New York Journal of Mathematics

New York J. Math. **27** (2021) 205–252.

Graded C^*-algebras and twisted groupoid C^*-algebras

Jonathan H. Brown, Adam H. Fuller, David R. Pitts and Sarah A. Reznikoff

Abstract. Let A be a C^*-algebra that is acted upon by a compact abelian group. We show that if the fixed-point algebra of the action contains a Cartan subalgebra D satisfying an appropriate regularity condition, then A is the reduced C^*-algebra of a groupoid twist. We further show that the embedding $D \hookrightarrow A$ is uniquely determined by the twist. These results generalize Renault's results on Cartan subalgebras of C^*-algebras.

Contents

1. Introduction

Abelian operator algebras are well understood: abelian C^*-algebras are all isomorphic to spaces of continuous functions on a locally compact Hausdorff space; abelian von Neumann algebras are all isomorphic to L^∞-spaces. The study of non-abelian operator algebras is often aided by the presence of appropriate abelian subalgebras. This idea was exemplified by Feldman and Moore's characterization of von Neumann algebras containing Cartan subalgebras in 1977 [17]. Cartan embeddings arise naturally in many examples, including finite dimensional von Neumann algebras and von Neumann

Received August 29, 2019.

2010 *Mathematics Subject Classification.* Primary 46L55, Secondary 46L05, 22A22.

Key words and phrases. groupoid, Cartan, graded C^*-algebras.

ISSN 1076-9803/2021

algebras constructed from free actions of discrete groups on abelian von Neumann algebras. Feldman and Moore [17] gave a complete classification of Cartan subalgebras in terms of measured equivalence relations.

To transfer Feldman and Moore's theory to the topological setting, Renault [33] defined Cartan subalgebras for C^*-algebras.

Definition 1.1. [33, Definition 5.1] *Let A be a C^*-algebra. A maximal abelian C^*-algebra $D \subseteq A$ is a* Cartan subalgebra *of A if*

(1) there exists a faithful conditional expectation $E\colon A \to D$;
(2) D contains an approximate unit for A;
(3) the set of normalizers of D, i.e. the $n \in A$ such that $nDn^* \subseteq D$ and $n^*Dn \subseteq D$, generate A as a C^*-algebra.

When D is a Cartan subalgebra of A, we call (A, D) a Cartan pair*.*

Renault [33], building on work by Kumjian [18], showed that there is a one-to-one correspondence between Cartan pairs of separable C^*-algebras and C^*-algebras of second countable twisted groupoids; that is, between Cartan pairs and the reduced C^*-algebra generated by an extension of groupoids

$$\mathbb{T} \times G^{(0)} \to \Sigma \to G.$$

In Renault's result, G must be topologically principal: Renault refers to G as the Weyl groupoid of the Cartan pair. It is reasonable to seek a larger class of inclusions $D \subseteq A$ with D abelian that can be used to construct twists.

This idea has recently been pursued successfully by several authors, and larger classes of inclusions have been shown to arise as C^*-algebras of twists. In particular, motivated by shift spaces and the work by Matsumoto and Matui [24, 25, 23], Brownlowe, Carlsen, and Whittaker [7] were able to construct a Weyl type groupoid from a general graph C^*-algebra and its canonical diagonal and use this construction to show that diagonal-preserving isomorphisms of these inclusions come precisely from isomorphisms of Weyl type groupoids. This led to work proving similar results for Leavitt path algebras [6] and Steinberg algebras [2].

The paper [2] in particular inspired this work. Steinberg algebras are algebraic analogues of groupoid C^*-algebras [37, 11]. In [2], the authors consider Steinberg algebras associated to groupoids G equipped with a homomorphism $c : G \to \Gamma$ where Γ is an abelian group and $c^{-1}(0)$ is topologically principal. The Steinberg algebra is then naturally graded by Γ; the authors use this grading to reconstruct G. It is well known that algebras graded by an abelian group Γ correspond in the C^*-algebraic theory to C^*-algebras endowed with a $\hat{\Gamma}$ action (for example see [39], [30]).

In this paper, we construct groupoids from inclusions of an abelian C^*-algebra D into a C^*-algebra A endowed with the action of a compact abelian group. In particular, the aim of our work is to generalize Renault's characterization of Cartan pairs by reduced C^*-algebras of twisted groupoids. Our results apply to examples appearing naturally in the study of higher-rank graph and twisted higher-rank graph C^*-algebras (See Example 7.2 below).

We start with a C^*-algebra A and a discrete abelian countable group Γ such that the dual group $\hat{\Gamma}$ acts continuously on A by automorphisms. Let $A^{\hat{\Gamma}}$ be the points in A fixed by the action of $\hat{\Gamma}$: this is a subalgebra of A called the fixed point algebra. Assume $A^{\hat{\Gamma}}$ contains a Cartan subalgebra D. If in addition the normalizers of D in A densely span A we call (A, D) a Γ-*Cartan pair*. We note that the normalizers of D in $A^{\hat{\Gamma}}$ are homogeneous of degree 0. In particular, if the action by $\hat{\Gamma}$ is trivial, then (A, D) is a Cartan pair.

If (A, D) is a Γ-Cartan pair, then following Kumjian's construction, we show how to create a twisted groupoid $(\Sigma; G)$ that is graded by Γ. This yields the following commutative diagram

$$\begin{tikzcd} \mathbb{T} \times G^{(0)} \arrow[r, "\iota"] & \Sigma \arrow[r, "q"] \arrow[dr, "c_\Sigma"'] & G \arrow[d, "c_G"] \\ & & \Gamma \end{tikzcd} \tag{1.1}$$

where c_Σ and c_G are homomorphisms. We prove in Theorem 4.19 that there is a natural isomorphism between (A, D) and the reduced crossed product $(C_r^*(\Sigma; G), C_0(G^{(0)}))$.

Next, if $\Sigma \to G$ is a twist satisfying the commutative diagram (1.1), we show that the inclusion $C_0(G^{(0)}) \hookrightarrow C_r^*(\Sigma; G)$ satisfies our hypotheses, so Theorem 4.19 allows us to construct a new twist from this inclusion. The natural question is: does our construction in Theorem 4.19 recover $\Sigma \to G$? We answer this affirmatively in Theorem 6.2. This second question is the main focus of [7], [6] [2], and [10] in the case that the twist is trivial.

The paper [10] by Carlsen, Ruiz, Sims and Tomforde is similar in scope to our present work. While [10] is also concerned with translating the results of [2] to a C^*-algebraic framework, their work avoids twists altogether, instead focusing on showing rigidity results along the lines of our Theorem 6.2. The results of [10] apply to C^*-algebras already known to arise from groupoids, however it does contain some remarkable innovations which allows the authors to address C^*-algebras endowed with co-actions of a possibly nonabelian group. Furthermore, Carlsen, Ruiz, Sims and Tomforde relax the requirement that the abelian subalgebra D must be Cartan in the fixed point algebra.

Whether or not a C^*-algebra satisfies the Universal Coefficient Theorem (UCT) remains the main stumbling block in the classification program for simple nuclear C^*-algebras. Indeed, Tikuisis, White and Winter [38] have shown that all separable, unital, simple, nuclear C^*-algebras with finite nuclear dimension satisfying the UCT are classifiable. Recent results of Barlak and Li [4] show that if A is a nuclear C^*-algebra containing a Cartan subalgebra, then A satisfies the UCT. We discuss in Example 7.3 how their results also apply to our setting.

This paper is organized as follows. We begin with preliminaries on twists (Section 2). In Section 3 we define Γ-Cartan pairs and review the relationship between topological grading and strong group actions.

In Section 4 we prove our main theorem, Theorem 4.19, which shows that a Γ-Cartan pair is isomorphic to the reduced C^*-algebra of a twist. In Section 5 we then provide a few basic results concerning a natural Γ-Cartan pair that arises in the presence of a twist. In Section 6 we prove our rigidity result, Theorem 6.2, which shows that if the inclusion in the previous section comes from a twist then our construction recovers the twist.

Section 7 gives some examples to which our theorems apply. Notably, in Example 7.2 we show that the twisted higher-rank graph C^*-algebras introduced in [21] and [22] give examples of Γ-Cartan pairs. Moreover, the groupoid description of twisted higher-rank graph C^*-algebras given in [22] yields groupoids isomorphic to ours.

Finally, in an appendix, we describe how we can obtain the results of Section 4 by using a coaction of a non-abelian group (instead of an action of an abelian group); note that in this case the grading on the C^*-algebra is by the group itself, rather than by its dual. The authors thank John Quigg for pointing out this alternative construction.

Acknowledgments. Whilst conducting this research, JHB and AHF made use of meeting space made available by the Columbus Metropolitan Library. We would like to thank CML for their important work in the community. AHF would like to thank Christopher Schafhauser for patiently answering his questions on [4].

This work was partially supported by grants from the Simons Foundation (#316952 to David Pitts and #360563 to Sarah Reznikoff) and by the American Institute of Mathematics SQuaREs Program.

2. Preliminaries

2.1. Étale groupoids. A groupoid G is a small category in which every morphism has an inverse. The unit space $G^{(0)}$ of G is the set of identity morphisms. The maps $s, r : G \to G^{(0)}$, given by $s(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$, are the *source* and *range* maps. For $S, T \subseteq G$ we denote

$$ST := \{\gamma\eta : \gamma \in S, \eta \in T, r(\eta) = s(\gamma)\}.$$

If either S or T is the singleton set $\{\gamma\}$ we remove the set brackets from the notation and write $S\gamma$ or γT.

A topological groupoid is a groupoid G endowed with a topology such that inversion and composition are continuous. An open set $B \subseteq G$ is a *bisection* if $r(B)$ and $s(B)$ are open and $r|_B$ and $s|_B$ are homeomorphisms onto their images. The groupoid G is *étale* if there is a basis for the topology on G consisting of bisections. When G is étale then $G^{(0)}$ is open and closed in G.

For $x \in G^{(0)}$, the *isotropy group at* x is $xGx := \{\gamma \in G : r(\gamma) = s(\gamma) = x\}$ and the *isotropy subgroupoid* is the set $G' = \{\gamma \in G : r(\gamma) = s(\gamma)\}$. A topological groupoid G is *topologically principal* if $\{x \in G^{(0)} : xGx = \{x\}\}$ is dense in $G^{(0)}$; it is *effective* if the interior of G' is $G^{(0)}$. If G is second countable these notions coincide [5, Lemma 3.1], but in the general (not necessarily second countable) case, effective is the more useful notion.

Unless explicitly stated otherwise, for the remainder of this paper, we make the following assumptions.

Standing Assumptions on Groupoids. *Throughout, all groupoids are:*

(1) locally compact and
(2) Hausdorff.

2.2. Twists. The main focus of this paper is on twists and their C^*-algebras. We provide a brief account of the necessary background here. Much of this background can also be found in [33]. We also encourage the reader to consult the recent expository article by Sims [34]. We now expand on a few details that are particularly relevant to our context.

A twist is the analog of a central extension of a discrete group by the circle $\mathbb{T}$. Here is the formal definition.

Definition 2.1 (see [34, Definition 5.1.1])**.** *Let Σ and G be topological groupoids with G étale, and let $\mathbb{T} \times G^{(0)}$ be the product groupoid. That is, $(z_1, x_1)(z_2, x_2)$ is defined if and only if $x_1 = x_2$, in which case the product is given by $(z_1, x_1)(z_2, x_2) = (z_1 z_2, x_1)$; inversion is $(z, x)^{-1} = (z^{-1}, x)$, and the topology is the product topology. The unit space of $\mathbb{T} \times G^{(0)}$ is $\{1\} \times G^{(0)}$.*

The pair (Σ, G) is a twist if there is an exact sequence

$$\mathbb{T} \times G^{(0)} \xrightarrow{\iota} \Sigma \xrightarrow{q} G$$

where

(1) ι and q are continuous groupoid homomorphisms with ι one-to-one and q onto;
(2) $\iota|_{\{1\}\times G^{(0)}}$ and $q|_{\Sigma^{(0)}}$ are homeomorphisms onto $\Sigma^{(0)}$ and $G^{(0)}$, respectively (identify $\Sigma^{(0)}$ and $G^{(0)}$ using q);
(3) $q^{-1}(G^{(0)}) = \iota(\mathbb{T} \times G^{(0)})$;
(4) for every $\gamma \in \Sigma$ and $z \in \mathbb{T}$, $\iota(z, r(\gamma))\gamma = \gamma\iota(z, s(\gamma))$; and
(5) for every $g \in G$ there is an open bisection U with $g \in U$ and a continuous function $\phi_U : U \to \Sigma$ such that $q \circ \phi_U = \mathrm{id}|_U$ and the map $\mathbb{T} \times U \ni (z \times h) \mapsto \iota(z, r(h))\,\phi_U(h)$ is a homeomorphism of $\mathbb{T} \times U$ onto $q^{-1}(U)$.

(Conditions (1–3) say the sequence is an extension, (4) says the extension is central, and (5) says G is étale and the extension is locally trivial.) A twist is often denoted simply by $\Sigma \to G$.

For $z \in \mathbb{T}$ and $\gamma \in \Sigma$ we will write

$$z \cdot \gamma := \iota(z, r(\gamma))\gamma \quad \textit{and} \quad \gamma \cdot z := \gamma\iota(z, s(\gamma))$$

for the action of $\mathbb{T}$ on Σ arising from the embedding of $\mathbb{T} \times G^{(0)}$ into Σ. Notice that this action of $\mathbb{T}$ on Σ is free.

Also, for $\gamma \in \Sigma$, we will often denote $q(\gamma)$ by $\dot{\gamma}$; indeed, we use the name $\dot{\gamma}$ for an arbitrary element of G.

Remark 2.2. *By* [41, Exercise 9K(3)] *the map $q : \Sigma \to G$ is a quotient map.*

The C^*-algebra of the twist is constructed from the completion of an appropriate function algebra $C_c(\Sigma; G)$. This algebra can be constructed in two different ways and both will be used in this note.

First description of $C_c(\Sigma; G)$: Sections of a line bundle. The first way to construct $C_c(\Sigma; G)$ is by considering sections of a complex line bundle L over G. Define L to be the quotient of $\mathbb{C} \times \Sigma$ by the equivalence relation on $\mathbb{C} \times \Sigma$ given by $(\lambda, \gamma) \sim (\lambda_1, \gamma_1)$ if and only if there exists $z \in \mathbb{T}$ such that $(\lambda_1, \gamma_1) = (\overline{z}\lambda, z \cdot \gamma)$. We sometimes write $L = (\mathbb{C} \times \Sigma)/\mathbb{T}$. Use $[\lambda, \gamma]$ to denote the equivalence class of (λ, γ). Observe that for any $z \in \mathbb{T}$,

$$[\lambda, z \cdot \gamma] = [z\lambda, \gamma]. \tag{2.1}$$

With the quotient topology, L is Hausdorff. The (continuous) surjection $P : L \to G$ is given by

$$P : [\lambda, \gamma] \mapsto \dot{\gamma}.$$

For $\dot{\gamma} \in G$ and $\gamma_0 \in q^{-1}(\dot{\gamma})$, the map $\mathbb{C} \ni \lambda \mapsto [\lambda, \gamma_0] \in P^{-1}(\dot{\gamma})$ is a homeomorphism, so L is a complex line bundle over G. In general, there is no canonical choice of γ_0. However, when $\dot{\gamma} \in G^{(0)}$, $\Sigma^{(0)} \cap q^{-1}(\dot{\gamma})$ is a singleton set, so there is a canonical choice: take γ_0 to be the unique element of $\Sigma^{(0)} \cap q^{-1}(\dot{\gamma})$. Thus, recalling that $\Sigma^{(0)}$ and $G^{(0)}$ have been previously identified (using $q|_{\Sigma^{(0)}}$), when $x \in G^{(0)} = \Sigma^{(0)}$, we sometimes identify $P^{-1}(x)$ with $\mathbb{C}$ via the map $\lambda \mapsto [\lambda, x] = \lambda \cdot [1, x]$.

Finally, there is a continuous map $\varpi : L \to [0, \infty)$ given by

$$\varpi([\lambda, \gamma]) := |\lambda|.$$

When $f : G \to L$ is a section and $\dot{\gamma} \in G$, we will sometimes write $|f(\dot{\gamma})|$ instead of $\varpi(f(\dot{\gamma}))$.

Since Σ is locally trivial, L is locally trivial as well. Indeed, given $\ell \in L$, let B be an open bisection of G containing $P(\ell)$. Let $\phi_B : B \to \Sigma$ be a continuous function satisfying the conditions of Definition 2.1(5). Then for every element $\ell_1 \in P^{-1}(B)$, there exist unique $\lambda \in \mathbb{C}$ and $\dot{\gamma} \in B$ so that $\ell_1 = [\lambda, \phi_B(\dot{\gamma})]$. It follows that the map $[\lambda, \phi_B(\dot{\gamma})] \mapsto (\lambda, \dot{\gamma})$ is a homeomorphism of $P^{-1}(B)$ onto $\mathbb{C} \times B$, so L is locally trivial.

There is a partially defined multiplication on L, given by

$$[\lambda, \gamma][\lambda', \gamma'] = [\lambda\lambda', \gamma\gamma'],$$

whenever γ and γ' are composable in Σ. When $[\lambda,\gamma],[\lambda',\gamma']\in L$ satisfy $\dot\gamma=\dot\gamma'$, let

$$[\lambda,\gamma]+[\lambda',\gamma']:=[\lambda+z\lambda',\gamma], \tag{2.2}$$

where z is the unique element of $\mathbb{T}$ so that $\gamma'=z\cdot\gamma$. There is also an involution on L given by

$$\overline{[\lambda,\gamma]}=[\overline{\lambda},\gamma^{-1}]. \tag{2.3}$$

We use the symbol $C_c(\Sigma;G)$ to denote the set of "compactly supported" continuous sections of L, that is,

$$\begin{aligned} C_c(\Sigma;G):=\{f:G\to L \mid f &\text{ is continuous,} \\ &P\circ f=\mathrm{id}|_G, \text{ and } \varpi\circ f \text{ has compact support}\}. \end{aligned} \tag{2.4}$$

Notation 2.3. For $f\in C_c(\Sigma;G)$, we denote the support of $\varpi\circ f$ by $\operatorname{supp}(f)$; we denote its open support by $\operatorname{supp}'(f)$. Further, let $C(\Sigma;G)$ and $C_0(\Sigma;G)$ be, respectively, the set of continuous sections and continuous sections vanishing at infinity of the bundle L.

We endow $C_c(\Sigma;G)$ with a $*$-algebra structure where addition is pointwise (using (2.2)), multiplication is given by convolution:

$$f*g(\dot\gamma)=\sum_{\dot\eta_1\dot\eta_2=\dot\gamma} f(\dot\eta_1)g(\dot\eta_2)=\sum_{r(\dot\eta)=r(\dot\gamma)} f(\dot\eta)g(\dot\eta^{-1}\dot\gamma), \tag{2.5}$$

and the involution is from (2.3):

$$f^*(\dot\gamma)=\overline{f(\dot\gamma^{-1})}.$$

Note that if f,g are supported on bisections B_1,B_2 and $\dot\eta_i\in B_i$ then $f*g(\dot\eta_1\dot\eta_2)=f(\dot\eta_1)g(\dot\eta_2)$. We can identify $C_0(G^{(0)})$ with a subalgebra of continuous sections of the line bundle L by

$$C_0(G^{(0)})\to C_0(\Sigma;G) \quad\text{by}\quad \phi\mapsto\left(\dot\gamma\mapsto\begin{cases}[\phi(\dot\gamma),\iota(1,\dot\gamma)] & \dot\gamma\in G^{(0)}\\ 0 & \text{otherwise}\end{cases}\right).$$

Note that this identification takes pointwise multiplication on $C_0(G^{(0)})$ to the convolution on $C_c(\Sigma;G)$.

