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Reducing subspaces of C_{00} contractions

Chafiq Benhida, Emmanuel Fricain and Dan Timotin

ABSTRACT. Using the Sz.-Nagy–Foias theory of contractions, we obtain general results about reducibility for a class of completely nonunitary contractions. These are applied to certain truncated Toeplitz operators, previously considered by Li–Yang–Lu and Gu. In particular, a negative answer is given to a conjecture stated by the latter.

CONTENTS

1. Introduction

We will denote by L^2 the Lebesgue space $L^2(\mathbb{T}, dm)$, where dm is normalized Lebesgue measure. The subspace of functions whose negative Fourier coefficients are zero is denoted by H^2; it is identified with the space of analytic functions in the unit disc with square summable Taylor coefficients. An inner function is an element of H^2 whose values have modulus 1 almost everywhere on $\mathbb{T}$.

If θ is an inner function, then the space $K_\theta = H^2 \ominus \theta H^2$ is usually called a model space; it has been the focus of much research, in function theory in the unit disc as well as in operator theory (see, for instance,[7, 6]; or [2] for a more recent account). In particular, in the last two decades several papers discuss the so-called *truncated Toeplitz operators*, introduced in [8], which are compressions to K_θ of multiplication operators on L^2.

Originating with work in [1], the question of reducibility of a certain class of truncated Toeplitz operators has been recently investigated in papers by Yi,

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Yang, and Lu [4, 5] and Gu [3]. Besides certain remarkable results, they also contain intriguing questions that have not yet found their solution.

The current paper has several purposes. First, we put the problem of reducibility of the truncated Toeplitz operators in a larger context, that of the Sz.-Nagy–Foias theory of completely nonunitary contractions [9], and show that some results in the above quoted papers may be generalized or given more transparent proofs. Secondly, we answer in the negative a conjecture stated in [3] and prove a statement that replaces it.

The plan of the paper is the following. After presenting in the next section the elements of Sz.-Nagy theory that interest us, we obtain in Section 3 some general results about reducibility for completely nonunitary contractions. These results are applied in Section 4 to a certain class of truncated Toeplitz operators. The connection to [5] is achieved in Section 5, while the relation to [3] is the content of the last section.

2. Sz.-Nagy–Foias dilation theory

The general reference for this section is the monograph [9], in particular chapters I, II, and VI.

2.1. Minimal isometric dilation. If $\mathcal{H}$ is a Hilbert space and $\mathcal{H}_1$ is a closed subspace, we will denote by $P_{\mathcal{H}_1}$ the orthogonal projection onto $\mathcal{H}_1$.

A closed subspace M of $\mathcal{H}$ is said to be *reducing* for an operator T if both M and $M^\perp$ are invariant with respect to T. A *completely nonunitary contraction* $T \in \mathcal{L}(\mathcal{H})$ is a linear operator that satisfies $\|T\| \le 1$, and there is no reducing subspace of T on which it is unitary. The defect of T is the operator $D_T = (I - T^*T)^{1/2}$, and the defect space is $\mathcal{D}_T = \overline{D_T\mathcal{H}}$.

We write $T \in C_{\cdot 0}$ if T^{*n} tends strongly to 0, and $T \in C_{00}$ if T and T^* are in $C_{\cdot 0}$, that is T^n and T^{*n} both tend strongly to 0. If $T \in C_{00}$, then it can be shown that $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*}$. The subclass of C_{00} for which this dimension is finite and equal to N is denoted by $C_0(N)$. We will mostly be interested by contractions in the class C_{00}.

An isometric dilation of T is an isometric operator $V \in \mathcal{L}(\mathcal{K})$, with $\mathcal{K} \supset \mathcal{H}$, such that $P_{\mathcal{H}}V^n|\mathcal{H} = T^n$ for any $n \in \mathbb{N}$. Note that if $T = P_{\mathcal{H}}V|\mathcal{H}$ and $VH^\perp \subset H^\perp$, then V is a dilation. An isometric dilation $V \in \mathcal{L}(\mathcal{K})$ is called minimal if $\mathcal{K} = \bigvee_{n=0}^\infty V^n\mathcal{H}$. This is uniquely defined, modulo a unitary isomorphism commuting with the dilations; in [9] there is a precise description of its geometric structure. This becomes simpler for contractions in $C_{\cdot 0}$; since this is the only case we are interested in, we will describe the minimal isometric dilation in this case.

We will say that a subspace $X \subset \mathcal{K}$ is *wandering* for V if $V^nX \perp V^mX$ for any $n \ne m$, and in this case we will denote $M_+(X) := \bigoplus_{n=0}^\infty V^nX$. Note that $M_+(X)$ is invariant with respect to V.

Lemma 2.1. *If T is a completely nonunitary contraction and V is its minimal isometric dilation, then $T \in C_{\cdot 0}$ if and only if there exist wandering subspaces*

$L, L_* \subset \mathcal{K}$ for V, with $\dim L = \dim \mathcal{D}_T$ and $\dim L_* = \dim \mathcal{D}_{T^*}$, such that

$$\mathcal{K} = M_+(L_*) = \mathcal{H} \oplus M_+(L). \tag{2.1}$$

In this case, the operators

$$\phi : D_T x \mapsto (V - T)x, \quad \phi_* : D_{T^*} x \mapsto x - VT^* x \tag{2.2}$$

extend to unitary operators $\mathcal{D}_T \to L$ and $\mathcal{D}_{T^*} \to L_*$.

2.2. Analytic vector valued functions. If $\mathcal{E}$ is a Hilbert space, then $H^2(\mathcal{E})$ is the Hilbert space of $\mathcal{E}$-valued analytic functions in $\mathbb{D}$ with the norms of the Taylor coefficients square summable. As in the scalar case, these functions have strong radial limits almost everywhere on $\mathbb{T}$, and so may be identified with their boundary values, defined on $\mathbb{T}$.

