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One-parameter isometry groups and inclusions between operator algebras

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Abstract. We make a careful study of one-parameter isometry groups on Banach spaces, and their associated analytic generators, as first studied by Cioranescu and Zsido. We pay particular attention to various, subtly different, constructions which have appeared in the literature, and check that all give the same notion of generator. We give an exposition of the "smearing" technique, checking that ideas of Masuda, Nakagami and Woronowicz hold also in the weak*-setting. We are primarily interested in the case of one-parameter automorphism groups of operator algebras, and we present many applications of the machinery, making the argument that taking a structured, abstract approach can pay dividends. A motivating example is the scaling group of a locally compact quantum group $\mathbb{G}$ and the fact that the inclusion $C_0(\mathbb{G}) \to L^\infty(\mathbb{G})$ intertwines the relevant scaling groups. Under this general setup, of an inclusion of a C^*-algebra into a von Neumann algebra intertwining automorphism groups, we show that the graphs of the analytic generators, despite being only non-self-adjoint operator algebras, satisfy a Kaplansky Density style result. The dual picture is the inclusion $L^1(\mathbb{G}) \to M(\mathbb{G})$, and we prove an "automatic normality" result under this general setup. The Kaplansky Density result proves more elusive, as does a general study of quotient spaces, but we make progress under additional hypotheses.

Contents

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1. Introduction

A one-parameter automorphism group of an operator algebra is $(\alpha_t)_{t\in\mathbb{R}}$ where each α_t is an automorphism, we have the group law $\alpha_t\circ\alpha_s = \alpha_{t+s}$, and a continuity condition on the orbit maps $a \mapsto \alpha_t(a)$ (either norm continuity for a C^*-algebra, or weak*-continuity for a von Neumann algebra). As for the more common notion of a semigroup of operators, such groups admit a "generator", an in general unbounded operator which characterises the group. This paper will be concerned with the *analytic generator*, formed by complex analytic techniques, which can loosely be thought of as the exponential of the more common infinitesimal generator.

The analytic generator was defined and studied in [9], see also [39, 40, 41], [27, Appendix F], [20]. There are immediate links with Tomita-Takesaki theory, [33, Chapter VIII] and [41], although we contrast the explicit use of generators in [41] with the more adhoc approach of [33]. Our principle interest comes from the operator algebraic approach to quantum groups, [23], and specifically the treatment of the antipode. For a quantum group, the antipode represents the group inverse, and is represented as an, in general unbounded, operator S on an operator algebra. This operator factorises as $S = R\tau_{-i/2}$ where R is the *unitary antipode*, an anti-$*$-homomorphism, and $\tau_{-i/2}$ which is an analytic continuation of a one-parameter automorphism group, the *scaling group* (τ_t). Furthermore, $S^2 = \tau_{-i}$ which is precisely the analytic generator.

We tend to think of the quantum group $\mathbb{G}$ as an "abstract object" which can be represented be a variety of operator algebras, in particular the reduced C^*-algebra $C_0(\mathbb{G})$, thought of as functions on $\mathbb{G}$ vanishing at infinity, and the von Neumann algebra $L^\infty(\mathbb{G})$, thought of as measurable functions on $\mathbb{G}$. There is a natural inclusion $C_0(\mathbb{G}) \to L^\infty(\mathbb{G})$, which intertwines the scaling group(s)— the scaling group is norm-continuous on $C_0(\mathbb{G})$ and weak*-continuous on $L^\infty(\mathbb{G})$. Much of this paper is concerned with this situation in the abstract: an inclusion of a C^*-algebra into a von Neumann algebra which intertwines automorphism groups. Such a situation also occurs in Tomita-Takesaki theory, where a convenient way to construct type III von Neumann algebras is to start with a KMS state on C^*-algebra and to apply the GNS construction, see [16] for example. One of our main results, Theorem 5.1, gives a Kaplansky density result for the graphs of the analytic generators in such a setting.

Using the coproduct we can turn the dual spaces into Banach algebras. This leads to the dual of $C_0(\mathbb{G})$, denoted $M(\mathbb{G})$ and thought of as a convolution algebra of measures, and also to the predual of $L^\infty(\mathbb{G})$, denoted $L^1(\mathbb{G})$ and thought of as the absolutely continuous measures. These do not carry a natural involution, because we would wish to use the antipode which is not everywhere defined, but there are natural dense $*$-subalgebras, $L^1_\sharp(\mathbb{G})$ and $M_\sharp(\mathbb{G})$, compare Section 7 below. Part of our motivation for writing this paper was to attempt to understand our result, with Salmi,

that when $\mathbb{G}$ is coamenable, there is a Kaplansky density result for the inclusion $L^1_\sharp(\mathbb{G}) \to M_\sharp(\mathbb{G})$; compare Proposition 7.5 below, where we are still unable to remove the coamenability condition. A positive general result is Theorem 7.4 which shows that if $\omega \in L^1(\mathbb{G})$ and $\omega^* \circ S$ is bounded on $D(S) \subseteq L^\infty(\mathbb{G})$, then $\omega \in L^1_\sharp(\mathbb{G})$. This is notable because it gives a criterion to be a member of $L^1_\sharp(\mathbb{G})$ which is not "graph-like": we do not suppose the existence of another member of $L^1(\mathbb{G})$ interacting with S in some way.

A further motivation for writing this paper was to make the case that considering the analytic generator (or rather, the process of analytic continuation) as a theory in its own right has utility; compare with the adhoc approach of [33] or [35]. In particular, we take a great deal of care to consider the various different topologies that have been used in the literature, and to verify that these lead to the same constructions:

- Either the weak, or norm, topology gives the same continuity assumption on the group (α_t) (this is well-known) but it is not completely clear that norm analytic continuation (as used in [27] for example) is the same as weak analytic continuation (which is the framework of [9]). Theorem 2.6 below in particular implies that it is.
- For a von Neumann algebra, [9] used weak*-continuity, but it is also common to consider the σ-strong* topology, [22, 24], or the strong topology, [12] for example. A priori, it is hence not possible to apply the results of [9] (for example) to the definition used in [22]. Theorem 2.16 below shows that these do however give the same analytic extensions.
- It is also possible to use duality directly; this approach is taken in [35] for example. Duality is explored in [39]; compare Theorem 2.17 below.

In Section 2 we give an introduction to one-parameter isometry groups on Banach spaces and explore and prove the topological results summarised above. We also explore some examples. Section 3 is devoted to the technique of "smearing", and in particular to the ideas of [27, Appendix F], which we find to be very powerful. We check that the ideas of [27, Appendix F] also work for weak*-continuous groups. These first two sections are deliberately expositionary in nature.

In Section 4 we present a variety of applications of the smearing technique. We give new proofs of some known results (for example, Zsido's result that the graph of the generator is an algebra, without using the machinery of spectral subspaces). In the direction of Tomita-Takesaki theory, as an example of the utility of taking a structured approach, we show how the main result of [6] follows almost immediately from the work of Cioranescu and Zsido in [9], and give another application of smearing to prove the remainder the results of [6]. We finish by making some remarks on considering the

graph of the generator as a Banach algebra: we believe there is interesting further work here.

In Section 5 we formulate and prove a Kaplansky Density result. Given a C^*-algebra A included in a von Neumann algebra M with A generating M, Kaplansky Density says that the unit ball of A is weak*-dense in the unit ball of M. If (α_t) is an automorphism group of M which restricts to a norm-continuous group on A, then we can consider the graphs of the generators, say $\mathcal{G}(\alpha^A_{-i})$ and $\mathcal{G}(\alpha^M_{-i})$, which are non-self-adjoint operator algebras. We have that $\mathcal{G}(\alpha^A_{-i}) \subseteq \mathcal{G}(\alpha^M_{-i})$ and is weak*-dense (see Proposition 4.2 for example). The main result here is that the unit ball of $\mathcal{G}(\alpha^A_{-i})$ is weak*-dense in the unit ball of $\mathcal{G}(\alpha^M_{-i})$. The key idea is to consider the bidual $\mathcal{G}(\alpha^A_{-i})^{**}$, and to identify $\mathcal{G}(\alpha^M_{-i})$ within this.

In Section 6, we consider the "adjoint" of the above situation, the inclusion $M_* \to A^*$. Our groundwork in Section 5 leads us to show Theorem 6.2 which shows that if $\omega \in M_*$ and $\omega \in D(\alpha^{A^*}_{-i})$ then automatically $\alpha^{A^*}_{-i}(\omega) \in M_*$, so that $\omega \in D(\alpha^{M_*}_{-i})$. The analogous result for the inclusion $A \to M$ is false, see Example 4.4. We make a study of quotients. For both dual spaces, and quotients, we seem to require extra hypotheses (essentially, forms of complementation). We finish by making some remarks about "implemented" automorphism groups, as studied further in [9, Section 6] and [41]. In the final section we apply our results to the study of locally compact quantum groups.

1.1. Notation. We use E, F for Banach spaces, and write E^* for the dual space of E. For $x \in E, \mu \in E^*$ we write $\langle \mu, x\rangle = \mu(x)$ for the pairing. Given a bounded linear map $T : E \to F$ we write T^* for the (Banach space) adjoint $T^* : F^* \to E^*$. This should not cause confusion with the Hilbert space adjoint. We use A for a Banach or C^*-algebra, and M for a von Neumann algebra, writing M_* for the predual of M.

If $E_0 \subseteq E$ is a closed subspace, then by the Hahn-Banach theorem we may identify the dual of E_0 with $E^*/E_0^\perp$, and identify $(E/E_0)^*$ with $E_0^\perp$, where

$$E_0^\perp = \{\mu \in E^* : \langle \mu, x\rangle = 0 \ (x \in E_0)\}.$$

Similarly, for a subspace $X \subseteq M$ we define ${}^\perp X = \{\omega \in M_* : \langle x, \omega\rangle = 0 \ (x \in X)\}$. The weak*-closure of X is $({}^\perp X)^\perp$, and if X is weak*-closed, then $M_*/{}^\perp X$ is the canonical predual of X.

By a *metric surjective* $T : E \to F$ we mean a surjective bounded linear map such that the induced isomorphism $E/\ker T \to F$ is an isometric isomorphism. By Hahn-Banach, this is if and only if $T^* : F^* \to E^*$ is an isometry onto its range (which is $(\ker T)^\perp$).

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2. One-parameter groups

A *one-parameter group of isometries* on a Banach space E is a family $(\alpha_t)_{t\in\mathbb{R}}$ of bounded linear operators on E such that α_0 is the identity, each α_t is a contraction, and $\alpha_t \circ \alpha_s = \alpha_{t+s}$ for $s, t \in \mathbb{R}$. Then α_{-t} is the inverse to α_t, and thus each α_t is actually an isometric isomorphism of E.

We want to consider one of a number of continuity conditions on (α_t):

(1) We say that (α_t) is *norm-continuous* if, for each $x \in E$, the orbit map $\mathbb{R} \to E; t \mapsto \alpha_t(x)$ is continuous, for the norm topology on E;
(2) We say that (α_t) is *weakly-continuous* if each orbit map is continuous for the weak topology on E. However, this condition implies already that (α_t) is norm-continuous; see [33, Proposition 1.2'] for a short proof.
(3) If E is the dual of a Banach space E_*, then (α_t) is *weak*-continuous* if each operator α_t is weak*-continuous, and the orbit maps are weak*-continuous.

Example 2.1. Consider the Banach spaces $c_0(\mathbb{Z})$ and $\ell^\infty(\mathbb{Z})$. Let α_t be the operator given by multiplication by $(e^{int})_{n\in\mathbb{Z}}$. Then (α_t) forms a one-parameter group of isometries which is norm-continuous on $c_0(\mathbb{Z})$, and which is weak*-continuous on $\ell^\infty(\mathbb{Z})$, but not norm-continuous on $\ell^\infty(\mathbb{Z})$ (consider the orbit of the constant sequence $(1) \in \ell^\infty(\mathbb{Z})$).

We shall mainly be interested in the case of a Banach algebra A. If each (α_t) is an algebra homomorphism, then we call (α_t) a *(one-parameter) automorphism group*. If A is a C^*-algebra, then we require that each α_t be a $*$-homomorphism, and, unless otherwise specified, we suppose that (α_t) is norm-continuous. When $A = M$ is actually a von Neumann algebra, unless otherwise specified, we assume that (α_t) is weak*-continuous. When M acts on a Hilbert space H, there are of course other natural topologies on M, and we shall make some comments about these later, see Theorem 2.16 below, for example.

In the classical theory of, say, C_0-semigroups (where we replace $\mathbb{R}$ by $[0, \infty)$) central to the theory is the notion of a generator. This paper will be concerned with a different idea, the *analytic generator*, which arises from complex analysis techniques. Here we follow [9]; see also [20] in the norm-continuous case, and the lecture notes [22, Section 5.3].

Definition 2.2. For $z \in \mathbb{C} \setminus \mathbb{R}$ define

$$S(z) = \{w \in \mathbb{C} : 0 \leq \operatorname{im} w / \operatorname{im} z \leq 1\}.$$

That is, $S(z)$ is the closed horizontal strip bounded by $\mathbb{R}$ and $\mathbb{R}+z$. For $t \in \mathbb{R}$ let $S(t) = \mathbb{R}$.

For a Banach space E, a function $f : S(z) \to E$ is *norm-regular* when f is continuous, and analytic in the interior of $S(z)$.

Notice that we make no boundedness assumption, but see Remark 2.4 below.

We remind the reader that for a domain $U \subseteq \mathbb{C}$ and $f : U \to E$, we have that f is analytic (in the sense of having an absolutely convergent power series, locally to any point in U) if and only if $\mu \circ f$ is complex differentiable, for each $\mu \in E^*$. If $E = (E_*)^*$ is a dual space, then it suffices that f be "weak*-differentiable", that is, we test only for $\mu \in E_*$. For a short proof see [33, Appendix A1], and for further details, see for example [1, 2].

When $E = (E_*)^*$ is a dual space, we say that $f : S(z) \to E$ is *weak*-regular* when f is weak*-continuous. By the above remarks, it does not matter which notion of "analytic" we consider on the interior of $S(z)$.

Definition 2.3. Let (α_t) be a norm-continuous, one-parameter group of isometries on E, and let $z \in \mathbb{C}$. Define a subset $D(\alpha_z) \subseteq E$ by saying that $x \in D(\alpha_z)$ when there is a norm-regular $f : S(z) \to E$ with $f(t) = \alpha_t(x)$ for each $t \in \mathbb{R}$; in this case, we set $\alpha_z(x) = f(z)$.

We make the same definition for a weak*-continuous isometry group, using a weak*-regular map f.

Suppose we have two regular maps $f, g : S(z) \to E$ with $f(t) = g(t) = \alpha_t(x)$ for each $t \in \mathbb{R}$. For $\mu \in E^*$ (or E_* in the weak*-continuous case) consider the map $h : S(z) \to \mathbb{C}; w \mapsto \langle \mu, f(w) - g(w) \rangle$. Then h is regular and vanishes on $\mathbb{R}$, and so by the reflection principle, and Morera's Theorem, we can extend h to an analytic function on the interior of $S(z) \cup S(-z)$ which vanishes on $\mathbb{R}$, and which hence vanishes on all of $S(z)$. As μ was arbitrary, this shows that $f(w) = g(w)$ for each $w \in S(z)$. We conclude that the regular map occurring in the definition of α_z is unique; we term f an analytic extension of the orbit map $t \mapsto \alpha_t(x)$.

It is easy to show that $D(\alpha_z)$ is a subspace of E, and that $\alpha_z : D(\alpha_z) \to E$ is a linear operator. We remark that [9] uses a vertical strip instead, but one can simply "rotate" the results to our convention. We have the familiar properties (see [20, Section 1], [9, Section 2]), all of which follow essentially immediately from uniqueness of analytic extensions:

(1) $\alpha_t \circ \alpha_z = \alpha_z \circ \alpha_z = \alpha_{z+t}$ for $t \in \mathbb{R}$; here using the usual notion of composition of not necessarily everywhere defined operators.
(2) if $w \in S(z)$ then $\alpha_z \subseteq \alpha_w$. It follows that $S(z) \to E; w \mapsto \alpha_w(x)$ is defined, and by uniqueness, is the analytic extension of the orbit map for x.
(3) $\alpha_{-z} = \alpha_z^{-1}$.
(4) $\alpha_{z_1} \circ \alpha_{z_2} \subseteq \alpha_{z_1+z_2}$, with equality if both z_1, z_2 lie on the same side of the real axis.

Furthermore, α_z is a closed operator (see [20, Theorem 1.20] for the norm-continuous case, and [9, Theorem 2.4] for the weak*-continuous case).

Remark 2.4. Contrary to some sources, we have not imposed any boundedness assumptions on our regular maps; however, in our setting, this is automatic. Let $z = t + is \in \mathbb{C}$ and $x \in D(\alpha_z)$. Then $x \in D(\alpha_{is})$ and $\alpha_z(x) = \alpha_t(\alpha_{is}(x))$ and so $\|\alpha_z(x)\| = \|\alpha_{is}(x)\|$. In the rest of this remark, we will assume without loss of generality that $s > 0$.

In the norm-continuous case, the map $[0, s] \to E; r \mapsto \alpha_{ir}(x)$ is norm-continuous, and so has bounded image. As (α_t) is an isometry group, it follows that $w \mapsto \alpha_w(x)$ is bounded on $S(z)$. By the Three-Lines Theorem, if we set

$$M = \max\Big(\sup_r \|\alpha_r(x)\|, \sup_r \|\alpha_{is+r}(x)\|\Big) = \max(\|x\|, \|\alpha_z(x)\|),$$

then $\|\alpha_w(x)\| \leq M$ for each $w \in S(z)$.

In the weak*-continuous case, for any $\mu \in E_*$, the map $[0, s] \to \mathbb{C}; r \mapsto \langle \alpha_{ir}(x), \mu\rangle$ is continuous and so bounded, and so, again, the Three-Lines Theorem shows that $|\langle \alpha_w(x), \mu\rangle| \leq M\|\mu\|$ for $w \in S(z)$. Taking the supremum over $\|\mu\| \leq 1$ shows that $\|\alpha_w(x)\| \leq M$ for $w \in S(z)$.

Similar remarks would also apply to weakly-continuous extensions, if we were to consider these.

The paper [9] works with general dual pairs of Banach spaces, which satisfy certain axioms. In particular, if (α_t) is norm-continuous on E, then it is weakly-continuous, and so we can consider weakly-regular extensions, to which the general theory of [9] applies.

Remark 2.5. In particular, the dual pairs of Banach spaces which [9] considers admit a "good" integration theory. We shall only consider the cases of weak*-continuous maps, for which we can just consider weak*-integrals; and weakly-continuous maps, for which the theory is less obvious. Indeed, let $f : \mathbb{R} \to E$ be weakly continuous with $\int_{\mathbb{R}} \|f(t)\|\, dt < \infty$. A naive definition of $\int_{\mathbb{R}} f(t)\, dt$ defines a member of E^{**}, but this integral actually converges in E, see [9, Proposition 1.4] and [4, Proposition 1.2]. Alternatively, if E is separable, we can use the Bochner integral and the Pettis Measurability Theorem.

Suppose $x \in E$ and $f : S(z) \to E$ is a weakly-regular extension of the orbit map for x. Then $t \mapsto f(t) = \alpha_t(x)$ is norm-continuous, and also $t \mapsto f(t + z) = \alpha_t(\alpha_z(x)) = \alpha_t(f(z))$ (by property (1) above) is norm-continuous. Further, on the interior of $S(z)$, we have that f is analytic, and hence norm-continuous. However, it is not immediately clear why f need be norm-continuous on all of $S(z)$. We now show that actually f is automatically norm-continuous; but below we give an example to show that under slightly weaker conditions, norm-continuity on all of $S(z)$ can fail, showing that this is more subtle than it might appear.