Second description of $C_c(\Sigma;G)$: Covariant functions. A function f on Σ is *covariant* if for every $z\in\mathbb{T}$ and $\gamma\in\Sigma$,

$$f(z\cdot\gamma)=\overline{z}\,f(\gamma).$$

The second way to describe $C_c(\Sigma;G)$ is as the set of compactly supported continuous covariant functions on Σ, that is,

$$C_c(\Sigma;G):=\{f\in C_c(\Sigma):\forall\gamma\in\Sigma\ \ \forall z\in\mathbb{T}\ \ f(z\cdot\gamma)=\overline{z}f(\gamma)\}. \tag{2.6}$$

Addition is pointwise, the involution is $f^*(\gamma) = \overline{f(\gamma^{-1})}$, and the convolution multiplication is given by

$$f * g(\gamma) = \sum_{\substack{\dot\eta \in G \\ r(\dot\eta)=r(\dot\gamma)}} f(\eta) g(\eta^{-1}\gamma), \tag{2.7}$$

where for each $\dot\eta$ with $r(\dot\eta) = r(\dot\gamma)$, only one representative η of $\dot\eta$ is chosen. It is easy to verify that this is well-defined.

Equivalence of the descriptions. To proceed, we need to be more explicit on how these two descriptions of $C_c(\Sigma; G)$ are the same. Take $f \in C_c(\Sigma)$ such that $f(z \cdot \gamma) = \overline{z} f(\gamma)$ for all $\gamma \in \Sigma$ and $z \in \mathbb{T}$. Let $\tilde f$ be the section of the line bundle given by

$$\tilde f(\dot\gamma) = [f(\gamma), \gamma].$$

Note that by the definition of the line bundle, this is well-defined.

On the other hand, consider a compactly supported continuous section $\tilde f : G \to L$. For $\gamma \in \Sigma$, the fact that $P \circ \tilde f = \mathrm{id}|_G$ yields $P\left([1,\gamma]^{-1}\tilde f(\dot\gamma)\right) = s(\dot\gamma)$. Hence there exists $\lambda_\gamma \in \mathbb{C}$ such that $[1,\gamma]^{-1}\tilde f(\dot\gamma) = \lambda_\gamma \cdot [1, s(\gamma)]$, that is,

$$\tilde f(\dot\gamma) = \lambda_\gamma \cdot [1,\gamma] = [\lambda_\gamma, \gamma].$$

Define $f : \Sigma \to \mathbb{C}$ by

$$f(\gamma) = \lambda_\gamma.$$

Then f is continuous and compactly supported since $\tilde f$ is and satisfies

$$f(z \cdot \gamma) = \overline{z} f(\gamma). \tag{2.8}$$

We have thus described a linear isomorphism between the spaces defining $C_c(\Sigma; G)$ given in (2.4) and (2.6). It is a routine matter to show this linear map is a $*$-algebra isomorphism, so that the two descriptions coincide. Notice that $\gamma \in \mathrm{supp}(f)$ if and only if $\dot\gamma \in \mathrm{supp}(\tilde f)$.

Remark 2.4. *Technically, the support of a function $f : \Sigma \to \mathbb{C}$ satisfying the covariance condition (2.8) is a subset of Σ, but (2.8) allows us to regard both* $\mathrm{supp}(f)$ *and* $\mathrm{supp}'(f)$ *as subsets of G. We shall do this. Thus the notions of support are the same whether f is viewed as a covariant function or as a section of the line bundle.*

To define the reduced groupoid C^*-algebra, we need to define regular representations. For $x \in G^{(0)}$, let $\mathcal{H}_x = \ell^2(\Sigma x, Gx)$ be the set of square summable sections of the line bundle $L|_{Gx}$; that is,

$$\mathcal{H}_x = \{\chi : Gx \to P^{-1}(Gx) \mid \text{ for } \dot\gamma \in Gx,\ P(\chi(\dot\gamma)) = \dot\gamma, \text{ and } \varpi \circ \chi \in \ell^2(Gx)\}.$$

Given $\chi_1, \chi_2 \in \mathcal{H}_x$ and $\dot\gamma \in Gx$, $P\left(\overline{\chi_2(\dot\gamma)}\chi_1(\dot\gamma)\right) = x \in G^{(0)}$, so that we obtain a unique $\lambda_{\dot\gamma} \in \mathbb{C}$ so that

$$\overline{\chi_2(\dot\gamma)}\chi_1(\dot\gamma)) = \lambda_{\dot\gamma} \cdot [1, x].$$

We may therefore define an inner product on $\mathcal{H}_x$: $\langle\chi_1,\chi_2\rangle$ is the unique element of $\mathbb{C}$ such that

$$\sum_{\dot\gamma\in Gx}\overline{\chi_2(\dot\gamma)}\chi_1(\dot\gamma)=\langle\chi_1,\chi_2\rangle\cdot[1,x]=[\langle\chi_1,\chi_2\rangle\,,x]. \tag{2.9}$$

The regular representation of $C_c(\Sigma;G)$ on $\mathcal{H}_x$ is then defined as follows. For $f\in C_c(\Sigma,G)$ and $\chi\in\mathcal{H}_x$,

$$(\pi_x(f)\chi)\,(\dot\gamma)=\sum_{\substack{\dot\eta\in G,\dot\zeta\in Gx\\ \dot\eta\dot\zeta=\dot\gamma}}f(\dot\eta)\chi(\dot\zeta)=\sum_{\substack{\dot\eta\in G\\ r(\dot\eta)=r(\dot\gamma)}}f(\dot\eta)\chi(\dot\eta^{-1}\dot\gamma)\qquad(\dot\gamma\in Gx).$$

The reduced C^*-algebra of $(\Sigma;G)$, denoted $C_r^*(\Sigma;G)$, is the completion of $C_c(\Sigma;G)$ under the norm $\|f\|_r=\sup_{x\in G^{(0)}}\|\pi_x(f)\|$.

Remark 2.5. *Viewing $C_c(\Sigma,G)$ as the space of compactly supported sections of the line bundle affords us an alternative way to describe the regular representations, as follows. Given $x\in G^{(0)}$, define a linear functional ϵ_x on $C_c(\Sigma,G)$ by defining $\epsilon_x(f)$ to be the unique scalar such that $f(x)=[\epsilon_x(f),x]$. Note that for $f\in C_c(\Sigma,G)$,*

$$\epsilon_x(f^*f)=\sum_{s(\dot\gamma)=x}\varpi(f(\dot\gamma))^2\geq 0,$$

so ϵ_x is positive. Morever, if also $g\in C_c(\Sigma,G)$, then [15, Proposition 3.10] *shows there exist a finite number of open bisections $U_1,\dots,U_n$ for G such that* $\operatorname{supp}(g)\subseteq\bigcup_{j=1}^n U_j$*, from which it follows that*

$$(\varepsilon_x(f^*g^*gf))^{1/2}\leq n\,\|g\|_\infty\,\varepsilon_x(f^*f)^{1/2}.$$

Thus the GNS construction may be applied to ε_x to produce a representation $(\pi_{\varepsilon_x},\mathcal{H}_{\varepsilon_x})$ of $C_c(\Sigma;G)$. Letting L_{ε_x} be the left kernel of ε_x, the map $C_c(\Sigma;G)/L_{\varepsilon_x}\ni g+L_{\varepsilon_x}\mapsto g|_{Gx}$ is isometric and so determines an isometry $W:\mathcal{H}_{\varepsilon_x}\to\mathcal{H}_x$. As G is étale, for $\dot\gamma\in Gx$, there is an open bisection U for G with $\dot\gamma\in U$, and hence we may find $g\in C_c(\Sigma;G)$ supported in U with $g(\dot\gamma)\neq 0$. Thus, if $h\in\ell^2(\Sigma x,Gx)$ has finite support, there exists $f\in C_c(\Sigma,G)$ with $f|_{Gx}=h$. This implies that W is onto, and a calculation shows that $W\pi_{\varepsilon_x}=\pi_xW$. This shows π_x and π_{ε_x} are unitarily equivalent representations of $C_c(\Sigma;G)$. Of course, the same applies when $C_c(\Sigma;G)$ is viewed as compactly supported continuous covariant functions on Σ: in this case $\varepsilon_x(f)=f(x)$.

For $x\in G^{(0)}$, it will be useful to have a fixed orthonormal basis for $\mathcal{H}_x$. For $\dot\eta\in Gx$, we select $\delta_{\dot\eta}\in\mathcal{H}_x$ such that

$$(\varpi\circ\delta_{\dot\eta})(\dot\gamma)=\begin{cases}1 & \text{if } \dot\gamma=\dot\eta\\ 0 & \text{otherwise.}\end{cases}$$

and insist in particular that $\delta_x(x)=[1,x]$. Then

$$\{\delta_{\dot\eta}:\dot\eta\in Gx\}$$

is an orthonormal basis for $\mathcal{H}_x$. In the sequel, we will have occasion to consider the element $\delta_{\dot\eta}(\dot\eta) \in L$. By choosing (and fixing) $\eta \in q^{-1}(\dot\eta)$ there exists a unique $\lambda_{\dot\eta} \in \mathbb{T}$ such that

$$\delta_{\dot\eta}(\dot\eta) = [\lambda_{\dot\eta}, \eta]. \tag{2.10}$$

It is sometimes useful to informally regard $\left\langle \pi_x(f)\delta_{\dot\eta}, \delta_{\dot\zeta} \right\rangle$ as a product of elements of L, and we now give a formula which provides this description. For $f \in C_c(\Sigma; G)$, $x \in G^{(0)}$ and $\dot\eta, \dot\zeta \in Gx$, the definition of $\pi_x(f)$ and the inner product on $\mathcal{H}_x$ yield

$$\left[\left\langle \pi_x(f)\delta_{\dot\eta}, \delta_{\dot\zeta} \right\rangle, x\right] = \overline{\delta_{\dot\zeta}(\dot\zeta)} f(\dot\zeta\dot\eta^{-1})\delta_{\dot\eta}(\dot\eta). \tag{2.11}$$

In particular, $\left[\left\langle \pi_x(f)\delta_x, \delta_{\dot\zeta} \right\rangle, x\right] = \overline{\delta_{\dot\zeta}(\dot\zeta)} f(\dot\zeta)$. Therefore,

$$f(\dot\zeta) = \left\langle \pi_x(f)\delta_x, \delta_{\dot\zeta} \right\rangle \cdot \delta_{\dot\zeta}(\dot\zeta) \quad \text{and} \quad \left|\left\langle \pi_x(f)\delta_x, \delta_{\dot\zeta} \right\rangle\right| = \varpi(f(\dot\zeta)). \tag{2.12}$$

Example 2.6. *Suppose that σ is a normalized continuous 2-cocycle on the étale groupoid G. This is a continuous function from the set of composable pairs $G^{(2)}$ into $\mathbb{T}$ such that $\sigma(\gamma, s(\gamma)) = 1 = \sigma(r(\gamma), \gamma)$ and for all composable triples, $(\gamma_1, \gamma_2, \gamma_3)$,*

$$\sigma(\gamma_2, \gamma_3)\overline{\sigma(\gamma_1\gamma_2, \gamma_3)}\sigma(\gamma_1, \gamma_2\gamma_3)\overline{\sigma(\gamma_1, \gamma_2)} = 1.$$

Define $\Sigma := \mathbb{T} \times_\sigma G$, where $\mathbb{T} \times_\sigma G$ is the Cartesian product of $\mathbb{T}$ and G with the product topology and multiplication defined by $(z_1, \gamma_1)(z_2, \gamma_2) = (z_1z_2\sigma(\gamma_1, \gamma_2), \gamma_1\gamma_2)$. In this case, L may be identified with $\mathbb{C} \times G$ by $\phi : [\lambda, (z, \dot\gamma)] \mapsto (\lambda z, \dot\gamma)$ and we identify sections of L with functions on G by

$$\tilde f = p_1 \circ \phi \circ f \quad \textit{where } f \in C_c(G; \Sigma)$$

where p_1 is the projection onto the first factor. Now for compactly supported sections f, g of L,

$$\begin{aligned} f * g(\dot\gamma) &= \sum f(\dot\eta)g(\dot\eta^{-1}\dot\gamma) = \sum(\tilde f(\dot\eta), \dot\eta)(\tilde g(\dot\eta^{-1}\dot\gamma), \dot\eta^{-1}\dot\gamma) \\ &= \sum[\tilde f(\dot\eta), (1, \dot\eta)][\tilde g(\dot\eta^{-1}\dot\gamma), (1, \dot\eta^{-1}\dot\gamma)] \\ &= \sum[\tilde f(\dot\eta)\tilde g(\dot\eta^{-1}\dot\gamma), (\sigma(\dot\eta, \dot\eta^{-1}\dot\gamma), \dot\gamma)] \\ &= \sum[\tilde f(\dot\eta)\tilde g(\dot\eta^{-1}\dot\gamma)\,\sigma(\dot\eta, \dot\eta^{-1}\dot\gamma), (1, \dot\gamma)] \\ &= \left(\sum \tilde f(\dot\eta)\tilde g(\dot\eta^{-1}\dot\gamma)\,\sigma(\dot\eta, \dot\eta^{-1}\dot\gamma), \dot\gamma)\right). \end{aligned}$$

This last sum is the convolution formula for $\tilde f, \tilde g$ in $C_c(\Sigma; G)$ used by Renault in [31]. In particular, if σ is trivial then we get the usual convolution formula for étale groupoid C^*-algebras.

We will use the following proposition to find useful subalgebras of the twisted groupoid C^*-algebra.

Lemma 2.7. *Let $\mathbb{T}\times G^{(0)} \xrightarrow{\iota} \Sigma \xrightarrow{q} G$ be a twist and H be an open subgroupoid of G. Define $\Sigma_H := q^{-1}(H)$. Then*

$$\mathbb{T} \times H^{(0)} \overset{\iota|_{H^{(0)}}}{\to} \Sigma_H \overset{q|_{\Sigma_H}}{\to} H$$

is a twist. Moreover the map $\kappa : C_c(\Sigma_H; H) \hookrightarrow C_c(\Sigma; G)$ defined by extending functions by zero extends to an inclusion of $C^*_r(\Sigma_H; H)$ into $C^*_r(\Sigma; G)$.

Proof. That $\mathbb{T} \times H^{(0)} \overset{\iota|_{H^{(0)}}}{\to} \Sigma_H \overset{q|_{\Sigma_H}}{\to} H$ is a twist comes from the facts that $q(\iota(\lambda, x)) = x$ and $\gamma \in \Sigma_H$ if and only if $\dot\gamma \in H$.

View elements of $C_c(\Sigma; G)$ and $C_c(\Sigma_H; H)$ as sections of line bundles. By definition, the respective line bundles are $L_{\Sigma_H} = (\mathbb{C} \times \Sigma_H)/\mathbb{T}$ and $L_\Sigma = (\mathbb{C} \times \Sigma)/\mathbb{T}$. Therefore, $L_{\Sigma_H} = L_\Sigma|_H$. Since H and Σ_H are open, we may define $\kappa : C_c(\Sigma_H; H) \hookrightarrow C_c(\Sigma; G)$ by extending functions by zero.

For each $x \in X$, let ε_x be defined as in Remark 2.5 and let $\varepsilon^H_x := \varepsilon_x \circ \kappa$. Then ε_x and ε^H_x extend to states on $C^*_r(\Sigma; G)$ and $C^*_r(\Sigma_H; H)$. Let $(\pi_x, \mathcal{H}_x)$ and $(\pi^H_x, \mathcal{H}^H_x)$ be their associated GNS representations and let $L_x \subseteq C^*_r(\Sigma; G)$ and $L^H_x \subseteq C^*_r(\Sigma_H; H)$ be the left kernels of ε_x and ε^H_x respectively. For $h \in C_c(\Sigma_H; H)$, $\varepsilon^H_x(h^* * h) = \varepsilon_x(\kappa(h)^* * \kappa(h))$, so the map on $C_c(\Sigma_H; H)$ defined by $(h + L^H_x) \mapsto (\kappa(h) + L_x)$ extends to an isometry $W_x : \mathcal{H}^H_x \to \mathcal{H}_x$. A calculation shows that for $h \in C_c(\Sigma_H; H)$, $W_x\pi^H_x(h) = \pi_x(\kappa(h))W_x$ so that

$$W_x\pi^H_x(h)W^*_x = \pi_x(\kappa(h))(W_xW^*_x).$$

Thus, for $h \in C_c(\Sigma_H; H)$,

$$\begin{aligned} \|h\|_{C^*_r(\Sigma_H;H)} = \sup_{x\in X} \left\|\pi^H_x(h)\right\| &= \sup_{x\in X} \|W_x\pi_x(\kappa(h))W^*_x\| \\ &\le \sup_x \|\pi_x(\kappa(h))\| = \|\kappa(h)\|_{C^*_r(\Sigma;G)}. \end{aligned} \tag{2.13}$$

Let $B = \overline{\kappa(C_c(\Sigma_H; H))}$, so B is a C^*-subalgebra of $C^*_r(\Sigma; G)$. By (2.13), the map $\kappa(h) \mapsto h$ extends to a $*$-epimorphism $\Theta : B \twoheadrightarrow C^*_r(\Sigma_H; H)$.

Now let $\Delta : C^*_r(\Sigma; G) \to C_0(X)$ be the faithful conditional expectation determined by $C_c(\Sigma; G) \ni f \mapsto f|_X$; likewise let $\Delta^H : C^*_r(\Sigma_H; H) \to C_0(X)$ be determined by $C_0(\Sigma_H; H) \ni h \mapsto h|_X$. For $h \in C_0(\Sigma_H; H)$, $\Delta(\kappa(h)) = \Delta^H(h)$. Therefore, for $b \in B$, $\Delta(b) = \Delta^H(\eta(b))$. So for $b \in B$, $\Theta(b^*b) = 0$ implies $b = 0$ by the faithfulness of Δ. It follows that Θ is a $*$-isomorphism of B onto $C^*_r(\Sigma_H; H)$. Therefore, Θ^{-1} is a $*$-isomorphism of $C^*_r(\Sigma_H; H)$ onto $\overline{\kappa(C_0(\Sigma_H; H))}$, which is what we needed to show. □

The following proposition allows us to view elements of $C^*_r(\Sigma; G)$ as functions in $C_0(\Sigma; G)$. This proposition was originally proved in the case of Example 2.6 above by Renault in [31, Proposition II.4.2]. Renault uses it without proof in the full generality of twists in [33]. As we know of no proof of [31, Proposition II.4.2] for twists, we provide a proof here at the level of generality we will require. Note that $C_0(\Sigma, G)$ can be made into a Banach space with $\|f\| = \sup_{\dot\gamma\in G} \varpi(f(\dot\gamma))$.