Denote by $\boldsymbol{T}_z^{\mathcal{E}}$ multiplication by z acting on $H^2(\mathcal{E})$; it is an isometric operator. If $\omega : \mathcal{E} \to \mathcal{E}'$ is unitary, then the notation $\tilde{\omega}$ will indicate the unique unitary extension $\tilde{\omega} : H^2(\mathcal{E}) \to H^2(\mathcal{E}')$ such that $\tilde{\omega}\boldsymbol{T}_z^{\mathcal{E}} = \boldsymbol{T}_z^{\mathcal{E}'}\tilde{\omega}$.

Suppose $X \subset \mathcal{K}$ is wandering for the isometry $V \in \mathcal{L}(\mathcal{K})$. Then the map $\mathfrak{F}_X$, defined by

$$\mathfrak{F}_X(\sum_{n=0}^{\infty} V^n x_n) = \sum_{n=0}^{\infty} \lambda^n x_n, \tag{2.3}$$

is unitary from $M_+(X)$ to $H^2(X)$.

Another class of functions that we have to consider take as values operators between two Hilbert spaces $\mathcal{E}, \mathcal{E}_*$. More precisely, we will be interested in *contractive analytic functions*; that is, functions $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}^*)$, which satisfy $\|\Theta(z)\| \le 1$ for all $z \in \mathbb{D}$. As in the scalar case, Θ has boundary values $\Theta(e^{it})$ almost everywhere on $\mathbb{T}$.

A contractive analytic function is called *pure* if $\|\Theta(0)x\| < \|x\|$ for any $x \in \mathcal{E}$, $x \neq 0$. Any contractive analytic function admits a decomposition in a direct sum $\Theta = \Theta_p \oplus \Theta_u$, where Θ_p is pure and Θ_u is a constant unitary operator; then Θ_p is called the *pure part* of Θ. A contractive analytic function will be called *bi-inner* if $\Theta(e^{it})$ is almost everywhere unitary. (We prefer this shorter word rather than call them inner and *-inner).

The appropriate equivalence relation for contractive analytic functions is that of coincidence: two analytic functions $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$, $\Theta' : \mathbb{D} \to \mathcal{L}(\mathcal{E}', \mathcal{E}'_*)$ are said to *coincide* if there exist unitary operators $\omega : \mathcal{E} \to \mathcal{E}'$, $\omega_* : \mathcal{E}'_* \to \mathcal{E}'_*$, such that $\Theta'(\lambda)\omega = \omega_*\Theta(\lambda)$ for all $\lambda \in \mathbb{D}$.

2.3. Functional model and characteristic function. The model theory of Sz.-Nagy and Foias associates to any completely nonunitary contraction T a pure contractive analytic function $\Theta_T(z)$, with values in $\mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*})$, defined by the formula

$$\Theta_T(z) = -T + zD_{T^*}(I - zT^*)^{-1}D_T|\mathcal{D}_T. \tag{2.4}$$

A functional model space and an associated model operator are constructed by means of Θ_T, and one can prove that T is unitarily equivalent to this model operator.

As we will be interested only in C_{00} contractions, we will describe the model only in this case, in which it takes a significantly simpler form. The reason is that $T \in C_{00}$ is equivalent to Θ_T bi-inner. The functional model associated to a bi-inner contractive analytic function $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}^*)$ is defined as follows: the *model space* is

$$\mathcal{H}_\Theta = H^2(\mathcal{E}_*) \ominus \Theta H^2(\mathcal{E}), \tag{2.5}$$

while the *model operator* $\boldsymbol{S}_\Theta$ is the compression to $\mathcal{H}_\Theta$ of $\boldsymbol{T}_z^{\mathcal{E}_*}$. If Θ is pure, then $\boldsymbol{T}_z^{\mathcal{E}_*}$ is precisely a minimal unitary dilation of $\boldsymbol{S}_\Theta$.

Note that (2.5) shows that $\boldsymbol{S}_\Theta$ satisfies the assumptions of Lemma 2.1 with $L = \Theta\mathcal{E}$, $L_* = \mathcal{E}_*$. In particular,

$$\dim \mathcal{D}_{\boldsymbol{S}_\Theta} = \dim \mathcal{E}, \qquad \dim \mathcal{D}_{\boldsymbol{S}_\Theta^*} = \dim \mathcal{E}_*. \tag{2.6}$$

Suppose $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ and $\Theta' : \mathbb{D} \to \mathcal{L}(\mathcal{E}', \mathcal{E}'_*)$ coincide, by means of the operators $\omega : \mathcal{E} \to \mathcal{E}'$, $\omega_* : \mathcal{E}'_* \to \mathcal{E}'_*$. Then the unitary $\tilde{\omega}_* : H^2(\mathcal{E}_*) \to H^2(\mathcal{E}'_*)$ satisfies $\tilde{\omega}(\mathcal{H}_\Theta) = \mathcal{H}_{\Theta'}$ and

$$\tilde{\omega}_* \boldsymbol{S}_\Theta = \boldsymbol{S}_{\Theta'} \tilde{\omega}_*.$$

Returning now to the contraction T and its characteristic function, the next lemma is a particular case of one of the basic results in [9, Chapter VI].

Lemma 2.2. *If $T \in C_{00}$, then the formula (2.4) defines a bi-inner pure analytic function with values in $\mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*})$, and T is unitarily equivalent to $\boldsymbol{S}_{\Theta_T}$. $\boldsymbol{S}_{\Theta_T}$ is called the* functional model *of T.*

There is a relation between the functional model and the geometrical structure of a minimal unitary dilation given by (2.1), as shown by the next result.