Theorem 2.6. *Let E be a Banach space, and let $f : S(z) \to E$ be a bounded, weakly-regular map. Assume further that $t \mapsto f(t)$ and $t \mapsto f(z+t)$ are norm continuous. Then f is norm-regular.*

Proof. Define $g : S(z) \to E$ by $g(w) = e^{-w^2} f(w)$. Then g is weakly-regular, and $t \mapsto g(t)$ and $t \mapsto g(z+t)$ are uniformly (norm) continuous.

We now use a "smearing" technique. For $n > 0$ define $g_n : S(z) \to E$ by

$$g_n(w) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} g(w+t)\ dt.$$

Here the integral is in the sense of Remark 2.5, or alternatively, as g is norm continuous on any horizontal line, we can use a Riemann integral. It follows easily that $g_n(t) \to g(t)$, uniformly in $t \in \mathbb{R}$, as $n \to \infty$; similarly $g_n(t+z) \to g(t+z)$ uniformly in t.

We claim that

$$g_n(w) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 (t-w)^2} g(t)\ dt.$$

We prove this by, for each $\mu \in E^*$, considering the scalar-valued function $w \mapsto \langle \mu, g_n(w) \rangle$, and using contour deformation, and continuity.

We now observe that $w \mapsto \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 (t-w)^2} g(t)\ dt$ is entire. In particular, g_n is norm continuous on $S(z)$. As $g_n \to g$ uniformly on $\mathbb{R}$ and $\mathbb{R} + z$, the Three-Lines Theorem implies uniform convergence on all of $S(z)$. We conclude that g is norm-regular, which implies also that f is norm-regular. □

Corollary 2.7. *Let (α_t) be norm-continuous on E. If we use norm-regular extensions, or weakly-regular extensions, then we arrive at the same operator α_z.*

Thus the approaches of [20] and [9] do give the same operators.

Example 2.8. If we weaken the hypotheses of Theorem 2.6 to only require that $t \mapsto f(t)$ be continuous, then f need not be norm-regular, as the following example shows. Set $E = c_0 = c_0(\mathbb{N})$, and define $F : \overline{\mathbb{D}} \to E$ by

$$F(z) = \big(F_n(z)\big)_{n \in \mathbb{N}} = \big(\exp(k_n(e^{-i\pi/n} z - 1))\big)_{n \in \mathbb{N}}.$$

Here (k_n) is a rapidly increasing sequence of integers. Notice that $|F_n(z)| = \exp(k_n(\mathrm{re}(e^{-i\pi/n} z) - 1)) \leq 1$. Then:

- for $z \in \mathbb{D}$ we have that $e^{-i\pi/n} z \in \mathbb{D}$ and so $\mathrm{re}(e^{-i\pi/n} z) - 1 < 0$ and hence $F_n(z) \to 0$ as $n \to \infty$;
- If $z = e^{it}$ for $t \notin 2\pi\mathbb{Z}$, then $\mathrm{re}(e^{-i\pi/n} z) - 1 = \cos(t - \pi/n) - 1 \to \cos(t) - 1 < 0$ and so $F_n(z) \to 0$;
- $|F_n(1)| = \exp(k_n(\cos(\pi/n) - 1)) \to 0$ so long as (k_n) increases fast enough.

Thus $(F_n(z)) \in c_0$ for all $z \in \overline{\mathbb{D}}$. Notice that each F_n is continuous, and analytic on $\mathbb{D}$.

We now use that $c_0^* = \ell^1$, and for any $a = (a_n) \in \ell^1$ we have that

$$\langle a, F(z) \rangle = \sum_{n=1}^{\infty} a_n F_n(z)$$

converges uniformly for $z \in \overline{\mathbb{D}}$. We conclude that F is weakly-regular, that is, analytic on $\mathbb{D}$ and weakly-continuous on $\overline{\mathbb{D}}$. However,

$$\|F(e^{i\pi/n}) - F(1)\| \geq |F_n(e^{i\pi/n}) - F_n(1)| = |1 - \exp(k_n(e^{-i\pi/n} - 1))|.$$

This will be large if (k_n) increases rapidly. Thus F is not norm-continuous.

Finally, we can use a Mobius transformation to obtain an example defined on the strip $S(i)$. Indeed, $z \mapsto w = i(1-z)/(1+z)$ maps $\mathbb{D}$ to the upper half-plane, and maps $\mathbb{T}$ to $\mathbb{R} \cup \{\infty\}$, and sends $1 \in \mathbb{T}$ to $0 \in \mathbb{R}$. We hence obtain $G : S(i) \to c_0$ which is weakly-regular, with $t \mapsto G(t+i)$ norm-continuous, but $t \mapsto G(t)$ not norm-continuous.

2.1. Analytic generators. We call the closed operator α_{-i} the *analytic generator* of (α_t). Note that the use of $-i$ is really convention, as we can always rescale and consider (α_{tr}) for any non-zero $r \in \mathbb{R}$. In particular, $\alpha_{-i/2}$ often appears in applications.

We have that α_{-i} is a closed, densely defined operator. The operator α_{-i} does determine (α_t), see for example Section 6.4 below, and indeed one can reconstruct (α_t) from α_{-i}, see [9, Section 4].

Example 2.9. Let us compute the analytic extensions of the group(s) from Example 2.1. If $x = (x_n) \in D(\alpha_z) \subseteq c_0(\mathbb{Z})$ then for each n, the map $t \mapsto e^{int} x_n$ has an analytic extension to $S(z)$, which by uniqueness must be the map $w \mapsto e^{inw} x_n$. Thus $\alpha_z(x) = (e^{inz} x_n) \in c_0(\mathbb{Z})$. Reversing this, if $(e^{inz} x_n) \in c_0(\mathbb{Z})$, then by the three-lines theorem, $(x_n) \in D(\alpha_z)$. In particular, we see that $x = (x_n) \in D(\alpha_{-i}) \subseteq c_0(\mathbb{Z})$ if and only if (x_n) is in $c_0(\mathbb{Z})$ and $(x_n e^n) \in c_0(\mathbb{Z})$.

Similar remarks apply to $\ell^\infty(\mathbb{Z})$. In particular, we see that $x = (x_n) \in D(\alpha_{-i}) \subseteq \ell^\infty(\mathbb{Z})$ if and only if (x_n) and $(x_n e^n)$ are bounded.

Consider $x_n = 0$ for $n < 0$ and $x_n = e^{-n}$ for $n \geq 0$. Then $x = (x_n) \in c_0(\mathbb{Z})$ but while $(x_n e^n)$ is bounded, it is not in $c_0(\mathbb{Z})$. It follows that $x \notin D(\alpha_{-i})$ for the group acting on $c_0(\mathbb{Z})$, but x is in $D(\alpha_{-i})$ for the group acting on $\ell^\infty(\mathbb{Z})$.

Example 2.10. If we consider a one-parameter isometry group on a Hilbert space H, then we have the familiar notion of a (strongly continuous) unitary group $(u_t)_{t \in \mathbb{R}}$. Stone's Theorem tells us that there is a self-adjoint (possibly unbounded) operator A on H with $u_t = e^{itA}$ for each $t \in \mathbb{R}$. Alternatively, we can consider the analytic generator u_{-i}. [9, Theorem 6.1] shows that u_{-i}, as a (possibly unbounded) operator on H is positive and injective, and

equal to e^A. Thus, informally, we can think of the analytic generator as the exponential of the infinitesimal generator.

We now consider the case when $E = A$ is a Banach algebra, or a C^*-algebra.

Proposition 2.11. *Let (α_t) be an automorphism group of a Banach algebra A. Then $D(\alpha_z)$ is a subalgebra of A and α_z a homomorphism.*

Proof. Let $a, b \in D(\alpha_z)$. We can pointwise multiply the analytic extensions $w \mapsto \alpha_w(a)$ and $w \mapsto \alpha_w(b)$. This is continuous, and analytic on the interior of $S(z)$; here we use the joint norm continuity of the product on A. Thus $ab \in D(\alpha_z)$ with $\alpha_z(ab) = \alpha_z(a)\alpha_z(b)$. □

Proposition 2.12. *Let (α_t) be an automorphism group of a C^*-algebra A. For $a \in D(\alpha_z)$ we have that $a^* \in D(\alpha_{\overline{z}})$ and $\alpha_{\overline{z}}(a^*) = \alpha_z(a)^*$.*

Proof. Let $f : S(z) \to A$ be the analytic extension of the orbit map for a. Then $g : S(\overline{z}) \to A; w \mapsto f(\overline{w})^*$ is regular (the complex conjugate and the involution "cancel" to show that g is analytic on the interior of $S(\overline{z})$), from which the result follows. □

These results become more transparent if we consider the *graph* of α_z,

$$\mathcal{G}(\alpha_z) = \big\{(a, \alpha_z(a)) : a \in D(\alpha_z)\big\},$$

which is a closed subspace of $A \oplus A$, as α_z is closed. Thus $\mathcal{G}(\alpha_z)$ is a subalgebra of $A \oplus A$, and in the C^*-algebra case, $\mathcal{G}(\alpha_{-i})$ has the (non-standard) involution

$$\mathcal{G}(\alpha_{-i}) \ni (a, b) \mapsto (b^*, a^*) \in \mathcal{G}(\alpha_{-i}).$$

Here we used that $\alpha_i = \alpha_{-i}^{-1}$.

A Banach algebra A which is the dual of a Banach space A_* in such a way that the product on A becomes separately weak*-continuous is a *dual Banach algebra*, [29]. The following result is shown in [39] using the idea of a *spectral subspace* from [4, 5, 13]. This allows us to find weak*-dense subspaces (in fact, subalgebras) on which (α_t) is norm continuous. We shall later give a different, easier proof, see Section 4.

Theorem 2.13 ([39, Theorem 1.6])**.** *Let A be a dual Banach algebra and let (α_t) be a weak*-continuous automorphism group of A. Then $D(\alpha_z)$ is a subalgebra of A, and α_z is a homomorphism.*

For a dual Banach algebra, we cannot simply copy the proof of Proposition 2.11, as in the weak*-topology, the product is only separately continuous. In particular, this remark applies to von Neumann algebras. The approach taken in [22], and implicitly in [24] for example, is to use the σ-strong*-topology; [12, Section 2.5] does the same, but with $M \subseteq \mathcal{B}(H)$ a concretely represented von Neumann algebra, and the use of the strong topology. Such approaches would allow the proof of Proposition 2.11 to now

work. Unfortunately, it is not clear if using the σ-strong*-topology instead of the weak*- (that is, σ-weak-) topology gives the same set $D(\alpha_z)$. Indeed, is the resulting α_z even closed? This issue is not addressed in [22]. We now show that, actually, we do obtain the same $D(\alpha_z)$.

Let M be a von Neumann algebra with predual M_*. For $\omega \in M_*^+$ we consider the seminorms

$$p_\omega : M \to [0,\infty),\ x \mapsto \langle x^*x, \omega\rangle^{1/2};$$
$$p'_\omega : M \to [0,\infty),\ x \mapsto \langle x^*x + xx^*, \omega\rangle^{1/2}.$$

The σ-strong topology is given by the seminorms $\{p_\omega : \omega \in M_*^+\}$, and similarly the σ-strong* topology is given by the seminorms $\{p'_\omega\}$.

Lemma 2.14. *Let $E = (E_*)^*$ be a dual Banach space, let p be a seminorm on E for which there exists $k > 0$ with $p(x) \leq k\|x\|$ for $x \in E$, and let $z \in \mathbb{C}$. Let $f : S(z) \to E$ be bounded and weak*-regular, and further suppose that $t \mapsto f(t)$ and $t \mapsto f(z+t)$ are continuous for p. Then f is continuous for p on all of $S(z)$.*

Proof. We seek to follow the proof of Theorem 2.6. Define $g(w) = e^{-w^2}f(w)$ so again g is weak*-regular and $t \mapsto g(t)$, $t \mapsto g(z+t)$ are uniformly continuous for p. For $n > 0$ we can again define $g_n : S(z) \to E$ by

$$g_n(w) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} g(w+t)\ dt,$$

the integral converging in the weak* sense. We see that $g_n(t) \to g(t)$ uniformly in t, for the seminorm p, and similarly for $g_n(t+z) \to g(t+z)$.

We again have the alternative expression $g_n(w) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t-w)^2) f(t)\ dt$. Thus g_n extends to an analytic function on $\mathbb{C}$; in particular g_n is locally given by a $\|\cdot\|$-convergent power series, which is hence also p-convergent. It follows that g_n is p-continuous on $S(z)$. As $p(g_n - g) \to 0$ uniformly on $\mathbb{R}$ and $\mathbb{R} + z$, the Three-Lines Theorem implies uniform convergence on all of $S(z)$. Thus g is p-continuous on $S(z)$, and the same is true of f. □

Lemma 2.15. *Let M be a von Neumann algebra and let (α_t) be a weak*-continuous automorphism group. For each $x \in M$ the map $\mathbb{R} \to M; t \mapsto \alpha_t(x)$ is σ-strong* continuous.*

Proof. Let $\omega \in M_*^+$ and $t \in \mathbb{R}$. Then for $x \in M$,

$$\begin{aligned}
&\lim_{t\to 0} \langle (\alpha_t(x) - x)^*(\alpha_t(x) - x), \omega\rangle \\
&\quad = \lim_{t\to 0} \langle \alpha_t(x^*x) - x^*\alpha_t(x) - \alpha_t(x^*)x + x^*x, \omega\rangle \\
&\quad = \lim_{t\to 0} \langle \alpha_t(x^*x) + x^*x, \omega\rangle - \langle \alpha_t(x), \omega x^*\rangle - \langle \alpha_t(x^*), x\omega\rangle \\
&\quad = 2\langle x^*x, \omega\rangle - \langle x, \omega x^*\rangle - \langle x^*, x\omega\rangle = 0,
\end{aligned}$$

where we used repeatedly that α_t is a $*$-homomorphism, and that M_* is an M-module, and of course that (α_t) is weak*-continuous. Similarly, $\langle(\alpha_t(x)-x)(\alpha_t(x)-x)^*,\omega\rangle \to 0$ as $t \to 0$. Thus $\alpha_t(x) \to x$ as $t \to 0$, in the σ-strong* topology. □

Theorem 2.16. *Let M be a von Neumann algebra, let (α_t) be a weak*-continuous automorphism group, let $x \in M$, and let $f : S(z) \to M$ be a weak*-regular extension of $t \mapsto \alpha_t(x)$. Then f is continuous for the σ-strong* (and so σ-strong) topology.*

Proof. By Lemma 2.15, $t \mapsto f(t) = \alpha_t(x)$ and $t \mapsto f(z+t) = \alpha_t(f(z))$ are σ-strong* continuous. The result now follows from Lemma 2.14 applied to the seminorms p'_ω for $\omega \in M_*^+$. □

We conclude that the definition of α_z from [22] does agree with the definition in [9], and we are free to use either the σ-strong* topology, or the weak* topology. If $M \subseteq \mathcal{B}(H)$ and we use the strong topology, the same remarks apply.

2.2. Duality. Let E be a Banach space and let (α_t) be a norm-continuous one-parameter group of isometries of E. For each t let $\alpha_t^* \in \mathcal{B}(E^*)$ be the Banach space adjoint. Then (α_t^*) is a weak*-continuous one-parameter group of isometries of E^*.

Similarly, let $E = (E_*)^*$ be a dual Banach space and let (α_t) be a weak*-continuous one-parameter group of isometries of E. For each t, as α_t is weak*-continuous it has a pre-adjoint $\alpha_{*,t}$. As

$$\langle \alpha_t(x), \mu\rangle = \langle x, \alpha_{*,t}(\mu)\rangle \qquad (x \in E, \mu \in E_*)$$

it is easy to see that $(\alpha_{*,t})$ is a one-parameter group of isometries of E_* which is weakly-continuous, and hence which is norm-continuous.

We recall that when $T : D(T) \subseteq E \to F$ is an operator between Banach spaces, then the *adjoint* of T is defined by setting $\mu \in D(T^*) \subseteq F^*$ when there exists $\lambda \in E^*$ with $\langle \mu, T(x)\rangle = \langle \lambda, x\rangle$ for $x \in D(T)$. In this case, we set $T^*(\mu) = \lambda$. This is more easily expressed in terms of graphs. Define $j : E \oplus F \to F \oplus E$ by $j(x,y) = (-y,x)$. Then $\mathcal{G}(T^*)$ is equal to

$$(j\mathcal{G}(T))^\perp = \{(\mu,\lambda) \in F^* \oplus E^* : \langle(\mu,\lambda),(-T(x),x)\rangle = 0 \ (x \in D(T))\}.$$

That $\mathcal{G}(T^*)$ is the graph of an *operator* is equivalent to T being densely defined; in this case, $\mathcal{G}(T^*)$ is always weak*-closed. We can reverse this construction, starting with an operator $S : D(S) \subseteq F^* \to E^*$ and forming $S_* : D(S_*) \subseteq E \to F$ by $\mathcal{G}(S_*) = {}^\perp(jD(S))$. Then S_* is an operator exactly when S is weak*-densely defined, and S_* is always closed. Thus, if T is closed and densely-defined, then $S = T^*$ is weak*-closed and densely defined, and $S_* = T$. We are actually unaware of a canonical reference for this construction (which clearly parallels the very well-known construction for Hilbert space operators) but see [18, Section 5.5, Chapter III] for example.

The following is shown in [39] using a very similar argument to the proof that the generator, of a weak*-continuous group, is weak*-closed. We give a different proof, which relies on the closure result, and which will be presented below in Section 4. In fact, given the discussion above, this theorem is effectively equivalent to knowing that the generator is closed.

Theorem 2.17 ([39, Theorem 1.1])**.** *Let (α_t) on E and (α_t^*) on E^* be as above. For any z, we form α_z using (α_t), and form $\alpha_z^{E^*}$ using (α_z^*). Then $\alpha_z^* = \alpha_z^{E^*}$.*

We remark that we have used this result before, e.g. [8, Appendix], but without sufficient justification as to why $\alpha_z^* = \alpha_z^{E^*}$. Similar ideas, but without the machinery of using (α_t^*), are considered in [20, Proposition 1.24, Proposition 2.44].

3. Smearing

We now want to present some ideas from the Appendix of [27], which only considered norm-continuous one-parameter groups. We shall verify that the ideas continue to work for weak*-continuous one-parameter groups. This is fairly routine, excepting perhaps Proposition 3.5, but we feel it is worth giving the details, as we think the techniques and results are interesting. We also streamline the proof of the main technical lemma, directly invoking the classical Wiener Theorem, instead of using Distribution theory. We remark that the use of convolution algebra ideas goes back to at least [4, 5] and [13].

Let (α_t) be a one-parameter group of isometries on E; we shall consider both the case when (α_t) is norm-continuous, and when $E = (E_*)^*$ is a dual space and (α_t) is weak*-continuous. Given $n > 0$ define $\mathcal{R}_n : E \to E$ by

$$\mathcal{R}_n(x) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2 t^2) \alpha_t(x) \; dt.$$

The integral converges in norm, or the weak*-topology, according to context. As in the proof of Theorem 2.6, a contour deformation argument shows that for any $z \in \mathbb{C}$, $\mathcal{R}_n(x) \in D(\alpha_z)$ with

$$\alpha_z(\mathcal{R}_n(x)) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2 (t-z)^2) \alpha_t(x) \; dt.$$

Furthermore, if already $x \in D(\alpha_z)$ then $\alpha_z(\mathcal{R}_n(x)) = \mathcal{R}_n(\alpha_z(x))$.

This concept of smearing is very standard in arguments involving analytic generators, but it is common to consider the limit as $n \to \infty$. For example, for any $x \in E$ we have that $\mathcal{R}_n(x) \to x$ as $n \to \infty$ (again, in norm or the weak*-topology) and so this shows that $D(\alpha_z)$ is dense. In the following, the point is to show that it is possible to work with $\mathcal{R}_n$ for a fixed n.

In the following, a subspace $X \subseteq E$ is (α_t)-*invariant* when $\alpha_t(x) \in X$ for each $x \in X, t \in \mathbb{R}$. The following is immediate from the construction of $\mathcal{R}_n$ as a vector-valued integral.