Proposition 2.8. *Let $(\Sigma; G)$ be a twist with G étale. Then the inclusion map $j : C_c(\Sigma; G) \to C_0(\Sigma; G)$ extends to a norm-decreasing injective linear map of $C^*_r(\Sigma; G)$ into $C_0(\Sigma; G)$. Moreover, the algebraic operations of adjoint and convolution on $C_c(\Sigma; G)$ extend to corresponding operations on $j(C^*_r(\Sigma; G))$: that is, for every $a, b \in C^*_r(\Sigma; G)$ and $\dot\gamma \in G$,*

$$j(a^*)(\dot\gamma) = \overline{j(a)(\dot\gamma^{-1})} \quad \textit{and} \quad j(ab)(\dot\gamma) = \sum_{r(\dot\eta)=r(\dot\gamma)} j(a)(\dot\eta)\, j(b)(\dot\eta^{-1}\dot\gamma). \tag{2.14}$$

Proof. The algebra $C_c(\Sigma; G)$ may be regarded as a subalgebra of $C^*_r(\Sigma; G)$ or as its image under j in $C_0(\Sigma; G)$. First we show that for $f \in C_c(\Sigma; G)$ we have $\|f\|_r \geq \|f\|_\infty$. To see this, for $\dot\gamma \in G$ consider $\delta_{s(\dot\gamma)}$. We have

$$\begin{aligned}\|f\|_r \geq \|\pi_{s(\dot\gamma)}(f)\| \geq \|\pi_{s(\dot\gamma)}(f)\delta_{s(\dot\gamma)}\| &= \langle \pi_{s(\dot\gamma)}(f)(\delta_{s(\dot\gamma)}), \pi_{s(\dot\gamma)}(f)(\delta_{s(\dot\gamma)})\rangle^{1/2} \\ &= \sqrt{\sum_{s(\dot\eta)=s(\dot\gamma)} |f(\eta)|^2} \geq |f(\dot\gamma)|.\end{aligned} \tag{2.15}$$

Thus j extends to a norm decreasing linear map $j : C^*_r(\Sigma; G) \to C_0(\Sigma; G)$.

We turn to showing that j is injective. Since j is norm-decreasing, the equalities in (2.12) extend to every element of $C^*_r(\Sigma; G)$. Therefore, for any $\dot\gamma \in Gx$, and $a \in C^*_r(\Sigma; G)$,

$$\begin{aligned}\|\pi_x(a)\delta_{\dot\gamma}\|^2 &= \sum_{\dot u\in Gx} |\langle \pi_x(a)\delta_{\dot\gamma}, \delta_{\dot u}\rangle|^2 = \sum_{\dot u\in Gx} |\pi_x(a)\delta_{\dot\gamma}(\dot u)|^2 \\ &= |\pi_x(a)\delta_{\dot\gamma}(\dot\gamma)|^2 = |j(a)(\dot\gamma\dot\gamma^{-1})|^2.\end{aligned}$$

So if $j(a) = 0$, then $\pi_x(a) = 0$ for every $x \in G^{(0)}$. Thus $a = 0$, so j is injective.

To verify the first equality in (2.14), observe that it holds for $a \in C_c(\Sigma; G)$. For general $a \in C^*_r(\Sigma; G)$, observe that for any $f \in C_c(\Sigma; G)$, the fact that j is contractive yields

$$\varpi(j(a^*(\dot\eta)) - \overline{j(a)(\dot\eta)}) \leq \varpi(j(a^* - f^*)(\dot\eta)) + \varpi(\overline{j(f-a)(\dot\eta^{-1})}) \leq 2\,\|a - f\|_r\,.$$

As the right-most term in this inequality can be made as small as desired by choosing f appropriately, we obtain the first equality.

Before establishing the second, for $a \in C^*_r(\Sigma; G)$ and $x \in G^{(0)}$, define

$$\|a\|_{2,x} := \|\pi_x(a)\delta_x\|\,.$$

Then $\max\{\|a\|_{2,x}\,, \|a^*\|_{2,x}\} \leq \|a\|_r$ and

$$\|a\|^2_{2,x} = \sum_{\dot\eta\in Gx} |\langle \pi_x(a)\delta_x, \delta_{\dot\eta}\rangle|^2 = \sum_{\dot\eta\in Gx} |j(a)(\dot\eta)|^2$$

and, using the first equality in (2.14),

$$\|a^*\|_{2,x}^2 = \sum_{\dot\eta \in xG} |j(a)(\dot\eta)|^2.$$

To establish the second equality in (2.14), first note it holds when $a, b \in C_c(\Sigma; G)$. Now let $a, b \in C_r^*(\Sigma; G)$ be arbitrary. Suppose $(f_i), (g_i)$ are nets in $C_c(\Sigma; G)$ such that $\|f_i - a\|_r \to 0$ and $\|g_i - b\|_r \to 0$. Then

$$\begin{aligned}
&\varpi\left(\sum_{r(\dot\eta)=r(\dot\gamma)} j(f_i)(\dot\eta)\, j(g_i)(\dot\eta^{-1}\dot\gamma) - \sum_{r(\dot\eta)=r(\dot\gamma)} j(a)(\dot\eta)\, j(b)(\dot\eta^{-1}\dot\gamma)\right)\\
&= \varpi\left(\sum_{r(\dot\eta)=r(\dot\gamma)} j(f_i)(\dot\eta)\, j(g_i - b)(\dot\eta^{-1}\dot\gamma) + \sum_{r(\dot\eta)=r(\dot\gamma)} j(f_i - a)(\dot\eta)\, j(b)(\dot\eta^{-1}\dot\gamma)\right)\\
&\leq \|f_i^*\|_{2,r(\dot\gamma)}\, \|g_i - b\|_{2,s(\dot\gamma)} + \|f_i^* - a^*\|_{2,r(\dot\gamma)}\, \|b\|_{2,s(\dot\gamma)}\\
&\leq \|f_i\|_r\, \|g_i - g\|_r + \|f_i - a\|_r\, \|b\|_r\,,
\end{aligned}$$

from which it follows that

$$\lim_{i\to\infty} \sum_{r(\dot\eta)=r(\dot\gamma)} j(f_i)(\dot\eta)\, j(g_i)(\dot\eta^{-1}\dot\gamma) = \sum_{r(\dot\eta)=r(\dot\gamma)} j(a)(\dot\eta)\, j(b)(\dot\eta^{-1}\dot\gamma).$$

Therefore, for every $\dot\gamma \in G$,

$$\begin{aligned}
j(ab)(\dot\gamma) &= \left\langle \pi_{s(\dot\gamma)}(ab)\delta_{s(\dot\gamma)}, \delta_{\dot\gamma}\right\rangle \delta_{\dot\gamma}(\dot\gamma) = \lim j(f_i g_i)(\dot\gamma)\\
&= \lim \sum_{r(\dot\eta)=r(\dot\gamma)} f_i(\dot\eta) g_i(\dot\eta^{-1}\dot\gamma) = \sum_{r(\dot\eta)=r(\dot\gamma)} a(\dot\eta) b(\dot\eta^{-1}\dot\gamma),
\end{aligned}$$

as desired. □

Definition 2.9. *Let G be an étale groupoid and Γ a discrete abelian group. A twist graded by Γ is a twist $\mathbb{T} \times G^{(0)} \hookrightarrow \Sigma \twoheadrightarrow G$ over G together with continuous groupoid homomorphisms $c_\Sigma : \Sigma \to \Gamma$ and $c_G : G \to \Gamma$ such that the diagram,*

$$\begin{tikzcd}
\mathbb{T} \times G^{(0)} \arrow[r] & \Sigma \arrow[r] \arrow[dr, "c_\Sigma"'] & G \arrow[d, "c_G"] \\
& & \Gamma
\end{tikzcd} \tag{2.16}$$

commutes. We will sometimes abbreviate (2.16) and simply say $\Sigma \to G$ is a Γ-graded twist.

For $\omega \in \hat\Gamma$ and $t \in \Gamma$ we denote the natural pairing $\omega(t)$ by $\langle \omega, t\rangle$. We will use additive notation for the group Γ and multiplicative notation for the group $\hat\Gamma$. We now show that the grading maps c_Σ and c_G induce an action of $\hat\Gamma$ on $C_r^*(\Sigma; G)$. This fact is well known to experts but we include a proof for completeness.

Lemma 2.10. *Suppose $\Sigma \to G$ is a Γ-graded twist. There exists a continuous action of $\hat{\Gamma}$ on $C_r^*(\Sigma; G)$ characterized by*

$$(\omega \cdot f)(\dot{\gamma}) = \langle \omega, c_G(\dot{\gamma}) \rangle f(\dot{\gamma})$$

where $\omega \in \hat{\Gamma}$ and $f \in C_c(\Sigma; G)$.

Proof. First we check that the action is multiplicative. For this we compute

$$\begin{aligned}
(\omega \cdot f) * (\omega \cdot g)(\dot{\gamma}) &= \sum_{r(\dot{\eta})=r(\dot{\gamma})} (\omega \cdot f)(\dot{\eta})(\omega \cdot g)(\dot{\eta}^{-1}\dot{\gamma}) \\
&= \sum_{r(\dot{\eta})=r(\dot{\gamma})} \langle \omega, c(\dot{\eta}) \rangle f(\dot{\eta}) \langle \omega, c(\dot{\eta}^{-1}\dot{\gamma}) \rangle g(\dot{\eta}^{-1}\dot{\gamma}) \\
&= \langle \omega, c(\dot{\gamma}) \rangle \sum_{r(\dot{\eta})=r(\dot{\gamma})} f(\dot{\eta}) g(\dot{\eta}^{-1}\dot{\gamma}) = (\omega \cdot (f * g))(\dot{\gamma}).
\end{aligned}$$

Now let L be the line bundle over G associated to Σ and for $x \in G^{(0)}$ let $L_x := L|_{Gx}$. Consider the regular representation π_x of $C_r^*(\Sigma; G)$ associated to $x \in G^{(0)}$.

For $\chi \in \mathcal{H}_x$ define $\chi_\omega \in \mathcal{H}_x$ by $\chi_\omega(\dot{\gamma}) := \overline{\langle \omega, c(\dot{\gamma}) \rangle} \chi(\dot{\gamma})$. Then $\|\chi_\omega\|^2 = \|\chi\|^2$, so the mapping $\chi \mapsto \chi_\omega$ is a unitary $W_\omega \in \mathcal{B}(\mathcal{H}_x)$.

So for $f \in C_c(\Sigma; G)$,

$$\begin{aligned}
\pi_x(\omega \cdot f)\chi(\dot{\gamma}) &= \sum_{r(\dot{\eta})=r(\dot{\gamma})} \langle \omega, c(\dot{\eta}) \rangle f(\dot{\eta}) \chi(\dot{\eta}^{-1}\dot{\gamma}) \\
&= \sum_{r(\dot{\eta})=r(\dot{\gamma})} \langle \omega, c(\dot{\gamma}) \rangle \overline{\langle \omega, c(\dot{\gamma}) \rangle \langle \omega, c(\dot{\eta}^{-1}) \rangle} f(\dot{\eta}) \chi(\dot{\eta}^{-1}\dot{\gamma}) \\
&= \langle \omega, c(\dot{\gamma}) \rangle \sum_{r(\dot{\eta})=r(\dot{\gamma})} f(\dot{\eta}) \chi_\omega(\dot{\eta}^{-1}\dot{\gamma}) = \langle \omega, c(\dot{\gamma}) \rangle \pi_x(f) \chi_\omega(\dot{\gamma}).
\end{aligned}$$

This then implies that $\|\pi_x(\omega \cdot f)\chi\| = \|\pi_x(f)\chi_\omega\|$. So now

$$\|\pi_x(\omega \cdot f)\| = \sup_{\|\chi\|=1} \|\pi_x(\omega \cdot f)\chi\| = \sup_{\|\chi\|=1} \|\pi_x(f)\chi_\omega\| = \sup_{\|\chi\|=1} \|\pi_x(f)\chi\| = \|\pi_x(f)\|$$

and since this holds for all x we get

$$\|\omega \cdot f\|_r = \|f\|_r$$

as desired.

Now suppose that we have nets $\omega_i \to \omega$ and $a_i \to a \in C_r^*(G; \Sigma)$. Consider $\omega_i \cdot a_i - \omega \cdot a = \omega_i \cdot a_i - \omega_i \cdot a + \omega_i \cdot a - \omega \cdot a$. Since $\|\omega \cdot a\|_r = \|a\|_r$, to show $\omega_i \cdot a_i \to \omega \cdot a$ it suffices to show $\omega_i \cdot a \to \omega \cdot a$. For i sufficiently large we can assume $\|a\|_r \sup |\langle \omega_i - \omega \cdot c(\dot{\eta}) \rangle| < \epsilon$. Now

$$\begin{aligned}\|\pi_x(\omega_i \cdot a - \omega \cdot a)\chi\| &= \|\sum_{r(\dot\eta)=r(\dot\gamma)} \langle \omega_i\omega^{-1}, c(\dot\eta)\rangle a(\dot\eta)\chi(\dot\eta^{-1}\dot\gamma)\| \\ &= \|\langle \omega_i\omega^{-1}, c(\dot\gamma)\rangle \sum_{r(\dot\eta)=r(\dot\gamma)} a(\dot\eta)\chi_{\omega_i\omega^{-1}}(\dot\eta^{-1}\dot\gamma)\| \\ &\leq |\langle \omega_i\omega^{-1}, c(\dot\gamma)\rangle|\|a\|_r < \epsilon.\end{aligned}$$

Since this holds for all $x \in G^{(0)}$ we get the result. □

Remark 2.11. *When elements of $C_c(\Sigma; G)$ are viewed as in (2.6), the action of $\hat\Gamma$ on $C_r^*(\Sigma; G)$ is characterized by*

$$(\omega \cdot f)(\gamma) = \langle \omega, c_\Sigma(\gamma)\rangle\, f(\gamma),$$

where $\omega \in \hat\Gamma$ and $f \in C_c(\Sigma)$ is covariant.

3. Γ-Cartan pairs and abelian group actions

In this section we define the main objects of our study, Γ-Cartan pairs, and explore the relationship between Γ-Cartan pairs and strongly continuous actions of compact abelian groups on C^*-algebras. We first give some preliminary results on topologically graded C^*-algebras.

Definition 3.1. *A C^*-algebra A is* topologically graded *by a (discrete abelian) group Γ if there exists a family of linearly independent closed linear subspaces $\{A_t\}_{t\in\Gamma}$ of A such that*

- *$A_tA_s \subseteq A_{t+s}$,*
- *$A_t^* = A_{-t}$,*
- *A is densely spanned by $\{A_t\}_{t\in\Gamma}$; and*
- *there is a faithful conditional expectation from A onto A_0.*

Definition 3.2. *Let A be a C^*-algebra topologically graded by a group Γ. We call an element $a \in A$ homogeneous if $a \in A_t$ for some t. Let $D \subseteq A_0$ be an abelian subalgebra. We denote the set of normalizers of D in A by $N(A, D)$ or simply N. Also, n is a homogeneous normalizer if it is both a normalizer and homogeneous: that is, n is a normalizer and $n \in A_t$ for some $t \in \Gamma$. We denote the set of homogeneous normalizers by $N_h(A, D)$ or simply N_h. Notice that for $n \in N_h$ and $d \in D$ we have $nd, dn \in N_h$.*

The term topologically graded was introduced by Exel [14]; see also [16].

An action of a compact abelian group on a C^*-algebra produces a topological grading, which we now describe in some detail.

Let Γ be a discrete abelian group and A a C^*-algebra. As is customary, we say $\hat\Gamma$ *acts strongly on A* if there is a strongly continuous group of automorphisms on the C^*-algebra A indexed by $\hat\Gamma$. That is, there is a map $\hat\Gamma \times A \to A$, written $(\omega, a) \mapsto \omega \cdot a$ such that:

(1) for every ω, $a \mapsto \omega \cdot a$ is an automorphism β_ω of A;

(2) the map $\omega \mapsto \beta_\omega$ is a homomorphism of $\hat{\Gamma}$ into $\mathrm{Aut}(A)$; and
(3) for each $a \in A$, the map $\omega \mapsto \omega \cdot a$ is norm continuous.

Let $A^{\hat{\Gamma}}$ be the fixed point algebra under this action. For t in Γ and $a \in A$ define

$$\Phi_t(a) := \int_{\hat{\Gamma}} (\omega \cdot a)\langle \omega^{-1}, t\rangle d\omega, \tag{3.1}$$

and let

$$A_t = \Phi_t(A)$$

be the range of Φ_t. Then for each $t \in \Gamma$, Φ_t is a completely contractive and idempotent linear map. The following simple fact is worth noting.

Lemma 3.3. *The map $\Phi_0 : A \to A^{\hat{\Gamma}} = A_0$ is a faithful conditional expectation.*

Sketch of Proof. That Φ_0 is a conditional expectation is clear, so it remains to show Φ_0 is faithful. If $\Phi_0(a^*a) = 0$, then for every state ρ on A, $\int_{\hat{\Gamma}} \rho(\omega \cdot (a^*a))\, d\omega = 0$. Thus $\rho(\omega \cdot (a^*a)) = 0$ for every state ρ and every $\omega \in \hat{\Gamma}$. Taking ω to be the unit element gives $\rho(a^*a) = 0$ for every state, so $a^*a = 0$. □

We now characterize the homogeneous elements of A. The following lemma is a generalization of [1, Lemma 5.2.10]. where it is proved for $\Gamma = \mathbb{Z}$.

Lemma 3.4. *Suppose $\hat{\Gamma}$ acts strongly on A. The following statements hold for all $t \in \Gamma, a, b \in A$.*

(1) $a \in A_t$ iff for every $\sigma \in \hat{\Gamma}$, $\omega \cdot a = \langle \omega, t\rangle a$.
(2) $a \in A_t$ iff $a^* \in A_{-t}$.
(3) If $a \in A_t$, $b \in A_s$ then $ab \in A_{t+s}$.
(4) If $a \in A_t$ and $s \in \Gamma$, then $\Phi_s(a) = \begin{cases} a & \text{if } s = t; \\ 0 & \text{otherwise.} \end{cases}$

Proof. Let $a \in A_t$ and $\sigma \in \hat{\Gamma}$. Then

$$\begin{aligned} \sigma \cdot a = \sigma \cdot \Phi_t(a) &= \int_{\hat{\Gamma}} ((\sigma\omega) \cdot a) \left\langle \omega^{-1}, t\right\rangle d\omega = \int_{\hat{\Gamma}} (\omega \cdot a) \left\langle \omega^{-1}, t\sigma\right\rangle d\omega \\ &= \langle \sigma, t\rangle \Phi_t(a) = \langle \sigma, t\rangle a. \end{aligned}$$

Conversely if $\sigma \cdot a = \langle \sigma, t\rangle a$ for every $\sigma \in \hat{\Gamma}$, then

$$\Phi_t(a) = \int_{\hat{\Gamma}} (\omega \cdot a)\langle \omega^{-1}, t\rangle d\omega = \int_{\hat{\Gamma}} a\langle \omega, t\rangle\langle \omega^{-1}, t\rangle d\omega = a.$$

Items (2) and (3) follow immediately since $\sigma \cdot (a^*) = (\sigma \cdot a)^* = \overline{\langle \sigma, t\rangle} a^* = \langle \sigma, -t\rangle a^*$ and $\sigma \cdot (ab) = (\sigma \cdot a)(\sigma \cdot b) = \langle \sigma, t\rangle\, a\langle \sigma, s\rangle b = \langle \sigma, t + s\rangle ab$.

Lastly for (4),

$$\Phi_s(a) = \int_{\hat{\Gamma}} (\omega \cdot a)\langle \omega^{-1}, s\rangle d\omega = a \int_{\hat{\Gamma}} \langle \omega, t\rangle\langle \omega^{-1}, s\rangle d\omega = \delta_{s,t} a. \qquad \square$$

The following lemma and its corollary show the linear span of the homogeneous spaces $\{A_t\}_{t\in\Gamma}$ is dense in A. We thank Ruy Exel for showing us the simple proof.