Lemma 2.3. *Suppose $T \in C_{00}$, $V \in \mathcal{L}(\mathcal{K})$ is a minimal isometric dilation of T, and L, L_* are wandering subspaces for V satisfying (2.1). Extend ϕ, ϕ_* in (2.2) to unitary operators $\tilde{\phi} : H^2(\mathcal{D}_T) \to H^2(L)$, $\tilde{\phi}_* : H^2(\mathcal{D}_{T^*}) \to H^2(L_*)$, and define $\Omega = \mathfrak{F}_{L_*}^* \Phi_*$.*

(i) *The map $\mathfrak{F}_{L_*}^* \Phi_* \Theta_T \Phi^* \mathfrak{F}_L$ is the inclusion of $M_+(L)$ into $M_+(L_*)$.*

(ii) *We have*

$$\begin{aligned} &\Omega\mathcal{H}_{\Theta_T} = \mathcal{H}, \quad \Omega(\mathcal{D}_{T^*}) = L_*, \quad \Omega\Theta_T(\mathcal{D}_T) = L, \\ &\Omega\boldsymbol{T}_z^{\mathcal{D}_{T^*}} = V\Omega, \quad \Omega\boldsymbol{S}_{\Theta_T} = T\Omega. \end{aligned} \tag{2.7}$$

(iii) *If $\Theta = \phi_* \Theta_T \phi^*$ is written $\Theta(\lambda) = \sum_{n=0}^\infty \lambda^n \Theta_n$ (with $\Theta_n : L \to L_*$), then*

$$\Theta_n = P_{L_*}(V^*)^n J, \tag{2.8}$$

where J denotes the embedding of L into $M_+(L_*)$.

3. Reducibility

In the sequel of the paper we will be interested by reducibility of certain contractions. Fortunately, this can be easily characterized through characteristic functions.

Lemma 3.1. *Suppose $T \in C_{00}$ has characteristic function $\Theta_T : \mathcal{D}_T \to \mathcal{D}_{T^*}$. Then the following are equivalent.*

(i) $T = T_1 \oplus T_2$.

(ii) *There exist nontrivial orthogonal decompositions $\mathcal{D}_T = E^1 \oplus E^2$, $\mathcal{D}_{T^*} = E^1_* \oplus E^2_*$ which diagonalize $\Theta_T(\lambda)$ for all $\lambda \in \mathbb{D}$; that is,*

$$\Theta_T(\lambda) = \begin{pmatrix} \Theta_1(\lambda) & 0 \\ 0 & \Theta_2(\lambda) \end{pmatrix}. \tag{3.1}$$

In this case $\dim \mathcal{D}_{T_i} = \dim E^i = \dim \mathcal{D}_{T_i^*} = \dim E^i_*$*, and Θ_{T_i} coincides with Θ_i.*

Proof. If $T = T_1 \oplus T_2$, then $\mathcal{D}_T = \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}$, $\mathcal{D}_{T^*} = \mathcal{D}_{T_1^*} \oplus \mathcal{D}_{T_2^*}$, and formula (2.4) splits according to these decompositions into $\Theta_T(\lambda) = \Theta_{T_1}(\lambda) \oplus \Theta_{T_2}(\lambda)$. So (3.1) is valid, taking $E^i = \mathcal{D}_{T_i}$, $E^i_* = \mathcal{D}_{T_i^*}$.

Conversely, if $\Theta(\lambda) := \Theta_T(\lambda) = \Theta_1(\lambda) \oplus \Theta_2(\lambda)$, then, according to (2.5), $\mathcal{H}_\Theta = \mathcal{H}_{\Theta_1} \oplus \mathcal{H}_{\Theta_2}$, and $\mathcal{H}_{\Theta_1}, \mathcal{H}_{\Theta_2}$ are invariant with respect to $\boldsymbol{S}_\Theta$. Since this last operator is unitarily equivalent to T, T is also reducible. Moreover, $\boldsymbol{S}_\Theta | \mathcal{H}_{\Theta_i}$ is unitarily equivalent to $\boldsymbol{S}_{\Theta_i}$, and the equality of the dimensions follows from (2.6). □

Corollary 3.2. *Suppose $T \in C_{00}$. Then T is reducible if and only if there exist nontrivial subspaces $E \subset \mathcal{D}_T$, $E_* \subset \mathcal{D}_{T^*}$, such that $\Theta_T(e^{it})E = E_*$ for almost all t.*

Proof. If nontrivial subspaces as assumed exist, then, since $\Theta_T(e^{it})$ is unitary almost everywhere, we also have $\Theta_T(e^{it})E^\perp = E_*^\perp$ for almost all t. The decompositions $\mathcal{D}_T = E \oplus E^\perp$, $\mathcal{D}_{T^*} = E_* \oplus E_*^\perp$ satisfy then (3.1). □

The following is a geometrical reformulation of Corollary 3.2 in terms of the spaces L, L_* appearing in an arbitrary minimal isometric dilation of T.

Corollary 3.3. *Suppose $T \in C_{00}$ and $V \in \mathcal{L}(\mathcal{K})$ is a minimal dilation of T, such that (2.1) is valid for L, L_* wandering subspaces for V. Let d be a finite positive integer or ∞. Then:*

(i) *If T has a nontrivial reducing subspace such that the restriction has d-dimensional defects, then there exist nontrivial subspaces $L^1 \subset L$, $L^1_* \subset L_*$, both of dimension d, such that*

$$L^1 \subset M_+(L^1_*). \tag{3.2}$$

(ii) *The converse also holds if $d < \infty$.*

Proof. (i). Suppose T has a reducing subspace with defect of dimension d. We apply Lemma 3.1, which gives decomposition (3.1), where $\Theta_i(\lambda) : E^i \to E^i_*$, and $\dim E^1 = \dim E^1_* = d$. So $\Theta_1 H^2(E^1) \subset H^2(E^1_*)$; in particular, if we look at E^1 as the constant functions in $H^2(E^1)$, we have

$$\Theta_1 E^1 \subset H^2(E^1_*). \tag{3.3}$$

Denote then $L^1 = \phi E^1$ and $L^1_* = \phi_* E^1_*$ (ϕ, ϕ_* in (2.2)). We consider the unitary operator Ω from Lemma 2.3. Formulas (2.7) yield also $\Omega E^1_* = L^1_*$, $\Omega\Theta_1(E^1) = L^1$, and $\Omega(H^2(E^1_*)) = M_+(L^1_*)$. Therefore (3.3) implies (3.2).