Lemma 3.1. *For each $x \in E$, we have that $\mathcal{R}_n(x)$ is contained in the smallest (α_t)-invariant, closed (norm or weak* as appropriate) subspace of E containing x.*

The following result is somewhat less expected.

Lemma 3.2. *For each $x \in E$ and $n > 0$, we have that x is contained in the smallest (α_t)-invariant, closed (norm or weak* as appropriate) subspace of E containing $\mathcal{R}_n(x)$.*

Proof. Choose $\mu \in E^*$ or E_* as appropriate with $\langle \mu, \alpha_t(\mathcal{R}_n(x))\rangle = 0$ for each $t \in \mathbb{R}$. By Hahn-Banach, it suffices to show that $\langle \mu, x\rangle = 0$.

Define $f, g : \mathbb{R} \to \mathbb{C}$ by

$$f(t) = \langle \mu, \alpha_t(x)\rangle, \quad g(t) = \langle \mu, \alpha_t(\mathcal{R}_n(x))\rangle \qquad (t \in \mathbb{R}).$$

Then f and g are bounded continuous functions, and

$$\begin{aligned} g(t) &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2 s^2)\langle \mu, \alpha_{t+s}(x)\rangle \, ds \\ &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2 (s-t)^2)\langle \mu, \alpha_s(x)\rangle \, ds \\ &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2 (s-t)^2) f(s) \, ds. \end{aligned}$$

Thus g is the convolution of φ with f, where $\varphi(s) = \frac{n}{\sqrt{\pi}} \exp(-n^2 s^2)$, so that $\varphi \in L^1(\mathbb{R})$.

So, we wish to show that if $\varphi * f = 0$ then $f = 0$. Given $F \in L^\infty(\mathbb{R})$ and $a, b \in L^1(\mathbb{R})$, a simple calculation shows that

$$\langle F \cdot a, b\rangle = \langle F, a * b\rangle = \langle F * \check{a}, b\rangle,$$

where here $F \cdot a$ is the usual dual module action of $L^1(\mathbb{R})$ on $L^\infty(\mathbb{R}) = L^1(\mathbb{R})^*$, and $\check{a} \in L^1(\mathbb{R})$ is the function defined by $\check{a}(t) = a(-t)$. As $f \in C^b(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, by Hahn-Banach, we see that $\varphi * f = 0$ is equivalent to $\langle f, \check{\varphi} * g\rangle = 0$ for each $g \in L^1(\mathbb{R})$. To conclude that $f = 0$ it hence suffices to show that $\{\check{\varphi} * g : g \in L^1(\mathbb{R})\}$ is dense in $L^1(\mathbb{R})$. This is equivalent to showing that the translates of $\check{\varphi}$ are linearly dense in $L^1(\mathbb{R})$. In turn, this follows immediately from Wiener's Theorem (see [36, Theorem II] or [28, Theorem 9.4]) as $\check{\varphi} = \varphi$ has a nowhere vanishing Fourier transform. We remark that a different approach to this result would be to use Eymard's Fourier algebra [11] (where a related result about the action of $A(G)$ on $VN(G)$ holds for all locally compact groups G) but as we need simply the most classical version, we shall not give further details. □

In the following, $n > 0$ is any (fixed) number.

Proposition 3.3. *Let $D \subseteq E$ be an (α_t)-invariant subspace. Then $\mathcal{R}_n(D) = \{\mathcal{R}_n(x) : x \in D\}$ and D have the same (norm, or weak*) closure.*

Proof. As α_t commutes with $\mathcal{R}_n$ for each t, it follows that $\mathcal{R}_n(D)$ is (α_t)-invariant. For each $x \in D$, the closure of $\mathcal{R}_n(D)$ contains the smallest closed (α_t)-invariant subspace containing $\mathcal{R}_n(x)$, so by Lemma 3.1, $x \in \overline{\mathcal{R}_n(D)}$, and hence $\overline{D} \subseteq \overline{\mathcal{R}_n(D)}$. The reverse inclusion follows similarly from Lemma 3.2. □

The following gives a criteria for being a member of the graph of α_z.

Proposition 3.4. *Let $x, y \in E$ and $z \in \mathbb{C}$ with $\alpha_z(\mathcal{R}_n(x)) = \mathcal{R}_n(y)$. Then $x \in D(\alpha_z)$ with $\alpha_z(x) = y$.*

Proof. Consider the graph $\mathcal{G}(\alpha_z) = \{(x, \alpha_z(x)) : x \in D(\alpha_z)\}$, a closed subspace of $E \oplus E$. The one-parameter group $\beta_t = \alpha_t \oplus \alpha_t$ on $E \oplus E$ leaves $\mathcal{G}(\alpha_z)$ invariant. The hypothesis is that $(\mathcal{R}_n(x), \mathcal{R}_n(y)) \in \mathcal{G}(\alpha_z)$, and a simple calculation shows that the "smearing operator" for β is $\mathcal{R}_n \oplus \mathcal{R}_n$. Thus Lemma 3.1 applied to (β_t) shows that $(x, y) \in \mathcal{G}(\alpha_z)$, as required. □

In the norm-continuous case, we equip $D(\alpha_z)$ with the graph norm, $\|x\|_{\mathcal{G}} = \|x\| + \|\alpha_z(x)\|$ (which is the ℓ^1 norm; but clearly any complete norm would work). In the weak*-continuous case, equip $D(\alpha_z)$ with the restriction of the weak*-topology on $E \oplus_1 E = (E_* \oplus_\infty E_*)^*$ (again here any suitable norm on $E_* \oplus E_*$ would suffice). In either case, we speak of the *graph topology* on $D(\alpha_z)$.

Proposition 3.5. *Let $D_1 \subseteq D_2 \subseteq E$ be subspaces with D_1 dense in D_2, and let $z \in \mathbb{C}$. Then $\mathcal{R}_n(D_1) \subseteq \mathcal{R}_n(D_2)$ is dense in the graph topology (or, equivalently, the closure of α_z restricted to $\mathcal{R}_n(D_1)$ agrees with the closure of α_z restricted to $\mathcal{R}_n(D_2)$).*

Proof. We show the weak*-continuous case, the norm-continuous case being easier (and already shown in [27]). Let $(\alpha_{*,t})$ be the one-parameter group on E_* given by (α_t), see the discussion in Section 2.2.

For $x \in D_2$ we seek a net $(y_i) \subseteq D_1$ with $\mathcal{R}_n(y_i) \to \mathcal{R}_n(x)$ weak*, and with $\alpha_z(\mathcal{R}_n(y_i)) \to \alpha_z(\mathcal{R}_n(x))$ weak*.

Let $M \subseteq E_*$ be a finite set, and $\epsilon > 0$. We seek $y \in D_1$ with

$$\Big| \int_{\mathbb{R}} e^{-n^2 t^2} \langle \alpha_t(x - y), \mu \rangle \, dt \Big| < \epsilon \qquad (\mu \in M),$$

and with

$$\Big| \int_{\mathbb{R}} e^{-n^2 (t-z)^2} \langle \alpha_t(x - y), \mu \rangle \, dt \Big| < \epsilon \qquad (\mu \in M).$$

These inequalities would follow if we can show that $|\langle \alpha_t(x - y), \mu \rangle| < \epsilon'$ for $|t| \leq K, \mu \in M$, where K, ϵ' depend only on ϵ (and on z which is fixed). This is equivalent to

$$|\langle x - y, \alpha_{*,t}(\mu) \rangle| < \epsilon' \qquad (\mu \in M, |t| \leq K).$$

Now, the set $\{\alpha_{*,t}(\mu) : |t| \leq K, \mu \in M\}$ is compact in E_*, because M is finite and $t \mapsto \alpha_{*,t}(\mu)$ is norm continuous. Thus D_1 being weak*-dense in D_2 is enough to ensure we can choose such a y as required. □

Theorem 3.6. *Let $D \subseteq E$ be an (α_t)-invariant subspace, let $z \in \mathbb{C}$, and suppose that $D \subseteq D(\alpha_z)$. If D is dense in E, then D is a core for α_z.*

Proof. As in the proof of Proposition 3.4 we shall again consider (β_t) acting on $\mathcal{G}(\alpha_z)$. As D is (α_t)-invariant, it follows that $\mathcal{G}(\alpha_z|_D) = \{(x, \alpha_z(x)) : x \in D\}$ is (β_t)-invariant. Set $D' = \{(x, \alpha_z(x)) : x \in D\}$. Applying Proposition 3.3 to (β_t), it follows that the closures of D' and $\mathcal{R}_n(D')$ agree. Equivalently, the closure of $\alpha_z|_D$ agrees with the closure of $\alpha_z|_{\mathcal{R}_n(D)}$. Apply this with $D = D(\alpha_z)$ itself to see that

$$\overline{\alpha_z|_{\mathcal{R}_n(D(\alpha_z))}} = \alpha_z.$$

As $\mathcal{R}_n(D(\alpha_z)) \subseteq \mathcal{R}_n(E) \subseteq D(\alpha_z)$, it follows that

$$\overline{\alpha_z|_{\mathcal{R}_n(E)}} = \alpha_z.$$

As D is dense in E, it now follows from Proposition 3.5 that $\mathcal{R}_n(D)$ is a core for α_z, because $\mathcal{R}_n(E)$ is a core. Then finally applying the first part of the proof again shows that D itself is a core for α_z, as required. □

We end this section with a result purely about weak*-continuous one-parameter groups.

Proposition 3.7. *Let (α_t) be a weak*-continuous group on a dual space $E = (E_*)^*$. For any n and $x \in E$, the map $\mathbb{R} \to E; t \mapsto \alpha_t(\mathcal{R}_n(x))$ is norm continuous.*

Proof. For any fixed n, notice that the Gaussian kernel $\varphi(t) = \frac{n}{\sqrt{\pi}} \exp(-n^2t^2)$ is in $L^1(\mathbb{R})$. As the translation action of $\mathbb{R}$ on $L^1(\mathbb{R})$ is strongly continuous, we see that

$$\lim_{s\to 0} \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} |\exp(-n^2t^2) - \exp(-n^2(t-s)^2)|\, dt = 0.$$

It then follows that

$$\|\mathcal{R}_n(x) - \alpha_s(\mathcal{R}_n(x))\| \leq \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} |\exp(-n^2t^2) - \exp(-n^2(t-s)^2)| \|\alpha_t(x)\|\, dt,$$

which converges to 0 as $s \to 0$, uniformly in $\|x\|$. □

4. Applications

The previous section drew some conclusions about the operators $\mathcal{R}_n$. We now wish to present a number of applications of these conclusions, which we think demonstrates the power of these ideas. We start by giving the proof that "the dual of the generator is the generator of the dual group".

Proof of Theorem 2.17. We fix $n > 0$, and then make the key, but easy, observation that the Banach space adjoint $\mathcal{R}_n^*$ of $\mathcal{R}_n$ is the "smearing operator" of the dual group (α_t^*). By Theorem 3.6, we know that $\mathcal{R}_n(E)$ is a core for α_z, that is, $\{(\mathcal{R}_n(x), \alpha_z\mathcal{R}_n(x)) : x \in E\}$ is (norm) dense in $\mathcal{G}(\alpha_z)$. Similarly, using the key observation, $\{(\mathcal{R}_n^*(\mu), \alpha_z^{E^*}\mathcal{R}_n^*(\mu)) : \mu \in E^*\}$ is weak*-dense in $\mathcal{G}(\alpha_z^{E^*})$. Notice further that if we define $\mathcal{T}_n = \alpha_z\mathcal{R}_n$, then $\mathcal{T}_n(x) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t-z)^2)\alpha_t(x)\, dt$, from which it follows that $\mathcal{T}_n^* = \alpha_z^{E^*}\mathcal{R}_n^*$.

Let $(\mu, \lambda) \in \mathcal{G}(\alpha_z^*)$. This is equivalent to $(-\lambda, \mu) \in \mathcal{G}(\alpha_z)^\perp$, which by the previous paragraph is equivalent to

$$0 = \langle -\lambda, \mathcal{R}_n(x)\rangle + \langle \mu, \alpha_z\mathcal{R}_n(x)\rangle = \langle -\mathcal{R}_n^*(\lambda) + \mathcal{T}_n^*(\mu), x\rangle \qquad (x \in E).$$

That is, equivalent to $\mathcal{T}_n^*(\mu) = \mathcal{R}_n^*(\lambda)$. By Proposition 3.4, this is equivalent to $(\mu, \lambda) \in \mathcal{G}(\alpha_z^{E^*})$, as required. □

We now consider Theorem 2.13 which shows that if (A, A_*) is a dual Banach algebra and (α_t) a weak*-continuous automorphism group of A, then $\mathcal{G}(\alpha_z)$ is a subalgebra of $A \oplus A$.

Lemma 4.1. *Let (A, A_*) be a dual Banach algebra and let $X \subseteq A$ be a (possibly not closed) subalgebra. Then the weak*-closure of X is a subalgebra.*

Proof. Let $\overline{X}$ be the weak*-closure of X. Then $\overline{X}$ is the dual of $A_*/{}^\perp X$, and $\overline{X} = ({}^\perp X)^\perp$. That A is a dual Banach algebras is equivalent to A_* being an A-bimodule for the natural actions coming from the product on A.

For $\mu \in {}^\perp X$ and $a, b \in X$, we have that $\langle b, \mu \cdot a\rangle = \langle ab, \mu\rangle = 0$ as X is a subalgebra. Thus $\mu \cdot a \in {}^\perp X$ for each $a \in X$, and similarly, $X \cdot {}^\perp X \subseteq {}^\perp X$. Now let $x \in \overline{X}$, so for $a \in X$, we have that $\langle a, x \cdot \mu\rangle = \langle ax, \mu\rangle = \langle x, \mu \cdot a\rangle = 0$, as $x \in ({}^\perp X)^\perp$. Thus $x \cdot \mu \in {}^\perp X$, and similarly $\mu \cdot x \in {}^\perp X$. Finally, for $x, y \in \overline{X}$ and $\mu \in {}^\perp X$, we have that $\langle xy, \mu\rangle = \langle y, \mu \cdot x\rangle = 0$. Thus $xy \in \overline{X}$ as required. □

Proof of Theorem 2.13. Fix $n > 0$ and let $\mathcal{R}$ be the smearing operator $\mathcal{R}_n$ defined on A using (α_t). For $a \in A$ we have that $\mathcal{R}(a)$ is analytic so in particular $w \mapsto \alpha_w(\mathcal{R}(a))$ is norm continuous. As in the proof of Proposition 2.11 it follows that for $a, b \in A$ we have that $w \mapsto \alpha_w(\mathcal{R}(a))\alpha_w(\mathcal{R}(b))$ is analytic and extends $t \mapsto \alpha_t(\mathcal{R}(a))\alpha_t(\mathcal{R}(b)) = \alpha_t(\mathcal{R}(a)\mathcal{R}(b))$. It follows that $\mathcal{R}(a)\mathcal{R}(b) \in D(\alpha_z)$ with $\alpha_z(\mathcal{R}(a)\mathcal{R}(b)) = \alpha_z(\mathcal{R}(a))\alpha_z(\mathcal{R}(b))$.

By Theorem 3.6 we know that $X = \{(\mathcal{R}(a), \alpha_z(\mathcal{R}(a))) : a \in A\}$ is weak*-dense in $\mathcal{G}(\alpha_z)$. We have just proved that X is a subalgebra of $A \oplus A$. If we consider, say, $A \oplus_\infty A$, then this is a dual Banach algebra with predual $A_* \oplus_1 A_*$. The result follows from Lemma 4.1. □

A recurring theme in much of the rest of the paper is the following setup. Let A be a C*-algebra which is weak*-dense in a von Neumann algebra M. Suppose that (α_t) is a one-parameter automorphism group of M which restricts to a (norm-continuous) one-parameter automorphism group of A.

To avoid confusion, we shall write (α_t^M) and (α_t^A), and similarly for the analytic extensions.

Proposition 4.2. *Let M, A and (α_t) be as above, and let $z \in \mathbb{C}$. Then $D(\alpha_z^A)$ is a core for $D(\alpha_z^M)$.*

Proof. As $D(\alpha_z^A)$ is norm dense in A, it is also weak*-dense in M. The result now follows immediately from Theorem 3.6, as clearly $\mathcal{D}(\alpha_z^A)$ is (α_t^M)-invariant, because it is (α_t^A)-invariant. □

Proposition 4.3. *Let M, A and (α_t) be as above, and let $z \in \mathbb{C}$. Let $a \in A$ be such that $a \in D(\alpha_z^M)$. Then $a \in D(\alpha_z^A)$ if and only if $\alpha_z^M(a) \in A$.*

In other words, if $\mathcal{G}^M \subseteq M \oplus M$ is the graph of α_z^M, and $\mathcal{G}^A \subseteq A \oplus A$ is the graph of α_z^A, then $\mathcal{G}^M \cap (A \oplus A) = \mathcal{G}^A$.

Proof. By the definition of analytic continuation, it follows that $\mathcal{G}^A \subseteq \mathcal{G}^M$ for the inclusion $A \oplus A \subseteq M \oplus M$. Thus, if $a \in D(\alpha_z^A)$ then $\alpha_z^M(a) = \alpha_z^A(a) \in A$.

Conversely, suppose that $a \in D(\alpha_z^M)$ with $b = \alpha_z^M(a) \in A$. As (α_t) is norm continuous on A, we have that both $\mathcal{R}_n(a), \mathcal{R}_n(b) \in A$, and we obtain the same elements if we consider a norm converging integral, or a weak*-converging integral. In M, we have that $\alpha_z^M(\mathcal{R}_n(a)) = \mathcal{R}_n(\alpha_z^M(a)) = \mathcal{R}_n(b)$. However, $\alpha_z^M(\mathcal{R}_n(a))$ is equal to another integral which we can consider converging in A. Thus Proposition 3.4 applied to A gives the result. □

A more abstract result about "inclusions" of general one-parameter groups could be formulated and proved in a similar way; compare also Proposition 4.6 below. We remark that "quotients" of one-parameter groups seems a more subtle issue, see Section 6.1 below.

Example 4.4. Consider Examples 2.1 and 2.9. There we considered a one-parameter isometric group acting on the C^*-algebra $c_0(\mathbb{Z})$ and the von Neumann algebra $\ell^\infty(\mathbb{Z})$. Of course, these groups were not automorphism groups.

Consider the Hilbert space $H = \ell^2(\mathbb{Z})$ with orthonormal basis $(\delta_n)_{n\in\mathbb{Z}}$. Let (p_n) be a sequence of non-zero positive numbers, and define the (in general unbounded) positive non-degenerate operator P on H by $P(\delta_n) = p_n\delta_n$. Then $P^{it}(\delta_n) = p_n^{it}\delta_n$ for $t \in \mathbb{R}$.

Now consider $\mathcal{B}(H \oplus H)$, the bounded operators on $H \oplus H$, which we identify with 2×2 matrices with entries in $\mathcal{B}(H)$. Let $u_t = \begin{pmatrix} P^{it} & 0 \\ 0 & 1 \end{pmatrix}$ a unitary on $H \oplus H$ with $u_t^* = u_{-t}$. Then $x \mapsto \tau_t(x) = u_t x u_{-t}$ defines a weak*-continuous automorphism group on $\mathcal{B}(H \oplus H)$. We have that

$$u_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} u_{-t} = \begin{pmatrix} P^{it} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P^{-it} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} P^{it}aP^{-it} & P^{it}b \\ cP^{-it} & d \end{pmatrix}.$$

Now, $c_0(\mathbb{Z})$ acts on $\ell^2(\mathbb{Z})$ by multiplication, and commutes with P, so

$$u_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} u_{-t} = \begin{pmatrix} a & \alpha_t(b) \\ \alpha_{-t}(c) & d \end{pmatrix} \qquad (a,b,c,d \in c_0(\mathbb{Z})),$$

where $\alpha_t(a) = (p_n^{it} a_n)$ for $t \in \mathbb{R}, a = (a_n) \in c_0(\mathbb{Z})$. Thus α_t is a generalisation of the group considered in Examples 2.1 and 2.9. So (τ_t) restricts to a (norm-continuous) automorphism group of $\mathbb{M}_2(c_0(\mathbb{Z}))$. We can clearly replace $c_0(\mathbb{Z})$ by $\ell^\infty(\mathbb{Z})$ if we also replace the norm topology by the weak* topology.