Lemma 3.5. *Suppose the compact abelian group $\hat{\Gamma}$ acts strongly on the C^*-algebra A, and $a \in A$. Then $a \in \overline{\operatorname{span}}\{\Phi_t(a) : t \in \Gamma\}$.*

Proof. Let $B := \overline{\operatorname{span}}\{\Phi_t(a) : t \in \Gamma\}$. Suppose ρ is a bounded linear functional on A which annihilates B. Define $g_a : \hat{\Gamma} \to \mathbb{C}$ by $g_a(\omega) = \rho(\omega \cdot a)$. Compute the Fourier transform of g_a: for $t \in \Gamma$,

$$\begin{aligned}\hat{g}_a(t) &= \int_{\hat{\Gamma}} g_a(\omega)\overline{\langle \omega, t\rangle}\, d\beta \\ &= \rho\left(\int_{\hat{\Gamma}} (\omega \cdot a)\overline{\langle \omega, t\rangle}\, d\omega\right) \\ &= \rho(\Phi_t(a)) = 0.\end{aligned}$$

Since the Fourier transform is one-to-one, $g_a = 0$. Taking $\omega = 1$, we get $\rho(a) = 0$. As this does not depend on the choice of ρ, by the Hahn-Banach theorem, $a \in B$ $\square$

As an immediate corollary we get that $\{A_t\}_{t\in\Gamma}$ has dense span in A.

Corollary 3.6. *Suppose the compact abelian group $\hat{\Gamma}$ acts on the C^*-algebra A. For $t \in \Gamma$, let $A_t := \{a \in A : \beta \cdot a = \langle \beta, t\rangle\, a$ for every $\beta \in \hat{\Gamma}\}$. Then $A = \overline{\operatorname{span}}\{A_t : t \in \Gamma\}$.*

Remark 3.7. *Lemmas 3.4 and 3.5 show that if $\hat{\Gamma}$ acts strongly on A, then A is topologically graded by Γ. In particular, when $\Sigma \to G$ is a Γ-graded twist, Lemma 2.10 shows that $C^*_r(\Sigma; G)$ is topologically graded by Γ. In* [30, Theorem 3] *the converse to Lemma 3.4 is proved: it is shown that if A is topologically graded by Γ, then there is a strongly continuous action of $\hat{\Gamma}$ on A such that $a \in A_t$ if and only if*

$$a = \int_{\hat{\Gamma}} (\omega \cdot a)\langle \omega^{-1}, t\rangle\; d\omega.$$

We now observe that the proof of Lemma 3.5 can be used to show that if $\overline{\operatorname{span}}\, N(A, D) = A$ then $\overline{\operatorname{span}}\, N_h(A, D) = A$. Here are the details.

Proposition 3.8. *Suppose $\hat{\Gamma}$ acts on A and that D is a MASA in A_0. If $n \in N$, then for every $t \in \Gamma$, $\Phi_t(n) \in N_h$ and $n \in \overline{\operatorname{span}}\{\Phi_t(n) : t \in \Gamma\}$.*

Proof. Fix $n \in N$. By Lemma 3.5 it suffices to show $\Phi_t(n) \in N_h$. Let $d \in D$. Then $\Phi_t(n)^* d\Phi_t(n) \in A_0$. For $e \in D$, and $\omega \in \hat{\Gamma}$, $\omega \cdot e = e$. So

$$\begin{aligned}
\Phi_t(n)^* d\Phi_t(nn^*n)e &= \Phi_t(n)^* d \int_{\hat{\Gamma}} \omega \cdot n(\omega \cdot (n^*ne))\langle \omega^{-1}, t\rangle d\omega \\
&= \Phi_t(n)^* d \int_{\hat{\Gamma}} \omega \cdot (nen^*n)\langle \omega^{-1}, t\rangle d\omega \\
&= \Phi_t(n)^* dnen^* \Phi_t(n) = \Phi_t(n)^* nen^* d\Phi_t(n) \\
&= \int_{\hat{\Gamma}} \omega \cdot (n^*nen^*)\langle \omega, t\rangle d\omega d\Phi_t(n) \\
&= n^*ne\Phi_t(n)^* d\Phi_t(n) = en^*n\Phi_t(n)^* d\Phi_t(n) \\
&= e\Phi_t(nn^*n)^* d\Phi_t(n).
\end{aligned}$$

This relation holds if we replace n^*n by a polynomial in n^*n and by taking limits we see that it holds if we replace n^*n by $(n^*n)^{1/k}$ for any $k \in \mathbb{N}$. Since $\lim_k n(n^*n)^{1/k} = n$, we find that $\Phi_t(n)^* d\Phi_t(n)$ commutes with every element of D. Since D is a MASA in A_0, $\Phi_t(n)^* d\Phi_t(n) \in D$. A similar argument shows that $\Phi_t(n) d\Phi_t(n)^* \in D$. So $\Phi_t(n) \in N_h$. ☐

We now define a main object of study.

Definition 3.9. *Let A be C^*-algebra topologically graded by a discrete abelian group Γ and D an abelian C^*-subalgebra of A_0. We say the pair (A, D) is Γ-Cartan if*

(1) D is Cartan in A_0,
(2) $N(A, D)$ spans a dense subset of A.

The following observations are simple but important. In particular, for Γ-Cartan pairs we may focus on homogeneous normalizers in place of more general normalizers.

Lemma 3.10. *Suppose (A, D) is a Γ-Cartan pair. The following statements hold.*

(1) The span of the homogeneous normalizers, $N_h(A, D)$, is dense in A.
(2) If (e_i) is an approximate unit for A_0, then (e_i) is an approximate unit for A.
(3) For any $n \in N(A, D)$, n^*n and nn^* belong to D.
(4) Any approximate unit for D is an approximate unit for A.

Proof. As noted in Remark 3.7, a topological grading arises from an action of a compact abelian group. By Proposition 3.8, $N_h(A, D)$ spans a dense subset of A.

Now suppose (e_i) is an (not necessarily countable) approximate unit for A_0. Let $n \in N_h$. Then nn^* and n^*n belong to A_0. Since (e_i) is an approximate unit for A_0,

$$(e_in - n)(e_in - n)^* = e_inn^*e_i - nn^*e_i - e_inn^* + nn^* \to 0, \tag{3.2}$$

whence $e_i n \to n$. Similarly, $ne_i \to n$. Hence for any $a \in \operatorname{span} N_h$, $e_i a \to a$ and $ae_i \to a$. Since $\operatorname{span} N_h$ is dense in A, (e_i) is an approximate unit for A.

Since (A_0, D) is a Cartan pair, D contains an approximate unit (e_i) for A_0. By part (2), (e_i) is also an approximate unit for A. Then for any $n \in N(A, D)$, $D \ni n^* e_i n \to n^* n$, so $n^* n \in D$. Likewise, $nn^* \in D$.

Finally, if (e_i) is an approximate unit for D and $n \in N$, (3.2) together with the fact that $nn^* \in D$, gives $e_i n \to n$; likewise $ne_i \to n$. As before, $\overline{\operatorname{span} N} = A$ implies (e_i) is an approximate unit for A. $\square$

4. Twists from Γ-Cartan pairs

Throughout this section, we consider a fixed Γ-Cartan pair (A, D). The purpose of this section is to define a twist $\hat{D} \times \mathbb{T} \to \Sigma \to G$ from the pair (A, D) so that $A \cong C_r^*(\Sigma; G)$ and $D \cong C_0(G^{(0)})$. This task is completed in Theorem 4.19. Our methods follow those found in Kumjian [18] and Renault [33], and also use techniques from Pitts [26]. (The methods in [26] have been extended and updated in Pitts [28].)

Renault and Kumjian construct a twist from the Weyl groupoid associated to a Cartan pair by first considering a groupoid G of germs and then using the multiplicative structure of the normalizers to construct the twist as an extension Σ of G by $\mathbb{T} \times G^{(0)}$. Finally, they recognize Σ as a family of linear functionals on A.

To a certain extent, we follow the Kumjian-Renault approach. We will define Σ and G in two ways. We first construct sets Σ and G using the Weyl groupoid (the topologies and groupoid operations come later). After doing so, we identify Σ as a family of linear functionals and G as as a family of (non-linear) functions on A. The product on Σ and G is obtained by translating the product on A to Σ utilizing the first approach, and the second approach makes defining the topologies on Σ and G straightforward. Viewing Σ and G as functions highlights the parallel between the Gelfand theory for commutative C^*-algebras and relationship of the twist and the pair (A, D) more transparent.

To begin, we fix some notation. Write

$$X := \hat{D}.$$

We generally identify D with $C_0(X)$; thus for $x \in X$ and $d \in D$, we write $d(x)$ instead of $\hat{d}(x)$.

Let E denote the faithful conditional expectation $E \colon A_0 \to D$. By [30] there is a corresponding strong action of $\hat{\Gamma}$ on A. We denote by Φ_t the completely contractive map $\Phi_t \colon A \to A_t$ as defined in Equation (3.1). Set

$$\Delta := E \circ \Phi_0.$$

By Lemma 3.3, Δ is a faithful conditional expectation of A onto D.

For $n \in N$, Lemma 3.10 gives $n^*n, nn^* \in D$; let

$$\operatorname{dom}(n) := \{x \in \hat{D} : n^*n(x) > 0\} \quad \text{and} \quad \operatorname{ran}(n) := \{x \in \hat{D} : nn^*(x) > 0\}.$$

By the definition of normalizer, $ndn^* \in D$ for all $d \in D$. So N_h acts on D by conjugation. As D is abelian, this induces a partial action α on the spectrum. The following result of Kumjian gives a precise description of this action.

Proposition 4.1. [18, Proposition 1.6] *Let $n \in N$. Then there exists a unique partial homeomorphism $\alpha_n : \operatorname{dom}(n) \to \operatorname{ran}(n)$ such that for each $d \in D$ and $x \in \operatorname{dom}(n)$,*

$$(n^*dn)(x) = d(\alpha_n(x))\,(n^*n)(x).$$

When the action is clear from the context, we will sometimes write

$$n.x := \alpha_n(x).$$

By [33, Lemma 4.10] (or [18, Corollary 1.7]), for $n, m \in N$ and $d \in D$ we have

$$\alpha_n \circ \alpha_m = \alpha_{mn}, \quad \alpha_{n^*} = \alpha_n^{-1}, \text{ and } \quad \alpha_d = \operatorname{id}_{\operatorname{supp}'(d)}.$$

The collection $\{\alpha_n : n \in N\}$ is an inverse semigroup, sometimes called the *Weyl semigroup* of the inclusion (A, D).

Dual to the Weyl semigroup is a collection of partial automorphisms $\{\theta_n : n \in N\}$ of D. Given $n \in N$, $\overline{nn^*D}$ and $\overline{n^*nD}$ are ideals of D whose Gelfand spaces may be identified with $\operatorname{ran}(n)$ and $\operatorname{dom}(n)$ respectively. By [27, Lemma 2.1], the map $nn^*D \ni d \mapsto n^*dn \in n^*Dn$ extends uniquely to a $*$-isomorphism $\theta_n : \overline{nn^*D} \to \overline{n^*nD}$ such that for every $d \in \overline{nn^*D}$,

$$dn = n\theta_n(d) \tag{4.1}$$

and for every $x \in \operatorname{dom}(n)$,

$$\theta_n(d)(x) = d(\alpha_n(x)). \tag{4.2}$$

Lemma 4.2. *Suppose $n \in N_h(A, D)$ and $x \in X$ such that $\Delta(n)(x) \neq 0$. Then x is in the interior of the set of fixed points of α_n and there exists $h \in D$ such that $h(x) = 1$ and $nh = hn \in D$.*

Proof. First note $n \in A_0$ because $0 \neq \Delta(n)(x) = E(\Phi_0(n))(x)$, and thus $\Phi_0(n) \neq 0$. Furthermore, $x \in \operatorname{dom}(n)$ and [27, Lemma 2.5] gives $\alpha_n(x) = x$. We claim that x is actually in the interior of the set of fixed points of α_n. If not, then there exists a net (x_i) in $\operatorname{dom}(n)$ such that $\alpha_n(x_i) \neq x_i$ and $x_i \to x$. Then $\Delta(n)(x_i) \to \Delta(n)(x) \neq 0$. However, by [27, Lemma 2.5] again, $\Delta(n)(x_i) = 0$ for all i, a contradiction.

Now let F be the interior of the set of fixed points of α_n and $J := \{d \in D : \operatorname{supp} d \subseteq F\}$. For $S \subseteq D$ let

$$S^\perp = \{a \in D \colon ax = 0 \text{ for all } x \in S\}.$$

Note that $J^{\perp\perp}$ is the fixed point ideal K_0 for n (see [27, Definition 2.13]). Then by [27, Lemma 2.15] there exists $h \in D$ with $h(x) = 1$ and $nh = hn \in D'$. But $n \in A_0$ and $h \in D$, so $nh \in A_0 \cap D' = D$ because D is maximal abelian in A_0. Thus $nh = hn \in D$. This completes the proof. ☐

The following is an interesting structural fact about the relationship between Δ and the action of N_h on D, which is used when defining the inverse operation on Σ.

Proposition 4.3. *For any $n \in N_h$ and $a \in A$,*

$$n^*\Delta(a)n = \Delta(n^*an).$$

Proof. To begin, we claim that for $m \in N_h$,

$$n^*\Delta(nm)n = n^*n\Delta(mn). \tag{4.3}$$

Since the terms on both sides of (4.3) belong to D, it suffices to show that for every $x \in X$

$$(n^*\Delta(nm)n)(x) = (n^*n\Delta(mn))(x). \tag{4.4}$$

As $n = \lim_{k\to\infty} n(n^*n)^{1/k}$, both sides of (4.4) vanish if $(n^*n)(x) = 0$. Thus to obtain (4.4) it suffices to prove that for $x \in \mathrm{dom}(n)$

$$\Delta(nm)(\alpha_n(x)) = \Delta(mn)(x), \tag{4.5}$$

and this is what we shall do.

Suppose first that $\Delta(nm)(\alpha_n(x)) \neq 0$. Lemma 4.2 shows there exists $k \in D$ with $k(\alpha_n(x)) = 1$ and $nmk = knm \in D$. Then

$$\begin{aligned}\Delta(nm)(\alpha_n(x)) &= (k\Delta(nm))(\alpha_n(x)) = \Delta(knm)(\alpha_n(x))\\ &= (knm)(\alpha_n(x)) = \frac{(n^*(knm)n)(x)}{(n^*n)(x)} = \frac{\Delta((n^*kn)mn)(x)}{(n^*n)(x)}\\ &= \frac{(n^*kn)(x)}{(n^*n)(x)}\Delta(mn)(x) = k(\alpha_n(x))\Delta(mn)(x) = \Delta(mn)(x).\end{aligned}$$

Next, suppose $\Delta(mn)(x) \neq 0$ and put $y = \alpha_n(x)$. We do a similar calculation. Another application of Lemma 4.2 produces $h \in D$ with $h(x) = 1$ and $mnh = hmn \in D$. As $x = \alpha_{n^*}(y)$,

$$\begin{aligned}\Delta(mn)(x) = (mnh)(\alpha_{n^*}(y)) &= \frac{(n(mnh)n^*)(y)}{(nn^*)(y)} = \frac{\Delta(nm(nhn^*))(y)}{(nn^*)(y)}\\ &= \Delta(nm)(y)\frac{(nhn^*)(y)}{(nn^*)(y)} = \Delta(nm)(\alpha_n(x))h(\alpha_{n^*}(y))\\ &= \Delta(nm)(\alpha_n(x))h(x) = \Delta(nm)(\alpha_n(x)).\end{aligned}$$

We have shown that $\Delta(mn)(x) \neq 0$ if and only if $\Delta(nm)(\alpha_n(x)) \neq 0$, and, when this occurs, $\Delta(mn)(x) = \Delta(nm)(\alpha_n(x))$. Thus (4.5) holds, completing the proof of the claim.

By varying m and using the facts that $n^*n \in D$ (Lemma 3.10(3)) and $\overline{\operatorname{span} N_h} = A$, (4.3) implies that for every $a \in A$, $n^*\Delta(na)n = (n^*n)\Delta(an) = \Delta(n^*(na)n)$. Therefore, for every $a \in \overline{nA}$,

$$n^*\Delta(a)n = \Delta(n^*an).$$

Given $k \in \mathbb{N}$, there exists a sequence of polynomials $\{p_j\}$ each of which vanish at the origin such that $(nn^*)^{1/k} = \lim_j p_j(nn^*)$. Thus, for $a \in A$ and $k \in \mathbb{N}$, $(nn^*)^{1/k}a(nn^*)^{1/k} \in \overline{nA}$. Hence for each $a \in A$,

$$\begin{aligned} n^*\Delta(a)n &= \lim_k n^*(nn^*)^{1/k}\Delta(a)(nn^*)^{1/k}n = \lim_k n^*\left(\Delta((nn^*)^{1/k}a(nn^*)^{1/k})\right)n \\ &= \lim_k \Delta(n^*(nn^*)^{1/k}a(nn^*)^{1/k}n) = \Delta(n^*an). \end{aligned}$$

This completes the proof. □

4.1. Local equivalence relations from homogeneous normalizers. Let

$$\mathcal{G} := \{(n,x) \in N_h \times X : n^*n(x) \neq 0\}.$$

We now define two equivalence relations on $\mathcal{G}$ arising as germs of the subsemigroup of the Weyl semigroup arising from homogeneous normalizers. While we shall define the groupoids Σ and G in the twist $\Sigma \to G$ as functions on A, the equivalence relations below will enable us to define the multiplicative structure on Σ and G.

Definition 4.4. *For $(n,x), (n',x') \in \mathcal{G}$, consider*

(1) $x = x'$,
(2_Σ) *there exist $d, d' \in C_c(X)$ such that $d(x) > 0$, $d'(x) > 0$ and $nd = n'd'$,*
(2_G) *there exist $d, d' \in C_c(X)$ such that $d(x) \neq 0$, $d'(x) \neq 0$ and $nd = n'd'$.*

By (4.1) and (4.2), the latter two conditions may equivalently be replaced with the following conditions.

($2_\Sigma{}'$) *There exist $d, d' \in C_c(X)$ such that $d(\alpha_n(x))) > 0$, $d'(\alpha_n(x))) > 0$ and $dn = d'n'$.*
($2_G{}'$) *There exist $b, b' \in C_c(X)$ such that $d(\alpha_n(x))) \neq 0$, $d'(\alpha_n(x))) \neq 0$ and $dn = d'n'$.*

Note that in conditions (2_Σ) and (2_G), we may assume that $d, d' \in \overline{n^*nD} \cap \overline{n'^*n'D}$; likewise we may assume $d, d' \in \overline{nn^*D} \cap \overline{n'n'^*D}$ in conditions ($2_\Sigma{}'$) and ($2_G{}'$).

Define $\sim_\Sigma$ as the relation given by (1) and (2_Σ) and $\sim_G$ as the relation given by (1) and (2_G). We omit the proof that these are equivalence relations. We denote the equivalence classes by $[n,x]_\Sigma, [n,x]_G$ respectively. We shall omit the subscript when the proof does not depend on which relation is used. Following Renault [33], define

$$\Sigma_{A,D,\Gamma} := \mathcal{G}/\sim_\Sigma \quad \text{and} \quad G_{A,D,\Gamma} := \mathcal{G}/\sim_G .$$

We omit the A, D, Γ from the notation and write Σ and G respectively when the inclusion and grading are clear from context.

Essentially, G is a modification of the groupoid of germs of the α action and Σ is a twist on this. The following is a useful observation.

Lemma 4.5. *For $i = 1, 2$ suppose $(n_i, x_i) \in \mathcal{G}$ and $[n_1, x_1] = [n_2, x_2]$. Then $n_1^* n_2 \in A_0$.*

Proof. We do this only for $\sim_G$, leaving the obvious modifications for $\sim_\Sigma$ to the reader. By definition of $\sim_G$, $x_1 = x_2 =: x$ and there exist $d_1, d_2 \in D$ with $d_i(x) \neq 0$ and $n_1 d_1 = n_2 d_2$. Then $n_i d_i \in N_h$ and $d_1^* n_1^* n_2 d_2 = d_1^* n_1^* n_1 d_1$ is a non-zero element of D. Since $n_i \in N_h$, there exists $t \in \hat{\Gamma}$ such that $n_1^* n_2 \in A_t$. But A_t is a D-bimodule, so $d_1^* n_1^* n_2 d_2 \in A_0 \cap A_t$, whence $t = 0$. □

It is useful to have an alternative description of the equivalence relations $\sim_\Sigma$ and $\sim_G$ before continuing.