(ii) Conversely, suppose we have the required spaces satisfying (3.2); therefore $M_+(L^1) \subset M_+(L^1_*)$. Define $\Theta'(\lambda) = \phi_*\Theta_T(\lambda)\phi^* : L \to L_*$. By using $\mathfrak{F}_{L_*}$, we obtain $\Theta' H^2(L^1) \subset H^2(L^1_*)$, which means that $\Theta'(e^{it})L^1 \subset L^1_*$ almost everywhere. Since $\dim L^1 = \dim L^1_* = d < \infty$, we have in fact $\Theta'(e^{it})L^1 = L^1_*$ almost everywhere. As in the proof of Corollary 3.2, it follows that $\Theta'(e^{it})L^{1\perp} \subset L^{1\perp}_*$ almost everywhere, whence we may obtain a decomposition similar to (3.1). This implies the reducibility of Θ', and thus the reducibility of Θ_T and of T. □

In particular, we obtain a nice result if we consider reducing subspaces with defects of dimension 1.

Corollary 3.4. *An operator $T \in C_{00}$ has a reducing subspace with defects of dimension 1 if and only if there exists $y \in L$, $y_* \in L_*$, $y, y' \neq 0$, and a scalar inner function u, such that $y = u(V)y_*$. In this case the characteristic function of the reduced operator is precisely u.*

Proof. By Corollary 3.3 applied to $d = 1$, the existence of a reducing subspace with defects of dimension 1 is equivalent to the existence of elements of norm 1 $y \in L, y_* \in L_*$, such that $y \in M_+(y_*)$. The Fourier representation $\mathfrak{F}_{y_*}$ maps $M_+(y_*)$ onto H^2; more precisely, from (2.3) it follows that $\mathfrak{F}_{y_*}(f(V)y_*) = f$. In particular, y is a wandering vector for V, which implies that $u := \mathfrak{F}_{y_*}y$ is an inner function.

If we denote by $\mathcal{H}_1$ the reducing subspace of dimension 1 obtained, then have $\mathcal{H}_1 = M^+(y_*) \ominus M^+(y)$. Through the Fourier representation $\mathfrak{F}_{y_*}$, this becomes $H^2 \ominus uH^2$. By comparing with the general formula for the functional model, we see that the characteristic function of the reduced operator is u. □

Remark 3.5. Part of the results in this section may be extended to more general contractions. Thus Lemma 3.1 is true for a general completely nonunitary contraction; we have then to use in the proof the more complicated general form of the functional model associated to T. Appropriately modified versions of Corollaries 3.2 and 3.3 can also be stated. However, since the statements are less neat, we have preferred to restrict ourselves to the case $T \in C_{00}$, which will be used in the applications in the sequel of the paper.

4. A class of contractions

In the rest of the paper we will work in the Hardy space H^2, applying the above results to a particular class of contractions. By T_φ we will denote the usual

Toeplitz operator on H^2, that is, the compression of the operator of multiplication by φ on H^2. Recall here that, for a scalar inner function $K_\theta = H^2 \ominus \theta H^2 = \mathcal{H}_\theta$ (see (2.5) with $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$).

Let then θ, B be two scalar inner functions that satisfy the basic assumption

$$\ker T_{\theta\overline{B}} = \{0\}. \tag{4.1}$$

Note that $f \in \ker T_{\theta\overline{B}}$ if and only if $\theta f \in \ker T_{\overline{B}} = K_B$, whence (4.1) is equivalent to $\theta H^2 \cap K_B = \{0\}$.

We will consider the operator $A_B^\theta \in \mathcal{L}(K_\theta)$, defined by

$$A_B^\theta = P_{K_\theta} T_B | K_\theta. \tag{4.2}$$

The operator A_B^θ is usually called the *truncated Toeplitz operator* on K_θ with symbol B. It is known [8] that truncated Toeplitz operators are *complex symmetric*; that is, there exist a complex conjugation C_θ on K_θ such that

$$(A_B^\theta)^* = C_\theta A_B^\theta C_\theta. \tag{4.3}$$

The next theorem identifies concretely a minimal isometric dilation of A_B^θ; it is a generalization of [5, Lemma 3.1].

Theorem 4.1. *Let B and θ two inner functions satisfying* (4.1)*. The operator $T_B \in \mathcal{L}(H^2)$ is a minimal isometric dilation of A_B^θ. For this minimal isometric dilation we have*

$$L = \theta K_B, \quad L_* = K_B. \tag{4.4}$$

Proof. T_B is an isometry on H^2, and $T_B(K_\theta^\perp) = T_B(\theta H^2) \subset \theta H^2 = K_\theta^\perp$. Thus it follows from (4.2) that T_B is a dilation of A_B^θ.

We show now by induction according to n that

$$K_\theta + BK_\theta + \cdots + B^n K_\theta = K_{B^n\theta}. \tag{4.5}$$

Equality (4.5) is obviously true for $n = 0$. Suppose that it is true up to $n-1$. We are left then to prove that

$$K_{B^{n-1}\theta} + B^n K_\theta = K_{B^n\theta}. \tag{4.6}$$

It is immediate from the definitions that the left hand side is contained in the right hand side. On the other hand, we have

$$K_{B^n\theta} = K_{B^{n-1}\theta} \oplus B^{n-1}\theta K_B = B^n K_\theta \oplus K_{B^n}.$$

If $f \in K_{B^n\theta}$ is orthogonal to $K_{B^{n-1}\theta}$ as well as to $B^n K_\theta$, it follows that

$$f \in (\theta B^{n-1} K_B) \cap K_{B^n}.$$

So $f = \theta B^{n-1} g$ with $g \in K_B$; and also $f \perp B^n H^2$, which means $\theta g \perp BH^2$, or $\theta g \in K_B$. It follows that $0 = T_{\overline{B}}(\theta g) = T_{\theta\overline{B}} g$. By (4.1), this implies $g = 0$, whence $f = 0$.

