ there exists $f_\lambda \in \mathcal{F}_\phi(\mathbb{C}^n)$ such that $\left\langle f_\lambda, k_\nu^\phi \right\rangle_\phi = \delta_{\lambda\nu}$ for all $\nu \in \Lambda$, and (ii) $\{f_\lambda\}_{\lambda\in\Lambda}$ is a Bessel sequence for $\mathcal{F}_\phi(\mathbb{C}^n)$, i.e., there exists a constant $C > 0$ such that

$$\sum_{\lambda\in\Lambda} |\langle f, f_\lambda\rangle_\phi|^2 \le C \|f\|_\phi^2,$$

for every $f \in \mathcal{F}_\phi(\mathbb{C}^n)$. Here, and throughout the paper, we use the standard notation $\delta_{\lambda\nu} = 1$ for $\nu = \lambda$ and 0 otherwise.

If the interpolation can be guaranteed only for the standard basis $\{\delta_\lambda\}_{\lambda\in\Lambda}$ of $l^2(\Lambda)$, with norm control of the approximants, then we say that Λ is a weak interpolation set. More precisely, $\Lambda = \{\lambda\} \subseteq \mathbb{C}^n$ is a *weak interpolation set* for $\mathcal{F}_\phi(\mathbb{C}^n)$, if the following two conditions hold: (i) for every $\lambda \in \Lambda$ there exists $f_\lambda \in \mathcal{F}_\phi(\mathbb{C}^n)$ such that $\left\langle f_\lambda, k_\nu^\phi \right\rangle_\phi = \delta_{\lambda\nu}$ for all $\nu \in \Lambda$, (ii) $\sup_{\lambda\in\Lambda} \|f_\lambda\|_\phi < \infty$. Note that, Λ is a weak interpolation set if and only if the corresponding sequence of normalized reproducing kernels $\{k_\lambda^\phi\}_{\lambda\in\Lambda}$ is uniformly minimal in $\mathcal{F}_\phi(\mathbb{C}^n)$.

It is easy to see that every interpolation set is a weak interpolation set. It was shown by Seip and Schuster [14] that in the one-dimensional case the converse is also true, i.e., these two classes of sets coincide. As far as we know, it is not known whether these classes of sets coincide in the higher dimensional case.

Another important result of Seip and Wallstén [16], [18] shows that in the classical one-dimensional case interpolation sets can be completely characterized in terms of the upper Beurling density $D^+(\Lambda)$ of Λ defined by,

$$D^+(\Lambda) := \lim_{r\to\infty} \sup \sup_{a\in\mathbb{C}} \frac{\#\{\Lambda \cap B(a,r)\}}{m(B(a,r))},$$

where $B(a,r)$ denotes an Euclidean ball centered at a with radius r. Namely, Λ is an interpolation set (or equivalently weak interpolation set) for $\mathcal{F}(\mathbb{C})$ if and only if Λ is uniformly discrete, and $D^+(\Lambda) < 1$.

It was proved by Berndtsson, Ortega-Cerdá and Seip in [1], [13] that in the weighted one-dimensional case, just as in the classical case, interpolation sets can be completely characterized by an appropriately defined weighted upper Beurling density

$$D^+_\phi(\Lambda) := \lim_{r\to\infty} \sup \sup_{a\in\mathbb{C}} \frac{\#\{\Lambda \cap B(a,r)\}}{m_\phi(B(a,r))},$$

where $dm_\phi(z) = \|K^\phi_z\|^2_\phi e^{-2\phi(z)} dm(z)$. These results have been extended in dimension one for an even more general class of weights [8], and the necessity of this density condition was also proved in higher dimensions [7], [3]. The corresponding necessary density condition for weak interpolation sets in the weighted case was proved very recently in [10].

Analogous density results for interpolation and weak interpolation sets have been proved in the Paley-Wiener space [5], [4], [11], in the Bergman space [17], and in de Branges space [9].

The goal of this paper is to provide a similar type necessary density condition for an even larger class of "interpolation sets". This type of sets (to be defined momentarily) were relatively recently introduced by Olevskii and Ulanovskii [12] in the Paley-Wiener setting, where it was shown that all such sets must satisfy a Beurling density condition similar to the one for usual interpolation sets. Our result can be viewed as a Bargmann-Fock space counterpart of their result.

We now precisely define the above mentioned larger classes of "interpolation sets".

Definition 1.1. *For a given $0 \le d < 1$ we will say that a countable set $\Lambda = \{\lambda\} \subseteq \mathbb{C}^n$ is d-approximate interpolation set for $\mathcal{F}_\phi(\mathbb{C}^n)$ if the following two conditions hold:*

(i) *For every $\lambda \in \Lambda$ there exists $h_\lambda \in \mathcal{F}_\phi(\mathbb{C}^n)$ such that*

$$\sum_{\nu\in\Lambda} \left| \left\langle h_\lambda, k^\phi_\nu \right\rangle_\phi - \delta_{\lambda\nu} \right|^2 \le d^2,$$

i.e., the $l^2(\Lambda)$ distance between the sequences $(\left\langle h_\lambda, k^\phi_\nu \right\rangle_\phi)$ and $(\delta_{\lambda\nu})$ is no greater than d.

(ii) $\{h_\lambda\}_{\lambda\in\Lambda}$ *from (i) is a Bessel sequence in* $\mathcal{F}_\phi(\mathbb{C}^n)$*, i.e., there exists a constant* $C > 0$ *such that*

$$\sum_{\lambda\in\Lambda} |\langle f, h_\lambda\rangle_\phi|^2 \le C\,\|f\|_\phi^2,$$

for any $f \in \mathcal{F}_\phi(\mathbb{C}^n)$.

Note that 0-approximate interpolation sets coincide with interpolation sets in $\mathcal{F}_\phi(\mathbb{C}^n)$.