Definition 4.6. *For $(n, x), (n', x') \in \mathcal{G}$, consider the properties*

(i) $x = x'$,

(ii_Σ) $\Delta(n^* n')(x) > 0$, and

(ii_G) $\Delta(n^* n')(x) \neq 0$.

Define $\approx_\Sigma$ to be the relation on $\mathcal{G}$ given by (i) and (ii_Σ), and define $\approx_G$ be the relation on $\mathcal{G}$ given by (i) and (ii_G).

Proposition 4.7. *The relations $\approx_\Sigma$ and $\sim_\Sigma$ are the same. Likewise, the relations $\approx_G$ and $\sim_G$ are the same.*

Proof. We prove $\approx_\Sigma$ and $\sim_\Sigma$ are the same. The proof for $\approx_G$ and $\sim_G$ is similar.

Suppose $(n, x) \approx_\Sigma (n', x')$. Then $x = x'$ and $\Delta(n^* n')(x) > 0$. Since n, n' are homogeneous normalizers, so is $n^* n'$. Lemma 4.2 implies there is an $h \in D$ such that $hn^* n' = n^* n' h \in D$ and $h(x) > 0$. Now consider the equalities,

$$n(n^* n' h)((n'h)^* n' h) = nn^*[((n'h)(n'h)^*) n' h] = n' h(nn^* \circ \alpha_{(n'h)^*})(n'h)^*(n'h).$$

Take $d = (n^* n' h)((n'h)^* n' h)$ and $d' = h(nn^* \circ \alpha_{(n'h)^*})(n'h)^*(n'h)$, so that $nd = n'd'$. Note

$$d(x) = n^* n'(x) h(x)((n'h)^* n' h)(x) > 0 \text{ and}$$
$$d'(x) = h(x)(nn^* \circ \alpha_{(n'h)^*})(x)((n'h)^*(n'h))(x) > 0.$$

Thus $(n, x) \sim_\Sigma (n', x)$. The converse follows immediately from the definitions. □

4.2. Viewing Σ as linear functionals. Our next goal is to show that Σ may be identified as a family of linear functionals on A and G as a family of functions on A. This highlights the role of the inclusion (A, D) in producing Σ and G and will allow us to easily define Hausdorff topologies on Σ and G. In addition, for $a \in A$ we will define $\hat{a} : \Sigma \to \mathbb{C}$ by $\hat{a}([n, x]) = [n, x](a)$. The main result of this section shows that the map $A \ni a \mapsto \hat{a} \in C_r^*(\Sigma; G)$ is an isomorphism which in a natural sense extends the the Gelfand transform.

We write $A^\#$ for the Banach space dual of A. For $f \in A^\#$, let $f^* \in A^\#$ be defined by $A \ni a \mapsto \overline{f(a^*)}$ and let $|f|$ be the function on A defined by $|f|(a) = |f(a)|$. For a non-empty subset $K \subseteq A^\#$, write $|K| := \{|f| : f \in K\}$. Equip K with the relative weak-$*$ topology and $|K|$ with the quotient topology arising from the surjective map, $K \ni f \mapsto |f|$. Then K and $|K|$ are Hausdorff.

Put

$$\mathfrak{S} := \{x \circ \Delta : x \in X\},$$

so $\mathfrak{S}$ consists of all states of the form $A \ni a \mapsto \Delta(a)(x)$. Then $\mathfrak{S}$ is a family of state extensions of pure states on D to all of A. We make the following observations.

Observations 4.8.

(1) With the relative weak-$*$ topology (i.e. the $\sigma(A^\#, A)$-topology) on $\mathfrak{S}$, the restriction map, $\mathfrak{S} \ni \psi \mapsto \psi|_D$ is a homeomorphism of $\mathfrak{S}$ onto X.

(2) Lemma 4.2 implies that if $\psi \in \mathfrak{S}$, then for every $n \in N_h$,

$$|\psi(n)|^2 \in \{0, \psi(n^*n)\}. \tag{4.6}$$

This condition is a variant of the notion of *compatible state* introduced in [27], the difference being that (4.6) is required to hold only for elements of N_h rather than all of N as in [27].

By [27, Proposition 4.4(iii)] and the Cauchy-Schwartz inequality, the compatibility condition (4.6) implies that in the GNS representation $(\pi_\psi, \mathcal{H}_\psi)$ associated to ψ, the set of vectors $\mathcal{V} := \{n + L_\psi : n \in N_h\}$ has the property that any two vectors in $\mathcal{V}$ are either orthogonal or parallel; here L_ψ is the left kernel of ψ, $L_\psi := \{a \in A : \psi(a^*a) = 0\}$. Notice also that $\operatorname{span} \mathcal{V}$ is dense in $\mathcal{H}_\psi$.

For any $(n, x) \in \mathcal{G}$, define an element of $A^\#$ by

$$\psi_{(n,x)}(a) := \frac{\Delta(n^*a)(x)}{|n|(x)}. \tag{4.7}$$

Simple calculations show that $\left\|\psi_{(n,x)}\right\| = 1$ and for $d_1, d_2 \in D$ and $a \in A$,

$$\psi_{(n,x)}(d_1 a d_2) = d_1(\alpha_n(x))\psi_{(n,x)}(a) d_2(x).$$

In other words, in the language of [12, Section 2], $\psi_{(n,x)}$ is a norm one eigenfunctional with source $s(\psi_{(n,x)}) = x$ and range $r(\psi_{(n,x)}) = \alpha_n(x)$. Furthermore, observe that $(n, x) \in \mathcal{G} \Leftrightarrow (n^*, \alpha_n(x)) \in \mathcal{G}$ and a calculation using Proposition 4.3 shows that for $(n, x) \in \mathcal{G}$,

$$\psi_{(n,x)}^* = \psi_{(n^*, \alpha_n(x))}. \tag{4.8}$$

For later use, notice that for $d \in D$ with $d(x) > 0$ and $z \in \mathbb{T}$,

$$\psi_{(nd,x)} = \psi_{(n,x)} \quad \text{and} \quad \psi_{(zn,x)} = \overline{z}\psi_{(n,x)}. \tag{4.9}$$

Let

$$\mathcal{E} := \{\psi_{(n,x)} : (n,x) \in \mathcal{G}\}. \tag{4.10}$$

Since a state ψ on a C^*-algebra B is uniquely determined by $|\psi|$, it also makes sense to define source and range maps on $|\mathcal{E}|$ by $s(|\psi_{(n,x)}|) = x$ and $r(|\psi_{(n,x)}|) = \alpha_n(x)$. Then the source and range maps carry $\mathcal{E}$ and $|\mathcal{E}|$ onto X.

Given $\psi \in \mathcal{E}$, write $\psi = \psi_{(n,x)} \in \mathcal{E}$, and choose $m \in N_h$ such that $\psi(m) \neq 0$. Notice that for any $a \in A$, we have

$$\Delta(a)(x) = \frac{\psi(ma)}{\psi(m)} \quad \text{and} \quad \Delta(a)(m.x) = \frac{\psi(am)}{\psi(m)}. \tag{4.11}$$

(Indeed, since $\psi(m) \neq 0$, Lemma 4.2 gives $n \sim_G m$, and a computation gives (4.11).) Setting

$$\mathfrak{s}(\psi) := \frac{\psi(ma)}{\psi(m)} \quad \text{and} \quad \mathfrak{r}(\psi) = \frac{\psi(am)}{\psi(m)}, \tag{4.12}$$

then $\mathfrak{s}(\psi) = s(\psi) \circ \Delta$ and $\mathfrak{r} = r(\psi) \circ \Delta$. Thus $\mathfrak{s}(\psi)$ and $\mathfrak{r}(\psi)$ are the (necessarily unique) elements of $\mathfrak{S}$ satisfying

$$\mathfrak{s}(\psi)|_D = s(\psi) \quad \text{and} \quad \mathfrak{r}(\psi)|_D = r(\psi).$$

Also, notice that $\mathfrak{S} \subseteq \mathcal{E}$, for if $\psi = x \circ \Delta \in \mathfrak{S}$, then $\psi = \psi_{(d,x)}$ for any $d \in D$ with $d(x) > 0$. Also, it follows easily (using Lemma 4.2) that

$$\mathfrak{S} = \{\psi_{(n,x)} : \Delta(n)(x) > 0\} = \{\psi_{(d,x)} : d \in D \text{ and } d(x) > 0\}. \tag{4.13}$$

We list a few additional properties of $\mathcal{E}$ and $|\mathcal{E}|$.

Lemma 4.9. *The following statements hold.*

1. *The map $\psi_{(n,x)} \mapsto [n,x]_\Sigma$ is a well-defined bijection of $\mathcal{E}$ onto Σ.*
2. *If $g \in \mathcal{E}$ and $m \in N_h$ satisfies $g(m) > 0$, then $g = \psi_{(m,s(g))}$.*
3. *$\mathcal{E} \cup \{0\}$ is weak-$*$ compact; in particular $\mathcal{E}$ is locally compact. Furthermore, s, r are continuous mappings of $\mathcal{E}$ onto X.*
4. *The map $|\psi_{(n,x)}| \mapsto [n,x]_G$ is a well-defined bijection of $|\mathcal{E}|$ onto G.*
5. *If $|\phi| \in |\mathcal{E}|$ and $m \in N_h$ satisfies $|\phi|(m) \neq 0$, then $|\phi| = |\psi_{(m,s(|\phi|))}|$.*
6. *$|\mathcal{E}|$ is locally compact and s, r are continuous mappings of $|\mathcal{E}|$ onto X.*

Proof. We prove statements (1), (2) and (3), leaving the others to the reader. To establish the first, it suffices to show $\psi_{(n_1,x_1)} = \psi_{(n_2,x_2)}$ if and only if $[n_1,x_1]_\Sigma = [n_2,x_2]_\Sigma$ and this is what we do.

Suppose $\psi_{(n_1,x_1)} = \psi_{(n_2,x_2)}$. Applying the source map gives $x_1 = x_2$; write $x := x_1 = x_2$. Now

$$0 < \psi_{(n_2,x)}(n_2) = \psi_{(n_1,x)}(n_2) = \frac{\Delta(n_1^* n_2)(x)}{|n_1|(x)},$$

which implies $\Delta(n_1^*n_2)(x) > 0$. Proposition 4.7 now gives $[n_1, x_1]_\Sigma = [n_2, x_2]_\Sigma$. Conversely, if $[n_1, x_1]_\Sigma = [n_2, x_2]_\Sigma$, then $x_1 = x_2 =: x$ and there exists $d_1, d_2 \in D$ with $d_1(x) > 0$ and $d_2(x) > 0$ such that $n_1d_1 = n_2d_2$. Then

$$\psi_{(n_1,x_1)} = \psi_{(n_1,x)} = \psi_{(n_1d_1,x)} = \psi_{(n_2d_2,x)} = \psi_{(n_2,x_2)}.$$

This gives statement (1).

For statement (2), write $\phi = \psi_{(n,x)}$ and apply Proposition 4.7 and part (1).

Turning now to statement (3), $\mathfrak{s}(\psi_{(n,x)}) = x \circ \Delta$ and $\mathfrak{r}(\psi_{(n,x)}) = \alpha_n(x) \circ \Delta$, so the maps $\mathfrak{s}, \mathfrak{r} : \mathcal{E} \to \mathfrak{S}$ are surjective. They are continuous by (4.12), so $s, r : \mathcal{E} \to X$ are also continuous surjections.

Next suppose that $\psi_{(n_i,x_i)}$ is a net in $\mathcal{E}$ converging weak-$*$ to $\phi \in A^\#$; write $\psi_i := \psi_{(n_i,x_i)}$. If $\phi = 0$, there is nothing to do. So suppose $\phi \neq 0$. By [12, Proposition 2.3], ϕ is an eigenfunctional. Thus if $x := s(\phi)$, continuity of s yields $x_i \to x$.

Since $\operatorname{span} N_h$ is dense in A, there exists $n \in N_h$ such that $\phi(n) > 0$. Since $n(n^*n)^{1/k} \to n$ as $k \to \infty$, we have $0 < \phi(n) = \lim_k \phi(n(n^*n)^{1/k}) = \phi(n) \lim_k (n^*n)^{1/k}(x)$. Thus $n^*n(x) \neq 0$ and so $(n, x) \in \mathcal{G}$. Since $\psi_i \to \phi$, $\psi_i(n)$ is eventually nonzero, so we may as well assume that $\psi_i(n) \neq 0$ for every λ. Proposition 4.7 implies $(n_i, x_i) \sim_G (n, x_i)$. Hence there exists $z_i \in \mathbb{T}$ such that $\psi_i = \psi_{(z_in,x_i)} = \overline{z_i} \cdot \psi_{(n,x_i)}$. Therefore,

$$0 < \phi(n) = \lim \overline{z_i} \frac{\Delta(n^*n)(x_i)}{|n|(x_i)} = \lim \overline{z_i}\, |n|(x_i).$$

As $|n|(x_i) \to |n|(x)$, we conclude $z_i \to 1$. It follows that $\phi = \lim \psi_i = \lim \overline{z_i} \cdot \psi_{(n,x_i)} = \psi_{(n,x)}$, so $\phi \in \mathcal{E}$. Thus $\mathcal{E} \cup \{0\}$ is a closed subset of the unit ball of $A^\#$, and hence is compact. □

Notation 4.10. We use the bijections of Lemma 4.9 to identify Σ with $\mathcal{E}$ (respectively G with $|\mathcal{E}|$) and will use $\mathcal{E}$ and Σ interchangeably (resp. G and $|\mathcal{E}|$) depending upon what is convenient for the context. Thus for $a \in A$, we will often write $[n, x]_\Sigma(a)$ and $[n, x]_G(a)$ instead of $\psi_{(n,x)}(a)$ and $|\psi_{(n,x)}|(a)$. Then Σ and G become Hausdorff topological spaces of functions on A. When convenient, we will also identify $\mathfrak{S}$ with X via the restriction mapping from 4.8(1).

4.3. The twist associated to a Γ-Cartan pair. We are now prepared to place groupoid structures on G and Σ. This is done exactly as in [26, Definition 8.10 and Theorem 8.12] or [28, Definition 7.17 and Theorem 7.18]; for convenience, we provide sketches of the proofs using the present notation.

Lemma 4.11. *Σ and G are Hausdorff topological groupoids under the following operations:*

- *Multiplication: $[m, \alpha_n(x)][n, x] = [mn, x]$;*
- *Inversion: $[n, x]^{-1} = [n^*, \alpha_n(x)]$.*

The map $x \mapsto [d, x]$ for $d \in D$ with $d(x) > 0$ identifies X with the unit space of Σ and G. Furthermore, under this identification $r([n, x]) = \alpha_n(x)$ and $s([n, x]) = x$.

Proof. We sketch the proof for Σ. The proof for G is left to the reader (details may be found in [26] or [28]). That inversion is well-defined and continuous follows from (4.7), (4.8), and Lemma 4.9. Also, it is clear that inversion is involutive.

Next we show multiplication is well-defined. Suppose $[m_1, y]_\Sigma = [m_2, y]_\Sigma$ and $[n_1, x]_\Sigma = [n_2, x]_\Sigma$. Using the bijection in Lemma 4.9 we can identify ψ and ϕ with $[m_1, y]_\Sigma = [m_2, y]_\Sigma$ and $[n_1, x]_\Sigma = [n_2, x]_\Sigma$ respectively. We have $y = \alpha_{n_i}(x)$. By the definition of $\sim_\Sigma$ we can assume $m_2 = m_1 d$ and $n_2 = n_1 d'$ where $d(y) > 0$ and $d'(x) > 0$. So to show that multiplication is well defined it suffices to show that $\psi_{(m_1 n_1, x)} = \psi_{(m_1 d n_1 d', x)}$. But this follows since $m_1 d n_1 d' = m_1 n_1 \theta_{n_1}(d) d'$ and we know from equation (4.9) that $\psi_{(\nu b, x)} = \psi_{(\nu, x)}$ for all $\nu \in N_h$, $x \in \operatorname{dom}(\nu)$ and $b \in D$ with $b(x) > 0$.

Multiplication is associative since multiplication in the C^*-algebra is.

Suppose $[m, x], [n, y] \in \Sigma$ are such that the composition $[m, x][n, y]$ is defined. Then $x = \alpha_n(y)$. We must show that

$$[m, x][n, y][n^*, \alpha_n(y)] = [m, x] \quad \text{and} \quad [m^*, \alpha_m(y)][m, x][n, y] = [n, y].$$

But these equalities follow from Lemma 4.9(2) because

$$[mnn^*, \alpha_n(y)](m) > 0 \quad \text{and} \quad [m^* mn, y](n) > 0.$$

This completes the proof that Σ is a groupoid when equipped with the indicated operations.

Since $\Sigma^{(0)} = \{[m, x]_\Sigma^{-1}[m, x]_\Sigma : (m, x) \in \mathcal{G}\}$ we obtain

$$\Sigma^{(0)} = \{[d, x]_\Sigma \in \Sigma : d \in D \text{ and } d(x) > 0\} = \mathfrak{G}.$$

It follows that the map $X \ni x \mapsto [d, x]_\Sigma$ where $d \in D$ is chosen so that $d(x) > 0$, is a bijection of X onto $\Sigma^{(0)}$. Similarly, the map $X \ni x \mapsto [d, x]_G$ where $d \in D$ satisfies $d(x) > 0$ (or merely satisfies $d(x) \neq 0$)) is a bijection of X onto $G^{(0)}$.

For $(n, x) \in \mathcal{G}$, $r([n, x]) = [nn^*, \alpha_n(x)]$ and $s([n, x]) = [n^* n, x])$. This gives the desired identification of the range and source maps.