Since

$$\Big(\bigvee_n K_{B^n\theta}\Big)^\perp = \bigcap_n B^n \theta H^2 = \{0\},$$

it follows that

$$H^2 = \bigvee_{n=0}^{\infty} T_B^n K_\theta. \tag{4.7}$$

Therefore T_B is a *minimal* isometric dilation of A_B^θ.

Then

$$H^2 = \bigoplus_{n=0}^{\infty} B^n K_B = \bigoplus_{n=0}^{\infty} T_B^n K_B = M_+(K_B), \tag{4.8}$$

whence $L_* = K_B$.

On the other hand, we have

$$K_{B\theta} = K_\theta \oplus \theta K_B = K_B \oplus B K_\theta, \tag{4.9}$$

Therefore

$$H^2 = \bigoplus_{n=0}^{\infty} B^n K_B = K_\theta \oplus \theta H^2 = K_\theta \oplus \bigoplus_{n=0}^{\infty} T_B^n \theta K_B = K_\theta \oplus M_+(\theta K_B), \tag{4.10}$$

whence $L = \theta K_B$. □

Corollary 4.2. *With the above assumptions, A_B^θ is in C_{00}.*

Proof. In view of equation (4.10), it follows from Lemma 2.1 that A_B^θ is in $C_{\cdot 0}$. On the other hand, it follows from (4.3) that

$$((A_B^\theta)^*)^n = C_\theta (A_B^\theta)^n C_\theta,$$

whence A_B^θ is also in $C_{0\cdot}$. □

Using the identification of a minimal unitary dilation in Theorem 4.1 we may compute the characteristic function of A_B^θ. The next theorem generalizes [3, Theorem 2.4] (see Section 6 below).

Theorem 4.3. *Let B and θ two inner functions satisfying (4.1). The characteristic function of A_B^θ is $\Phi : \mathbb{D} \to \mathcal{L}(K_B)$ defined by*

$$\Phi(\lambda) = A^B_{\frac{\theta}{1-\lambda\overline{B}}}. \tag{4.11}$$

Proof. We have identified in Theorem 4.1 L, L_* with $\theta K_B, K_B$ respectively. We intend to apply Lemma 2.3 (iii). Since we want to consider the characteristic function of A as an analytic function with values in $\mathcal{L}(K_B)$, the embedding J is precisely multiplication by θ. Then, if $\Phi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \Phi_n$, (2.8) yields

$$\Phi_n f = P_{K_B} \overline{B}^n \theta f$$

for $f \in K_B$. Thus $\Phi_n = A^B_{\theta \overline{B}^n}$. Therefore

$$\Phi(\lambda) = \sum_{n=0}^{\infty} \lambda^n A^B_{\theta\overline{B}^n} = A^B_{\theta \sum_{n=0}^{\infty} \lambda^n \overline{B}^n} = A^B_{\frac{\theta}{1-\lambda\overline{B}}}.$$ □

We may also obtain a more precise form of Corollary 3.4.

Corollary 4.4. *Let B and θ two inner functions satisfying (4.1). Then the following assertions are equivalent:*

(i) *The operator A_B^θ has a reducing subspace such that the restriction has one-dimensional defects.*

(ii) *There exist u inner and $h_1, h_2 \in K_B$, $h_1, h_2 \neq 0$, such that*

$$\theta = \frac{h_2}{h_1}(u \circ B), \tag{4.12}$$

(iii) *There exist u, v_1, v_2 inner, with*

$$\ker T_{v_1 \bar{B}} \cap \ker T_{v_2 \bar{B}} \neq \{0\},$$

such that

$$\theta = \frac{v_2}{v_1}(u \circ B). \tag{4.13}$$

Proof. The equivalence of (i) and (ii) follows by applying in this case Corollary 3.4. We have $L_* = K_B$, $L = \theta K_B$, and so the existence of the required reducing subspace is equivalent to the existence of $h_1, h_2 \in K_B$, $h_1, h_2 \neq 0$ and an inner function u, such that $\theta h_1 = u(V)h_2$. Since $V = T_B$, $u(V)$ is multiplication by $u \circ B$, and we have

$$\theta h_1 = h_2(u \circ B). \tag{4.14}$$

If (ii) is true, then we must have $h_i = v_i g$ for some inner functions v_1, v_2 and g outer, so (4.13) is true. Note that, if v is an analytic and bounded function in $\mathbb{D}$, then

$$vh \in K_B \Leftrightarrow h \in \ker T_{v\bar{B}}. \tag{4.15}$$

So $v_1 g, v_2 g \in K_B$ is equivalent to $g \in \ker T_{v_1 \bar{B}} \cap \ker T_{v_2 \bar{B}}$.

The implication (iii) $\Longrightarrow$ (ii) follows easily by reversing the steps. □

Note that the function u in (ii) and (iii) of the previous corollary is non constant because otherwise $\theta h_1 \in K_B$, and thus $h_1 \in \ker T_{\theta \overline{B}}$ which contradicts hypothesis (4.1).

5. A particular case

Let us consider now the particular case when B is a finite Blaschke product. Denote $\phi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha} z)$. If B has roots (counting with multiplicities) $w_1, \dots, w_k$, it is known that

$$K_B = \left\{ \frac{p(z)}{\prod_{i=1}^k (1 - \bar{w}_i z)} \; : \; p \text{ polynomial of degree } \leq k - 1 \right\}. \tag{5.1}$$

In this case condition (4.1) has a simple equivalent form.

Lemma 5.1. *If B is a finite Blaschke product, then (4.1) is satisfied if and only if*

$$\dim K_B \leq \dim K_\theta. \tag{5.2}$$

Proof. Indeed, first assume that (5.2) is satisfied, and let $f \in \ker T_{\theta\overline{B}}$. Then $\theta f \in \ker T_{\overline{B}} = K_B$, whence $f = T_{\overline{\theta}}(\theta f) \in T_{\overline{\theta}} K_B \subset K_B$. If $f \not\equiv 0$, then $\theta = \frac{\theta f}{f}$ is a quotient of two polynomials of degree at most $\deg B - 1$, which contradicts assumption (5.2).