Definition 1.2. *For a given* $0 \le d < 1$ *we will say that a countable set* $\Lambda = \{\lambda\} \subseteq \mathbb{C}^n$ *is d-approximate weak interpolation set for* $\mathcal{F}_\phi(\mathbb{C}^n)$ *if the following two conditions hold:*

(i) *For every* $\lambda \in \Lambda$ *there exists* $f_\lambda \in \mathcal{F}_\phi(\mathbb{C}^n)$ *such that*

$$\sum_{\nu\in\Lambda} \left| \left\langle f_\lambda, k_\nu^\phi \right\rangle_\phi - \delta_{\lambda\nu} \right|^2 \le d^2,$$

i.e., the $l^2(\Lambda)$ *distance between the sequences* $(\left\langle f_\lambda, k_\nu^\phi \right\rangle_\phi)$ *and* $(\delta_{\lambda\nu})$ *is no greater than d.*

(ii) $\sup_{\lambda\in\Lambda} \|f_\lambda\|_\phi < \infty$.

Again, 0-approximate weak interpolation sets coincide with weak interpolation sets in $\mathcal{F}_\phi(\mathbb{C}^n)$. Note that the two classes of approximately interpolating sets differ only in the second condition. As the terminology suggests every d-approximate interpolation set is a d-approximate weak interpolation set. We don't know if the converse is true even in the classical one-dimensional setting.

Our first result is the following, which gives a necessary upper density condition on d-approximate weak interpolation sets in the classical Bargmann-Fock space $\mathcal{F}(\mathbb{C}^n)$.

Theorem 1.3. *Let* $0 \le d < 1$*. Suppose* $\Lambda \subseteq \mathbb{C}^n$ *is a uniformly discrete set that is a d-approximate weak interpolation set for* $\mathcal{F}(\mathbb{C}^n)$*. Then*

$$D^+(\Lambda) \le \frac{1}{1-d^2}.$$

This result can be easily extended to all classical weights of the form $\phi(z) = \frac{\alpha}{2}|z|^2 + \frac{n}{2}\log\frac{\pi}{\alpha}, \alpha > 0$. In the general weighted case, we can only prove the corresponding result for d-approximate interpolation sets.

Theorem 1.4. *Let* $0 \le d < 1$*. Suppose* $\Lambda \subseteq \mathbb{C}^n$ *is a uniformly discrete set that is a d-approximate interpolation set for* $\mathcal{F}_\phi(\mathbb{C}^n)$*. Then*

$$D_\phi^+(\Lambda) \le \frac{1}{1-d^2}.$$

2. Preliminaries

In this section, we collect some preliminary results that will play an important role in the proofs of our main results.

2.1. Reproducing kernels. In the classical case, due to the explicit formulas for the reproducing kernel, it is easy to see that $\|K_z\| = e^{\frac{\pi}{2}|z|^2}$ for all $z \in \mathbb{C}^n$, and $|\langle k_z, k_w\rangle| = e^{-\frac{\pi}{2}|z-w|^2}$ for all $z, w \in \mathbb{C}^n$, i.e., each normalized reproducing kernel $k_z \in \mathcal{F}(\mathbb{C}^n)$ is sharply concentrated around its indexing point $z \in \mathbb{C}^n$. In the weighted case, we still have the following similar estimates proved in [15] and [2] respectively:

$$\left\|K_z^\phi\right\|_\phi \simeq e^{\phi(z)}, \tag{2}$$

$$\left|\left\langle k_z^\phi, k_w^\phi\right\rangle_\phi\right| \leq Ce^{-c|z-w|}, \tag{3}$$

for every $z, w \in \mathbb{C}^n$, and some constants which only depend on the implied constants in (1). These estimates will be of crucial importance in all our proofs. As a simple consequence of (2) we have $dm_\phi \simeq dm$, and therefore $D^+(\Lambda) \simeq D^+_\phi(\Lambda)$. Also, since the Lebesgue measure m satisfies the annular decay property so does the measure m_ϕ, i.e., for any $\rho > 0$,

$$\limsup_{r\to\infty} \sup_{a\in\mathbb{C}^n} \frac{m_\phi(B(a, r+\rho))}{m_\phi(B(a,r))} = 1.$$

2.2. Uniform discreteness and its consequences. Recall that a set $\Lambda \subseteq \mathbb{C}^n$ is said to be *uniformly discrete* if $\delta := \inf\{|\lambda - \nu| : \lambda \neq \nu \in \Lambda\} > 0$. The constant $\delta > 0$ is called the separation constant of Λ. We will need the following two, well-known, simple properties of uniformly discrete sets. First, as simple counting argument shows, for any Euclidean ball $B(a,r) \subseteq \mathbb{C}^n$ with radius $r > 1$ we have $\#\{\Lambda \cap B(a,r)\} \leq (1+2/\delta)^{2n} r^{2n}$, where $\delta > 0$ is the separation constant of Λ. In particular, $\#\{\Lambda \cap B(a,r)\}$ is finite, and the upper Beurling density of Λ satisfies $D^+_\phi(\Lambda) \simeq D^+(\Lambda) \leq \frac{n!}{\pi^n}(1+\frac{2}{\delta})^{2n} < \infty$. Our main goal is to show that under additional interpolation assumptions on Λ this trivial density upper bound can be significantly improved (especially when $\delta > 0$ is very small).

The second consequence of the uniform discreteness of Λ, that will be used in our proofs, is that any uniformly discrete $\Lambda \subseteq \mathbb{C}^n$ generates a Bessel sequence of normalized reproducing kernels $\{k_\lambda^\phi\}_{\lambda\in\Lambda}$ in $\mathcal{F}_\phi(\mathbb{C}^n)$, i.e., there exists a constant $C_\delta > 0$ such that

$$\sum_{\lambda\in\Lambda} |\langle f, k_\lambda^\phi\rangle_\phi|^2 \leq C_\delta \|f\|_\phi^2,$$

for all $f \in \mathcal{F}_\phi(\mathbb{C}^n)$ (see [6], Proposition 3.2.5). This is a simple consequence of the mean value inequality.

2.3. Concentration operators. The following class of operators, usually called concentration operators (or sometimes Toeplitz operators), will play an important role in our proofs. For any Borel set $B \subseteq \mathbb{C}^n$ with finite Lebesgue measure, define a *concentration operator* $T_B : \mathcal{F}_\phi(\mathbb{C}^n) \to \mathcal{F}_\phi(\mathbb{C}^n)$ by

$$T_B f = \int_B \left\langle f, k_z^\phi\right\rangle_\phi k_z^\phi dm_\phi(z),$$

where the right-hand side is defined in the weak sense, i.e., as the unique element in $\mathcal{F}_\phi(\mathbb{C}^n)$ such that

$$\langle T_B f, g\rangle_\phi = \int_B \left\langle f, k_z^\phi\right\rangle_\phi \left\langle k_z^\phi, g\right\rangle_\phi dm_\phi(z),$$

for all $g \in \mathcal{F}_\phi(\mathbb{C}^n)$.