We have already observed that inversion is continuous and we now verify that multiplication is continuous. Let $\mathcal{E}^{(2)}$ be the set of composable pairs, that is, the collection $(\psi, \phi) \in \mathcal{E} \times \mathcal{E}$ with $s(\psi) = r(\phi)$. Suppose $(\phi_i)_{i \in I}$ and $(\psi_i)_{i \in I}$ are nets in $\mathcal{E}$ converging to $\phi, \psi \in \mathcal{E}$ respectively, and such that $(\phi_i, \psi_i) \in \mathcal{E}^{(2)}$ for all λ. Since s and r are continuous, we find that $s(\phi) = \lim_i s(\phi_i) = \lim_i r(\psi_i) = r(\psi)$, so $(\phi, \psi) \in \mathcal{E}^{(2)}$. Let $n, m \in N_h$ be such that $\phi(n) > 0$ and $\psi(m) > 0$. There exists i_0, so that $i \geq i_0$ implies $\phi_i(n)$ and $\psi_i(m)$ are non-zero. For each $i \geq i_0$, there exist scalars $\lambda_i, \lambda_i' \in \mathbb{T}$ such that $\phi_i = \lambda_i [n, s(\phi_i)]$ and $\psi_i = \lambda_i' [m, s(\psi_i)]$. Since

$$\lim_i \phi_i(n) = \phi(n) = \lim_i [n, s(\phi_i)](n)$$

and

$$\lim_i \psi_i(n) = \psi(n) = \lim_i [n, s(\psi_i)](n),$$

we conclude that $\lim \lambda_i = 1 = \lim \lambda_i'$. So for any $a \in A$,

$$\begin{aligned}(\phi\psi)(a) &= \frac{s(\psi)((nm)^*a)}{(s(\psi)((nm)^*(nm)))^{1/2}} = \lim_i \frac{s(\psi_i)((nm)^*a)}{(s(\psi_i)((nm)^*(nm)))^{1/2}} \\ &= \lim_i [n, s(\phi_i)][m, s(\psi_i)] = \lim_i (\phi_i\psi_i)(a),\end{aligned}$$

giving continuity of multiplication. □

Define $q \colon \Sigma \to G$ and $\iota : \mathbb{T} \times G^{(0)} \to \Sigma$ by

$$q([n,x]_\Sigma) := [n,x]_G \quad \text{and} \quad \iota(\lambda, [d,x]_G) = [\lambda|d|, x]_\Sigma.$$

Then q and ι are continuous groupoid homomorphisms with q surjective and ι injective. Moreover,

$$q^{-1}(G^{(0)}) = \{[d,x]_\Sigma \colon d \in D \text{ and } d(x) \neq 0\} = \iota(\mathbb{T} \times G^{(0)}).$$

Furthermore, for $(n,x) \in \mathcal{G}$, and $\lambda \in \mathbb{T}$,

$$\iota(\lambda, [nn^*, \alpha_n(x)]_G)\,[n,x]_\Sigma = [\lambda n, x]_\Sigma = [n,x]_\Sigma\,\iota(\lambda, [n^*n, x]_G).$$

We thus have a central extension of groupoids,

$$\mathbb{T} \times G^{(0)} \overset{\iota}{\hookrightarrow} \Sigma \overset{q}{\to} G.$$

Also, for $\lambda \in \mathbb{T}$ and $(n,x) \in \mathcal{G}$,

$$\lambda \cdot [n,x]_\Sigma = [\lambda n, x]_\Sigma. \tag{4.14}$$

As $G^{(0)}$ may be identified with X, we usually identify $\iota(\mathbb{T} \times G^{(0)})$ with $\mathbb{T} \times X$ by

$$[d,x]_\Sigma \mapsto \left(\frac{d(x)}{|d|(x)}, x\right). \tag{4.15}$$

Under this identification, the extension of groupoids above becomes

$$\mathbb{T} \times X \hookrightarrow \Sigma \overset{q}{\to} G.$$

Remark 4.12. *We have already seen an action of $\mathbb{T}$ on Σ: $\lambda \cdot [n,x]_\Sigma = [\lambda n, x]_\Sigma$. When elements of Σ are identified with their corresponding elements of $\mathcal{E}$ via the map in Lemma 4.9, there is another action of $\mathbb{T}$ on Σ, namely scalar multiplication of linear functionals. These actions differ: if scalar multiplication of linear functionals is denoted by juxtaposition, then*

$$\overline{\lambda}[n,x]_\Sigma = \lambda \cdot [n,x]_\Sigma.$$

For $n \in N_h$, let

$$Z(n) := \{[n,x]_G : x \in \operatorname{dom}(n)\}.$$

Lemma 4.13. *For each $n \in N_h$, $Z(n)$ is an open bisection for G and $\{Z(n) : n \in N_h\}$ is a base for the topology on G. Moreover, $q^{-1}(Z(n))$ is homeomorphic to $\mathbb{T} \times Z(n)$. In particular, G is an étale groupoid and the bundle $\Sigma \to G$ is locally trivial.*

Proof. The relevant definitions and an application of Lemma 4.7 yield

$$q^{-1}(Z(n)) = \{[m,y]_\Sigma \in \Sigma : [m,y](n) \neq 0\},$$

which is an open subset of Σ. Thus $Z(n)$ is an open subset of G. We claim $r|_{Z(n)}$ and $s|_{Z(n)}$ are homeomorphisms of $Z(n)$ onto $\operatorname{ran}(n)$ and $\operatorname{dom}(n)$ respectively. As r is the composition of the source map with the inversion map, it suffices to show this for s only. First note that $s|_{Z(n)} : Z(n) \to \operatorname{dom}(n)$ is a bijection by definition. By Lemma 4.9(6), $s, r : G \to X$ are continuous. Next we show $(s|_{Z(n)})^{-1}$ is a continuous function from $\operatorname{dom} n$ to $Z(n)$. If $x_i \in \operatorname{dom} n$ is a net and $x_i \to x \in \operatorname{dom} n$, then $\psi_{(n,x_i)} \to \psi_{(n,x)}$ (by definition of the weak-* topology), so $[n,x_i]_G \to [n,x]_G$ by Lemma 4.9(4). Thus the claim holds, and $Z(n)$ is therefore an open bisection.

Let $U \subseteq G$ be open and choose $[n,x]_G \in U$. Then $V := U \cap Z(n)$ is an open bisection, so $s(V)$ is an open subset of X containing x. Let $d \in D$ be such that $\operatorname{supp} d \subseteq s(V)$ and $d(x) = 1$. Since $\operatorname{dom}(nd) = \operatorname{dom}(n) \cap \operatorname{supp}'(d) \subseteq s(V)$, we find

$$[n,x]_G \in Z(nd) = s^{-1}(\operatorname{dom}(nd)) \subseteq V \subseteq U.$$

Thus, $\{Z(n) : n \in N_h\}$ is a base for the topology on G. As $\{Z(n) : n \in N_h\}$ covers G, G is étale.

Consider the map $\tau : \mathbb{T} \times \operatorname{dom}(n) \to q^{-1}(Z(n))$ defined by $(z,x) \mapsto [zn,x]_\Sigma$. This map is a homeomorphism, and as $s|_{Z(n)} : Z(n) \to \operatorname{dom}(n)$ is a homeomorphism, we see $\Sigma \to G$ is locally trivial. $\square$

The following summarizes our discussion so far.

Proposition 4.14. *Both Σ and G are locally compact Hausdorff topological groupoids, G is étale, and $\mathbb{T} \times X \hookrightarrow \Sigma \overset{q}{\twoheadrightarrow} G$ is a twist.*

Define a map $\operatorname{gr} : N_h \to \Gamma$ by by taking $n \in A_t$ to t. This induces maps $c_\Sigma : \Sigma \to \Gamma$ and $c_G : G \to \Gamma$ given by

$$c_\Sigma([n,x]_\Sigma) = \operatorname{gr}(n) \quad \text{and} \quad c_G([n,x]_G) = \operatorname{gr}(n). \tag{4.16}$$

Notice that the definition of the topologies and the groupoid multiplications imply that c_Σ and c_G are continuous homomorphisms. We therefore have produced the graded twist,

$$\begin{tikzcd} \mathbb{T} \times G^{(0)} \arrow[r] & \Sigma \arrow[r] \arrow[dr, "c_\Sigma"'] & G \arrow[d, "c_G"] \\ & & \Gamma. \end{tikzcd}$$

4.4. Every Γ-Cartan pair is a twisted groupoid C^*-algebra. For $a \in A$, define a function $\hat{a} : \Sigma \to \mathbb{C}$ by

$$[n,x]_\Sigma \mapsto \frac{\Delta(n^*a)(x)}{(n^*n)^{1/2}(x)}, \quad \text{that is,} \quad \hat{a}([n,x]_\Sigma) = \psi_{(n,x)}(a).$$

By construction, $\hat{a}$ is a continuous function on Σ, and for $z \in \mathbb{T}$,

$$\hat{a}(z \cdot [n,x]_\Sigma) = \overline{z}\hat{a}([n,x]_\Sigma),$$

so $\hat{a}$ may be regarded as a continuous section of the line bundle over the twist $\Sigma \to G$. Thus we may regard the open support of $\hat{a}$ as a subset of G, as in Remark 2.4.

Lemma 4.15. *Suppose $n \in N_h$, then the open support of $\hat{n}$, $\operatorname{supp}'(\hat{n})$, is the set $Z(n)$*

Proof. Consider $\hat{n}$ for $n \in N_h$. Then

$$\hat{n}[m,y] = \frac{\Delta(m^*n)(y)}{(m^*m)^{1/2}(y)}.$$

This is zero unless $\Delta(m^*n) \neq 0$. Now Proposition 4.7 gives $[m,y]_G = [n,y]_G$, that is $[m,y]_G \in Z(n)$. ☐

Lemma 4.16. *The map $\Psi : A \to C(\Sigma; G)$ given by $a \mapsto \hat{a}$ is linear and injective.*

Proof. This map is linear since Δ is. Injectivity will follow since $\operatorname{span} N_h$ is dense in A. Indeed, suppose $\hat{a} \equiv 0$. Then for all $n \in N_h$,

$$\Delta(n^*a)(y) = 0$$

for all $y \in \operatorname{Dom}(n)$. Thus $\Delta(n^*a) = 0$ for all $n \in N_h$. By assumption $a \in \overline{\operatorname{span}}(N_h)$; take a net $\nu_i \in \operatorname{span}(N_h)$ such that $\nu_i \to a$. By linearity,

$$\forall i\, \Delta(\nu_i^* a) = 0.$$

Thus by continuity $\Delta(a^*a) = 0$. Since Δ is faithful, $a = 0$. ☐

Now let

$$N_{h,c} := \{n \in N_h : \operatorname{supp} \hat{n} \text{ is compact}\} \quad \text{and} \quad A_c := \operatorname{span} N_{h,c} \text{ (no closure)}.$$

Note that A_c is a $*$-algebra and by Lemma 4.15, for $a \in A_c$, $\hat{a} \in C_c(\Sigma; G)$.

Lemma 4.17. *Let (A, D) be a Γ-Cartan pair. Then*

(1) $N_{h,c}$ is dense in N_h.
(2) A_c is dense in A.
(3) $\Psi : a \mapsto \hat{a}$ sends A_c bijectively onto $C_c(\Sigma; G)$ and $D_c = D \cap A_c$ bijectively onto $C_c(X)$.
(4) Ψ is a $*$-algebra homomorphism.

Proof. For (1), let (e_i) be an approximate unit for D with $e_i \in C_c(X)$ for every i. By Lemma 3.10(4), (e_i) is also an approximate unit for A. Thus $n = \lim n e_i$. So to prove $N_{h,c}$ is dense it suffices to show that $nd \in N_{h,c}$ for all $d \in C_c(X)$. Given $d \in C_c(X)$, let $d_1 \in C_c(X)$ be such that $\operatorname{supp}'(d_1) \supseteq \operatorname{supp}(d)$. By Lemma 4.15, $\operatorname{supp}'(\widehat{nd_1}) = Z(nd_1)$. Recalling that $s|_{Z(nd_1)}$ is a homeomorphism of $Z(nd_1)$ onto $\operatorname{dom}(nd_1)$, we see that $\overline{Z(nd)}$ is compact because $\overline{Z(nd)} \subseteq Z(nd_1)$ and $\mathrm{dom}(nd) = \mathrm{supp}'(n^*n) \cap \mathrm{supp}'(d)$ has compact closure in $\mathrm{dom}(nd_1)$.

Now (2) follows immediately from (1).

Lemma 4.16 shows that Ψ is injective and $\Psi(D_c) = C_c(\mathfrak{S}) \simeq C_c(X)$, so to obtain (3), we must show $\Psi(A_c) = C_c(\Sigma, G)$. Now $C_c(\Sigma; G)$ is the span of sections of the line bundle supported on sets of the form $Z(n)$, as the $Z(n)$ form a basis for G and we can use a partition of unity argument. Thus it suffices to show that for $n \in N_h$, every $f \in C_c(\Sigma; G)$ with support in $Z(n)$ is in the image of Ψ. To proceed note the following.

i. The line bundle is trivial over $Z(n)$: this is true because $q^{-1}(Z(n)) = \{z \cdot [n, x]_\Sigma : x \in \mathrm{dom}(n), z \in \mathbb{T}\}$ and the map $\mathbb{T} \times Z(n) \ni (z, [n, x]_G) \mapsto [zn, x]_\Sigma$ is a homeomorphism of $\mathbb{T} \times Z(n)$ onto $q^{-1}(Z(n))$.
ii. The source map of G sends $Z(n)$ homeomorphically to $\{x : n^*n(x) \neq 0\}$ because $Z(n)$ is an open bisection.

Now let f be a section of the line bundle supported on $Z(n)$. By the first item above we can view f as a function. By item (ii), $f = d \circ (s|_{Z(n)})$ for some $d \in D$. Now take $a = \frac{n\bar{d}}{(n^*n)^{1/2}}$. We show $\hat{a} = f$. Indeed,

$$\hat{a}[m, x] = \frac{\Delta(dn^*m)(x)}{(n^*n)^{1/2}(x)(m^*m)^{1/2}(x)},$$

which is 0 unless the germ of m is the same as n. So we can assume that $n = m$. Hence the above becomes

$$\frac{d(x)n^*n(x)}{n^*n(x)} = d(x) = f([n, x])$$

Thus $\hat{a} = f$ and the claim holds.

Part (3) now follows.

It remains to show (4). By linearity it is enough to check that $\widehat{mn} = \hat{m} * \hat{n}$ and $\widehat{m^*} = (\hat{m})^*$ for $m, n \in N_{h,c}$. Using (2.7) we compute:

$$\hat{m} * \hat{n}([\nu, x]_\Sigma) = \sum_{[v,y]_\Sigma [w,x]_\Sigma = [\nu,x]_\Sigma} \hat{m}([v, y]_\Sigma)\, \hat{n}([w, x]_\Sigma)$$

(again, for each factorization $[v, y]_G[w, x]_G = [\nu, x]_G$ only one factorization $[v, y]_\Sigma[w, x]_\Sigma = [\nu, x]_\Sigma$ is chosen). But $\hat{m}([v, y]_\Sigma)\, \hat{n}([w, x]_\Sigma) = 0$ unless $[v, y]_G = [m, y]_G$ and $[w, x]_G = [n, x]_G$. When this occurs, there exist $z_v, z_w \in \mathbb{T}$ so that $[v, y]_\Sigma = [z_v m, y]_\Sigma$ and $[w, x]_\Sigma = [z_w n, x]_\Sigma$. As $[v, y]_\Sigma[w, x]_\Sigma = [\nu, x]_\Sigma$, we have $y = \alpha_n(x)$ and $[\nu, x]_\Sigma = [z_v z_w mn, x]_\Sigma$.

So

$$\begin{aligned}\hat{m}*\hat{n}([\nu,x]_\Sigma) &= \hat{m}([z_v m,y]_\Sigma)\hat{n}([z_w n,x]_\Sigma)\\ &= \overline{z_v z_w}\sqrt{(m^*m)(y)(n^*n)(x)}\\ &= \overline{z_v z_w}\left(\frac{(n^*m^*mn)(x)}{(n^*n)(x)}(n^*n)(x)\right)^{1/2}\\ &= \widehat{mn}([z_v z_w mn,x]_\Sigma) = \widehat{mn}([\nu,x]_\Sigma),\end{aligned}$$

as desired.

To see $\widehat{m^*} = (\hat{m})^*$, use Lemma 4.9 and (4.8):

$$\begin{aligned}\widehat{m^*}([n,x]_\Sigma) &= \psi_{(n,x)}(m^*) = \overline{\psi^*_{(n,x)}(m)} = \overline{\psi_{(n^*,\alpha_n(x))}(m)}\\ &= \overline{\hat{m}([n,x]_\Sigma^{-1})} = (\hat{m})^*([n,x]_\Sigma).\end{aligned}$$ □

Lemma 4.18. *When $C_c(\Sigma;G)$ is equipped with the reduced norm, $\Psi|_{A_c} : A_c \to C_c(\Sigma;G)$ is an isometric $*$-isomorphism.*

Proof. Lemma 4.17 gives $\Psi|_{A_c}$ is a $*$-isomorphism of A_c onto $C_c(\Sigma;G)$.

Fix $x \in X$. Using Remark 2.5, we may regard π_x as the GNS representation of $C_c(\Sigma;G)$ arising from the functional ε_x. On the other hand, the state $\rho_x := x \circ \Delta$ determines the GNS representation $(\pi_{\rho_x}, \mathcal{H}_{\rho_x})$ of A. Let $L_x \subseteq C_r^*(\Sigma;G)$ and $L_{\rho_x} \subseteq A$ be the left kernels of ε_x and ρ_x respectively.

We claim that for $n,m \in N_{h,c}$,

$$\rho_x(mn) = \varepsilon_x(\hat{m}*\hat{n}). \tag{4.17}$$

To see this, choose $d \in D$ with $d(x)=1$, so that $[d,x]_\Sigma \in \Sigma^{(0)}$. For $[n,x]_\Sigma$ with $[n,x]_\Sigma^{-1}[n,x]_\Sigma = [d,x]_\Sigma$, a computation similar to that used in the proof of Lemma 4.17(4) gives

$$\begin{aligned}\varepsilon_x(\hat{m}*\hat{n}) &= \hat{m}*\hat{n}([d,x]_\Sigma) = \hat{m}([n^*,\alpha_n(x)])\hat{n}([n,x]_\Sigma)\\ &= \frac{\Delta(nm)(\alpha_n(x))}{\sqrt{(nn^*)(\alpha_n(x))}}(n^*n)(x)^{1/2} = \Delta(nm)(\alpha_n(x))\\ &\overset{(4.5)}{=} \Delta(mn)(x) = \rho_x(mn).\end{aligned}$$

For $a \in A_c$, (4.17) gives

$$\rho_x(a^*a) = \varepsilon_x(\widehat{a^*}*\hat{a}).$$

Thus for $a \in A_c$, the map $a + L_{\rho_x} \mapsto \hat{a} + L_x$ extends to an isometry $W_x : \mathcal{H}_{\rho_x} \to \mathcal{H}_x$. Lemma 4.17(3) implies that W_x is onto. For $m,n \in N_h$,

$$W_x\pi_{\rho_x}(m)(n+L_{\rho_x}) = W_x(mn+L_{\rho_x}) = \hat{m}*\hat{n}+L_x = \pi_x(\hat{m})W_x(n+L_{\rho_x}).$$

It follows that $W_x\pi_{\rho_x}(m)W_x^* = \pi_x(\hat{m})$. Hence for $a \in A_c$,

$$W_x\pi_{\rho_x}(a)W_x^* = \pi_x(\Psi(a)).$$

Finally, for $a \in A_c$,

$$\|\Psi(a)\|_{C_r^*(\Sigma;G)} = \sup_x \|\pi_x(\Psi(a))\| = \sup_x \|\pi_{\rho_x}(a)\| = \|a\|_A ,$$

with the last equality following from the fact that Δ is faithful (as in the proof of Lemma 4.16). $\square$

We now come to the main result of this section.

Theorem 4.19. *Let (A, D) be a Γ-Cartan pair. Then there exists a graded twist*

$$\begin{tikzcd} \mathbb{T} \times G^{(0)} \arrow[r] & \Sigma \arrow[r] \arrow[dr, "c_\Sigma"'] & G \arrow[d, "c_G"] \\ & & \Gamma \end{tikzcd}$$

and a $\hat{\Gamma}$-covariant $*$-isomorphism $\Psi : A \to C_r^*(\Sigma; G)$ such that $\Psi(D) = C_0(G^{(0)})$.

Proof. Lemmas 4.17 and 4.18 show that Ψ determines a $*$-isomorphism of A onto $C_r^*(\Sigma; G)$ and the construction of the graded twist shows Ψ is $\hat{\Gamma}$-covariant. It remains to show that $\Psi(D) = C_0(X)$.