We have hence embedded the one-parameter *isometry* group (α_t) into the one-parameter *automorphism* group (τ_t). In particular, Example 2.9 shows that Proposition 4.3 is false if we drop the condition that $\alpha_z^M(a) \in A$ (that is, $A \cap D(\alpha_z^M)$ can be strictly larger than $D(\alpha_z^A)$).

The reader should compare this counter-example with Theorem 6.2 below.

Let (u_t) be a strongly continuous unitary group on a Hilbert space H, and define $\tau_t(x) = u_t x u_{-t}$ for $x \in \mathcal{B}(H)$, so that (τ_t) is a weak*-continuous automorphism group. Such groups were studied in [9, Section 6].

Theorem 4.5 ([9, Theorem 6.2])**.** *With $\tau_t(x) = u_t x u_{-t}$ acting on $\mathcal{B}(H)$, we have that $x \in D(\tau_z) \subseteq \mathcal{B}(H)$ if and only if $D(u_z x u_{-z})$ is a core for u_{-z} and $u_z x u_{-z}$ is bounded. If $x \in D(\tau_z)$ then $D(u_z x u_{-z}) = D(u_{-z})$ and $\tau_z(x)$ is the closure of $u_z x u_{-z}$.*

We recall that $D(u_z x u_{-z}) = \{\xi \in D(u_{-z}) : x u_{-z}\xi \in D(u_z)\}$. If $M \subseteq \mathcal{B}(H)$ is a von Neumann algebra, and $\tau_t(M) \subseteq M$ for each $t \in \mathbb{R}$, then we obtain the restricted automorphism group (τ_t^M). If we are given an automorphism group (α_t) on M, and (u_t) on H, then a criteria for when (α_t) arises as the restriction of (τ_t), given in terms of u_{-i} and α_{-i}, is [39, Corollary 2.5]. Alternatively, for a criteria for when $\tau_t(M) \subseteq M$, given in terms of M and u_{-i}, see [41, Theorem 3.5], which follows [37, 38].

Let us record that the above characterisation also applies to $\mathcal{D}(\tau_z^M)$; notice that the conclusion is stronger than Proposition 4.3.

Proposition 4.6. *Consider (τ_t^M) as above. Then $x \in D(\tau_z^M)$ if and only if $x \in M$ with $D(u_z x u_{-z})$ is a core for u_{-z} and $u_z x u_{-z}$ is bounded. If $x \in D(\tau_z^M)$ then $D(u_z x u_{-z}) = D(u_{-z})$ and $\tau_z^M(x)$ is the closure of $u_z x u_{-z}$.*

Proof. This should be compared with [39, Corollary 2.5] mentioned above. Given such an x, let y be the closure of $u_z x u_{-z}$. By the previous theorem, there is a weak*-regular map $f : S(z) \to B(H)$ with $f(t) = \tau_t^M(x)$ for $t \in \mathbb{R}$, and with $f(z) = y$. For any $\omega \in {}^\perp M \subseteq \mathcal{B}(H)_*$ we have that $S(z) \to \mathbb{C}; w \mapsto \langle f(w), \omega\rangle$ is regular, and identically 0 on $\mathbb{R}$, and so vanishes everywhere. Thus f maps $S(z)$ into $({}^\perp M)^\perp = M$ and so $y \in M$, so $(x,y) \in \mathcal{G}(\tau_z^M)$ as required. □

4.1. Tomita-Takesaki theory. We now make some remarks about Tomita-Takesaki theory. Let M be a von Neumann algebra with φ a normal semi-finite faithful weight on M, see [33, Chapter VII]. Let $\mathfrak{n}_\varphi = \{x \in M :$

$\varphi(x^*x) < \infty\}$ and let $\Lambda : \mathfrak{n}_\varphi \to H$ be the GNS map. Then $\mathfrak{A} = \Lambda(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ is a full left Hilbert algebra, and Tomita-Takesaki theory gives rise to the modular conjugation J on H, and the modular operator Δ which implements the modular automorphism group $\sigma_t(\cdot) = \Delta^{it}(\cdot)\Delta^{-it}$.

There is a direct link between σ_{-i} and φ, which we quote for the sake of interest.

Proposition 4.7 (See [15, Section 3] or [33, Theorem 3.25, Chapter VIII])**.** *For $a, b \in M$ the following are equivalent:*

(1) $(a, b) \in \mathcal{G}(\sigma_{-i})$;
(2) $a\mathfrak{n}_\varphi^* \subseteq \mathfrak{n}_\varphi^*, \mathfrak{n}_\varphi b \subseteq \mathfrak{n}_\varphi$ *and* $\varphi(ax) = \varphi(xb)$ *for* $x \in \mathfrak{n}_\varphi^*\mathfrak{n}_\varphi$.

Proposition 4.8. *Let M be a von Neumann algebra with a nsf weight φ on M, with GNS construction (H, Λ, π), and modular automorphism group (σ_t). Let $\mathfrak{A}_0 \subseteq H$ be the Tomita algebra, with modular automorphism group (σ_t^0) and representation $\pi_L : \mathfrak{A}_0 \to M$, so that $\pi_L \circ \sigma_t^0 = \sigma_t \circ \pi_L$. Then $\pi_L(\mathfrak{A}_0)$ is a core for σ_z on M.*

Proof. As $\pi_L(\mathfrak{A}_0)$ generates M, this follows immediately from Theorem 3.6. □

As an illustration of the utility of the ideas developed and summarised so far, we now wish to give a short proof of the results of [6], where careful calculation with functional calculus and unbounded operator techniques were used. We will abstract the setting of [6] away from Markov operators, and instead work in the following setting: we have Hilbert spaces H, K and positive non-degenerate (unbounded) operators Δ_H, Δ_K on H and K respectively. Thus $(\Delta_H^{it})_{t\in\mathbb{R}}$ is a one-parameter (strongly-continuous) unitary group on H, and similarly for (Δ_K^{it}) on K. Suppose we have a bounded operator $T : H \to K$ with $T\Delta_H^{it} = \Delta_K^{it}T$ for all $t \in \mathbb{R}$.

Proposition 4.9 ([6, Theorem 1.1])**.** *With the above setup, we have that $\Delta_K^{-t}T\Delta_H^t$ is densely-defined, and bounded, with closure T, for each $t \in \mathbb{R}$.*

Proof. This follows almost immediately from [9, Theorem 6.2], compare Theorem 4.5. Indeed, we define a weak*-continuous one-parameter isometry group on $\mathcal{B}(H, K)$ by $\alpha_t(x) = \Delta_K^{-it}x\Delta_H^{it}$. The hypothesis on T is precisely that $\alpha_t(T) = T$ for all t, and from this it follows that T is analytic for (α_t) and $\alpha_z(T) = T$ for all z. In particular, $T \in D(\alpha_{-it})$ with $\alpha_{-it}(T) = T$, so from [9, Theorem 6.2], it follows that $D(\Delta_K^{-t}T\Delta_H^t) = D(\Delta_H^t)$ and $\Delta_K^{-t}T\Delta_H^t$ (which is thus densely-defined) has bounded closure equal to T, as required. □

Along the way, [6] proves more, and in particular [6, Theorem 3.1], in our more abstract setting, is the following result, which we think is interesting in its own right.

Theorem 4.10. *With the above setup, for any $z \in \mathbb{C}$, we have that $T\Delta_H^z$ is closeable, with closure $\Delta_K^z T$.*

Proof. From [9, Theorem 6.2] (as applied in the above proof), we know that $D(\Delta_K^{-z}T\Delta_H^z) = D(\Delta_H^z)$ and $\Delta_K^{-z}T\Delta_H^z \subseteq T$. Equivalently, that for $\xi \in D(\Delta_H^z)$, we have that $T\Delta_H^z\xi \in D(\Delta_K^{-z})$ with $\Delta_K^{-z}T\Delta_H^z\xi = T\xi$. Equivalently, $T\Delta_H^z \subseteq \Delta_K^z T$. As $\Delta_K^z T$ is always closed, we have in fact that $\overline{T\Delta_H^z} \subseteq \Delta_K^z T$.

Consider the unitary group (Δ_K^{it}) and for $n > 0$ form $R_K = \mathcal{R}_n$ on K. Similarly form R_H. As $T\Delta_H^{it} = \Delta_K^{it}T$, it follows that $R_K T = TR_H$. Furthermore, for any $\xi \in H$,

$$\begin{aligned} T\Delta_H^z R_H\xi &= \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2(t-z)^2} T\Delta_H^{it}\xi \; dt \\ &= \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2(t-z)^2} \Delta_K^{it}T\xi \; dt = \Delta_K^z R_K T\xi, \end{aligned}$$

Let $(\xi,\eta) \in \mathcal{G}(\Delta_K^z T)$, that is, $(T\xi,\eta) \in \mathcal{G}(\Delta_K^z)$. Thus $(TR_H\xi, R_K\eta) = (R_K T\xi, R_K\eta) \in \mathcal{G}(\Delta_K^z)$, that is, $\Delta_K^z R_K T\xi = R_K\eta$, but from above,

$$\Delta_K^z R_K T\xi = T\Delta_H^z R_H\xi,$$

and so

$$(R_H\xi, R_K\eta) = (R_H\xi, T\Delta_H^z R_H\xi) \in \mathcal{G}(T\Delta_H^z) \subseteq \mathcal{G}(\overline{T\Delta_H^z}).$$

By Lemma 3.2 we see that (ξ,η) is in the norm-closed $(\Delta_H^{it}, \Delta_K^{it})$-invariant subspace of $H \oplus K$ generated by $(R_H\xi, R_K\eta)$. However, this shows that $(\xi,\eta) \in \mathcal{G}(\overline{T\Delta_H^z})$. In turn, this shows that $\Delta_K^z T \subseteq \overline{T\Delta_H^z}$.

Hence we have that $\Delta_K^z T = \overline{T\Delta_H^z}$ as claimed. □

The setup of [6] is actually as follows: $\Phi : (N,\rho) \to (M,\varphi)$ is a (ρ,φ)-Markov map and T is defined by $Tx\xi_\rho = \Phi(x)\xi_\varphi$ for $x \in N$. The second part of [6, Theorem 1.1] shows that T also intertwines the modular conjugations J_ρ and J_φ. This follows readily, as

$$J_\varphi T J_\rho x\xi_\rho = J_\varphi T\sigma_{i/2}^\rho(x)^*\xi_\rho = J_\varphi\Phi(\sigma_{i/2}^\rho(x)^*)\xi_\rho.$$

As $\Phi\sigma_t^\rho = \sigma_t^\varphi\Phi$ for each $t \in \mathbb{R}$, an analytic continuation argument shows that $\Phi\sigma_{i/2}^\rho \subseteq \sigma_{i/2}^\varphi\Phi$. Thus

$$J_\varphi T J_\rho x\xi_\rho = J_\varphi\sigma_{i/2}^\varphi(\Phi(x)^*)\xi_\rho = \Phi(x)\xi_\rho = Tx\xi_\rho.$$

Thus $J_\varphi T J_\rho = T$.

4.2. The graph as a Banach algebra. When A is a Banach algebra and (α_t) a one-parameter automorphism group, we have seen that $\mathcal{G}(\alpha_z) \subseteq A \oplus A$ is a (closed) subalgebra; and when A is a dual Banach algebra and (α_t) weak*-continuous, then $\mathcal{G}(\alpha_z)$ is also a dual Banach algebra.

If further A is a Banach $*$-algebra, then let us consider $\mathcal{G}(\alpha_{-i})$ (here any member of $i\mathbb{R}$ would lead to similar conclusions). Given $a \in D(\alpha_{-i})$, by Proposition 2.12, we have that $a^* \in D(\alpha_i)$ with $\alpha_i(a^*) = \alpha_{-i}(a)^*$. In particular, $\alpha_{-i}(a)^* \in D(\alpha_{-i})$ with $\alpha_{-i}(\alpha_{-i}(a)^*) = a^*$. It follows that $(a,b) \in \mathcal{G}(\alpha_{-i})$ if and only if $(b^*,a^*) \in \mathcal{G}(\alpha_{-i})$. For $a \in D(\alpha_{-i})$ write $a^\natural = \alpha_{-i}(a)^*$.

Thus $\mathcal{G}(\alpha_{-i})$ becomes a Banach $*$-algebra. Similar considerations apply to the dual Banach algebra case.

To our knowledge, there has been little systematic study of these Banach $*$-algebras. There is an intriguing result stated without proof in [40], which gives a characterisation of which algebras $\mathcal{G}(\alpha_{-i})$ can arise, in the case of a weak*-continuous one-parameter automorphism group (α_t) on a von Neumann algebra M. In particular, for $x \in M$ invertible, there is a (unique) unitary $u \in M$ such that $ux, (ux)^{-1}$ are both member of $D(\alpha_{-i})$ with $\alpha_{-i/2}(ux), \alpha_{-i/2}((ux)^{-1})$ both positive. A proof of this factorisation result may be found in [41, Section 3], which in turn uses ideas from [37].

Example 4.11. Let $A = C_0(\mathbb{R})$ and let (α_t) be the "translation group" defined by $\alpha_t(f)(s) = f(s-t)$ for $s, t \in \mathbb{R}, f \in C_0(\mathbb{R})$. Suppose that $f \in D(\alpha_{-i})$ with analytic extension $F : S(-i) \to C_0(\mathbb{R})$. Define $g : S(i) \to \mathbb{C}$ by $g(w) = F(-w)(0)$ so that $g(t) = F(-t)(0) = \alpha_{-t}(f)(0) = f(0-(-t)) = f(t)$ for $t \in \mathbb{R}$. Thus g is (scalar-valued) regular and extends f, and $F(-i)(t) = \alpha_{-t}(F(-i))(0) = F(-i-t)(0) = g(i+t)$ so $\alpha_{-i}(f) = (g(i+t))_{t\in\mathbb{R}}$. As F is continuous, g must satisfying the "uniformly in C_0 condition" that, for $\epsilon > 0$, there is $K > 0$ so that $|g(x+iy)| < \epsilon$ if $|x| > K$ (for any $0 \leq y \leq 1$).

Conversely, suppose that $f \in C_0(\mathbb{R})$ admits such an extension g to $S(i)$ (so g is "uniformly in C_0"). Define $F : S(-i) \to C_0(\mathbb{R})$ by $F(w) = (g(-w+t))_{t\in\mathbb{R}}$. Then each $F(w) \in C_0(\mathbb{R})$ and $F(t) = (f(-t+s))_{s\in\mathbb{R}} = \alpha_t(f)$. Furthermore, F is norm-continuous (from the condition on g). To show that F is analytic on the interior of $S(-i)$, we need only show that $\mu \circ F$ is (scalar) analytic for each $\mu \in X$ where $X \subseteq C_0(\mathbb{R})^*$ is any norming subspace. If we take X to be the closed span of the point-mass measures (so $X = \ell^1(\mathbb{R})$) this follows immediately from g being analytic. Thus F is regular and so $f \in D(\alpha_{-i})$ with $\alpha_{-i}(f)(t) = F(-i)(t) = g(t+i)$ for $t \in \mathbb{R}$.

Thus $\mathcal{G}(\alpha_{-i})$ may be identified with a space of scalar-valued regular functions on the strip $S(i)$, which we could think of as some sort of "generalised Hardy space".

Similarly, (α_t) extends to a weak*-continuous automorphism group on $L^\infty(\mathbb{R})$. A slightly more involved argument, making use of the smearing technique, similarly allows us to regard $\mathcal{G}(\alpha_{-i})$ as being the subspace of $L^\infty(S(i))$ consisting of functions analytic on the interior of $S(i)$, and having suitable boundary values.

There are related Banach algebras which have been more studied. We first quickly recall Arveson's notion of spectral subspace from [4, Section 2]. In our setting, these are studied in [9, Section 5] and [39], see in particular the comment at the bottom on page 86 in [39]. These are subspaces of elements which are analytic for (α_t), and which have certain growth rates at infinity.

To be more precise, for example, let M be a von Neumann algebra and (α_t) a weak*-continuous automorphism group of M. In particular, following [39], we define $M^\alpha([1,\infty))$ to be the collection of $x \in M$ such that $x \in D(\alpha_{in})$

for $n = 1, 2, \cdots$ and $\limsup_n \|\alpha_{in}(x)\|^{1/n} \leq 1$. This space is often denoted by $H^\infty(\alpha)$; indeed, it is shown in [19, Proposition 2.1] that $x \in M^\alpha([1,\infty))$ if and only if $t \mapsto \langle \alpha_{-t}(x), \omega\rangle$ is in $H^\infty(\mathbb{R})$, for each $\omega \in M_*$. We say that M is α-*finite* when the collection $\{\omega \in M_*^+ : \omega \circ \alpha_t = \omega \ (t \in \mathbb{R})\}$ separates the points of M_+. In this case, $H^\infty(\alpha)$ is a *maximal subdiagonal algebra* in the sense of [3]. For more on this topic, see [17, 25] for example. Maximal subdiagonal algebras have been widely studied as non-commutative analogues of Hardy spaces.

Example 4.12. Consider $L^\infty(\mathbb{R})$ with the shift group, as in Examples 4.11. Then $H^\infty(\alpha)$ is simply the classical Hardy space of the upper-half-plane $H^\infty(\mathbb{R})$, see [25, Introduction].

As in Example 4.4, let $P(\delta_n) = p_n\delta_n$ on $\ell^2 = \ell^2(\mathbb{N})$, and define $\alpha_t(x) = P^{it}xP^{-it}$ for $x \in \mathcal{B}(\ell^2)$. Consider the matrix unit e_{jk} which sends δ_k to δ_j. Then $\alpha_t(e_{jk}) = p_k^{-it}p_j^{it}e_{jk}$ and so $\alpha_{in}(e_{jk}) = p_k^n p_j^{-n}e_{jk}$ for each $n = 1, 2, \cdots$. It follows that $e_{jk} \in H^\infty(\alpha)$ if and only if $p_k/p_j \leq 1$. If (p_n) is an increasing sequence, then $e_{jk} \in H^\infty(\alpha)$ exactly when $k \leq j$. A more involved calculation shows that $H^\infty(\alpha)$ consists exactly of the lower-triangular matrices.

While $\mathcal{G}(\alpha_{-i})$ is clearly different from $H^\infty(\alpha)$, there are some intriguing similarities. For example, the factorisation result of Zsido mentioned above, [40], is very similar to Arveson's factorisation result, [3, Section 4.2], showing that if $x \in M$ is invertible then there is $a \in H^\infty(\alpha)$ with $a^{-1} \in H^\infty(\alpha)$, and a unitary $u \in M$, with $x = ua$. We wonder if there is further to be developed here; in particular, is there a notion of L^p space for $\mathcal{G}(\alpha_{-i})$, similar to that for subdiagonal algebras, compare [26]?

Remark 4.13. Consider $\mathcal{A} = \mathcal{G}(\alpha_{-i})$ as a weak*-closed subalgebra of $M \oplus_\infty M$. Let $(x, y) \in \mathcal{A} \cap \mathcal{A}^*$ so $(x^*, y^*) \in \mathcal{A}$ so $(y, x) \in \mathcal{A}$ (given the above remarks). There are hence weak*-regular maps $f, g : S(-i) \to M$ with $f(t) = \alpha_t(x), g(t) = \alpha_t(y)$ for $t \in \mathbb{R}$ and $f(-i) = y, g(-i) = x$. It follows that $f(t - i) = g(t)$ and so "glueing" these maps together we obtain $h : S(-2i) \to M$ which by Morera's Theorem is regular, has $h(t) = \alpha_t(x)$ and $h(t - 2i) = g(-i) = x = h(t)$ for $t \in \mathbb{R}$. By "tiling" we can extend h to an entire map on $\mathbb{C}$ which is bounded, and hence constant. This shows that $x = y$ and $\alpha_t(x) = x$ for all t. We conclude that $\mathcal{A} \cap \mathcal{A}^* = \{(x, x) : x \text{ is } (\alpha_t)\text{-invariant}\}$.