The following well-known result contains all the basic properties of concentration operators that we will need. For proof see Corollary 2.3.7 in [6].

Proposition 2.1. *For any Borel set $B \subseteq \mathbb{C}^n$ with finite Lebesgue measure, the corresponding concentration operator $T_B : \mathcal{F}_\phi(\mathbb{C}^n) \to \mathcal{F}_\phi(\mathbb{C}^n)$ is a positive compact self-adjoint operator of trace class. Moreover, its trace and Hilbert-Schmidt norm satisfy the following identities:*

$$\|T_B\|_{Tr} = Tr(T_B) = m_\phi(B) = \int_{\mathbb{C}^n}\int_B \left|\left\langle k_z^\phi, k_w^\phi\right\rangle_\phi\right|^2 dm_\phi(z)dm_\phi(w) \tag{4}$$

$$\|T_B\|_{HS}^2 = \int_B\int_B \left|\left\langle k_z^\phi, k_w^\phi\right\rangle_\phi\right|^2 dm_\phi(z)dm_\phi(w). \tag{5}$$

2.4. Two lemmas. To prove our Theorem 1.3, we will adopt the proof strategy of Olevskii and Ulanovskii [12]. The argument has two crucial ingredients. The first one (essentially going back to Landau [5]) says that any subspace of $\mathcal{F}_\phi(\mathbb{C}^n)$ consisting entirely of elements that are concentrated on a finite measure set cannot have arbitrarily large dimension. The precise formulation of this statement is as follows. Given a number $0 < c < 1$, we say that a subspace $\mathcal{G}$ of the weighted Bargmann-Fock space $\mathcal{F}_\phi(\mathbb{C}^n)$ is *c-concentrated* on the set $B \subseteq \mathbb{C}^n$, if

$$c\,\|f\|_\phi^2 \le \int_B \left|\left\langle f, k_z^\phi\right\rangle_\phi\right|^2 dm_\phi(z) = \langle T_B f, f\rangle_\phi\,,$$

for every $f \in \mathcal{G}$, where T_B is the concentration operator.

Lemma 2.2. *Suppose $B \subseteq \mathbb{C}^n$ is a Borel set in $\mathbb{C}^n$ with finite Lebesgue measure and $0 < c < 1$. If $\mathcal{G}$ is a subspace of the weighted Bargmann-Fock space $\mathcal{F}_\phi(\mathbb{C}^n)$ which is c-concentrated on B, then*

$$dim\mathcal{G} \le \frac{m_\phi(B)}{c}.$$

Proof. Let T_B be the concentration operator over B. By Proposition 2.1, T_B is a positive compact self-adjoint operator, so we can denote all its eigenvalues in the decreasing order by $l_1 \ge \cdots \ge l_k \ge \cdots$, where entries are repeated with multiplicity. Let $\mathcal{G}'$ be an arbitrary finite-dimensional subspace of $\mathcal{G}$ and $k = dim\mathcal{G}'$. By the min-max principle,

$$l_k = \max_{dim\mathcal{H}=k}\ \min_{f\in\mathcal{H},\|f\|_\phi=1} \langle T_B f, f\rangle_\phi \ge \min_{f\in\mathcal{G}',\|f\|_\phi=1} \langle T_B f, f\rangle_\phi \ge c.$$

Then using $Tr(T_B)/k \geq l_k \geq c$ and (4), we obtain

$$dim\mathcal{G}' \leq \frac{m_\phi(B)}{c}.$$

Since $\mathcal{G}'$ was an arbitrary finite-dimensional subspace of $\mathcal{G}$, we obtain that $\mathcal{G}$ is finite-dimensional and the same estimate holds for $\mathcal{G}$. □

The following lemma is the second important ingredient in our proofs. It allows us to generate a subspace of fairly high-dimension which is concentrated so that we can apply Lemma 2.2. This second result is a finite-dimensional result which can be applied in our proofs only when we restrict the set Λ to a ball.

Lemma 2.3 ([12], Lemma 2)**.** *Let $\{\mathbf{u}_j\}_{1\leq j\leq N}$ be an orthonormal basis in an N-dimensional complex Euclidean space $\mathcal{U}$. Given $0 < d < 1$, suppose that $\{\mathbf{v}_j\}_{1\leq j\leq N}$ is a set of vectors in $\mathcal{U}$ satisfying*

$$\|\mathbf{v}_j - \mathbf{u}_j\|^2 \leq d^2,\ 1 \leq j \leq N.$$

Then for any b with $1 < b < 1/d$, there is a subspace X of $\mathbb{C}^N$, such that

(i) $(1 - b^2d^2)N - 1 < dimX$;

(ii) *the estimate*

$$(1 - \frac{1}{b})^2 \sum_{j=1}^{N} |c_j|^2 \leq \|\sum_{j=1}^{N} c_j\mathbf{v}_j\|^2,$$

holds for any vector $\mathbf{c} = (c_1, c_2, ..., c_N) \in X$.

3. Proof of Theorem 1.3

We now prove Theorem 1.3. Throughout this proof, we will use the following notation. For $\alpha > 0$ we will denote by $\mathcal{F}_\alpha(\mathbb{C}^n)$ the weighted Bargmann-Fock space associated with $\phi(z) = \frac{\alpha}{2}|z|^2 + \frac{n}{2}\log\frac{\pi}{\alpha}$. We will also denote by $\|\cdot\|_\alpha$, K_z^α, and k_z^α the norm, the reproducing kernel, and the normalized reproducing kernel (at z) of this space. Then $\|K_z^\alpha\|_\alpha = e^{\frac{\alpha}{2}|z|^2}$. Finally, we denote by m_α the measure corresponding to m_ϕ for this particular choice of ϕ, i.e., $dm_\alpha = \frac{\alpha^n}{\pi^n}dm$. Recall that in the classical case $\alpha = \pi$, by convention, we drop the sub(super)scripts in the above notation.

Notice if we could prove Theorem 1.3 for $0 < d < 1$, trivially, the theorem will hold for $d = 0$.