Suppose now that $\dim K_B > \dim K_\theta$. Then θH^2 has finite codimension in H^2 strictly smaller than $\dim K_B$, whence $\theta H^2 \cap K_B \neq \{0\}$. Applying (4.15) in case $v = \theta$, it follows that $\ker T_{\theta\overline{B}} \neq \{0\}$, contradicting (4.1). □

Condition (5.2) is precisely the one considered in [3] and [5]. To discuss this case, we need one more elementary lemma.

Lemma 5.2. *Suppose h_1, h_2 are two polynomials of degree at most $k-1$ and*

$$|h_1| = |h_2| \text{ a.e. on } \mathbb{T}. \tag{5.3}$$

Then,

$$\frac{h_2}{h_1} = \frac{B_2}{B_1},$$

where B_i are Blaschke products with $\deg B_1 + \deg B_2 \le k-1$.

Proof. First, a general remark. Suppose that h is a polynomial and write $h(z) = z^p g(z)$, with $p \in \mathbb{N} \cup \{0\}$ and $g(0) \neq 0$. Denote the roots (counting with multiplicities) of g by $\alpha_1, \dots, \alpha_\ell$. Then, h^o, the outer part of h, is a polynomial of degree $\deg g$, which has roots $Z_o(h) \cup Z_i(h)$, where $Z_o(h) := \{\alpha_i : |\alpha_i| \ge 1\}$ and $Z_i(h) := \{1/\bar{\alpha}_i : 0 < |\alpha_i| < 1\}$.

We may assume that h_1, h_2 have no common roots (otherwise we cancel them). It also follows then that h_1 and h_2 have no roots on $\mathbb{T}$ (since this would be a common root by (5.3)). Also, only one of them may have the root 0; suppose it is h_1, and write, as above, $h_1(z) = z^p g_1(z)$, with $g_1(0) \neq 0$.

Assumption (5.3) implies that the outer parts of g_1 and h_2 coincide. Since g_1 and h_2 have no common roots, but ${g_1}^o = h_2^o$, we must have $Z_o(g_1) = Z_i(h_2)$ and $Z_i(g_1) = Z_o(h_2)$. Then we can write $h_2/h_1 = B_2/B_1$, with

$$B_1 = z^p \prod_{\alpha_i \in Z_i^\sharp(h_1)} \phi_{\alpha_i}, \quad B_2 = \prod_{\alpha_i \in Z_i^\sharp(h_2)} \phi_{\alpha_i},$$

where $Z_i^\sharp(p) = \{\alpha_i : p(\alpha_i) = 0, 0 < |\alpha_i| < 1\} = \{1/\bar{\alpha}_i : \alpha_i \in Z_i(p)\}$. Since we have

$$\deg B_1 + \deg B_2 = p + |Z_i(g_1)| + |Z_i(h_2)| = p + |Z_i(g_1)| + |Z_o(g_1)| \le k-1,$$

the lemma is proved. □

The next theorem generalizes [5, Theorem 1.4].

Theorem 5.3. *Suppose B is a finite Blaschke product, while θ is an inner function with* $\deg\theta \ge \deg B$. *Then the operator A_B^θ has a reducing subspace such that the restriction has one-dimensional defects if and only if*

$$\theta = \frac{B_2}{B_1}(u \circ B), \tag{5.4}$$

where u is a non constant inner function, while B_1, B_2 are finite Blaschke products with $\deg B_1 + \deg B_2 \le \deg B - 1$.

Proof. We apply to this case Corollary 4.4 (ii). The existence of the required reducing subspace is then equivalent to the existence of $h_1, h_2 \in K_B$ and an inner function u, such that

$$\theta h_1 = h_2(u\circ B). \tag{5.5}$$

By (5.1), it is equivalent to assume in this equality that h_i are polynomials of degree $\le k-1$, where $k = \deg B$. Taking absolute values, we obtain, since θ and $u\circ B$ are inner, that $|h_1| = |h_2|$ on $\mathbb{T}$. We may then apply Lemma 5.2 to obtain the desired formula (5.4).

The converse is immediate, since (5.4) implies (5.5), with the degrees of h_1 and h_2 at most $k-1$. If we further write $g_i(z) = \frac{h_i(z)}{\prod_{i=1}^k (1-\bar{w}_i z)}$, we obtain

$$\theta g_1 = g_2(u\circ B).$$

Since $g_i \in K_B$, this is equivalent, by Corollary 4.4, to the existence of the required reducing subspace. □

The condition becomes simpler if θ is singular.

Theorem 5.4. *Let θ be a singular inner function and let B be a finite Blaschke product. Then the operator A_B^θ has a reducing subspace such that the restriction has one-dimensional defects if and only if*

$$\theta = S\circ B, \tag{5.6}$$

for some singular inner function S.

Proof. According to Theorem 5.3, it is sufficient to prove that (5.6) and (5.4) are equivalent. The implication (5.6) $\Longrightarrow$ (5.4) is clear. Assume now that (5.4) is satisfied, that is we can write

$$B_1\theta = B_2(u\circ B),$$

where B_1 and B_2 are finite Blaschke products with $\deg B_1 + \deg B_2 \le N-1$ and $N = \deg B$.

Since θ is singular, B_2 must be a factor of B_1 and may be canceled. So we may assume $B_2 = 1$, or $B_1\theta = u\circ B$, where $\deg B_1 \le N-1$.