Now consider when $\mathcal{A}+\mathcal{A}^*$ is weak*-dense in $M \oplus M$. If $(\omega, \tau) \in {}^\perp(\mathcal{A}+\mathcal{A}^*)$ then $(\omega, \tau) \in {}^\perp\mathcal{A} \cap {}^\perp\mathcal{A}^*$ so $(-\tau, \omega), (-\tau^*, \omega^*) \in \mathcal{G}(\alpha_{-i}^{M_*})$. Arguing as in the previous paragraph, this is if and only if $\omega = -\tau$ is $(\alpha_t^{M_*})$-invariant. Now, ω is $(\alpha_t^{M_*})$-invariant if and only if $\omega \in {}^\perp X$ where X is the weak*-closed linear span of $\{x - \alpha_t(x) : x \in M, t \in \mathbb{R}\}$. It follows that (x, y) is in the weak*-closure of $\mathcal{A}+\mathcal{A}^*$ if and only if $\langle (x, y), (\omega, -\omega)\rangle = 0$ for each $\omega \in {}^\perp X$, that is, $x - y \in ({}^\perp X)^\perp = X$.

For $\mathcal{A}$ to be a (finite, maximal) subdiagonal algebra of $M \oplus M$ we would want that $\mathcal{A} \cap \mathcal{A}^*$ to be the range of a faithful normal conditional expectation,

and we'd want X to be all of M (equivalently, there to be no non-zero $(\alpha_t^{M_*})$-invariant functionals). If (α_t) is trivial, then this is obviously not the case. For the shift-group on $L^\infty(\mathbb{R})$, however, we do have that $\mathcal{A}+\mathcal{A}^*$ is weak*-dense in $M \oplus M$, and $\mathcal{A} \cap \mathcal{A}^*$ is $\mathbb{C}(1,1)$, but there are no *normal* conditional expectations $M \oplus M$ to $\mathbb{C}(1,1)$ which are *multiplicative* on $\mathcal{G}(\alpha_{-i})$.

We finish this section with one general Banach algebraic result.

Proposition 4.14. *Let A be a Banach algebra with a bounded approximate identity bounded by $M \geq 1$. Let (α_t) be a (norm-continuous) automorphism group on A. For any z we have that $\mathcal{G}(\alpha_z)$ has a bounded approximate identity bounded by $M \geq 1$.*

Proof. We just give a sketch, as this could be proved exactly as [20, Proposition 2.26] (which is attributed to Van Daele and Verding); compare also the proof of [10, Theorem 12]. Indeed, as A has a bounded approximate identity, it admits a theory of multiplier algebras paralleling that of C^*-algebras. Instead of developing this theory, we give a direct proof.

Let (e_i) be a bounded approximate identity with $\|e_i\| \leq M$ for each i. The key idea is to consider $\mathcal{R}_n(e_i)$ with $n > 0$ *small* and not large. This will ensure that $\|\alpha_z \mathcal{R}_n(e_i)\|$ will be close to M. For $a \in A$, as $t \mapsto \alpha_t(a)$ is norm-continuous, for any $K > 0$ the set $\{\alpha_{-t}(a) : |t| \leq K\}$ is compact, and so $e_i \alpha_{-t}(a) \to \alpha_{-t}(a)$ uniformly for $|t| \leq K$. It follows that $\alpha_t(e_i)a = \alpha_t(e_i \alpha_{-t}(a)) \to \alpha_t(\alpha_{-t}(a)) = a$ uniformly for $|t| \leq K$. By the integral form of $\mathcal{R}_n$ and $\alpha_z \mathcal{R}_n$, it follows that if i is sufficiently large, then $\|\mathcal{R}_n(e_i)a - a\|$ and $\|\alpha_z(\mathcal{R}_n(e_i))a - a\|$ will be small. In this way, we can construct a bounded approximate identity in $\mathcal{G}(\alpha_z)$ with the required bound. □

5. A Kaplansky density type result

We again consider the case of a C^*-algebra generating a von Neumann algebra M, with a one-parameter automorphism group on M restricting to A. The Kaplansky Density Theorem tells us that the unit ball of A is weak*-dense in the unit ball of M. This section is devoted to proving the following; recall that Proposition 4.2 shows that $\mathcal{G}(\alpha_z^A)$ is weak*-dense in $\mathcal{G}(\alpha_z^M)$.

Theorem 5.1. *With $A, M, (\alpha_t)$ as before, let $z \in \mathbb{C}$, let α_z^A be the analytic extension on A, and α_z^M that on M. In $M \oplus_\infty M$, or $M \oplus_1 M$, the unit ball of $\mathcal{G}(\alpha_z^A)$ is weak*-dense in the unit ball of $\mathcal{G}(\alpha_z^M)$.*

Let M_* be the predual of M. By restricting functionals in M_* to $A \subseteq M$, we define a map $\iota : M_* \to A^*$. By Kaplansky Density, this map is an isometry. It is easy to see that it preserves the A-module actions, and so M_* is identified with a closed A-subbimodule of A^*. By [32, Section 2, Chapter III] there is a central projection $p \in A^{**}$ with $pA^* = A^*p = M_*$. In fact, we construct p by noticing that $M_*^\perp = \{x \in A^{**} : \langle x, \omega \rangle = 0 \ (\omega \in M_* \subseteq A^*)\}$ is a weak*-closed ideal in A^{**} and so $M_*^\perp = A^{**}p'$ for some

central projection $p' \in A^{**}$; we then set $p = 1 - p'$. We furthermore have that

$$A^* \cong pA^* \oplus_1 (1-p)A^*, \qquad A^{**} \cong pA^{**} \oplus_\infty (1-p)A^{**}.$$

Lemma 5.2. *Let β be a $*$-automorphism of A, and suppose that $\beta^*(M_*) \subseteq M_*$. Then $\beta^{**}(p) = p$. In particular, $\alpha_t^{**}(p) = p$ for all t.*

Proof. We have that β^{**} is a $*$-automorphism of A^{**}, and so $q = \beta^{**}(p)$ is a central projection. Then $(1-q)A^{**} = \beta^{**}((1-p)A^{**}) = \beta^{**}(M_*^\perp) = M_*^\perp$ as $\beta^*(M_*) = M_*$. Thus $(1-q)A^{**} = M_*^\perp$ and so $q = p$. □

In the following lemma, we identify A with a subspace of A^{**} in the canonical way.

Lemma 5.3. *For $a \in A$ and $s, t \in \mathbb{R}$ we have that $\alpha_s^{**}(p\alpha_t(a)) = p\alpha_{s+t}(a)$.*

Proof. As α_s^{**} is an automorphism, and using Lemma 5.2, we have that $\alpha_s^{**}(p\alpha_t(a)) = p\alpha_s^{**}(\alpha_t(a))$. A simple calculation shows that for $b \in A$, we have that $\alpha_s^{**}(b)$ is equal to the image of $\alpha_s(b) \in A$ in A^{**}. The result follows. □

To easy notation, fix $z \in \mathbb{C}$ and let $\mathcal{G} = \mathcal{G}(\alpha_z^A)$ regarded as a subspace of $A \oplus_\infty A$. Similarly let $\mathcal{G}^M = \mathcal{G}(\alpha_z^M)$ regarded as a subspace of $M \oplus_\infty M$. Notice that the dual space of $A \oplus_\infty A$ is $A^* \oplus_1 A^*$, and the bidual is $A^{**} \oplus_\infty A^{**}$. Then (p,p) is a central projection in $A^{**} \oplus_\infty A^{**}$. By the Hanh-Banach theorem, we can identify the dual space of $\mathcal{G}$ with the quotient $(A^* \oplus_1 A^*)/\mathcal{G}^\perp$, and in turn identify the dual of this quotient with $\mathcal{G}^{\perp\perp}$. Thus $\mathcal{G}^{**} = \mathcal{G}^{\perp\perp}$.

Theorem 5.4. *We have that $(p,p)\mathcal{G}^{\perp\perp} \subseteq \mathcal{G}^{\perp\perp} \subseteq A^{**} \oplus A^{**}$.*

Proof. Let $a \in A$, let $n > 0$, and define $f : S(z) \to A^{**}$ by

$$f(w) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t-w)^2) p\alpha_t(a)\ dt.$$

As $t \mapsto \alpha_t(a)$ is norm-continuous, also $t \mapsto p\alpha_t(a)$ is norm-continuous, and so the integral defining f is norm convergent, and f is norm-regular. In fact, we have that $f(w) = p\alpha_w(\mathcal{R}_n(a))$. From Lemma 5.3, we have that $\alpha_s^{**}(f(w)) = f(w+s)$ for $w \in S(z), s \in \mathbb{R}$.

Let $(-\lambda, \mu) \in \mathcal{G}^\perp$, which is equivalent to $\mu \in D(\alpha_z^{A^*})$ with $\alpha_z^{A^*}(\mu) = \lambda$. Thus there is $g : S(z) \to A^*$ a weak*-regular function with $g(t) = \alpha_t^*(\mu)$ for each $t \in \mathbb{R}$, and with $g(z) = \lambda$.

Define $h : S(z) \to \mathbb{C}$ by $h(w) = \langle f(w), g(z-w)\rangle$. Then

$$h(t) = \langle f(t), g(z-t)\rangle = \langle \alpha_t^{**}(f(0)), \alpha_{-t}^*(\lambda)\rangle = \langle f(0), \lambda\rangle \qquad (t \in \mathbb{R}).$$

Thus h is constant on $\mathbb{R}$. Furthermore, for $w \in S(z)$,

$$h(w) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t-w)^2) \langle p\alpha_t(a), g(z-w)\rangle\ dt,$$

here again using that the integral defining f is norm-convergent. Now, $\langle p\alpha_t(a), g(z-w)\rangle = \langle pg(z-w), \alpha_t(a)\rangle$, and so

$$\begin{aligned} h(w) &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t-w)^2)\langle pg(z-w), \alpha_t(a)\rangle \, dt \\ &= \langle pg(z-w), \alpha_w(\mathcal{R}_n(a))\rangle. \end{aligned}$$

As $w \mapsto \alpha_w(\mathcal{R}_n(a))$ is a norm-continuous map, and $w \mapsto pg(z-w)$ is bounded and weak*-continuous, it follows that h is continuous on $S(z)$. On the interior of $S(z)$, we have that h is the pairing between two functions given locally by power series. We conclude that h is regular. As h is constant on $\mathbb{R}$, h must be constant on $S(z)$.

Thus

$$\begin{aligned} \langle (p,p)(\mathcal{R}_n(a), \alpha_z(\mathcal{R}_n(a))), (-\lambda, \mu)\rangle &= \langle -p\lambda, \mathcal{R}_n(a)\rangle + \langle p\mu, \alpha_z(\mathcal{R}_n(a))\rangle \\ &= -\langle f(0), \lambda\rangle + \langle f(z), \mu\rangle = -h(0) + h(z) = 0. \end{aligned}$$

By Theorem 3.6, $\{(\mathcal{R}_n(a), \alpha_z(\mathcal{R}_n(a))) : a \in A\}$ is norm dense in $\mathcal{G}$, and as $(-\lambda, \mu) \in \mathcal{G}^\perp$ was arbitrary, the above calculation shows that $(p,p)\mathcal{G} \subseteq \mathcal{G}^{\perp\perp}$. By weak*-continuity, we conclude that $(p,p)\mathcal{G}^{\perp\perp} \subseteq \mathcal{G}^{\perp\perp}$ as claimed. □

Lemma 5.5. *Let $\mathfrak{A}$ be a dual Banach algebra, let $X \subseteq \mathfrak{A}$ be a weak*-closed subspace, let $p \in \mathfrak{A}$ be an idempotent (so $p^2 = p$) and suppose that $pX \subseteq X$. Then pX is weak*-closed.*

Proof. Let (x_i) be a net in X with $px_i \to a \in \mathfrak{A}$ weak*. We aim to show that $a \in pX$. Now, $px_i = p^2x_i \to pa$ as $\mathfrak{A}$ is a dual Banach algebra. Thus $a = pa$. Now, also $px_i \in pX \subseteq X$, by hypothesis, and as X is weak*-closed, $a \in X$. Thus $a = pa \in pX$ as required. □

As above, as $A \subseteq M$, restriction of functionals gives $\iota : M_* \to A^*$, which is an isometric inclusion by Kaplansky density. Furthermore, we have that $pA^* = \iota(M_*)$, and so we have an inverse map $\iota^{-1} : pA^* \to M_*$ and so the Banach space adjoint is a map $(\iota^{-1})^* : M \to (pA^*)^* \cong pA^{**}$. We give a word of warning: the composition of the isometries $A \to M \cong pA^{**} \to A^{**}$ is *not* the canonical map $A \to A^{**}$, but is rather the map $a \mapsto pa \in A^{**}$.

Lemma 5.6. *Identifying $M \oplus M$ with $pA^{**} \oplus pA^{**}$, and regarding $\mathcal{G}$ as a subspace of $A^{**} \oplus A^{**}$ in the canonical way, we have that $(p,p)\mathcal{G} \subseteq \mathcal{G}^M$.*

Proof. Denote by ϕ the corestriction of ι, so ϕ is an isometric isomorphism $M_* \to pA^*$, and hence $\phi^* : pA^{**} \to M$ is an isomorphism, the inverse of $(\iota^{-1})^*$. Similarly, let $\psi : A \to M$ be the inclusion. For $a \in A$ and $\omega \in M_*$,

$$\langle \phi^*(pa), \omega\rangle = \langle pa, \phi(\omega)\rangle = \langle p\phi(\omega), a\rangle = \langle p\iota(\omega), a\rangle = \langle \iota(\omega), a\rangle = \langle \psi(a), \omega\rangle.$$

This $\phi^*(pa) = \psi(a)$. As $(\psi \oplus \psi)\mathcal{G} \subseteq \mathcal{G}^M$, the result follows. □

Theorem 5.7. *Identifying $M \oplus M$ with $pA^{**} \oplus pA^{**}$, we have $(p,p)\mathcal{G}^{\perp\perp} = \mathcal{G}^M$.*

Proof. Given $(\alpha, \beta) \in \mathcal{G}^{\perp\perp}$ there is a bounded net (a_i, b_i) in $\mathcal{G}$ converging weak* to α. Then $(p\alpha, p\beta)$ is the weak*-limit of the net (pa_i, pb_i), and by Lemma 5.6 we know that $pa_i \in \mathcal{G}^M$ for each i. As $\mathcal{G}^M \subseteq pA^{**} \oplus pA^{**}$ is weak*-closed, we conclude that $(p,p)\mathcal{G}^{\perp\perp} \subseteq \mathcal{G}^M$.

We apply Lemma 5.5 to $A^{**} \oplus A^{**}$ and the idempotent (p,p), with the subspace $\mathcal{G}^{\perp\perp}$. By Theorem 5.4, the hypothesis of Lemma 5.5 holds, and so $(p,p)\mathcal{G}^{\perp\perp}$ is weak*-closed.

Given $(x,y) \in \mathcal{G}^M$, by Proposition 4.2, there is a net (perhaps *not* bounded) (a_i, b_i) in $\mathcal{G}$ converging weak* to (x,y) in $M \oplus M$. As $M \to pA^{**}$ is weak*-continuous, it follows that the net (pa_i, pb_i) converges weak* to $(x,y) \in \mathcal{G}^M \subseteq pA^{**} \oplus pA^{**}$. This net is in $(p,p)\mathcal{G} \subseteq (p,p)\mathcal{G}^{\perp\perp}$, and as $(p,p)\mathcal{G}^{\perp\perp}$ is weak*-closed, we conclude that $(x,y) \in (p,p)\mathcal{G}^{\perp\perp}$. Thus $\mathcal{G}^M \subseteq (p,p)\mathcal{G}^{\perp\perp}$ and we have equality. □

Our main theorem now follows easily.

Proof of Theorem 5.1. Given a member of the unit ball of $\mathcal{G}^M$, we regard $\mathcal{G}^M$ as being $(p,p)\mathcal{G}^{\perp\perp} \subseteq \mathcal{G}^{\perp\perp}$, and so we have a member of the unit ball of $\mathcal{G}^{\perp\perp} = \mathcal{G}^{**}$. By Hahn-Banach (that is, the Goldstine theorem) there is a net in the unit ball of $\mathcal{G}$ converging weak* to our element of $\mathcal{G}^M$, as we want.

To deal with the $\oplus_1$ normed case, we simply follow the same proof through, using $pA^* \oplus_\infty (1-p)A^*$ and $pA^{**} \oplus_1 (1-p)A^{**}$. While $pA^{**} \oplus_1 (1-p)A^{**}$ is not a C^*-algebra, it is still a Banach algebra, and so Lemma 5.5 still holds, and the rest follows. □

We finish with a result about stronger topologies.

Corollary 5.8. *With the hypotheses of Theorem 5.1, the unit ball of $\mathcal{G}(\alpha_z^A)$, in $M \oplus_\infty M$, is σ-strong*-dense in the unit ball of $\mathcal{G}(\alpha_z^M)$.*

Proof. This follows immediately from [32, Theorem 2.6(iv)] that in a von Neumann algebra N, for a convex subset K we have that the weak* and σ-strong* closures of K agree. □

6. Duals of automorphism groups

In this section, we shall look at the "dual" situation to the previous section. We again consider the case of a C^*-algebra generating a von Neumann algebra M, with a one-parameter automorphism group (α_t^M) on M restricting to A, say to given (α_t^A). Then the preadjoint gives a (norm-continuous) one-parameter isometry group $(\alpha_t^{M_*})$ on M_*, and the adjoint gives a (weak*-continuous) one-parameter isometry group $(\alpha_t^{A^*})$ on A^*. A simple calculation shows that the inclusion $\iota : M_* \to A^*$ intertwines these groups.

Proposition 6.1. *For $z \in \mathbb{C}$, we have that $D(\alpha_z^{M_*})$ is a (weak*) core for $D(\alpha_z^{A^*})$.*

Proof. Follows exactly as the proof of Proposition 4.2. □

The following is a stronger version of Proposition 4.3. We recall that the analogous result for the inclusion $A \to M$ is false, see Example 4.4.

Theorem 6.2. *Let $\omega \in M_*$ be such that $\iota(\omega) \in D(\alpha_z^{A^*})$. Then $\omega \in D(\alpha_z^{M_*})$.*

Proof. We continue with the notations of the previous section. Theorem 5.4 shows that $(p,p)\mathcal{G}^{\perp\perp} \subseteq \mathcal{G}^{\perp\perp}$ and so any $(\alpha,\beta) \in \mathcal{G}^{\perp\perp}$ is equal to

$$(p\alpha, p\beta) + ((1-p)\alpha, (1-p)\beta),$$

where both summands are members of $\mathcal{G}^{\perp\perp}$.

Let $(\mu,\lambda) \in \mathcal{G}(\alpha_z^{A^*})$, equivalently, $(-\lambda,\mu) \in \mathcal{G}^{\perp}$, or equivalently,

$$\langle (\alpha,\beta), (-\lambda,\mu)\rangle = 0$$

for all $(\alpha,\beta) \in \mathcal{G}^{\perp\perp}$. Given the above discussion, this in turn is equivalent to

$$\langle (p\alpha,p\beta), (-\lambda,\mu)\rangle = \langle ((1-p)\alpha,(1-p)\beta), (-\lambda,\mu)\rangle = 0 \quad ((\alpha,\beta) \in \mathcal{G}^{\perp\perp}).$$

That is,

$$\langle (\alpha,\beta), (-p\lambda,p\mu)\rangle = \langle (\alpha,\beta), (-(1-p)\lambda,(1-p)\mu)\rangle = 0 \quad ((\alpha,\beta) \in \mathcal{G}^{\perp\perp}).$$

Reversing this argument shows that $(\mu,\lambda) \in \mathcal{G}(\alpha_z^{A^*})$ if and only if both $(p\mu,p\lambda) \in \mathcal{G}(\alpha_z^{A^*})$ and $((1-p)\mu,(1-p)\lambda) \in \mathcal{G}(\alpha_z^{A^*})$.