Proof. By $\Lambda = \{\lambda\} \subseteq \mathbb{C}^n$ is a d-approximate weak interpolation set for $\mathcal{F}(\mathbb{C}^n)$, there exists a bounded sequence $\{f_\lambda\}_{\lambda\in\Lambda}$ such that

$$\sum_{\nu\in\Lambda} \left|\langle f_\lambda, k_\nu\rangle - \delta_{\lambda\nu}\right|^2 \leq d^2, \quad \forall\lambda \in \Lambda.$$

Let $\varepsilon > 0$. For any $\lambda \in \Lambda$ define $g_\lambda(z) := f_\lambda(z) k_\lambda^\varepsilon(z)$. Clearly, $g_\lambda : \mathbb{C}^n \to \mathbb{C}$ is entire as a product of two entire functions. Also, since $|k_\lambda^\varepsilon(z)|^2 \le e^{\varepsilon|z|^2}$ for all $z, \lambda \in \mathbb{C}^n$, we have

$$\int_{\mathbb{C}^n} |g_\lambda(z)|^2 e^{-(\pi+\varepsilon)|z|^2} dm(z) \le \int_{\mathbb{C}^n} |f_\lambda(z)|^2 e^{-\pi|z|^2} dm(z) < \infty.$$

Therefore, $g_\lambda \in \mathcal{F}_{\pi+\varepsilon}(\mathbb{C}^n)$ for all $\lambda \in \Lambda$. Moreover,

$$\begin{aligned}
\sum_{\nu\in\Lambda} \left| \left\langle g_\lambda, k_\nu^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon} - \delta_{\lambda\nu} \right|^2 &= \sum_{\nu\in\Lambda} \left| \left\langle g_\lambda, K_\nu^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon} \left\| K_\nu^{\pi+\varepsilon} \right\|_{\pi+\varepsilon}^{-1} - \delta_{\lambda\nu} \right|^2 \\
&= \sum_{\nu\in\Lambda} \left| \langle f_\lambda, K_\nu \rangle \left\langle k_\lambda^\varepsilon, K_\nu^\varepsilon \right\rangle_\varepsilon \left\| K_\nu^{\pi+\varepsilon} \right\|_{\pi+\varepsilon}^{-1} - \delta_{\lambda\nu} \right|^2 \\
&= \sum_{\nu\in\Lambda} \left| \langle f_\lambda, k_\nu \rangle \left\langle k_\lambda^\varepsilon, k_\nu^\varepsilon \right\rangle_\varepsilon - \delta_{\lambda\nu} \right|^2 \\
&\le \sum_{\nu\in\Lambda} \left| \langle f_\lambda, k_\nu \rangle - \delta_{\lambda\nu} \right|^2 \le d^2,
\end{aligned}$$

for any $\lambda \in \Lambda$. Note that in this simple computation we used that $\left\| K_\nu^{\pi+\varepsilon} \right\|_{\pi+\varepsilon} = \|K_\nu\| \, \|K_\nu^\varepsilon\|_\varepsilon$. The analog of this identity doesn't hold for more general weights which forces us to use a somewhat different strategy in the proof of Theorem 1.4.

Let $B(a, r)$ be an arbitrary open ball in $\mathbb{C}^n$. Since Λ is uniformly discrete, $\Lambda \cap B(a, r)$ is finite set. Suppose $\Lambda \cap B(a, r)$ is not an empty set and let $\Lambda \cap B(a, r) = \{\lambda_1, \lambda_2, ..., \lambda_N\}$. Consider the following vectors in $\mathbb{C}^N$

$$\mathbf{v}_j := (\left\langle g_{\lambda_j}, k_{\lambda_1}^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon}, ..., \left\langle g_{\lambda_j}, k_{\lambda_N}^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon}), 1 \le j \le N,$$

and the standard basis $\mathbf{u}_j := (\delta_{\lambda_j\lambda_1}, ..., \delta_{\lambda_j\lambda_N})$, $1 \le j \le N$.

By the above inequality we have

$$\|\mathbf{v}_j - \mathbf{u}_j\|^2 \le \sum_{\nu\in\Lambda} \left| \left\langle g_{\lambda_j}, k_\nu^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon} - \delta_{\lambda_j\nu} \right|^2 \le d^2, \ 1 \le j \le N.$$

By Lemma 2.3, for any $1 < b < 1/d$, there exists a subspace X of $\mathbb{C}^N$, such that

(i)
$$(1 - b^2 d^2)N - 1 < dim X; \tag{6}$$

(ii) the inequality
$$(1 - \frac{1}{b})^2 \sum_{j=1}^N |c_j|^2 \le \| \sum_{j=1}^N c_j \mathbf{v}_j \|^2, \tag{7}$$

holds for any vector $\mathbf{c} = (c_1, ..., c_N) \in X$.

Let $\mathcal{G} := \{\sum_{j=1}^N c_j g_{\lambda_j} | \mathbf{c} = (c_1, ..., c_N) \in X\} \subseteq \mathcal{F}_{\pi+\varepsilon}$. Since

$$\| \sum_{j=1}^N c_j \mathbf{v}_j \|^2 \ge (1 - \frac{1}{b})^2 \sum_{j=1}^N |c_j|^2,$$

holds for any vector $\mathbf{c} = (c_1, ..., c_N) \in X$, we have

$$dim X \leq dim \mathcal{G}. \tag{8}$$

Let $g = \sum_{j=1}^{N} c_j g_{\lambda_j} \in \mathcal{G}$ for some $(c_1, ..., c_N) \in X$. Using that $\{k_\lambda^{\pi+\varepsilon}\}_{\lambda\in\Lambda}$ is a Bessel sequence (due to the uniform discreteness of Λ) and (7), we have

$$\begin{aligned} \|g\|_{\pi+\varepsilon}^2 \geq C \sum_{\lambda\in\Lambda} \left| \left\langle g, k_\lambda^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon} \right|^2 \geq C \sum_{i=1}^{N} \left| \left\langle \sum_{j=1}^{N} c_j g_{\lambda_j}, k_{\lambda_i}^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon} \right|^2 \\ = C \| \sum_{j=1}^{N} c_j \mathbf{v}_j \|^2 \geq C(1-1/b)^2 \sum_{j=1}^{N} |c_j|^2 = C_1 \sum_{j=1}^{N} |c_j|^2, \end{aligned} \tag{9}$$

for any $g \in \mathcal{G}$, where C only depends on ε and the separation constant δ of Λ. Therefore, $C_1 := C(1-1/b)^2$ is independent of a and r as well (this will be used below).