Write then $u = B_3 S$, where B_3 is a Blaschke product and S is the singular part of u. Thus we have

$$B_1\theta = (B_3\circ B)(S\circ B).$$

We have $\deg(B_3\circ B) = \deg B_3 \deg B$; so, if B_3 is not constant, then

$$\deg(B_3\circ B) \ge \deg B = N > \deg B_1.$$

The contradiction obtained implies that B_3 is constant, and so

$$B_1\theta = S\circ B.$$

Since the right hand side is singular, it follows that B_1 is constant, which proves the theorem. □

6. The case $B(z) = z^N$

The case $B(z) = z^N$ is investigated at length in [3]. In particular, the characteristic function of $A^{\theta}_{z^N}$ is computed; let us show how Gu's result follows from Theorem 4.3 above.

We use the canonical basis of K_B formed by $1, z, \dots z^{N-1}$. To obtain the matrix of $A^B_{\frac{\theta}{1-\lambda\overline{B}}}$, consider first $A^B_{\frac{z^n}{1-\lambda\overline{B}}}$. We have

$$\frac{z^n}{1-\lambda\overline{B}} = \sum_{j=0}^{\infty} \lambda^j z^{n-jN} = \sum_{j=0}^{\infty} \lambda^j z^{N(n'-j)+m},$$

where $n = Nn' + m$, with $0 \le m \le N-1$. Since $A^B_{z^p}$ is nonzero only for $-(N-1) \le p \le N-1$, we have to consider in the above sum only two terms, corresponding to $j = n'$ and $j = n'+1$. Thus

$$A^B_{\frac{z^n}{1-\lambda\overline{B}}} = A^B_{\lambda^{n'} z^m + \lambda^{n'+1} z^{m-N}}.$$

Its matrix with respect to the canonical basis is

$$A^B_{\lambda^{n'} z^m + \lambda^{n'+1} z^{m-N}} = \begin{pmatrix} \ddots & \ddots & \lambda^{n'+1} & \ddots & \ddots \\ \ddots & \ddots & \ddots & \lambda^{n'+1} & \ddots \\ \lambda^{n'} & \ddots & \ddots & \ddots & \ddots \\ \ddots & \lambda^{n'} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{6.1}$$

with two nonzero constant diagonals (one in case $m = 0$), corresponding to entries a_{ij} with $i - j = m$ or $i - j = m - N$.

Therefore, if we decompose

$$\theta(z) = \theta_0(z^N) + z\theta_1(z^N) + \cdots + z^{N-1}\theta_{N-1}(z^N),$$

then

$$A^B_{\frac{\theta}{1-\lambda\overline{B}}} = \begin{pmatrix} \theta_0(\lambda) & \lambda\theta_{N-1}(\lambda) & \lambda\theta_{N-2}(\lambda) & \dots & \lambda\theta_1(\lambda) \\ \theta_1(\lambda) & \theta_0(\lambda) & \lambda\theta_{N-1}(\lambda) & \dots & \lambda\theta_2(\lambda) \\ \theta_2(\lambda) & \theta_1(\lambda) & \theta_0(\lambda) & \dots & \lambda\theta_3(\lambda) \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \theta_{N-1}(\lambda) & \theta_{N-2}(\lambda) & \theta_{N-3}(\lambda) & \dots & \theta_0(\lambda) \end{pmatrix}. \tag{6.2}$$

This is precisely the form given by [3, Theorem 2.4].

In the sequel we will solve a conjecture about $A^{\theta}_{z^N}$ left open in [3]. This appears as Conjecture 3.5 therein, and has the following statement.

Conjecture 6.1. *Suppose $B(z) = z^N$. Then the following are equivalent:*

(i) *A^{θ}_B has a reducing subspace such that the restriction has one-dimensional defects.*

(ii) *$\theta(z) = b(z)u(z^N)$ for some inner function u, while either $b \equiv 1$ or*

$$b(z) = \prod_{i=1}^{l} \psi_{\alpha_i, J_i}, \tag{6.3}$$

where $l \leq N-1$, $J_i \subset \{0, \dots, N-1\}$, and $\psi_{\alpha,J}$ is defined by

$$\psi_{\alpha,J}(z) = \prod_{i \in J} \phi_{\omega^i \alpha}(z). \tag{6.4}$$

[3, Theorem 3.4] shows that (i) $\Longrightarrow$ (ii), while (ii) $\Longrightarrow$ (i) is proved only for $N = 3$ in [3, Section 5].

Theorem 6.2. *Conjecture 6.1 is false for $N = 4$.*

Proof. Take two different nonzero values $\alpha, \beta \in \mathbb{D}$, and define

$$\theta(z) = \frac{(z^2 - \alpha^2)(z^2 - \beta^2)}{(1 - \bar{\alpha}^2 z^2)(1 - \bar{\beta}^2 z^2)}.$$

We have then

$$\theta(z) = \psi_{\alpha,J} \psi_{\beta,J}$$

with $J = \{0, 2\} \subset \{0, 1, 2, 3\}$, so θ satisfies condition (ii) of Conjecture 6.1.

On the other hand, if θ would satisfy condition (i), it would follow by Theorem 5.3 that one should have

$$B_2(z)\theta(z) = B_1(z)u(z^4), \tag{6.5}$$

with u inner and B_1, B_2 finite Blaschke products with $\deg B_1 + \deg B_2 \leq 3$.

Obviously u has also to be a finite Blaschke product. Equating the degrees in both sides yields

$$\deg B_1 + 4 = \deg B_2 + 4 \deg u.$$

First, $\deg B_1 = 3$ would imply $\deg B_2 = 0$, so $7 = 4 \deg u$: a contradiction. So the degree of the left hand side of (6.5) is between 4 and 6, which implies $\deg u = 1$. Again equating the degrees yields $\deg B_1 = \deg B_2 = 0$ or 1.

Now $u(z^4)$ has either the root 0 of multiplicity 4, or four distinct roots. Both possibilities are incompatible with the fact that the left hand side of (6.5) has either 2 or three roots. We have obtained the desired contradiction, so θ does not satisfy (i) of Conjecture 6.1. □

In fact, we may replace Conjecture 6.1 with a precise result. We will need the next lemma, also proved in [3, Theorem 3.4].