In particular, if $(\mu,\lambda) \in \mathcal{G}(\alpha_z^{A^*})$ with $\mu \in M_*$, that is, $p\mu = \mu$, then $(\mu,p\lambda) \in \mathcal{G}(\alpha_z^{A^*})$, but as this is a graph, it follows that $\lambda = p\lambda$, that is, $\lambda \in M_*$. The result now follows as in the proof of Proposition 4.3. □

6.1. Quotients. Let E be a Banach space and (α_t) a norm-continuous one-parameter group of isometries. Suppose that $F \subseteq E$ is a closed subspace with $\alpha_t(F) \subseteq F$ for each t. It is easy to see that $E/F \to E/F; x+F \mapsto \alpha_t(x)+F$ is a well-defined contraction for each t. We hence obtain a norm-continuous one-parameter group of isometries $(\alpha_t^{E/F})$ on E/F. By considering analytic continuations, it is easy to see that if $(x,y) \in \mathcal{G}(\alpha_z)$ then $(x+F, y+F) \in \mathcal{G}(\alpha_z^{E/F})$.

Proposition 6.3. *For any z we have that $D(\alpha_z)+F \subseteq E/F$ is a core for $\alpha_z^{E/F}$.*

Proof. As $D(\alpha_z)$ is dense in E, it follows that $D(\alpha_z)+F$ is dense in E/F. As $D(\alpha_z)+F$ is also $(\alpha_t^{E/F})$-invariant, the result follows immediately from Theorem 3.6. □

To say more, we consider a duality argument (that is, use the Hahn-Banach theorem). The dual space of $\mathcal{G}(\alpha_z) \subseteq E \oplus_\infty E$ is

$$E^* \oplus_1 E^*/\mathcal{G}(\alpha_z)^{\perp} \quad \text{where} \quad \mathcal{G}(\alpha_z)^{\perp} = \{(-\lambda,\mu) : (\mu,\lambda) \in \mathcal{G}(\alpha_z^*)\}.$$

Thus, the Banach space adjoint of the map $\mathcal{G}(\alpha_z) \to \mathcal{G}(\alpha_z^{E/F})$ is

$$\pi : F^{\perp} \oplus_1 F^{\perp}/\mathcal{G}(\alpha_z^{E/F})^{\perp} = (E/F)^* \oplus_1 (E/F)^*/\mathcal{G}(\alpha_z^{E/F})^{\perp} \to E^* \oplus_1 E^*/\mathcal{G}(\alpha_z)^{\perp}.$$

The proposition above implies that π is injective. In fact, this also follows using the argument in the proof of Proposition 4.3. Indeed, suppose that $\lambda, \mu \in F^\perp$ with $\pi((\mu,\lambda) + \mathcal{G}(\alpha_z^{E/F})^\perp) = 0$ so that $(\mu,\lambda) \in \mathcal{G}(\alpha_z)^\perp$, that is, $(-\lambda,\mu) \in \mathcal{G}(\alpha_z^*)$. As $(\alpha_t^{F^\perp})$ is the restriction of (α_t^*) from E^* to $F^\perp$, for all t, we see that $(-\lambda,\mu) \in \mathcal{G}(\alpha_z^{F^\perp})$. That is, $(\mu,\lambda) \in \mathcal{G}(\alpha_z^{E/F})^\perp$, from which it follows that π is injective.

However, we see no reason why π need be bounded below, or an isometry, in general. We wish now to give a condition under which π will be an isometry.

Lemma 6.4. *With E, F and (α_t) as above, suppose there is a norm-one projection $e : E^* \to F^\perp$. Then there is a norm-one projection $p : E^* \to F^\perp$ with with $p\alpha_t^* = \alpha_t^* p$ for each t.*

Proof. Consider $\mathbb{R}_d$, the real numbers considered as a discrete group under addition. This group is amenable, so there is a state $\Lambda \in \ell^\infty(\mathbb{R})^*$ which is shift-invariant. For $t \in \mathbb{R}$ define $e_t : E^* \to E^*$ by $e_t(\mu) = (\alpha_{-t}^* \circ e \circ \alpha_t^*)(\mu)$. Given $\mu \in F^\perp$, as $\alpha_t^*(\mu) \in F^\perp$ and so $e\alpha_t^*(\mu) = \alpha_t^*(\mu)$, it follows that $e_t(\mu) = \mu$. Thus e_t is a norm-one projection onto $F^\perp$. For $\mu \in E^*$ and $x \in E$, as $t \mapsto \langle e_t(\mu), x\rangle$ is bounded, the value $\langle \Lambda, (\langle e_t(\mu), x\rangle)\rangle$ is well-defined. Then $x \mapsto \langle \Lambda, (\langle e_t(\mu), x\rangle)\rangle$ is linear and bounded, and so defines $p(\mu) \in E^*$.

For $\mu \in F^\perp$ we have that $\langle p(\mu), x\rangle = \langle \Lambda, (\langle e_t(\mu), x\rangle)\rangle = \langle \Lambda, (\langle \mu, x\rangle)\rangle = \langle \mu, x\rangle$ and so $p(\mu) = \mu$. For any $\mu \in E^*$ and $x \in F$, as $\langle e_t(\mu), x\rangle = 0$ for all t, it follows that $p(\mu) \in F^\perp$. Thus p is a norm-one projection $E^* \to F^\perp$.

Finally, for $s \in \mathbb{R}$ and arbitrary μ, x we have that

$$\begin{aligned}\langle p\alpha_s^*(\mu), x\rangle &= \langle \Lambda, (\langle \alpha_{-t}^* e \alpha_{t+s}^*(\mu), x\rangle)\rangle = \langle \Lambda, (\langle \alpha_{-(t-s)}^* e \alpha_t^*(\mu), x\rangle)\rangle \\ &= \langle \Lambda, (\langle \alpha_{-t}^* e \alpha_t^*(\mu), \alpha_s(x)\rangle)\rangle = \langle p(\mu), \alpha_s(x)\rangle.\end{aligned}$$

Thus $p\alpha_s^* = \alpha_s^* p$ as required. □

Proposition 6.5. *With E, F and (α_t) as above, suppose there is a norm-one projection $p : E^* \to F^\perp$. Then π is an isometry, and so $\mathcal{G}(\alpha_z) \to \mathcal{G}(\alpha_z^{E/F})$ is a metric surjection.*

Proof. By the lemma, we may suppose that $p\alpha_t^* = \alpha_t^* p$ for each t. Let $\mu, \lambda \in F^\perp$ with $\|(\mu,\lambda) + \mathcal{G}(\alpha_z)^\perp\| < 1$. We aim to show that $\|(\mu,\lambda) + \mathcal{G}(\alpha_z^{E/F})^\perp\| \leq 1$. The hypothesis is that there is $\phi \in D(\alpha_z^*)$ with $\|\mu - \alpha_z^*(\phi)\| + \|\lambda + \phi\| < 1$.

For $n > 0$ form $\mathcal{R}_n$ on E^* using (α_t^*). As $\mathcal{R}_n$ is norm-decreasing, we have that $\|\mathcal{R}_n(\mu) - \mathcal{R}_n(\alpha_z^*(\phi))\| + \|\mathcal{R}_n(\lambda) + \mathcal{R}_n(\phi)\| < 1$. Set $\phi' = \mathcal{R}_n(\phi)$, and recall that $\mathcal{R}_n(\alpha_z^*(\phi)) = \alpha_z^*(\phi')$.

By Proposition 3.7, we know that $t \mapsto \alpha_t^*(\phi')$ is norm-continuous, and similarly $t \mapsto \alpha_t^*\alpha_z^*(\phi') = \alpha_t^*(\mathcal{R}_n(\alpha_z^*(\phi)))$ is norm-continuous. Let $f : S(z) \to E^*$ be the analytic extension of $t \mapsto \alpha_t^*(\phi')$ so f is weak*-regular and norm-continuous on $\mathbb{R}$ and $z + \mathbb{R}$. By Lemma 2.14, it follows that f is norm-regular

(this could also be proved by adapting the proof of Proposition 3.7 to show that $z \mapsto \alpha_z(\mathcal{R}_n(\phi))$ is norm-continuous.) Hence also $w \mapsto p(f(w))$ is norm-regular. It follows that $p(\phi') \in D(\alpha_z^{F^\perp})$ with $\alpha_z^{F^\perp}(\phi') = p(f(z)) = p(\alpha_z^*(\phi'))$.

As p is a contraction, we have that

$$\|\mathcal{R}_n(\mu) - \alpha_z^{F^\perp}(\phi'')\| + \|\mathcal{R}_n(\lambda) + \phi''\| < 1,$$

for $\phi'' = p(\phi') \in F^\perp$. This shows that $\|(\mathcal{R}_n(\mu), \mathcal{R}_n(\lambda)) + \mathcal{G}(\alpha_z^{E/F})^\perp\| < 1$, that is, the norm of $(\mathcal{R}_n(\mu), \mathcal{R}_n(\lambda))$ in $\mathcal{G}(\alpha_z^{E/F})^*$ is at most 1. As $\mathcal{R}_n(\mu) \to \mu$ weak*, as $n \to \infty$, and the same for λ, by taking weak*-limits we conclude that the norm of (μ, λ) in $\mathcal{G}(\alpha_z^{E/F})^*$ is at most 1, as required.

That $\mathcal{G}(\alpha_z) \to \mathcal{G}(\alpha_z^{E/F})$ is a metric surjection follows from the Hahn-Banach theorem. □

6.2. Kaplansky-like results. Motivated by Proposition 6.1 and the results of Section 5, we might wonder if the unit ball of $\mathcal{G}(\alpha_z^{M_*})$ is weak*-dense in the unit ball of $\mathcal{G}(\alpha_z^{A^*})$. This unfortunately seems subtle, and we can only give a partial answer.

Let us norm $\mathcal{G}(\alpha_z^{A^*})$ as a subspace of $A^* \oplus_\infty A^*$; similar remarks would apply to other choices of norm. Then $\mathcal{G}(\alpha_z^{A^*})$ is the dual space $A \oplus_1 A/X_A$ where $X_A = {}^\perp\mathcal{G}(\alpha_z^{A^*}) = \{(-b, a) : (a, b) \in \mathcal{G}(\alpha_z^A)\}$. Similarly, $\mathcal{G}(\alpha_z^{M_*})^* = M \oplus_1 M/X_M$ where $X_M = \{(-y, x) : (x, y) \in \mathcal{G}(\alpha_z^M)\}$. The Hahn-Banach theorem thus shows that the unit ball of $\mathcal{G}(\alpha_z^{M_*})$ is weak*-dense in the unit ball of $\mathcal{G}(\alpha_z^{A^*})$ if and only if $\mathcal{G}(\alpha_z^{M_*})$ norms $A \oplus_1 A/X_A$. This in turn is equivalent to $A \oplus_1 A/X_A \to M \oplus_1 M/X_M$ being an isometry.

Lemma 6.6. *Let $A_0 \subseteq A$ be a dense subset. The following are equivalent:*

(1) *$A \oplus_1 A/X_A \to M \oplus_1 M/X_M$ is an isometry;*
(2) *whenever $a \in A_0, (x, y) \in \mathcal{G}(\alpha_z^M)$ are such that $\|a - y\| + \|x\| < 1$ there is $(b, c) \in \mathcal{G}(\alpha_z^A)$ with $\|a - c\| + \|b\| < 1$.*

Proof. Suppose that (1) holds, and that we have $a, (x, y)$ as in (2). Then $\|(a, 0) + X_M\| < 1$ so by (1), we have that also $\|(a, 0) + X_A\| < 1$, and hence there are $(b, c) \in \mathcal{G}(\alpha_z^A)$ with $\|a - c\| + \|b\| < 1$, as required.

Conversely, suppose that (2) holds. As $A \oplus_1 A/X_A \to M \oplus_1 M/X_M$ is always norm-decreasing, it follows easily that (2) implies that $\|(a, 0) + X_M\| = \|(a, 0) + X_A\|$ for $a \in A_0$. It hence suffices to show that $\{(a, 0) + X_A : a \in A_0\}$ is norm dense in $A \oplus_1 A/X_A$. Choose $(a, b) \in A \oplus A$ and $\epsilon > 0$. There is n with $\|\mathcal{R}_n(b) - b\| < \epsilon$. Then $(\alpha_z\mathcal{R}_n(b), -\mathcal{R}_n(b)) \in X_A$ and so $(a, b) + X_A = (a + \alpha_z\mathcal{R}_n(b), b - \mathcal{R}_n(b)) + X_A$. As A_0 is dense, there is $a_0 \in A_0$ with $\|a + \alpha_z\mathcal{R}_n(b) - a_0\| < \epsilon$. It follows that $\|(a, b) - (a_0, 0) + X_A\| < 2\epsilon$, as required. □

Proposition 6.7. *Let $A_0 \subseteq A$ be a dense subset. Suppose that for each $a \in A_0$ and $\epsilon > 0$ there are contractive linear maps $T, S : M \to A$ with $\|S(a) - a\| < \epsilon$ and with $(T(x), S(y)) \in \mathcal{G}(\alpha_z^A)$ for each $(x, y) \in \mathcal{G}(\alpha_z^M)$. Then the unit ball of $\mathcal{G}(\alpha_z^{M_*})$ is weak*-dense in the unit ball of $\mathcal{G}(\alpha_z^{A^*})$.*

Proof. We verify condition (2) in Lemma 6.6. For $a \in A_0$ and $(x,y) \in \mathcal{G}(\alpha_z^M)$ with $\|a-y\|+\|x\| < 1$, choose $\epsilon > 0$, and pick T, S as in the hypothesis. Then $(T(x), S(y)) \in \mathcal{G}(\alpha_z^A)$ and

$$\|a-S(y)\|+\|T(x)\| \leq \|a-S(a)\|+\|S(a-y)\|+\|T(x)\| < \epsilon+\|a-y\|+\|x\|.$$

For $\epsilon > 0$ sufficiently small, we have $\|a-c\|+\|b\| < 1$ for

$$(b,c) = (T(x), S(y)) \in \mathcal{G}(\alpha_z^A)$$

hence showing condition (2). □

Let us make links with the machinery developed in Section 5. Firstly, another way to prove the main theorem in that section would be to use the central projection $p \in A^{**}$ to define maps T, S with the properties in Proposition 6.7. Secondly, we showed that if M_* is identified with pA^*, so identifying M with pA^{**}, then $\mathcal{G}(\alpha_z^M)$ can be identified with $(p \oplus p)\mathcal{G}(\alpha_z^A)^{\perp\perp} \subseteq \mathcal{G}(\alpha_z^A)^{\perp\perp}$. One can easily show that then

$$\|(a,b)+X_M\| = \|(pb,-pa)+\mathcal{G}(\alpha_z^A)^{\perp\perp}\|,$$

the latter norm being on $A^{**} \oplus_1 A^{**}/\mathcal{G}(\alpha_z^A)^{\perp\perp} = (A \oplus_1 A/\mathcal{G}(\alpha_z^A))^{**}$. Indeed, if $(z,w) \in \mathcal{G}(\alpha_z^A)^{\perp\perp}$ with $\|pb+z\|+\|pa-w\| < 1$ then also $\|pb+pz\|+\|pa-pw\| < 1$. Then (pz,pw) is identified with $(x,y) \in \mathcal{G}(\alpha_z^M)$, so $(-y,x) \in X_M$, and $\|a-y\|+\|b+x\| < 1$, so $\|(a,b)+X_M\| < 1$; and one can reverse this argument.

Note that the map $A \to A^{**}; a \mapsto pa$ is an isometry (as $A \to M$ is an isometry) but in general this is not the canonical map $A \to A^{**}$. Thus, showing that $A \oplus_1 A/X_A \to M \oplus_1 M/X_M$ is an isometry is equivalent to showing that $\|(a,b)+\mathcal{G}(\alpha_z^A)\| = \|(a,b)+\mathcal{G}(\alpha_z^A)^{\perp\perp}\| = \|(pa,pb)+\mathcal{G}(\alpha_z^A)^{\perp\perp}\|$ for all $a,b \in A$. This in turn requires us to have knowledge of $\|(p^\perp a, p^\perp b)+\mathcal{G}(\alpha_z^A)^{\perp\perp}\|$ where $p^\perp = 1-p$. The link with Proposition 6.7 is that the maps T, S there could be assembled into a net, and then a weak*-limit taken, thus obtaining $T, S : M = pA^{**} \to A^{**}$ with $S(pa) = a$ for $a \in A$, and mapping $\mathcal{G}(\alpha_z^M)$ to $\mathcal{G}(\alpha_z^A)^{\perp\perp}$. We do not see a way to push this line of argument further in general.

6.3. Implemented automorphism groups. Let M be a von Neumann algebra. We recall the notion of a *standard form* for M, [14], [33, Chapter IX], which we shall denote by $(M, L^2(M), J_M, L^2(M)^+)$. By [14, Theorem 3.2] for any (weak*-continuous) automorphism α of M, there is a unique u, a unitary on $L^2(M)$, with $\alpha(x) = uxu^*$ and $J_M = uJ_Mu^*, u(L^2(M)^+) = L^2(M)^+$. Furthermore, if (α_t) is a one-parameter automorphism group of M and (u_t) the resulting unitaries, then (u_t) is strongly continuous, [14, Corollary 3.6].

The following is a generalisation of a similar result of ours, [10, Lemma 3]; but that proof is not correct, as it requires taking a linear span. Indeed, the following could also be shown by adapting the (corrected) proof of [10, Lemma 3].

Proposition 6.8. *With $M, (\alpha_t), (u_t)$ as above, consider*

$$D = \lin\{\omega_{\xi,\eta} : \xi \in D(u_{i/2}), \eta \in D(u_{-i/2})\} \subseteq M_*.$$

Then D is a core for $\mathcal{G}(\alpha^{M_*}_{-i/2})$.

Proof. We first note that $D \subseteq D(\alpha^{M_*}_{-i/2})$. Indeed, if

$$\xi \in D(u_{i/2}), \quad \eta \in D(u_{-i/2}), \quad x \in D(\alpha_{-i/2}),$$

then by Proposition 4.6 we have that $D(u_{-i/2}xu_{i/2}) = D(u_{i/2})$ and $u_{-i/2}xu_{i/2}$ is closable with closure $y = \alpha_{-i/2}(x)$. Then

$$\begin{aligned}\langle(-y, x), (\omega_{\xi,\eta}, \omega_{u_{i/2}\xi,u_{-i/2}\eta})\rangle &= (u_{-i/2}\eta|xu_{i/2}\xi) - (\eta|y\xi)\\ &= (\eta|u_{-i/2}xu_{i/2}\xi) - (\eta|y\xi) = 0.\end{aligned}$$

This shows that $(\omega_{\xi,\eta}, \omega_{u_{i/2}\xi,u_{-i/2}\eta}) \in \mathcal{G}(\alpha^{M_*}_{-i/2})$ as required.

As $u_t u_z = u_z u_t$ for any $t \in \mathbb{R}, z \in \mathbb{C}$, it follows that D is (u_t)-invariant. As $D(u_{-i/2})$ and $D(u_{i/2})$ are dense in H, it follows that D is dense in M_*. The result now follows from Theorem 3.6. □

It would be interesting to characterise all of $\mathcal{G}(\alpha^{M_*}_{-i/2})$ (in a similar way) and not just a core. As M is in standard form, we know that $M_* = \{\omega_{\xi,\eta} : \xi, \eta \in L^2(M)\}$, with no linear span required. It is tempting to believe that this should allow us to improve the above result by removing the linear span. However, the following example shows that, naively, this will not work (though in the special setting of the proposition, the result might still hold– we have been unable to decide this).

Example 6.9. We construct a one-parameter isometry group (α_t) on $\ell^1 = (\ell^\infty)_*$ and a dense set $D \subseteq \ell^1$ which is (α_t)-invariant, with $D \subseteq D(\alpha_{-i})$ but such that $D' = \{(x, \alpha_{-i}(x)) : x \in D\}$ is not dense in $\mathcal{G}(\alpha_{-i})$. As in Example 4.4 we can embed this example into the predual of an automorphism group on a von Neumann algebra.