Fix a small $\sigma > 0$. A simple application of the Cauchy-Schwarz inequality and the already mentioned identity $\left\|K_z^{\pi+\varepsilon}\right\|_{\pi+\varepsilon} = \|K_z\| \, \|K_z^\varepsilon\|_\varepsilon$ yields

$$\begin{aligned} &\int_{B(a,r+\sigma r)^c} \left|g(z)\right|^2 e^{-(\pi+\varepsilon)|z|^2 - n\log(\frac{\pi}{\pi+\varepsilon})} dm(z) \\ =&\int_{B(a,r+\sigma r)^c} \left|g(z)\right|^2 e^{-(\pi+\varepsilon)|z|^2} dm_{\pi+\varepsilon}(z) \\ =&\int_{B(a,r+\sigma r)^c} \left| \sum_{j=1}^{N} c_j f_{\lambda_j}(z) k_{\lambda_j}^\varepsilon(z) \right|^2 e^{-(\pi+\varepsilon)|z|^2} dm_{\pi+\varepsilon}(z) \\ \leq& \sum_{j=1}^{N} \left|c_j\right|^2 \int_{B(a,r+\sigma r)^c} \sum_{j=1}^{N} \left| \left\langle f_{\lambda_j}, k_z \right\rangle \left\langle k_{\lambda_j}^\varepsilon, k_z^\varepsilon \right\rangle_\varepsilon \right|^2 dm_{\pi+\varepsilon}(z). \end{aligned}$$

We now estimate the integral term from the line above. Applying $\sup_{\lambda\in\Lambda} \|f_\lambda\| < \infty$ and doing a simple change of variables, we obtain

$$\begin{aligned} &\int_{B(a,r+\sigma r)^c} \sum_{j=1}^{N} \left| \left\langle f_{\lambda_j}, k_z \right\rangle \left\langle k_{\lambda_j}^\varepsilon, k_z^\varepsilon \right\rangle_\varepsilon \right|^2 dm_{\pi+\varepsilon}(z) \\ \leq& C \sum_{j=1}^{N} \int_{B(\lambda_j,\sigma r)^c} \left| \left\langle k_{\lambda_j}^\varepsilon, k_z^\varepsilon \right\rangle_\varepsilon \right|^2 dm_{\pi+\varepsilon}(z) \\ =& CN \int_{B(\mathbf{0},\sigma r)^c} \left| \left\langle k_{\mathbf{0}}^\varepsilon, k_z^\varepsilon \right\rangle_\varepsilon \right|^2 dm_{\pi+\varepsilon}(z). \end{aligned}$$

Again, using the uniform discreteness of Λ in the form $N = \#\{\Lambda \cap B(a,r)\} \leq (1+2/\delta)^{2n} r^{2n}$, we obtain that the last expression is bounded by

$$C(1+\frac{2}{\delta})^{2n}r^{2n}\int_{B(\mathbf{0},\sigma r)^c} e^{-\varepsilon|z|^2}dm_{\pi+\varepsilon}(z)=C'r^{2n}\int_{\sigma r}^{\infty} e^{-\varepsilon t^2}t^{2n-1}dt,$$

where C' depends only on n and δ, and not on a and r. Denote the last expression by $C_2 = C_2(r)$. Observe that $C_2 \xrightarrow{\text{unif.}} 0$ for any $a \in \mathbb{C}^n$ as $r \to \infty$(to be used in a moment). Using the last derivations, we obtain

$$\int_{B(a,r+\sigma r)^c} \left|g(z)\right|^2 e^{-(\pi+\varepsilon)|z|^2-n\log(\frac{\pi}{\pi+\varepsilon})}dm(z) \le C_2\sum_{j=1}^{N}\left|c_j\right|^2, \tag{10}$$

for every $g \in \mathcal{G}$.

Let $T_{B(a,r+\sigma r)}$ be the concentration operator over the ball $B(a, r+\sigma r)$. Combining (9) and (10), we obtain

$$(1-\frac{C_2}{C_1})\left\|g\right\|^2_{\pi+\varepsilon} \le \int_{B(a,r+\sigma r)} \left|g(z)\right|^2 e^{-(\pi+\varepsilon)|z|^2-n\log(\frac{\pi}{\pi+\varepsilon})}dm(z)$$
$$=\int_{B(a,r+\sigma r)} \left|\left\langle g, k_z^{\pi+\varepsilon}\right\rangle_{\pi+\varepsilon}\right|^2 dm_{\pi+\varepsilon}(z)=\left\langle T_{B(a,r+\sigma r)}g, g\right\rangle_{\pi+\varepsilon},$$

for every $g \in \mathcal{G}$.

Let $0 < \eta < 1$. Since C_1 doesn't depend on a and r, and $C_2 \xrightarrow{\text{unif.}} 0$ for any $a \in \mathbb{C}^n$ as $r \to \infty$. We clearly have $C_2/C_1 \xrightarrow{\text{unif.}} 0$ for any $a \in \mathbb{C}^n$ as $r \to \infty$. So there exists $R > 0$ such that $(1-\eta)\left\|g\right\|^2_{\pi+\varepsilon} \le \left\langle T_{B(a,r+\sigma r)}g, g\right\rangle_{\pi+\varepsilon}$ for every $g \in \mathcal{G}$ whenever $r > R$. In other words, the subspace $\mathcal{G}$ is $(1-\eta)$-concentrated on $B(a, r+\sigma r)$ whenever $r > R$.

By Lemma 2.2, we obtain $dim\mathcal{G} \le m_{\pi+\varepsilon}(B(a, r+\sigma r))/(1-\eta)$. Combining (6) and (8), we obtain

$$(1-b^2d^2)\#\{\Lambda\cap B(a,r)\}-1 < \frac{m_{\pi+\varepsilon}(B(a,r+\sigma r))}{1-\eta},$$

for any $a \in \mathbb{C}^n$ when $r > R$.