For $d \in D$ and $[n, x]_\Sigma \in \Sigma$,

$$\hat{d}([n, x]_\Sigma) = \frac{\Delta(n^*)(x)}{|n|(x)} d(x).$$

Changing perspective to viewing $\hat{d}$ as a section of the line bundle instead of as a covariant function and recalling that $[n, x]_G \in G^{(0)}$ if and only if $\Delta(n)(x) \neq 0$, we get

$$\begin{aligned} \hat{d}([n, x]_G) &= \left[\frac{\Delta(n^*)(x)}{|n|(x)} d(x), [n, x]_\Sigma\right] \\ &\overset{(2.1), (4.14)}{=} \begin{cases} 0 & \text{if } [n, x]_G \notin G^{(0)} \\ \left[d(x), [\frac{\Delta(n^*)(x)}{|n|(x)} n, x]_\Sigma\right] & \text{if } [n, x]_G \in G^{(0)}. \end{cases} \end{aligned}$$

As $[\frac{\Delta(n^*)(x)}{|n|(x)} n, x]_\Sigma \in \Sigma^{(0)}$, under the identification of X with $\Sigma^{(0)}$,

$$\left[d(x), [\frac{\Delta(n^*)(x)}{|n|(x)} n, x]_\Sigma\right]$$

and $[d(x), x]$ represent the same element of the line bundle L. Thus

$$\hat{d}([n, x]_G) = \begin{cases} 0 & \text{if } [n, x]_G \notin G^{(0)} \\ [d(x), x]] & \text{if } [n, x]_G \in G^{(0)}, \end{cases}$$

showing that $\Psi(d) \in C_0(X)$. On the other hand, if $f \in C_c(\Sigma; G)$ vanishes off $G^{(0)}$, define $d \in D$ as follows. For $x \in X$, choose $n \in N_h$ so that $(n^*n)(x) \neq 0$; then let $d(x)$ be the unique scalar satisfying $f([n^*n, x]_G) =$

$[d(x), x] \in L$. Then $\hat{d}(x) = f$. It follows that $C_0(X) = \Psi(D)$ and the proof is complete. □

5. Γ-Cartan pairs from Γ-graded twists

In the previous section, we associated a graded twist (Σ, G, Γ) to a Γ-Cartan pair (A, D) and showed that (A, D) can be recovered from (Σ, G, Γ). The purpose of this section is to produce a Γ-Cartan pair from any suitable twist graded by the abelian group Γ.

Throughout this section we assume the following:

Assumptions 5.1. We fix a Γ-graded twist

$$\begin{tikzcd} \mathbb{T} \times G^{(0)} \arrow[r] & \Sigma \arrow[r, "q"] \arrow[dr, "c_\Sigma"'] & G \arrow[d, "c_G"] \\ & & \Gamma, \end{tikzcd} \tag{5.1}$$

with G étale (and Hausdorff) where the diagram commutes and

(1) Γ is a discrete abelian group;
(2) c_G and c_Σ are (continuous) groupoid homomorphisms; and
(3) $c_G^{-1}(0)$ is effective.

The homomorphisms c_G and c_Σ are often called *cocycles* in the literature as they are elements of the first groupoid cohomology group with coefficients. We will persist in referring to c_Σ and c_G as cocycles here.

For notational convenience, let

$$\mathcal{P} := c_\Sigma^{-1}(0) \quad \text{and} \quad R := c_G^{-1}(0).$$

The commutativity of (5.1) yields

$$\mathcal{P} = q^{-1}(R) \quad \text{and} \quad G^{(0)} = R^{(0)}.$$

Also, the continuity of c_G ensures R is a clopen subgroupoid of G. Thus we obtain the twist

$$\mathbb{T} \times G^{(0)} \hookrightarrow \mathcal{P} \overset{q|_{\mathcal{P}}}{\twoheadrightarrow} R. \tag{5.2}$$

Since R is étale and effective, it follows from Renault's work in [33, Section 4] that $C_0(G^{(0)})$ is a Cartan MASA in $C_r^*(\mathcal{P}; R)$. (Renault makes the assumption that $R^{(0)}$ is second countable, but that assumption is not required to show that $(C_r^*(\mathcal{P}; R), C_0(R^{(0)}))$ is a Cartan pair. A close inspection of [33] shows that he uses R effective instead of R topologically principal, but these notions coincide when $R^{(0)}$ is second countable.)

By Lemma 2.7, the inclusion $C_c(\mathcal{P}; R) \hookrightarrow C_c(\Sigma; G)$ given by extension by zero extends to a $*$-monomorphism

$$i : C_r^*(\mathcal{P}; R) \hookrightarrow C_r^*(\Sigma; G).$$

Proposition 5.2. *The image of $C_r^*(\mathcal{P};R)$ under i is the fixed point algebra of the action of $\hat{\Gamma}$ on $C_r^*(\Sigma;G)$, that is, $i(C_r^*(\mathcal{P};R)) = C_r^*(\Sigma;G)^{\hat{\Gamma}}$.*

Proof. First notice that $i(C_r^*(\mathcal{P};R)) \subseteq C_r^*(\Sigma;G)^{\hat{\Gamma}}$: indeed, if $f \in C_c(\mathcal{P};R)$ and $\dot{\gamma} \in G$,

$$\begin{aligned}\omega \cdot i(f)(\dot{\gamma}) = \langle \omega, c_G(\dot{\gamma})\rangle i(f)(\dot{\gamma}) &= \begin{cases} 0 & \text{if } \dot{\gamma} \notin R \\ \langle \omega, 0\rangle i(f)(\dot{\gamma}) = i(f)(\dot{\gamma}) & \text{if } \dot{\gamma} \in R \end{cases} \\ &= i(f)(\dot{\gamma}).\end{aligned}$$

We now turn to showing $C_r^*(\Sigma;G)^{\hat{\Gamma}} \subseteq i(C_r^*(\mathcal{P};R))$. First suppose $a \in C_r^*(\Sigma;G)^{\hat{\Gamma}} \cap C_c(\Sigma;G)$. Then for $\dot{\gamma} \in G$,

$$a(\dot{\gamma}) = \int_{\hat{\Gamma}} (\omega \cdot a)(\dot{\gamma})\, d\omega \overset{(2.10)}{=} \int_{\hat{\Gamma}} \langle \omega, c_G(\dot{\gamma})\rangle\, a(\dot{\gamma})\, d\omega = a(\dot{\gamma}) \int_{\hat{\Gamma}} \langle \omega, c_G(\dot{\gamma})\rangle\ d\omega,$$

which vanishes unless $\dot{\gamma} \in R$. Thus $a \in i(C_c(\mathcal{P}, R))$.

For general $a \in C_r^*(\Gamma;G)^{\hat{\Gamma}}$, choose a net $f_i \in C_c(\Sigma;G)$ so that $f_i \to a$. Then $\Phi_0(f_i) \to \Phi_0(a) = a$. Also note that $\Phi_0(C_c(\Sigma;G)) \subseteq C_c(\Sigma;G)$, so $\Phi_0(f_i) \in i(C_c(\mathcal{P};R))$. It follows that $a \in i(C_r^*(\mathcal{P};R))$. ☐

Here is the main result of this section.

Proposition 5.3. *The pair $(C_r^*(\Sigma;G), C_0(G^{(0)}))$ is a Γ-Cartan pair.*

Proof. It is well known that $C_0(G^{(0)})$ is an abelian subalgebra of $C_r^*(\Sigma;G)$ that contains an approximate unit for $C_r^*(\Sigma;G)$ [32, Lemma 3.2]. Lemma 2.10 gives an action of $\hat{\Gamma}$ on $C_r^*(\Sigma;G)$. We have already observed that $C_0(G^{(0)})$ is a Cartan MASA in $C_r^*(\mathcal{P};R)$, so Lemma 5.2 shows that $C_0(G^{(0)})$ is a Cartan MASA in $C_r^*(\Sigma;G)^{\hat{\Gamma}}$. Since G is étale, $\operatorname{span} N$ is dense in $C_r^*(\Sigma;G)$. Thus $(C_r^*(\Sigma;G), C_0(G^{(0)}))$ is Γ-Cartan. ☐

We close this section with a result describing the supports of homogeneous normalizers. This is necessary for the proof of Lemma 6.1 below.

Lemma 5.4. *Let $a \in C_r^*(\Sigma;G)$ and S_a be the open support of a. Then a is a homogeneous normalizer if and only if S_a is a bisection in $c^{-1}(t)$ for some t in Γ.*

Proof. An element $a \in C_r^*(\Sigma;G)$ is homogeneous of degree t if and only if

$$\begin{aligned}a &= \int_{\hat{\Gamma}} \omega \cdot a\overline{\langle \omega, t\rangle} d\omega \\ &\Leftrightarrow a(\gamma) = \int_{\hat{\Gamma}} \langle \omega, c(\gamma)\rangle a(\gamma)\overline{\langle \omega, t\rangle} d\omega = a(\gamma)\int_{\hat{\Gamma}} \langle \omega, c(\gamma)\rangle\overline{\langle \omega, t\rangle} d\omega \\ &\Leftrightarrow t = c(\gamma) \text{ for all } \gamma \in S_a.\end{aligned}$$

Thus $a \in A_t$ if and only if $S_a \subseteq c^{-1}(t)$.

By the argument in [33, Proposition 4.7], if a is a normalizer, then $S_a^{-1}S_a \subseteq \mathrm{Iso}(G)$. Thus, when a is a homogeneous normalizer, $S_a \subseteq c^{-1}(t)$ for some t, whence $S_a^{-1}S_a \subseteq \mathrm{Iso}(G) \cap R$. As R is effective and $S_a^{-1}S_a$ is open, we obtain $S_a^{-1}S_a \subseteq G^{(0)}$. This implies that S_a is a bisection. $\square$

6. Analysis of $C_r^*(\Sigma; G)$

Throughout this section we fix a Γ-graded twist (Σ, G, Γ) satisfying Assumptions 5.1. Let (A, D) be the Γ-Cartan pair constructed from (Σ, G, Γ) in Proposition 5.3. An application of Theorem 4.19 to (A, D) yields another Γ-graded twist (Σ_1, G_1, Γ) also satisfying Assumptions 5.1. Our goal is to show that (Σ_1, G_1, Γ) and (Σ, G, Γ) are isomorphic in the sense that there are topological groupoid isomorphisms $\Upsilon_\Sigma : \Sigma \to \Sigma_1$ and $\Upsilon_G : G \to G_1$ such that the diagram,

$$\begin{tikzcd}
\mathbb{T} \times G^{(0)} \arrow[r, "\iota"] \arrow[dd, "\mathrm{id}\times\Upsilon|_{G^{(0)}}"'] & \Sigma \arrow[rr, "q"] \arrow[dr, "c_\Sigma"] \arrow[dd, "\Upsilon_\Sigma"'] & & G \arrow[dl, "c_G"'] \arrow[dd, "\Upsilon_G"] \\
& & \Gamma & \\
\mathbb{T} \times G_1^{(0)} \arrow[r, "\iota_1"'] & \Sigma_1 \arrow[rr, "q_1"'] \arrow[ur, "c_{\Sigma_1}"] & & G_1 \arrow[ul, "c_{G_1}"']
\end{tikzcd} \tag{6.1}$$

commutes.

Throughout, we will use the notation established in Section 4 for (A, D): thus $X = G^{(0)}$, $D = C(X)$, $\Delta = E \circ \Phi_0$, etc. Further, notice that for $a \in C_c(\Sigma; G)$, $\Delta(a)$ is nothing more than $a|_X$. Thus, for every $a \in A$, $\Delta(a)(x) = \varepsilon_x(a)$. Lastly, recall from Section 4 that $\mathcal{E} = \{\psi_{(n,x)} : (n, x) \in \mathcal{G}\}$ is a family of linear functionals on A which becomes a topological groupoid when equipped with the weak-$*$ topology, product $\psi_{(m,\alpha_n(x))}\psi_{(n,x)} = \psi_{(mn,x)}$, and inverse $\psi_{(n,x)}^{-1} = \psi_{(n^*,\alpha_n(x))}$. From Section 4 we have $\Sigma_1 = \mathcal{E}$ and $G_1 = |\mathcal{E}|$.

To begin, for $\gamma \in \Sigma$, consider the linear functional ε_γ on A determined by Proposition 2.8, described as follows. For $a \in A$, there is a unique scalar $\varepsilon_\gamma(a)$ such that $j(a)(\dot\gamma) \in L$ is represented by $(\varepsilon_\gamma(a), \gamma) \in \mathbb{T} \times \Sigma$, that is,

$$j(a)(\dot\gamma) = [\varepsilon_\gamma(a), \gamma].$$

Alternatively, if a is viewed as a covariant function on Σ,

$$\varepsilon_\gamma(a) = a(\gamma).$$

Note that ε_γ is a norm one linear functional on A.

Lemma 6.1. *The map $\Upsilon_\Sigma : \Sigma \to A^\#$ given by $\gamma \mapsto \varepsilon_\gamma$ is a homeomorphism of Σ onto $\mathcal{E}$. Furthermore, Υ_Σ is an isomorphism of topological groupoids.*

Proof. Fix $\gamma \in \Sigma$, put $x := s(\gamma)$ and choose $n \in N_h$ such that $\varepsilon_\gamma(n) > 0$. By Lemma 5.4, n is supported on a homogeneous bisection, whence $\dot\gamma$ is the unique element of $\operatorname{supp} n$ whose source is x. Thus $(n^*n)(x) =$

$\sum_{\sigma_1\sigma_2=x}\overline{n(\sigma_2)}n(\sigma_2)=\overline{n(\gamma)}n(\gamma)>0$, so $(n,x)\in\mathcal{G}$. To show $\varepsilon_\gamma=\psi_{(n,x)}$, it suffices to show $\varepsilon_\gamma(m)=\psi_{(n,x)}(m)$ for every $m\in N_h$. Choosing $m\in N_h$, we have $\Delta(n^*m)(x)=(n^*m)(x)=\sum_{\dot\sigma\in Gx}\overline{n(\sigma)}m(\sigma)$. As the terms in this sum are zero unless $\sigma\in\operatorname{supp}n$, and $\operatorname{supp}n\cap Gx=\{\dot\gamma\}$, we have

$$\psi_{(n,x)}(m)=\frac{\Delta(n^*m)(x)}{|n|(x)}=\frac{\overline{n(\gamma)}m(\gamma)}{|n(\gamma)|}=\varepsilon_\gamma(m)$$

because $n(\gamma)>0$. Thus $\varepsilon_\gamma=\psi_{(n,x)}$, as desired.

Now suppose $(n,x)\in\mathcal{G}$. Since n is supported on an open bisection by Lemma 5.4, there is a unique element of $\operatorname{supp}n$ whose source is x. Therefore, there is a unique element $\gamma\in\Sigma$ satisfying $s(\gamma)=x$ and $\varepsilon_\gamma(n)>0$. The argument of the previous paragraph shows $\psi_{(n,x)}=\varepsilon_\gamma$. We have thus shown that $\Upsilon_\Sigma(\Sigma)=\mathcal{E}$. Notice that our work also shows that Υ_Σ is bijective.

Recall that G is étale, $C_c(\Sigma;G)$ is dense in A, and elements of $\mathcal{E}$ are norm one linear functionals on A. So if (γ_i) is a net in Σ and $\gamma\in\Sigma$,

$$\begin{aligned}\gamma_i\to\gamma\Leftrightarrow\ &\text{for every } a\in C_c(\Sigma;G),\ \varepsilon_{\gamma_i}(a)\to\varepsilon_\gamma(a)\\ \Leftrightarrow\ &\text{for every } a\in A,\ \varepsilon_{\gamma_i}(a)\to\varepsilon_\gamma(a)\\ \Leftrightarrow\ &(\varepsilon_{\gamma_i})\text{ converges weak-}*\text{ to }\varepsilon_\gamma.\end{aligned}$$

Thus Υ_Σ is a homeomorphism.

We now observe that Υ_Σ preserves the groupoid operations. First, suppose $\gamma\in\Gamma$ and $n\in N_h$ is such that $\varepsilon_\gamma(n)>0$. Then $\varepsilon_\gamma=\psi_{(n,s(\gamma))}$. For $d\in D$, we have $\varepsilon_\gamma(dn)=d(r(\gamma))n(\gamma)$. On the other hand, $\psi_{(n,s(\gamma))}(dn)=d(\alpha_n(s(\gamma)))\sqrt{(n^*n)(s(\gamma))}=d(\alpha_n(s(\gamma)))|n(\gamma)|$. But as this holds for every $d\in D$ and $n(\gamma)>0$, we conclude that

$$r(\gamma)=\alpha_n(s(\gamma)).\tag{6.2}$$

Suppose the product of $\gamma_1,\gamma_2\in\Sigma$ is defined. For $i=1,2$, choose $n_i\in N_h$ so that $\varepsilon_{\gamma_i}(n_i)>0$. As n_i are supported in open bisections of G (Lemma 5.4),

$$\varepsilon_{\gamma_1\gamma_2}(n_1n_2)=(n_1n_2)(\gamma_1\gamma_2)=n_1(\gamma_1)n_2(\gamma_2)=\varepsilon_{\gamma_1}(n_1)\varepsilon_{\gamma_2}(n_2)>0.$$

We therefore obtain

$$\Upsilon_\Sigma(\gamma_1\gamma_2)=\psi_{(n_1n_2,s(\gamma_2))}\overset{(6.2)}{=}\psi_{(n_1,r(\gamma_2))}\psi_{(n_2,s(\gamma_2))}=\Upsilon_\Sigma(\gamma_1)\Upsilon_\Sigma(\gamma_2).$$

For $\gamma\in\Sigma$ and $n\in N_h$ such that $\varepsilon_\gamma(n)>0$, we have $\varepsilon_{\gamma^{-1}}(n^*)=n^*(\gamma^{-1})=\overline{n(\gamma)}>0$, so $\varepsilon_{\gamma^{-1}}=\psi_{(n^*,r(\gamma))}$. As $r(\psi_{(n,s(\gamma))})=\alpha_n(s(\gamma))=r(\gamma)=\alpha_n(s(\gamma))$, we obtain

$$\Upsilon_\Sigma(\gamma^{-1})=(\Upsilon_\Sigma(\gamma))^{-1}.$$

Finally, suppose $z\in\mathbb{T}$ and $\gamma\in\Sigma$. Choose $n\in N$ so that $\varepsilon_\gamma(n)>0$. Then $\varepsilon_{z\cdot\gamma}(zn)=(zn)(z\cdot\gamma)=\overline{z}(zn)(\gamma)=n(\gamma)>0$. Thus, $\varepsilon_{z\cdot\gamma}=\psi_{(zn,s(\gamma))}$. So by Remark 4.12,

$$\Upsilon_\Sigma(z\cdot\gamma)=z\cdot\Upsilon_\Sigma(\gamma).\qquad\square$$

Writing $\Sigma_1 := \mathcal{E}$ and $G_1 := |\mathcal{E}|$, we thus have defined the two left vertical arrows in (6.1). It follows that if $\dot\gamma \in G$, then for $\sigma_1, \sigma_2 \in q^{-1}(\dot\gamma)$, $q_2(\Upsilon_\Sigma(\sigma_1)) = q_2(\Upsilon_\Sigma(\sigma_2))$. Therefore the map $\Upsilon_G : G \to G_1$ given by

$$\Upsilon_G(\dot\gamma) := q_1(\Upsilon_\Sigma(\gamma))$$

is a well-defined isomorphism of groupoids. That Υ_G is a homeomorphism follows from the fact that $\Sigma \to G$ and $\Sigma_1 \to G_1$ are locally trivial and Υ_Σ is a homeomorphism (or use the fact that q, q_1 are quotient maps and Υ_Σ is a homeomorphism).

Now suppose $\dot\gamma \in G$ and $c_G(\dot\gamma) = t$. Then for $\gamma \in q^{-1}(\dot\gamma)$, $c_\Sigma(\gamma) = t$. Choosing $n \in N_h$ with $\varepsilon_\gamma(n) > 0$, we obtain $\operatorname{supp} n \subseteq c^{-1}(t)$. So by (4.16) we obtain $c_{\Sigma_1}(\psi_{(n,s(\gamma))}) = c_{G_1}(|\psi_{(n,s(\gamma))}|) = t$. It follows that (6.1) commutes. Thus we have proved the following theorem.