Define $\alpha_t(x) = (e^{int}x_n)$ for $x = (x_n)_{n\geq 1} \in \ell^1$. Thus $D(\alpha_{-i}) = \{x = (x_n) \in \ell^1 : \sum_n e^n|x_n| < \infty\}$. Let $(m(k))_{k\geq 1}$ be a strictly increasing sequence with $m(1) > 1$. Define $D \subseteq \ell^1$ by saying that $x = (x_n) \in D$ if and only if $x \in D(\alpha_{-i})$ and there exists N so that $|x_1| = e^{1+m(N)}|x_{1+m(N)}|$. Clearly D is (α_t)-invariant.

Given $(x_n) \in \ell^1$ of finite support, that is, there is $K \geq 1$ with $x_n = 0$ for $n > K$, then define $y = (y_n)$ by $y_n = x_n$ for $n \leq K$, $y_{1+m(K)} = e^{-1-m(K)}y_1$, and $y_n = 0$ otherwise. As $(m(k))$ is strictly increasing and $m(1) > 1$, we have that $1 + m(K) > K$ so (y_n) is well-defined. As $\sum_n |y_n|e^n = \sum_{n\leq K} |x_n|e^n + |y_1| < \infty$ we see that $y \in D$. Clearly $\|x - y\| = e^{-1-m(K)}|x_1|$ which is arbitrarily small (by choosing K large). We conclude that D is dense in ℓ^1.

Let $\delta_1 \in \ell^1$ be the sequence which is 1 at 1 and 0 otherwise. Suppose towards a contradiction that there is $x \in D$ with $\|(\delta_1, \alpha_{-i}(\delta_1)) - (x, \alpha_{-i}(x))\| < \epsilon$. Then $|x_1 - 1| < \epsilon$ and $\|e^1\delta_1 - \alpha_{-i}(x)\| < \epsilon$. As $x \in D$ there is N with

$|x_{1+m(N)}| = e^{-1-m(N)}|x_1|$ so $e^{1+m(N)}|x_{1+m(N)}| = |x_1| > 1-\epsilon$ showing that $\|e^1\delta_1 - \alpha_{-i}(x)\| > 1-\epsilon$ which is a contradiction if $\epsilon < 1/2$.

6.4. Tensor products. While not directly related to duality, we wish to briefly consider tensor products. This was also done in [20, Section 4], and so we shall just given an overview. Our aim is to demonstrate how "uniqueness" results for analytic generators, compare [27, Proposition F1] or [15, Lemma 4.4] for example, can be shown using the smearing technique, instead of Carlson's Lemma from complex analysis.

For ease, we shall simply work with the Banach space projective tensor product, see [31, Chapter 2] for example; see [20, Section 4] for more general considerations. Let E, F be Banach spaces, and $(\alpha_t), (\beta_t)$ be one-parameter isometry groups on E, F respectively. Then on the projective tensor product $E\widehat{\otimes}F$, it is clear that $\gamma_t = \alpha_t \otimes \beta_t$ defines a one-parameter isometry group. By considering analytic extensions, it follows that if $x \in D(\alpha_z), y \in D(\beta_z)$ then $x \otimes y \in D(\gamma_z)$ with $\gamma_z(x \otimes y) = \alpha_z(x) \otimes \beta_z(y)$. It follows that the algebraic tensor product $D(\alpha_z) \otimes D(\beta_z)$ is a subset of $D(\gamma_z)$. As $D(\alpha_z) \otimes D(\beta_z)$ is (γ_t)-invariant and dense in $E\widehat{\otimes}F$, Theorem 3.6 shows that $D(\alpha_z) \otimes D(\beta_z)$ is a core for γ_z.

We state the following for a linear map, but there is an obvious extension (following [27, Proposition F1]) to multi-linear maps.

Proposition 6.10. *Let $\theta : E \to F$ be a bounded linear map with $\theta\alpha_{-i} \subseteq \beta_{-i}\theta$. Then $\theta\alpha_t = \beta_t\theta$ for all $t \in \mathbb{R}$.*

Proof. We consider (β^*_t) on F^*, so, again, $\mathcal{G}(\beta^*_{-i}) = \{(-\lambda, \mu) : (\mu, \lambda) \in \mathcal{G}(\beta_{-i})^\perp\}$. Let $F_0 \subseteq F^*$ be the collection of μ such that $t \mapsto \beta^*_t(\mu)$ is norm continuous. This is readily seen to be a closed subspace, and by Proposition 3.7, F_0 is weak*-dense in F^*. Let (β^0_t) be the restriction of (β^*_t) to F_0, which forms a norm continuous isometry group.

Our hypothesis is that if $x \in D(\alpha_{-i})$ then $\theta(x) \in D(\beta_{-i})$ and $\beta_{-i}\theta(x) = \theta\alpha_{-i}(x)$, that is, $(\theta(x), \theta\alpha_{-i}(x)) \in \mathcal{G}(\beta_{-i})$. If $\mu \in D(\beta^0_i)$ with $\lambda = \beta^0_i(\mu)$, then $(\lambda, \mu) \in \mathcal{G}(\beta^0_{-i})$ and so $(\lambda, \mu) \in \mathcal{G}(\beta^*_{-i})$ so $(-\mu, \lambda) \in \mathcal{G}(\beta_{-i})^\perp$. It follows that $\langle \mu, \theta(x)\rangle = \langle \lambda, \theta\alpha_{-i}(x)\rangle$.

Let $\gamma_t = \alpha_t \otimes \beta^0_{-t}$ on $E\widehat{\otimes}F_0$, so that $D(\alpha_{-i}) \otimes D(\beta^0_i)$ is a core for γ_{-i}. Define $T : E\widehat{\otimes}F_0 \to \mathbb{C}$ by $T(x \otimes \mu) = \langle \mu, \theta(x)\rangle$. Then $T(x \otimes \mu) = T(\alpha_{-i}(x) \otimes \beta^0_i(\mu))$ for all $x \in D(\alpha_{-i}), \mu \in D(\beta^0_i)$. Thus $T(u) = T(v)$ for all $(u, v) \in \mathcal{G}(\gamma_{-i})$. As $T \in (E\widehat{\otimes}F_0)^*$, this means that $(T, T) \in \mathcal{G}(\gamma^*_{-i})$. Exactly as in Remark 4.13, this means we can find an entire, bounded, extension of the orbit map $t \mapsto \gamma^*_t(T)$, and so $\gamma^*_t(T) = T$ for all t.

It follows that $\langle \beta^0_{-t}(\mu), \theta\alpha_t(x)\rangle = \langle \mu, \theta(x)\rangle$ for each $x \in E, \mu \in F_0$. As F_0 is weak*-dense in F^*, and $\beta^0_{-t} = \beta^*_{-t}$ on F_0, it follows that $\beta_{-t}\theta\alpha_t(x) = \theta(x)$ for each $x \in E$, that is, $\beta_t\theta = \theta\alpha_t$, as required. □

Corollary 6.11. *Let $E = (E_*)^*, F = (F_*)^*$ be dual spaces, and $(\alpha_t), (\beta_t)$ be weak*-continuous. If $\theta : E \to F$ is weak*-continuous with $\theta\alpha_{-i} \subseteq \beta_{-i}\theta$ then $\theta\alpha_t = \beta_t\theta$.*

Proof. There is $\theta_* : F_* \to E_*$ with $\theta = (\theta_*)^*$. That $\theta\alpha_{-i} \subseteq \beta_{-i}\theta$ is equivalent to $(x, y) \in \mathcal{G}(\alpha_{-i}) \implies (\theta(x), \theta(y)) \in \mathcal{G}(\beta_{-i})$. Let $(\mu, \lambda) \in \mathcal{G}(\beta_{-i}^{F_*})$ so that $(-\lambda, \mu) \in {}^{\perp}\mathcal{G}(\beta_{-i})$. Thus, for $(x, y) \in \mathcal{G}(\alpha_{-i})$, we have that

$$\langle(\theta(x), \theta(y)), (-\lambda, \mu))\rangle = 0 \implies \langle x, \theta_*(\lambda)\rangle = \langle y, \theta_*(\mu)\rangle,$$

and so $(\theta_*(\mu), \theta_*(\lambda)) \in \mathcal{G}(\alpha_{-i}^{E_*})$. We have hence shown that $\theta_*\beta_{-i}^{F_*} \subseteq \alpha_{-i}^{E_*}\theta_*$. Hence $\theta_*\beta_t^{F_*} = \alpha_t^{E_*}\theta_*$ for all t, so taking adjoints gives the required conclusion. □

Applied with $E = F$ and θ the identity map, this result shows that the generator α_{-i} uniquely determines (α_t).

7. Locally compact quantum groups

We give a brief introduction to locally compact quantum groups, [22, 24, 23, 27, 35]. We write $\mathbb{G}$ for the abstract object thought of as a locally compact quantum group, which has a concrete operator-algebraic realisation as either the von Neumann algebra $L^\infty(\mathbb{G})$ or the C^*-algebra $C_0(\mathbb{G})$. We write Δ for the coproduct, either a unital normal injective $*$-homomorphism $L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$, or a non-degenerate $*$-homomorphism $C_0(\mathbb{G}) \to M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$. The left Haar weight, via the GNS construction, gives rise to a Hilbert space $L^2(\mathbb{G})$ on which $L^\infty(\mathbb{G})$ and $C_0(\mathbb{G})$ act. We denote the dual quantum group by $\widehat{\mathbb{G}}$, and identify $L^2(\mathbb{G})$ with $L^2(\widehat{\mathbb{G}})$. We recall the fundamental multiplicative unitary $W \in M(C_0(\mathbb{G}) \otimes C_0(\widehat{\mathbb{G}})) \subseteq L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\widehat{\mathbb{G}})$ which implements the coproduct as $\Delta(x) = W^*(1 \otimes x)W$. We can recover $C_0(\mathbb{G})$ as the norm closure of $\{(\mathrm{id} \otimes \omega)W : \omega \in \mathcal{B}(L^2(\mathbb{G}))_*\}$, and similarly $C_0(\widehat{\mathbb{G}})$ as the norm closure of $\{(\omega \otimes \mathrm{id})W : \omega \in \mathcal{B}(L^2(\mathbb{G}))_*\}$.

We write $L^1(\mathbb{G})$ for the predual of $L^\infty(\mathbb{G})$, and write $M(\mathbb{G})$ for the dual of $C_0(\mathbb{G})$. These both become Banach algebras for the "convolution product" induced by the coproduct. Furthermore, $M(\mathbb{G})$ is a dual Banach algebra, and the isometric inclusion $L^1(\mathbb{G}) \to M(\mathbb{G})$ is a homomorphism.

The group inverse operation, for a quantum group, is represented by the antipode, which in general is an unbounded operator S. Two related objects are R, the unitary antipode, which is a $*$-antiautomorphism of $C_0(\mathbb{G})$ which extends to a normal map on $L^\infty(\mathbb{G})$, and (τ_t) the scaling group, which is a one-parameter automorphism group on $C_0(\mathbb{G})$ which extends to a weak*-continuous automorphism group of $L^\infty(\mathbb{G})$. Thus we are in precisely the situation considered elsewhere in this paper, and furthermore, the scaling group exactly governs the unboundedness of the antipode, because $S = R\tau_{-i/2}$. We recall that R and (τ_t) commute, from which it follows that $R\tau_{-i/2} = \tau_{-i/2}R$. Let us think briefly about what exactly we mean by $S = R\tau_{-i/2}$:

- In [23, Definition 5.21], S is defined to be $R\tau_{-i/2}$, here acting on $C_0(\mathbb{G})$. As we are considering norm-continuous (τ_t) there is essentially no risk of ambiguity.
- In [24, Page 74], S is defined to be $R\tau_{-i/2}$, and it is not entirely clear what is meant by $\tau_{-i/2}$. Part of our motivation for developing the material in Sections 2.1 and 2.2 was to show that, actually, the particular definition of $\tau_{-i/2}$ is unimportant.
- In [35, Definition 2.23], S is defined to be $R\tau_{-i/2}$. This paper takes *as definition* that $\tau_{-i/2}$ is the adjoint of $\tau_{*,-i/2}$ where $(\tau_{*,t})$ is the one-parameter isometry group on $L^1(\mathbb{G})$. Of course, by Theorem 2.17, this agrees with the usual meaning of $\tau_{-i/2}$.

As $S = R\tau_{-i/2}$ and R and (τ_t) commute, it follows that $D(S) = D(\tau_{-i/2})$. As the inclusion $C_0(\mathbb{G}) \to L^\infty(\mathbb{G})$ intertwines R, it follows easily that questions about S can immediately be reduced to questions about $\tau_{-i/2}$. For example, the following is immediate from Theorem 5.1.

Theorem 7.1. *The unit ball of $\mathcal{G}(S) \subseteq C_0(\mathbb{G}) \oplus_\infty C_0(\mathbb{G})$ is weak*-dense in the unit ball of $\mathcal{G}(S) \subseteq L^\infty(\mathbb{G}) \oplus_\infty L^\infty(\mathbb{G})$.*

As $\Delta\tau_t = (\tau_t \otimes \tau_t)\Delta$ it follows that $(\tau_t^{L^1(\mathbb{G})})$ is an automorphism group for $L^1(\mathbb{G})$, and similarly $(\tau_t^{M(\mathbb{G})})$ is a weak*-continuous automorphism group for $M(\mathbb{G})$. The natural way to induce an involution on $L^1(\mathbb{G})$ is to use the antipode and the involution on $L^\infty(\mathbb{G})$, leading to definition of $L^1_\sharp(\mathbb{G})$ as those $\omega \in L^1(\mathbb{G})$ such that there is $\omega^\sharp \in L^1(\mathbb{G})$ with $\overline{\langle S(x)^*, \omega\rangle} = \langle x, \omega^\sharp\rangle$ for all $x \in D(S)$. The map $\omega \mapsto \omega^\sharp$ becomes an involution on $L^1_\sharp(\mathbb{G})$. It is shown in [21, Proposition 3.1] that then $L^1_\sharp(\mathbb{G}) \to C_0(\widehat{\mathbb{G}}); \omega \mapsto (\omega \otimes \mathrm{id})(W)$ is a $*$-homomorphism.

The following is the natural extension of this definition to $M(\mathbb{G})$.

Definition 7.2. We define $M_\sharp(\mathbb{G})$ to be the collection of $\mu \in M(\mathbb{G})$ such that there is $\mu^\sharp \in M(\mathbb{G})$ with $\overline{\langle \mu, S(a)^*\rangle} = \langle \mu^\sharp, a\rangle$ for $a \in D(S)$.

For $\mu \in M(\mathbb{G})$ we write $\mu^* \in M(\mathbb{G})$ for the functional $a \mapsto \overline{\langle \mu, a^*\rangle}$. Given $\mu \in M_\sharp(\mathbb{G})$ define $\lambda = R^*(\mu^*)$. For $a \in D(\tau_{-i/2})$ we have

$$\langle \mu^\sharp, a\rangle = \overline{\langle \mu, S(a)^*\rangle} = \langle \mu^*, S(a)\rangle = \langle \lambda, \tau_{-i/2}(a)\rangle.$$

Thus $(\mu^\sharp, -\lambda) \in \mathcal{G}(\tau_{-i/2})^\perp$ so $(\lambda, \mu^\sharp) \in \mathcal{G}(\tau^{M(\mathbb{G})}_{-i/2})$. We can reverse this argument, thus establishing that $\mu \in M_\sharp(\mathbb{G})$ if and only if $R^*(\mu^*) \in D(\tau^{M(\mathbb{G})}_{-i/2})$ and then $\mu^\sharp = \tau^{M(\mathbb{G})}_{-i/2}(R^*(\mu^*))$. An analogous argument holds for $L^1_\sharp(\mathbb{G})$.

The definitions of both $L^1_\sharp(\mathbb{G})$ and $M_\sharp(\mathbb{G})$ are both "graph like", in that given, say, $\mu \in M(\mathbb{G})$, we have that $\mu \in M_\sharp(\mathbb{G})$ when there exists $\mu^\sharp \in M(\mathbb{G})$ with a certain property. The next two results show that we can instead impose conditions purely on μ. The first result is an application of Hahn-Banach, but the second result is somewhat less obvious.

Proposition 7.3. *Let $\mu \in M(\mathbb{G})$ be such that $\mu^* \circ S : D(S) \subseteq C_0(\mathbb{G}) \to \mathbb{C}$ is bounded. Then $\mu \in M_\sharp(\mathbb{G})$.*

Proof. Let $\mu_0 = \mu^* \circ S$ and take a Hahn-Banach extension to an element $\mu^\sharp \in M(\mathbb{G})$ (or simply extend by continuity, as $D(S)$ is dense in $C_0(\mathbb{G})$). Thus, for $a \in D(S)$,

$$\langle \mu^\sharp, a \rangle = \langle \mu^*, S(a) \rangle = \overline{\langle \mu, S(a)^* \rangle},$$

so by definition, $\mu \in M_\sharp(\mathbb{G})$. □

Theorem 7.4. *Let $\omega \in L^1(\mathbb{G})$ be such that either:*

(1) *$\omega^* \circ S : D(S) \subseteq C_0(\mathbb{G}) \to \mathbb{C}$ is bounded; or*
(2) *$\omega^* \circ S : D(S) \subseteq L^\infty(\mathbb{G}) \to \mathbb{C}$ is bounded.*

Then $\omega \in L^1_\sharp(\mathbb{G})$.

Proof. Let us write $S_0 = R_0 \circ \tau^0_{-i/2}$ for S on $C_0(\mathbb{G})$, and $S_\infty = R_\infty \circ \tau^\infty_{-i/2}$ for S on $L^\infty(\mathbb{G})$. As the inclusion $C_0(\mathbb{G}) \to L^\infty(\mathbb{G})$ intertwines R_0 and R_∞, and intertwines $\tau^0_{-i/2}$ and $\tau^\infty_{-i/2}$, we see that it intertwines S_0 and S_∞.

If (1) holds, then by the previous proposition, $\omega \in M_\sharp(\mathbb{G})$, that is, $R_0^*(\omega^*) \in D(\tau^{M(\mathbb{G})}_{-i/2})$. We then apply Theorem 6.2 to see that $R_0^*(\omega^*) \in D(\tau^{L^1(\mathbb{G})}_{-i/2})$. However, $R_0^*(\omega^*)$ is equal to the image of $(R_\infty)_*(\omega^*) \in L^1(\mathbb{G})$ in $M(\mathbb{G})$, and so $\omega \in L^1_\sharp(\mathbb{G})$, as required.

Now suppose that (2) holds. Then the composition $D(S_0) \to D(S_\infty) \to \mathbb{C}$ is bounded, that is, (1) holds. The claim follows. □

In [10, Proposition 14] the author and Salmi showed the following, via "Banach algebraic" techniques. We wish here to quickly record how to use the more abstract approach of Section 6.2. We recall that $\mathbb{G}$ is coamenable when $L^1(\mathbb{G})$ has a bounded approximate identity, [7].

Proposition 7.5. *Let $\mathbb{G}$ be coamenable. For any $\mu \in D(\tau_z^{M(\mathbb{G})})$ there is a net (ω_i) in $D(\tau_z^{L^1(\mathbb{G})})$ with $\omega_i \to \mu$ weak* and with $\|\omega_i\| \leq \|\mu\|$ and $\|\tau_z^{L^1(\mathbb{G})}(\omega_i)\| \leq \|\tau_z^{M(\mathbb{G})}(\mu)\|$ for each i.*

Proof. We will use Proposition 6.7. Let

$$A_0 = \{(\mathrm{id} \otimes \phi)(W) : \phi \in \mathcal{B}(L^2(\mathbb{G}))_*\} \subseteq C_0(\mathbb{G})$$

a dense subset (actually, subalgebra).