Notice the above inequality still holds when $\Lambda \cap B(a, r)$ is an empty set. Therefore,

$$\begin{aligned} D^+(\Lambda) &= \limsup_{r\to\infty}\sup_{a\in\mathbb{C}^n}\frac{\#\{\Lambda\cap B(a,r)\}}{m(B(a,r))} \\ &\le \limsup_{r\to\infty}\sup_{a\in\mathbb{C}^n}\frac{m_{\pi+\varepsilon}(B(a,r+\sigma r))}{(1-\eta)(1-b^2d^2)m(B(a,r))}+\frac{1}{(1-b^2d^2)m(B(a,r))} \\ &= \frac{(\pi+\varepsilon)^n(1+\sigma)^{2n}}{\pi^n(1-\eta)(1-b^2d^2)}. \end{aligned}$$

Thus, since $\varepsilon > 0, \sigma > 0, \eta > 0, b > 1$ are arbitrary,

$$D^+(\Lambda) \le \frac{1}{1-d^2}.$$

□

4. Proof of Theorem 1.4

Let $T_{B(a,r)} : \mathcal{F}_\phi(\mathbb{C}^n) \to \mathcal{F}_\phi(\mathbb{C}^n)$ be the concentration operator over the ball $B(a,r)$ defined, as above, by

$$T_{B(a,r)}f = \int_{B(a,r)} \left\langle f, k_z^\phi \right\rangle_\phi k_z^\phi dm_\phi(z).$$

Again, we will denote all its eigenvalues in the decreasing order by

$$1 \geq l_1(T_{B(a,r)}) \geq \cdots \geq l_i(T_{B(a,r)}) \geq \cdots,$$

where entries are repeated with multiplicity.

Lemma 4.1. *For any $\epsilon > 0$, there exists $R > 0$ such that*

$$(1-\epsilon)\sum_{i=1}^{\infty} l_i(T_{B(a,r)}) \leq \sum_{i=1}^{\infty} l_i(T_{B(a,r)})^2,$$

for any $a \in \mathbb{C}^n$ and all $r > R$.

Proof. Since $\sum_{i=1}^\infty l_i(T_{B(a,r)}) = Tr(T_{B(a,r)}) = m_\phi(B(a,r))$, it's sufficient to show

$$\limsup_{r\to\infty} \sup_{a\in\mathbb{C}^n} \frac{1}{m_\phi(B(a,r))}\left(\sum_{i=1}^{\infty} l_i(T_{B(a,r)}) - \sum_{i=1}^{\infty} l_i(T_{B(a,r)})^2\right) = 0.$$

By Proposition 2.1 we have

$$\sum_{i=1}^{\infty} l_i(T_{B(a,r)}) - \sum_{i=1}^{\infty} l_i(T_{B(a,r)})^2 = \int_{B(a,r)}\int_{B(a,r)^c} \left|\left\langle k_z^\phi, k_w^\phi\right\rangle_\phi\right|^2 dm_\phi(w)dm_\phi(z).$$

Then by (3)

$$\sum_{i=1}^{\infty} l_i(T_{B(a,r)}) - \sum_{i=1}^{\infty} l_i(T_{B(a,r)})^2 \lesssim \int_{B(a,r)}\int_{B(a,r)^c} e^{-2c|z-w|} dm_\phi(w)dm_\phi(z).$$

Let $\rho > 0$. We break the double integral above as follows

$$\int_{B(a,r)}\int_{B(a,r+\rho)^c} + \int_{B(a,r)}\int_{B(a,r+\rho)\setminus B(a,r)},$$

and proceed to estimate each term separately. In both estimates, we will divide by $m_\phi(B(a,r))$ and use $m_\phi(B(a,r)) \simeq m(B(a,r))$ with the implied constants independent of $a \in \mathbb{C}^n$ and $r > 0$.

For the first term we have

$$\begin{aligned}
&\frac{1}{m_\phi(B(a,r))}\int_{B(a,r)}\int_{B(a,r+\rho)^c} e^{-2c|z-w|}dm_\phi(w)dm_\phi(z)\\
\lesssim&\frac{1}{m_\phi(B(a,r))}\int_{B(a,r)}\int_{B(z,\rho)^c} e^{-2c|z-w|}dm(w)dm(z)\\
=&\frac{1}{m_\phi(B(a,r))}\int_{B(a,r)}\int_{B(\mathbf{0},\rho)^c} e^{-2c|w|}dm(w)dm(z)\\
=&\frac{m(B(a,r))}{m_\phi(B(a,r))}\int_\rho^\infty e^{-2ct}\int_{\partial B(\mathbf{0},t)} dSdt \simeq \int_\rho^\infty e^{-2ct}t^{2n-1}dt.
\end{aligned}$$

So for any $\varepsilon > 0$, we can find a positive ρ such that

$$\limsup_{r\to\infty}\sup_{a\in\mathbb{C}^n}\frac{1}{m_\phi(B(a,r))}\int_{B(a,r)}\int_{B(a,r+\rho)^c} e^{-2c|z-w|}dm_\phi(w)dm_\phi(z) < \varepsilon. \tag{11}$$

We now estimate the second term, using the positive $\rho > 0$ from above.

$$\begin{aligned}
&\frac{1}{m_\phi(B(a,r))}\int_{B(a,r)}\int_{B(a,r+\rho)\setminus B(a,r)} e^{-2c|z-w|}dm_\phi(w)dm_\phi(z)\\
\lesssim&\frac{1}{m_\phi(B(a,r))}\int_{B(a,r+\rho)\setminus B(a,r)}\int_{\mathbb{C}^n} e^{-2c|z-w|}dm(z)dm(w)\\
=&\frac{1}{m_\phi(B(a,r))}\int_{B(a,r+\rho)\setminus B(a,r)}\int_{\mathbb{C}^n} e^{-2c|z|}dm(z)dm(w)\\
\simeq&\frac{m(B(a,r+\rho)\setminus B(a,r))}{m(B(a,r))}\int_0^\infty e^{-2ct}t^{2n-1}dt\\
\simeq&\frac{m(B(a,r+\rho)\setminus B(a,r))}{m(B(a,r))}\xrightarrow{\text{unif.}} 0,
\end{aligned}\tag{12}$$

for any $a \in \mathbb{C}^n$ as $r \to \infty$ (by the annular decay property of m).