Lemma 6.3. *If $\alpha \in \mathbb{D}$, then*

$$\phi_{\alpha^N}(z^N) = \prod_{i=0}^{N-1} \phi_{\omega^i \alpha}(z).$$

Theorem 6.4. *Suppose $B(z) = z^N$. Then the following are equivalent:*

(i) *A_B^θ has a reducing subspace such that the restriction has one-dimensional defects.*

(ii) *$\theta(z) = b(z)u(z^N)$ for some inner function u, while b is either 1 or a finite Blaschke product given by (6.3), where $l \le N-1$, $J_i \subset \{0, \dots, N-1\}$, $\psi_{\alpha,J}$ are defined by (6.4), and, moreover,*

$$\sum_{i=1}^{l} \min\{|J_i|, N - |J_i|\} \le N - 1. \tag{6.6}$$

Proof. (i) $\Longrightarrow$ (ii). From Theorem 5.3 we know that θ is given by (5.4), where B_1 and B_2 have no common roots. We may denote the roots of B_1 (counting multiplicities) by

$$\{\alpha_1^1, \dots, \alpha_{s_1}^1; \alpha_1^2, \dots, \alpha_{s_2}^2; \dots; \alpha_1^p, \dots, \alpha_{s_p}^p\},$$

where, for each $i = 1, \dots, p$, the values $\alpha_1^i, \dots, \alpha_{s_i}^i$ are all distinct, and

$$(\alpha_1^i)^N = \cdots = (\alpha_{s_i}^i)^N.$$

Similarly, we denote the roots of B_2 by

$$\{\beta_1^1, \dots, \beta_{r_1}^1; \beta_1^2, \dots, \beta_{r_2}^2; \dots; \beta_1^q, \dots, \beta_{r_q}^q\},$$

where, for each $i = 1, \dots, q$, the values $\beta_1^i, \dots, \beta_{r_i}^i$ are all distinct, and

$$(\beta_1^i)^N = \cdots = (\beta_{s_i}^i)^N.$$

Note that the condition $\deg B_1 + \deg B_2 \le N - 1$ is transcribed as

$$s_1 + \cdots + s_p + r_1 + \cdots + r_q \le N - 1. \tag{6.7}$$

In particular, $p + q \le N - 1$.

Now, it is easy to see that, for each $i = 1, \dots, q$, the Blaschke product

$$\phi_{\beta_1^i} \dots \phi_{\beta_{r_i}^i}$$

is equal to $\psi_{\beta_1^i, J_i}$ for some $J_i \subset \{0, \dots, N-1\}$. So

$$B_2 = \prod_{i=1}^{q} \psi_{\beta_1^i, J_i}. \tag{6.8}$$

The matter is more subtle as concerns B_1: it appears at the denominator, which we do not want. We have, similarly,

$$B_1 = \prod_{i=1}^{p} \psi_{\alpha_1^i, J_i'} \tag{6.9}$$

for some $J_i' \subset \{0, \dots, N-1\}$.

The factor $\phi_{\alpha_1^1}(z)$ must be canceled by a factor in $u(z^N)$, so α_1^1 must be a root of $u(z^N)$. But then $u(z^N)$ must also have the roots $\omega^j \alpha_1^1$ for $j = 1, \dots, N-1$, and so

$$u(z^N) = \prod_{j=0}^{N-1} \phi_{\omega^j \alpha_1^1}(z) u_1(z^N).$$

Since

$$\frac{\phi_{\omega^j\alpha_1^1}(z)}{\psi_{\alpha_1^1,J_1'}} = \psi_{\alpha_1^1,J_1''}$$

with $J_1'' = \{0,\dots,N-1\}\setminus J_1'$, we have

$$\frac{u(z^N)}{\psi_{\alpha_1^1,J_1'}} = \psi_{\alpha_1^1,J_1''}u_1(z^N).$$

We may continue the argument (or use an appropriate induction) to obtain

$$\frac{u(z^N)}{B_1(z)} = \prod_{i=1}^{p}\psi_{\alpha_1^i,J_i''}u'(z^N) \tag{6.10}$$

for an inner function u', where $J_i'' = \{0,\dots,N-1\}\setminus J_i'$. From (5.4), (6.8), and (6.10) it follows that

$$\theta(z) = \prod_{i=1}^{q}\psi_{\beta_1^i,J_i}\prod_{i=1}^{p}\psi_{\alpha_1^i,J_i''}\,u'(z^N).$$

This is exactly the form given by (6.3). Moreover $\min\{|J_i|,N-|J_i|\}\le r_i$ and $\min\{|J_i''|,N-|J_i''|\}\le s_i$, so (6.7) implies (6.6).

(ii) $\Longrightarrow$ (i). Suppose $b(z)$ is given by (6.3), with (6.6) satisfied. Define

$$B_2 = \prod_{\min\{|J_i|,N-|J_i|\}=|J_i|}\psi_{\alpha_i,J_i}$$

and

$$B_1 = \prod_{\min\{|J_i|,N-|J_i|\}=N-|J_i|}\psi_{\alpha_i,N\setminus J_i}.$$

Then

$$\theta(z) = \frac{B_2(z)}{B_1(z)}u_1(z^N),$$

where

$$u_1(z) = u(z)\prod_{\min\{|J_i|,N-|J_i|\}=N-|J_i|}\phi_{\alpha_i^N}(z);$$

note that we have used Lemma 6.3. □

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(Chafiq Benhida) Laboratoire Paul Painlevé, Université Lille 1, 59 655 Villeneuve d'Ascq Cédex, France
chafiq.benhida@univ-lille.fr

(Emmanuel Fricain) Laboratoire Paul Painlevé, Université Lille 1, 59 655 Villeneuve d'Ascq Cédex, France
emmanuel.fricain@univ-lille.fr

(Dan Timotin) Simion Stoilow Institute of Mathematics of the Romanian Academy, PO Box 1–764, Bucharest 014700, Romania
Dan.Timotin@imar.ro

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