For $\omega \in L^1(\mathbb{G})$ consider the map $P_\omega : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ given by $P_\omega(x) = (\mathrm{id} \otimes \omega)\Delta(x)$. This actually maps into $C^b(\mathbb{G})$, see for example the proof of [30, Theorem 2.4]. Let $(x, y) \in \mathcal{G}(\tau_z^{L^\infty(\mathbb{G})})$ and $(\omega, \phi), (\alpha, \beta) \in \mathcal{G}(\tau_z^{L^1(\mathbb{G})})$. Then

$$\begin{aligned}\langle (P_\phi(x), P_\omega(y)), (-\beta, \alpha) \rangle &= \langle (\mathrm{id} \otimes \omega)\Delta(y), \alpha \rangle - \langle (\mathrm{id} \otimes \phi)\Delta(x), \beta \rangle \\ &= \langle y, \alpha\omega \rangle - \langle x, \beta\phi \rangle = \langle (y, -x), (\alpha\omega, \beta\phi) \rangle = 0,\end{aligned}$$

because $(\alpha\omega, \beta\phi) \in \mathcal{G}(\tau_z^{L^1(\mathbb{G})})$ by Proposition 2.11. As (α, β) was arbitrary, this shows that $(P_\phi(x), P_\omega(y)) \in \mathcal{G}(\tau_z^{L^\infty(\mathbb{G})})$.

As $\mathbb{G}$ is coamenable, $L^1(\mathbb{G})$ has a contractive approximate identity, and so by Proposition 4.14, also $\mathcal{G}(\tau_z^{L^1(\mathbb{G})})$ has a contractive approximate identity, say (ω_i, ϕ_i). For the moment, suppose that $\mathbb{G}$ is compact, so $C_0(\mathbb{G}) = C^b(\mathbb{G}) = C(\mathbb{G})$. For each i, the pair $(P_{\phi_i}, P_{\omega_i})$ are contractive maps $L^\infty(\mathbb{G}) \to C(\mathbb{G})$ which map $\mathcal{G}(\tau_z^{L^\infty(\mathbb{G})})$ to $\mathcal{G}(\tau_z^{L^\infty(\mathbb{G})}) \cap (C_0(\mathbb{G}) \oplus C_0(\mathbb{G})) = \mathcal{G}(\tau_z^{C(\mathbb{G})})$, by Proposition 4.3. To invoke Proposition 6.7 it hence remains to show that $P_{\phi_i}(a) \to a$ in norm, for each $a = (\mathrm{id} \otimes \phi)(W) \in A_0$. However, then

$$\begin{aligned} P_{\phi_i}(a) &= (\mathrm{id} \otimes \phi_i)\Delta\big((\mathrm{id} \otimes \phi)(W)\big) = (\mathrm{id} \otimes \phi_i \otimes \phi)(W_{13}W_{23}) \\ &= (\mathrm{id} \otimes \phi)(W(1 \otimes \lambda(\phi_i))) = (\mathrm{id} \otimes \lambda(\phi_i)\phi)(W) \end{aligned}$$

where $\lambda(\phi_i) = (\phi_i \otimes \mathrm{id})(W) \in C_0(\widehat{\mathbb{G}})$. As (ϕ_i) is a bounded approximate identity (bai) in $L^1(\mathbb{G})$, it follows that $\lambda(\phi_i)$ is a bai for $C_0(\widehat{\mathbb{G}})$, as $\lambda(L^1(\mathbb{G}))$ is dense in $C_0(\widehat{\mathbb{G}})$. For any bai $(\widehat{a}_i)$ in $C_0(\widehat{\mathbb{G}})$ where have that $\widehat{a}_i\phi \to \phi$ in norm, for $\phi \in L^1(\mathbb{G})$. Thus $P_{\phi_i}(a) \to a$ in norm, as required.

To deal with the non-compact case, we apply Proposition 4.14 to find a contractive approximate identity (e_i, f_i) in $\mathcal{G}(\tau_z^{C_0(\mathbb{G})})$. Then, for any i, j, we have that $x \mapsto e_i P_{\phi_j}(x)$ maps $L^\infty(\mathbb{G})$ to $C_0(\mathbb{G})$, and again for $a \in A_0$, for sufficiently large i, j we have that $e_i P_{\phi_j}(a)$ is close to a. The proof now follows as before. □

Of course, if for example (τ_t) is trivial (for example, $\mathbb{G}$ is a Kac algebra) then we certainly do not need $\mathbb{G}$ to be coamenable. In this case, the conditions of Lemma 6.6 follow immediately from the triangle-inequality. We continue to wonder if the result above is really true for any $\mathbb{G}$.

We know that the left Haar weight φ is relatively invariant under (τ_t), that is, there is $\nu > 0$, the scaling constant, such that $\varphi(\tau_t(x)) = \nu^{-t}\varphi(x)$ for $x \in L^\infty(\mathbb{G})_+$. Denote $\mathfrak{n}_\varphi = \{x \in L^\infty(\mathbb{G}) : \varphi(x^*x) < \infty\}$ and let $\Lambda : \mathfrak{n}_\varphi \to L^2(\mathbb{G})$ be the GNS map. We may hence define a one-parameter unitary group (P^{it}) on $L^2(\mathbb{G})$ by $P^{it}\Lambda(x) = \nu^{t/2}\Lambda(\tau_t(x))$ for $x \in \mathfrak{n}_\varphi, t \in \mathbb{R}$. Then $\tau_t(x) = P^{it}xP^{-it}$ and so we are in the setting of Section 6.3. We remark that one can easily adapt the proof of [14, Proposition 3.7] to show that (P^{it}) is the canonical unitary implementation of (τ_t), in the sense of Section 6.3. The following is now immediate from Proposition 6.8, and corrects [10, Lemma 3] by requiring the linear span.

Proposition 7.6. *The set* $D = \mathrm{lin}\{\omega_{\xi,\eta} : \xi \in D(P^{1/2}), \eta \in D(P^{-1/2})\}$ *is a core for* $L^1_\sharp(\mathbb{G})$.

We finish with an application of Proposition 6.5. We recall that $\mathbb{G}$ is *amenable* when there is a state $m \in L^\infty(\mathbb{G})^*$ with $\langle m, (\mathrm{id} \otimes \omega)\Delta(x)\rangle = \langle m, x\rangle\langle 1, \omega\rangle$ for $\omega \in L^1(\mathbb{G}), x \in L^\infty(\mathbb{G})$, see [7]. From [7, Theorem 3.2] we know that if $\widehat{\mathbb{G}}$ is coamenable then $\mathbb{G}$ is amenable; the converse is a

well-known open question. By [7, Theorem 3.3] we know that when $\mathbb{G}$ is amenable, $L^\infty(\widehat{\mathbb{G}})$ is injective, [34, Chapter XVI], that is, there is a contractive projection $\mathcal{B}(L^2(\mathbb{G})) \to L^\infty(\widehat{\mathbb{G}})$.

Define $\tau_t^{\mathcal{B}}(x) = P^{it}xP^{-it}$ for $x \in \mathcal{B}(L^2(\mathbb{G}))$, which gives a weak*-continuous automorphism group. The pre-adjoint of $(\tau_t^{\mathcal{B}})$ gives a norm-continuous one-parameter isometry group $(\tau_t^{\mathcal{B}_*})$ on $\mathcal{B}(L^2(\mathbb{G}))_*$. Let K be the kernel of the quotient $\mathcal{B}(L^2(\mathbb{G}))_* \to L^1(\mathbb{G})$, so that $K^\perp = L^\infty(\mathbb{G})$ and hence $K = {}^\perp L^\infty(\mathbb{G})$ is $(\tau_t^{\mathcal{B}_*})$-invariant. Thus we are in the setting of Section 6.1. Notice that $(\tau_t^{\mathcal{B}_*/K})$ is simply $(\tau_t^{L^1(\mathbb{G})})$. Thus Proposition 6.5 shows that $\mathcal{G}(\tau_z^{\mathcal{B}_*}) \to \mathcal{G}(\tau_z^{L^1(\mathbb{G})})$ is a quotient map, in the case when $\widehat{\mathbb{G}}$ is amenable (in particular, when $\mathbb{G}$ is coamenable). This result is interesting, as it parallels the quotient map $\mathcal{B}(L^2(\mathbb{G}))_* \to L^1(\mathbb{G})$; we wonder if there is some analogue of a "standard form" for $\mathcal{G}(\tau_z^{L^1(\mathbb{G})})$, compare the comments in Section 4.2. We again remark that if (τ_t) is trivial, then of course $\mathcal{G}(\tau_z^{\mathcal{B}_*}) \to \mathcal{G}(\tau_z^{L^1(\mathbb{G})})$ is a quotient map, without any amenability condition.

References

[1] Arendt, Wolfgang; Nikolski, Nikolai K. Vector-valued holomorphic functions revisited. *Math. Z.* **234** (2000), no. 4, 777–805. MR1778409, Zbl 0976.46030, doi: 10.1007/s002090050008. 169

[2] Arendt, Wolfgang; Nikolski, Nikolai K. Addendum: "Vector-valued holomorphic functions revisited". *Math. Z.* **252** (2006), no. 3, 687–689. MR2207764, Zbl 0976.46030, doi: 10.1007/s00209-005-0858-x. 169

[3] Arveson, William B. Analyticity in operator algebras. *Amer. J. Math.* **89** (1967), 578–642. MR0223899, Zbl 0183.42501, doi: 10.2307/2373237. 186

[4] Arveson, William B. On groups of automorphisms of operator algebras. *J. Functional Analysis* **15** (1974), 217–243. MR0348518, Zbl 0296.46064, doi: 10.1016/0022-1236(74)90034-2. 170, 173, 176, 185

[5] Arveson, William B. The harmonic analysis of automorphism groups. *Operator algebras and applications, Part I* (Kingston, Ont., 1980), pp. 199–269, Proc. Sympos. Pure Math., 38, *Amer. Math. Soc., Providence, RI*, 1982. MR0679706, Zbl 0506.46047. 173, 176

[6] Bannon, Jon P.; Cameron, Jan M.; Mukherjee, Kunal K. The modular symmetry of Markov maps. *J. Math. Anal. Appl.* **439** (2016), no. 2, 701–708. MR3475946, Zbl 1355.46053, doi: 10.1016/j.jmaa.2016.03.013. 166, 183, 184

[7] Bédos, Erik; Tuset, Lars. Amenability and co-amenability for locally compact quantum groups. *Internat. J. Math.* **14** (2003), no. 8, 865–884. MR2013149, Zbl 1051.46047, doi: 10.1142/S0129167X03002046. 199, 200, 201

[8] Brannan, Michael P. ; Daws, Matthew; Samei, Ebrahim. Completely bounded representations of convolution algebras of locally compact quantum groups. Münster J. Math. **6** (2013), no. 2, 445–482. MR3148219, Zbl 1303.46038, arXiv:1107.2094. 176

[9] Ciorănescu, Ioana; Zsidó, László. Analytic generators for one-parameter groups. *Tohoku Math. J. (2)* **28** (1976), no. 3, 327–362. MR0430867, Zbl 0361.47014, doi: 10.2748/tmj/1178240775. 165, 166, 167, 168, 169, 170, 171, 172, 175, 182, 183, 184, 185

[10] Daws, Matthew; Salmi, Pekka. Completely positive definite functions and Bochner's theorem for locally compact quantum groups. *J. Funct. Anal.*

264 (2013), no. 7, 1525–1546. MR3019722, Zbl 1320.46056, arXiv:1210.5231, doi: 10.1016/j.jfa.2013.01.017. 187, 194, 199, 200

[11] Eymard, Pierre. L'algèbre de Fourier d'un groupe localement compact. *Bull. Soc. Math. France* **92** (1964), 181–236. MR0228628, Zbl 0169.46403, doi: 10.24033/bsmf.1607. 177

[12] Fima, Pierre; Vainerman, Leonid. Twisting and Rieffel's deformation of locally compact quantum groups: deformation of the Haar measure. *Comm. Math. Phys.* **286** (2009), no. 3, 1011–1050. MR2472024, Zbl 1179.46057, arXiv:0801.1906, doi: 10.1007/s00220-008-0559-5. 166, 173

[13] Forelli, Frank. Analytic and quasi-invariant measures. *Acta Math.* **118** (1967), 33–59. MR0209771, Zbl 0171.34201, doi: 10.1007/BF02392475. 173, 176

[14] Haagerup, Uffe. The standard form of von Neumann algebras. *Math. Scand.* **37** (1975), no. 2, 271–283. MR0407615, Zbl 0304.46044, doi: 10.7146/math.scand.a-11606. 194, 200

[15] Haagerup, Uffe. Operator-valued weights in von Neumann algebras. I. *J. Functional Analysis* **32** (1979), no. 2, 175–206. MR0534673, Zbl 0426.46046, doi: 10.1016/0022-1236(79)90053-3. 183, 196

[16] Izumi, Msaki. The flow of weights and the Cuntz–Pimsner algebras. *Comm. Math. Phys.* **357** (2018), no. 1, 203–229. MR3764567, Zbl 1397.46045, doi: 10.1007/s00220-017-2948-0. 165

[17] Jørgensen, Palle E. T. Ergodic properties of one-parameter automorphism groups of operator algebras. *J. Math. Anal. Appl.* **87** (1982), no. 2, 354–372. MR0658019, Zbl 0444.47038, doi: 10.1016/0022-247X(82)90129-9. 186

[18] Kato, Tosio. Perturbation theory for linear operators. Reprint of the 1980 edition. Classics in Mathematics. *Springer-Verlag, Berlin*, 1995. xxii+619 pp. ISBN: 3-540-58661-X. MR1335452, Zbl 0836.47009, doi: 10.1007/978-3-642-66282-9. 175

[19] Kawamura, Shinzô; Tomiyama, Jun. On subdiagonal algebras associated with flows in operator algebras. *J. Math. Soc. Japan* **29** (1977), no. 1, 73–90. MR0454650, Zbl 0339.46046, doi: 10.2969/jmsj/02910073. 186

[20] Kustermans, Johan. One-parameter representations on C^*-algebras. Preprint, 1997. arXiv:funct-an/9707009v1. 165, 168, 169, 170, 171, 176, 187, 196

[21] Kustermans, Johan. Locally compact quantum groups in the universal setting. *Internat. J. Math.* **12** (2001), no. 3, 289–338. MR1841517, Zbl 1111.46311, arXiv:math/9902015 doi: 10.1142/S0129167X01000757. 198

[22] Kustermans, Johan. Locally compact quantum groups. *Quantum independent increment processes. I*, 99–180, Lecture Notes in Math., 1865, *Springer, Berlin*, 2005. MR2132094, Zbl 1072.46048, doi: 10.1007/b105131. 166, 168, 173, 174, 175, 197

[23] Kustermans, Johan; Vaes, Stefaan. Locally compact quantum groups. *Ann. Sci. École Norm. Sup. (4)* **33** (2000), no. 6, 837–934. MR1832993, Zbl 1034.46508, doi: 10.1016/S0012-9593(00)01055-7. 165, 197, 198

[24] Kustermans, Johan; Vaes, Stefaan. Locally compact quantum groups in the von Neumann algebraic setting. *Math. Scand.* **92** (2003), no. 1, 68–92. MR1951446, Zbl 1034.46067, arXiv:math/0005219, doi: 10.7146/math.scand.a-14394. 166, 173, 197, 198

[25] Loebl, Richard I.; Muhly, Paul S. Analyticity and flows in von Neumann algebras. *J. Functional Analysis* **29** (1978), no. 2, 214–252. MR0504460, Zbl 0381.46036, doi: 10.1016/0022-1236(78)90007-1. 186

[26] Marsalli, Michael; West, Graeme. Noncommutative H^p spaces. *J. Operator Theory* **40** (1998), no. 2, 339–355. MR1660390, Zbl 1001.46041. 186

[27] Masuda, Tetsuya.; Nakagami, Yoshiomi.; Woronowicz, Stanisław. L. A C^*-algebraic framework for quantum groups. *Internat. J. Math.* **14** (2003), no. 9, 903–1001. MR2020804, Zbl 1053.46050, doi: 10.1142/S0129167X03002071. 165, 166, 176, 178, 196, 197

[28] Rudin, Walter. Functional analysis. Second edition. International Series in Pure and Applied Mathematics. *McGraw-Hill, Inc., New York*, 1991. xviii+424 pp. ISBN: 0-07-054236-8. MR1157815, Zbl 0867.46001 177

[29] Runde, Volker. Amenability for dual Banach algebras. *Studia Math.* **148** (2001), no. 1, 47–66. MR1881439, Zbl 1003.46028, doi: 10.4064/sm148-1-5. 173

[30] Runde, Volker. Uniform continuity over locally compact quantum groups. *J. Lond. Math. Soc. (2)* **80** (2009), no. 1, 55–71. MR2520377, Zbl 1188.46048, doi: 10.1112/jlms/jdp011. 199

[31] Ryan, Raymond A. Introduction to tensor products of Banach spaces. Springer Monographs in Mathematics. *Springer-Verlag London, Ltd., London*, 2002. xiv+225 pp. ISBN: 1-85233-437-1. MR1888309, Zbl 1090.46001, doi: 10.1007/978-1-4471-3903-4. 196

[32] Takesaki, Masamichi. Theory of operator algebras. I. Reprint of the first (1979) edition. Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-commutative Geometry, 5. *Springer-Verlag, Berlin*, 2002. xx+415 pp. ISBN: 3-540-42248-X. MR1873025, Zbl 0990.46034. 187, 190

[33] Takesaki, Masamichi. Theory of operator algebras. II. Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. *Springer-Verlag, Berlin*, 2003. xxii+518 pp. ISBN: 3-540-42914-X. MR1943006, Zbl 1059.46031, doi: 10.1007/978-3-662-10451-4. 165, 166, 168, 169, 182, 183, 194

[34] Takesaki, Masamichi. Theory of operator algebras. III. Encyclopaedia of Mathematical Sciences, 127. Operator Algebras and Non-commutative Geometry, 8. *Springer-Verlag, Berlin*, 2003. xxii+548 pp. ISBN: 3-540-42913-1. MR1943007, Zbl 1059.46032, doi: 10.1007/978-3-662-10453-8. 201

[35] Van Daele, Alfons. Locally compact quantum groups. A von Neumann algebra approach. *SIGMA Symmetry Integrability Geom. Methods Appl.* **10** (2014), Paper 082, 41 pp. MR3261862, Zbl 1312.46055, doi: 10.3842/SIGMA.2014.082. 166, 197, 198

[36] Wiener, Norbert. Tauberian theorems. *Ann. of Math. (2)* **33** (1932), no. 1, 1–100. MR1503035, Zbl 0004.05905, doi: 10.2307/1968102. 177

[37] Woronowicz, Stanisław L. On the purification of factor states. *Comm. Math. Phys.* **28** (1972), 221–235. MR0310657, Zbl 0244.46075, doi: 10.1007/BF01645776. 182, 185

[38] Woronowicz, Stanisław L. Operator systems and their application to the Tomita–Takesaki theory. *Algèbres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq.*, Marseille, 1977), pp. 459–527, Colloq. Internat. CNRS, 274, *CNRS, Paris*, 1979. MR0560657, Zbl 0456.46047. 182

[39] Zsidó, László. Spectral and ergodic properties of the analytic generators. *J. Approximation Theory* **20** (1977), no. 1, 77–138. MR0463974, Zbl 0352.47019, doi: 10.1016/0021-9045(77)90021-1. 165, 166, 173, 176, 182, 185

[40] Zsidó, László. The characterization of the analytic generator of *-automorphism groups. *Operator algebras and applications, Part II* (Kingston, Ont.), 1980, pp. 381-383, Proc. Sympos. Pure Math., 38, *American Mathematical Society, Providence, R.I.*, 1982. MR0679491, Zbl 0529.46052. 165, 185, 186

[41] Zsidó, László. On Woronowicz's approach to the Tomita–Takesaki theory. *Operator algebras and quantum groups*, 455–486, Banach Center Publ., 98, *Polish Acad. Sci. Inst. Math., Warsaw*, 2012. MR3059673, Zbl 1269.46039, doi: 10.4064/bc98-0-19 165, 167, 182, 185

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