Combining (11) and (12), we obtain

$$\limsup_{r\to\infty}\sup_{a\in\mathbb{C}^n}\frac{1}{m_\phi(B(a,r))}\left(\sum_{i=1}^\infty l_i(T_{B(a,r)}) - \sum_{i=1}^\infty l_i(T_{B(a,r)})^2\right) \lesssim \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get the desired equality. □

We now proceed with the proof of Theorem 1.4. Again, we only need to prove the theorem for $0 < d < 1$.

Proof of Theorem 1.4. The beginning of the proof is very similar to the proof of Theorem 1.3. Since $\Lambda = \{\lambda\} \subseteq \mathbb{C}^n$ is a d-approximate interpolation set for $\mathcal{F}_\phi(\mathbb{C}^n)$, there exists a Bessel sequence $\{h_\lambda\}_{\lambda\in\Lambda}$ such that

$$\sum_{\nu\in\Lambda}\left|\left\langle h_\lambda, k_\nu^\phi\right\rangle_\phi - \delta_{\lambda\nu}\right|^2 \le d^2, \quad \forall\lambda\in\Lambda.$$

Let $B(a,r)$ be an arbitrary open ball in $\mathbb{C}^n$. By Λ is uniformly discrete, $\Lambda \cap B(a,r)$ is finite set. Suppose $\Lambda \cap B(a,r)$ is not an empty set and let $\Lambda \cap B(a,r) = \{\lambda_1, \lambda_2, ..., \lambda_N\}$. Consider the following vectors in $\mathbb{C}^N$,

$$\mathbf{v}_j := (\left\langle h_{\lambda_j}, k^{\phi}_{\lambda_1} \right\rangle_{\phi}, \cdots, \left\langle h_{\lambda_j}, k^{\phi}_{\lambda_N} \right\rangle_{\phi}), 1 \leq j \leq N,$$

and the standard basis of $\mathbb{C}^N$, $\mathbf{u}_j := (\delta_{\lambda_j \lambda_1}, \cdots, \delta_{\lambda_j \lambda_N})$, $1 \leq j \leq N$.

Notice that

$$\|\mathbf{v}_j - \mathbf{u}_j\|^2 \leq \sum_{\nu \in \Lambda} \left| \left\langle h_{\lambda_j}, k^{\phi}_{\nu} \right\rangle_{\phi} - \delta_{\lambda_j \nu} \right|^2 \leq d^2, \ 1 \leq j \leq N.$$

By Lemma 2.3, for any $1 < b < 1/d$, there exists a subspace X of $\mathbb{C}^N$, such that

(i) $(1 - b^2 d^2)N - 1 < dimX$;

(ii) the inequality

$$(1 - \frac{1}{b})^2 \sum_{j=1}^{N} |c_j|^2 \leq \| \sum_{j=1}^{N} c_j \mathbf{v}_j \|^2, \tag{13}$$

holds for any vector $\mathbf{c} = (c_1, ..., c_N) \in X$.

Let $\mathcal{G} := \{\sum_{j=1}^{N} c_j h_{\lambda_j} | \mathbf{c} = (c_1, ..., c_N) \in X\}$. As in Theorem 1.3 we have

$$dim\mathcal{G} \geq dimX > (1 - b^2 d^2)N - 1. \tag{14}$$

Let $g = \sum_{j=1}^{N} c_j h_{\lambda_j} \in \mathcal{G}$ for some $(c_1, ..., c_N) \in X$. Using that $\{h_\lambda\}_{\lambda \in \Lambda}$ is a Bessel sequence and (13), we get

$$\begin{aligned}\sum_{i=1}^{N} \left| \left\langle g, k^{\phi}_{\lambda_i} \right\rangle_{\phi} \right|^2 &= \sum_{i=1}^{N} \left| \left\langle \sum_{j=1}^{N} c_j h_{\lambda_j}, k^{\phi}_{\lambda_i} \right\rangle_{\phi} \right|^2 = \| \sum_{j=1}^{N} c_j \mathbf{v}_j \|^2 \\ &\geq (1 - \frac{1}{b})^2 \sum_{j=1}^{N} |c_j|^2 \geq (1 - \frac{1}{b})^2 C \| \sum_{j=1}^{N} c_j h_{\lambda_j} \|^2_{\phi} = C_1 \|g\|^2_{\phi},\end{aligned} \tag{15}$$

for any $g \in \mathcal{G}$, where $C_1 := (1 - 1/b)^2 C$ is independent of a and r.

Let δ be the separation constant of Λ. Then $B(\lambda, \delta/2) \cap B(\nu, \delta/2) = \emptyset$ for any $\lambda \neq \nu \in \Lambda$. It follows from the mean value inequality that

$$\begin{aligned}\sum_{i=1}^{N} \left| \left\langle g, k^{\phi}_{\lambda_i} \right\rangle_{\phi} \right|^2 &\leq \sum_{i=1}^{N} C_\delta \int_{B(\lambda_i, \frac{\delta}{2})} \left| \left\langle g, k^{\phi}_{z} \right\rangle_{\phi} \right|^2 dm_\phi(z) \\ &\leq C_\delta \int_{B(a, r + \frac{\delta}{2})} \left| \left\langle g, k^{\phi}_{z} \right\rangle_{\phi} \right|^2 dm_\phi(z),\end{aligned} \tag{16}$$

for any $g \in \mathcal{G}$, where C_δ is independent of a and r.

Let $T_{B(a,r+\delta/2)}$ be the concentration operator over the ball $B(a,r+\delta/2)$. Combining (15) and (16), we obtain

$$c\,\|g\|_\phi^2 \leq \int_{B(a,r+\frac{\delta}{2})} \left|\left\langle g, k_z^\phi\right\rangle_\phi\right|^2 dm_\phi(z) = \left\langle T_{B(a,r+\frac{\delta}{2})} g, g\right\rangle_\phi, \tag{17}$$

for any $g \in \mathcal{G}$, where $0 < c := C_1/C_\delta < 1$ is independent of a and r.

Denote all the eigenvalues of $T_{B(a,r+\delta/2)}$ by $\{l_i(T_{B(a,r+\delta/2)})\}_{i=1}^\infty$ indexed in decreasing order. Let $k = dim\mathcal{G}$ ($k < \infty$, see Lemma 2.2). By the min-max principle and (17),

$$\begin{aligned} l_k(T_{B(a,r+\frac{\delta}{2})}) &= \max_{dim\mathcal{H}=k}\ \min_{g\in\mathcal{H},\|g\|_\phi=1} \left\langle T_{B(a,r+\frac{\delta}{2})} g, g\right\rangle_\phi \\ &\geq \min_{g\in\mathcal{G},\|g\|_\phi=1} \left\langle T_{B(a,r+\frac{\delta}{2})} g, g\right\rangle_\phi \geq c. \end{aligned}$$

For any $0 < \varepsilon < 1 - c$, we have

$$\begin{aligned} dim\mathcal{G} = k &\leq \#\{i : l_i(T_{B(a,r+\frac{\delta}{2})} \geq c\} \\ &= \#\{i : l_i(T_{B(a,r+\frac{\delta}{2})}) > 1-\varepsilon\} + \#\{i : c \leq l_i(T_{B(a,r+\frac{\delta}{2})}) \leq 1-\varepsilon\} \\ &\leq \sum_{l_i>1-\varepsilon} \frac{l_i(T_{B(a,r+\frac{\delta}{2})})}{1-\varepsilon} + \sum_{c\leq l_i\leq 1-\varepsilon} \frac{l_i(T_{B(a,r+\frac{\delta}{2})})}{c} \\ &\leq \frac{1}{1-\varepsilon}\sum_{i=1}^\infty l_i(T_{B(a,r+\frac{\delta}{2})}) + \frac{1}{c}\sum_{l_i\leq 1-\varepsilon} l_i(T_{B(a,r+\frac{\delta}{2})}). \end{aligned} \tag{18}$$

Let $\sigma = c\varepsilon^2$. By Lemma 4.1, there exists $R > 0$ such that for any $a \in \mathbb{C}^n$ and all $r > R$, we have

$$\begin{aligned} (1-\sigma)\sum_{i=1}^\infty l_i(T_{B(a,r)}) &\leq \sum_{i=1}^\infty l_i(T_{B(a,r)})^2 \\ = \sum_{l_i\leq 1-\varepsilon} l_i(T_{B(a,r)})^2 &+ \sum_{l_i>1-\varepsilon} l_i(T_{B(a,r)})^2 \\ \leq (1-\varepsilon)\sum_{l_i\leq 1-\varepsilon} l_i(T_{B(a,r)}) &+ \sum_{l_i>1-\varepsilon} l_i(T_{B(a,r)}) \\ = \sum_{i=1}^\infty l_i(T_{B(a,r)}) &- \varepsilon\sum_{l_i\leq 1-\varepsilon} l_i(T_{B(a,r)}). \end{aligned}$$

It follows that for any $a \in \mathbb{C}^n$ and all $r > R$,

$$\frac{1}{c}\sum_{l_i\leq 1-\varepsilon} l_i(T_{B(a,r)}) \leq \varepsilon\sum_{i=1}^\infty l_i(T_{B(a,r)}). \tag{19}$$

Combining (18) and (19), we obtain that for any $a \in \mathbb{C}^n$ and all $r > R$, we have

$$dim\mathcal{G} \leq \frac{1+\varepsilon-\varepsilon^2}{1-\varepsilon}\sum_{i=1}^{\infty} l_i(T_{B(a,r+\frac{\delta}{2})}).$$

Therefore, by (14) and Proposition 2.1, for any $a \in \mathbb{C}^n$ and all $r > R$, we have

$$(1-b^2d^2)\#\{\Lambda \cap B(a,r)\} - 1 < \frac{1+\varepsilon-\varepsilon^2}{1-\varepsilon} m_\phi(B(a, r+\frac{\delta}{2})).$$

Notice the last inequality still holds when $\Lambda \cap B(a,r)$ is an empty set. Finally, using the annular decay property of m_ϕ we obtain

$$\begin{aligned} D^+_\phi(\Lambda) &= \limsup_{r\to\infty} \sup_{a\in\mathbb{C}^n} \frac{\#\{\Lambda \cap B(a,r)\}}{m_\phi(B(a,r))} \\ &\leq \limsup_{r\to\infty} \sup_{a\in\mathbb{C}^n} \frac{(1+\varepsilon-\varepsilon^2)m_\phi(B(a,r+\frac{\delta}{2}))}{(1-\varepsilon)(1-b^2d^2)m_\phi(B(a,r))} + \frac{1}{(1-b^2d^2)m_\phi(B(a,r))} \\ &= \frac{(1+\varepsilon-\varepsilon^2)}{(1-\varepsilon)(1-b^2d^2)}. \end{aligned}$$

Thus, since $\varepsilon > 0, b > 1$ are arbitrary,

$$D^+_\phi(\Lambda) \leq \frac{1}{1-d^2}.$$

□

5. Final remarks

Several questions remain unanswered. The first is the question of sharpness of our results. Even though we strongly believe that our results are sharp (up to strictness in the inequalities), we were unfortunately unable to find a proof. Olevskii and Ulanovskii [12] prove the sharpness of their results by exploiting the fact that the Paley-Wiener space possesses an orthonormal basis of normalized reproducing kernels. It is well-known that such orthonormal or even Riesz bases are missing in the classical Bargmann-Fock space. (This was recently proved to be the case even in the weighted setting [3].) Therefore, it is unlikely that such a simple example can be constructed in the Bargmann-Fock space. On the other hand, in this direction, we believe that our results can be strengthened to strict inequalities in analogy with the above mentioned results for classical interpolation sets, i.e., 0-approximate interpolation sets.

Another interesting question that remains to be explored is the relationship between weak and strong d-approximate interpolation sets. As mentioned in the introduction, it was proved by Schuster and Seip that for $d = 0$ these classes coincide in the classical one-dimensional setting. Their proof extends to the weighted one-dimensional setting as well. However, nothing similar is known in the higher-dimensional case. The proof that Schuster and Seip utilize in dimension one doesn't extend, since it depends in a crucial way on the sufficiency of the Seip's density condition for interpolation which is known to be false in the higher dimensional setting. This at least suggests that these classes may differ in the higher dimensional setting, but, as far as we know, no examples have been found so far. It is very likely that a similar state of affairs will persist for d-approximate interpolation sets.

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This paper is available via http://nyjm.albany.edu/j/2021/27-61.html.