Theorem 6.2. *Let $\Sigma \to G$ be a Γ-graded twist satisfying Assumptions 5.1. Let $\Sigma_1 \to G_1$ be the twist constructed from $(C^*_r(\Sigma; G), C_0(G^{(0)}))$. For each $\gamma \in \Sigma$, choose a homogeneous normalizer $n \in C^*_r(\Sigma, G)$ such that $n(\gamma) > 0$. Then the map*

$$\Upsilon_\Sigma : \Sigma \to \Sigma_1 \quad \textit{given by} \quad \gamma \mapsto [n, s(\gamma)]_{\Sigma_1}$$

descends to a well-defined isomorphism of twists such that the diagram (6.1) commutes.

Corollary 6.3. *Suppose $\Sigma \to G$ and $\Sigma' \to G'$ are Γ-graded twists satisfying Assumptions 5.1. Suppose further that*

$$\Xi : C^*_r(\Sigma; G) \to C^*_r(\Sigma'; G')$$

is an isomorphism of C^*-algebras such that

(1) Ξ is equivariant for the induced $\hat\Gamma$ actions; and

(2) $\Xi|_{C_0(G^{(0)})} : C_0(G^{(0)}) \to C_0(G'^{(0)})$ is an isomorphism.

Then there exists groupoid isomorphisms $\upsilon_\Sigma, \upsilon_G$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{T} \times G^{(0)} \arrow[r, "\iota"] \arrow[dd, "id \times \upsilon_G|_{G^{(0)}}"'] & \Sigma \arrow[rr, "q"] \arrow[dd, "\upsilon_\Sigma"'] \arrow[dr, "c_\Sigma"] & & G \arrow[dl, "c_G"'] \arrow[dd, "\upsilon_G"] \\
& & \Gamma & \\
\mathbb{T} \times G'^{(0)} \arrow[r, "\iota'"'] & \Sigma' \arrow[rr, "q'"'] \arrow[ur, "c'_\Sigma"] & & G' \arrow[ul, "c'_G"']
\end{tikzcd}$$

Proof. By Theorem 6.2, there are isomorphisms $\Upsilon_\Sigma : \Sigma \to \Sigma_1$, $\Upsilon_G : G \to G_1$, $\Upsilon_{\Sigma'} : \Sigma' \to \Sigma'_1$, $\Upsilon_{G'} : G' \to G'_1$. Since Ξ is an equivariant isomorphism it takes $C^*_r(\Sigma; G)^{\hat\Gamma}$ isomorphically onto $C^*_r(\Sigma'; G')^{\hat\Gamma}$. Thus by construction, $\Sigma_1 \cong \Sigma'_1$, $G_1 \cong G'_1$. The result then follows from composition of isomorphisms. $\square$

7. Examples

Example 7.1. *Let G be a finite discrete abelian group. Take $A = C^*(G)$. Then*

$$C^*(G) = \operatorname{span}\{\delta_g : g \in G\}.$$

As pointed out in [10], if $|G| = |H|$, then $C^*(G) \cong C^*(H)$. So it is surprising that we would be able to recover G using Theorem 6.2. However, as this example illustrates, the induced action of $\hat{\Gamma}$ required in Theorem 6.2 plays a crucial role.

Suppose $c : G \to \Gamma$ is a homomorphism of G into a discrete abelian group, with $c^{-1}(0)$ topologically principal. Then $c^{-1}(0) = \{0\}$, so c is injective and $c(G)$ is isomorphic to G as a subgroup of Γ. So $A_0 = \mathbb{C}\delta_0$ and we consider the inclusion $D = A_0 \subseteq A$.

Notice that for $\omega \in \hat{G}$, $\omega\cdot\delta_g(h) = \langle \omega, c(h)\rangle\, \delta_g(h)$ so that $\omega\cdot\delta_g = \langle\omega, c(g)\rangle\delta_g$. Thus, by Lemma 3.4, $\delta_g \in A_{c(g)}$ and furthermore $\delta_g \notin A_t$ for $t \neq c(g)$. Since $C^*(G) = \operatorname{span}\{\delta_g : g \in G\}$ we have

$$A_t = \begin{cases} \mathbb{C}\delta_g & t = c(g) \\ 0 & \textit{otherwise.} \end{cases}$$

Thus the homogeneous normalizers of A_0 are all of the form $\lambda\delta_g$ for some $\lambda \in \mathbb{C}$. Take $X = \{*\} = \hat{A}_0$. Now

$$\begin{aligned} [\lambda\delta_g, *]_\Sigma = [\lambda'\delta_{g'}, *]_\Sigma &\Leftrightarrow g = g' \textit{ and } \overline{\lambda}\lambda' > 0, \\ [\lambda\delta_g, *]_G = [\lambda'\delta_{g'}, *]_G &\Leftrightarrow g = g'. \end{aligned}$$

So here $\Upsilon_G : g \mapsto [\delta_g, *]_G$ and $\Upsilon_\Sigma(z, g) \mapsto [z\delta_g, *]_\Sigma$ where $\Upsilon_\Sigma : \mathbb{T} \times G \to \Sigma_W$ are the desired isomorphisms from Theorem 6.2.

Example 7.2. *Let Λ be a k-graph. That is, Λ is a small category endowed with a functor, the degree map, $d : \Lambda \to \mathbb{N}^k$, that satisfies the following unique factorization property: if $\lambda \in \Lambda$ and $d(\lambda) = m+n$ there exists unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m, d(\nu) = n$ and $\lambda = \mu\nu$. We assume Λ has no sources in that for all objects v and all $m \in \mathbb{N}^k$ there exists μ with $r(\mu) = v$ and $d(\mu) = m$.*

In [22], Kumjian, Pask, and Sims introduce categorical cohomology on a k-graph Λ. In particular, they define a 2-cocycle with coefficients in $\mathbb{T}$ to be a function $\phi : \Lambda * \Lambda \to \mathbb{T}$ such that

$$\phi(\lambda_1, \lambda_2) + \phi(\lambda_1\lambda_2, \lambda_3) = \phi(\lambda_2, \lambda_3) + \phi(\lambda_1, \lambda_2\lambda_3)$$

where $\Lambda * \Lambda := \{(\mu, \nu) : s(\mu) = r(\nu)\}$ and λ_i are defined so that all of the compositions above make sense. Denote the set of these 2-cocycles by $Z_2(\Lambda, \mathbb{T})$. They prove in [22, Theorem 4.15]*, that there is an isomorphism from the second cubical cohomology group they defined in [21] to this categorical cohomology group.*

They define in [22, Definition 5.2] *the twisted k-graph C^*-algebra by $\phi \in Z_2(\Lambda, \mathbb{T})$ to be the universal C^*-algebra $C^*(\Lambda, \phi)$ generated by elements t_μ, $\mu \in \Lambda$ of a C^*-algebra satisfying the following.*

(1) *The t_v for $v \in d^{-1}(0)$ are mutually orthogonal projections,*
(2) *$t_\mu t_\nu = \phi(\mu, \nu) t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$,*
(3) *$t_\lambda^* t_\lambda = t_{s(\lambda)}$, and*
(4) *for all $v \in d^{-1}(0)$ and $n \in \mathbb{N}^k$, $t_v = \sum_{\substack{r(\lambda)=v \\ d(\lambda)=n}} t_\lambda t_\lambda^*$.*

By the universal property of $C^*(\Lambda, \phi)$, $d : \Lambda \to \mathbb{N}^k$ induces an action of $\mathbb{T}^k$ on $C^*(\Lambda, \phi)$ characterized by $z \cdot t_\mu t_\nu^* = z^{d(\mu)-d(\nu)} t_\mu t_\nu^*$.

Let

$$C = \overline{\operatorname{span}}\{t_\mu t_\nu^* \colon d(\mu) = d(\nu)\},$$

and let

$$D = \overline{\operatorname{span}}\{t_\mu t_\mu^*\}.$$

By [22, Lemma 7.4] *C is the fixed point algebra $C^*(\Lambda, \phi)^{\mathbb{T}^k}$, for this action. Moreover, as elements of the generating set $\{t_\mu t_\nu^* \colon d(\mu) = d(\nu)\}$ are all normalizers for D, to show D is Cartan in C it suffices to show D is maximal abelian and that there is a conditional expectation from C onto D.*

The conditional expectation P from C onto D is given by, for $\mu, \nu \in \Lambda$ with $d(\mu) = d(\nu)$

$$P(t_\mu t_\nu^*) = \delta_{\mu,\nu} t_\mu t_\nu^*.$$

The C^*-algebra C is an AF-algebra. This is shown in [22, Proposition 7.6]. *We recap and reframe some of those details to show D is Cartan in C.*

For each $n \in \mathbb{N}^k$ let

$$C_n = \overline{\operatorname{span}}\{t_\mu t_\nu^* \colon \mu, \nu \in \Lambda^n\},$$

and let

$$D_n = \overline{\operatorname{span}}\{t_\mu t_\mu^* \colon \mu \in \Lambda^n\}.$$

When $m \leq n$ we embed C_m in C_n using condition (4) *in the definition of $C^*(\Lambda, \phi)$ above: if $\mu, \nu \in \Lambda^m$ then*

$$t_\mu t_\nu^* = \sum_{\lambda \in \Lambda^{n-m} s(\mu)} t_\mu t_\lambda t_\lambda^* t_\nu^* \in C_n.$$

Note that this embedding also gives $D_m \subseteq D_n$. We have then that

$$C = \overline{\bigcup_{n \in \mathbb{N}^k} C_n},$$

and

$$D = \overline{\bigcup_{n \in \mathbb{N}^k} D_n}.$$

For each $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ denote by $K(\Lambda^n v)$ the compact operators on the Hilbert space $\ell^2(\Lambda^n v)$. Using a matrix unit argument, it is observed in [22, Proposition 7.6(1)] *that*

$$C_n = \bigoplus_{v \in \Lambda^0} K(\Lambda^n v).$$

As D_n is formed by the self-adjoint matrix units, D_n is a maximal abelian subalgebra of C_n. Further, there is a faithful conditional expectation P_n on C_n. We can describe this conditional expectation by

$$P_n(a) = \sum_{\mu \in \Lambda^n} (t_\mu t_\mu^*) a (t_\mu t_\mu^*),$$

where the series converges in the strong operator topology. Using this formula we can extend P_n to all of $B(H)$. A simple calculation shows that $P_n(B(H)) = D_n'$, i.e., P_n is a conditional expectation onto D_n'.

We further note that the embeddings of C_m into C_n for $m \leq n$, give $P_n|_{C_m} = P_m$. The conditional expectation $P\colon C \to D$ can then be described as the direct limit of the maps $\{P_n\}$ (see e.g. [29, Proposition A.8]*).*

To show that D is maximal abelian in C we use an argument similar to that found in [36, Chapter 1]. *Suppose $a \in D' \cap C$. Since $a \in C$ there is a net (a_n) with $a_n \in C_n$ such that*

$$\lim_n \|a_n - a\| = 0.$$

Further, since $a \in D'$, we have that $a \in D_n'$ for each $n \in \mathbb{N}^k$, and thus $P_n(a) = a$ for each $n \in \mathbb{N}^k$. Hence

$$\|P_n(a_n) - a\| = \|P_n(a_n - a)\| \leq \|a_n - a\|.$$

And thus the net $(P_n(a_n))_n$ converges to a. Since $P_n(a_n) \in D_n$ it follows that $a \in D$, and therefore D is maximal abelian in C.

Thus $(C^*(\Lambda, \phi), D)$ is a $\mathbb{Z}^k$-Cartan pair. Hence by Theorem 4.19 there exists a twist $\Sigma_W \to G_W$ such that $C^*(\Lambda, \phi) \cong C_r^*(\Sigma_W; G_W)$. Notice that here Σ_W and G_W consist of elements of the form $[s_\mu s_\nu^*, x]$ with $\mu, \nu \in \Lambda$ and $x \in \Lambda^\infty$.

In [22]*, the authors construct a groupoid G_Λ and a continuous cocycle ς such that $C^*(\Lambda, \phi) \cong C_r^*(G_\Lambda, \varsigma)$* [22, Theorem 6.7]. *By Theorem 6.2, $\Sigma_W \cong \mathbb{T} \times_\varsigma G_\Lambda$ and $G_W \cong G_\Lambda$, that is Theorem 4.19 recovers the construction in* [22]. *We provide some details of the isomorphisms of twists given above, but to proceed we need to provide a few details of the construction in* [22].

The groupoid construction in [22] *is standard and goes back to* [19, 31, 20]. *We say $\Lambda^\infty := \{x : \mathbb{N} \times \mathbb{N} \to \Lambda, x$ is a degree preserving functor$\}$ and $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(x)(m, n) = x(m + p, n + p)$.*

$$G_\Lambda := \{(x, \ell - m, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : \ell, m \in \mathbb{N}^k, \sigma^\ell x = \sigma^m y\}$$

with the topology on G_Λ given by basic open sets

$$Z(\mu,\nu) := \{(\mu x, d(\mu) - d(\nu), \nu x) : x \in \Lambda^\infty, r(x) = s(\mu) = s(\nu)\}.$$

It turns out that under our hypotheses each $Z(\mu,\nu)$ is compact and open and so there exists a subset $\mathfrak{P} \subseteq \Lambda \times \Lambda$ such that $\{Z(\mu,\nu) : (\mu,\nu) \in \mathfrak{P}\}$ is a partition of G_Λ. Thus for each $\gamma \in G_\Lambda$ there exists $(\mu_\gamma, \nu_\gamma) \in \mathfrak{P}$ such that $\gamma \in Z(\mu_\gamma, \nu_\gamma)$.

Now by [22, Lemma 6.3] *for $\gamma, \eta \in G_\Lambda$ composable we can find $y \in \Lambda^\infty$, $\alpha, \beta, \zeta \in \Lambda$ such that*

$$\gamma = (\mu_\gamma \alpha y, d(\mu_\gamma) - d(\nu_\gamma), \nu_\gamma \alpha y), \quad \eta = (\mu_\eta \beta y, d(\mu_\eta) - d(\nu_\eta), \nu_\eta \beta y),$$
$$\gamma\eta = (\mu_{\gamma\eta} \zeta y, d(\mu_{\gamma\eta}) - d(\nu_{\gamma\eta}), \nu_{\gamma\eta} \zeta y)$$

and

$$\varsigma_\phi(\gamma,\eta) := (\phi(\mu_\gamma,\alpha) - \phi(\nu_\gamma,\alpha)) + (\phi(\mu_\eta,\beta) - \phi(\nu_\eta,\beta)) - (\phi(\mu_{\gamma\eta},\zeta) - \phi(\nu_{\gamma\eta},\zeta))$$

is a well-defined continuous groupoid cocycle (see [31] *for the definition). We can then define*

$$\Sigma_{\Lambda,\phi} = \mathbb{T} \times_\varsigma G_\Lambda.$$

and then the twist is

$$\mathbb{T} \times G_\Lambda^{(0)} \to \Sigma_{\Lambda,\phi} \to G_\Lambda.$$

Now the isomorphisms in Theorem 6.2 are given by

$$\Upsilon_\Sigma : (z, (\mu_\gamma x, d(\mu_\gamma) - d(\nu_\gamma), \nu_\gamma x)) \mapsto [s_{\mu_\gamma} s^*_{\nu_\gamma}, \nu_\gamma x]_\Sigma$$
$$\Upsilon_G : (\mu_\gamma x, d(\mu_\gamma) - d(\nu_\gamma), \nu_\gamma x) \mapsto [s_{\mu_\gamma} s^*_{\nu_\gamma}, \nu_\gamma x]_G.$$

Example 7.3. *A main result in* [4] *is that any separable, unital, nuclear C^*-algebra which contains a Cartan subalgebra satisfies the UCT. In fact, more is shown. It is shown in* [4, Theorem 3.1] *that if $\Sigma \to G$ is a twist, where G is an étale Hausdorff locally compact second countable groupoid where the reduced C^*-algebra $C^*_r(\Sigma; G)$ is nuclear, then $C^*_r(\Sigma; G)$ satisfies the UCT. Hence we have the following corollary to Theorem 4.19 and* [4, Theorem 3.1].

Corollary 7.4. *Let A be a separable and nuclear C^*-algebra. If A contains an abelian subalgebra D such that (A, D) is a Γ-Cartan pair for a some discrete abelian group Γ, then A satisfies the UCT.*

Appendix A. Nonabelian groups

In the previous sections we assumed we had gradings by discrete abelian groups and actions by the dual group, as this case is familiar and doesn't involve the introduction of coactions. However all of the results of the paper can be extended to gradings by nonabelian groups, by replacing actions with coactions: in this short appendix we outline how to extend our results to the nonabelian case for those readers already familiar with coactions. For those readers interested in more information on coactions we recommend [13, Appendix A].

Throughout this appendix, all tensor products of C^*-algebras are spatial tensor products.

(A.1) Uses of Commutativity. The alert reader will no doubt have noticed that we used the commutativity of Γ in a few key places:

(1) to define a Γ grading on a C^*-algebra A when an action of $\hat{\Gamma}$ on A is given;
(2) to define maps $\Phi_t : A \to A_t$, including the faithful conditional expectation Φ_0 onto A_0, the fixed point algebra (Lemma 3.4);
(3) to define an action of $\hat{\Gamma}$ on $C^*_r(\Sigma; G)$ where $c : G; \Sigma \mapsto \Gamma$ (Lemma 2.10); and
(4) to show the fixed point algebra of the action above contains $C_0(G^{(0)})$ as a Cartan subalgebra (Proposition 5.2).

Now suppose that Γ is a not necessarily abelian discrete group whose identity we denote by e. For $s, t \in \Gamma$, we will use δ_s to denote the indicator function of the set $\{s\}$ and $\delta_{s,t}$ for the Kronecker δ (so $\delta_{s,t} = 1$ if $s = t$ and 0 if $s \neq t$). Let $\Lambda : C^*_r(\Gamma) \to B(\ell^2(\Gamma))$ be the left regular representation of Γ. Also, the map $\delta_s \mapsto \delta_s \otimes \delta_s \in C^*_r(\Gamma) \otimes C^*_r(\Gamma)$ extends to a $*$-homomorphism $\nu_\Gamma : C^*_r(\Gamma) \to C^*_r(\Gamma) \otimes C^*_r(\Gamma)$.

Let $\nu : A \to M(A \otimes C^*_r(\Gamma))$ be a (reduced) coaction. This means that ν is a non-degenerate $*$-homomorphism of A into $M(A \otimes C^*_r(\Gamma))$ such that

(i) $\nu(A)(I \otimes C^*_r(\Gamma)) \subseteq A \otimes C^*_r(\Gamma)$; and
(ii) $(\nu \otimes \mathrm{id}_\Gamma) \circ \nu = (\mathrm{id}_A \otimes \nu_\Gamma) \circ \nu$ (these maps belong to $\mathcal{B}(A, M(A \otimes C^*_r(\Gamma) \otimes C^*_r(\Gamma))))$.

Notice that since Γ is discrete, $I \otimes \delta_e$ is the identity of $M(A \otimes C^*_r(\Gamma))$. Thus condition (i) implies that actually,

$$\nu : A \to A \otimes C^*_r(\Gamma).$$

Furthermore, the fact that Γ is discrete implies that ν is non-degenerate in the sense that $\nu(A)(I \otimes C^*_r(\Gamma))$ is dense in $A \otimes C^*_r(\Gamma)$; see [3] or [13, Remark A.22(3)].

For $t \in \Gamma$, there is a slice map $S_t : A \otimes C^*_r(\Gamma) \to A$ characterized by

$$a \otimes b \mapsto a\langle \Lambda(b)\delta_e, \delta_t \rangle,$$

(see [40] or [13, §A.4]). Further, it follows from the second statement of [13, Lemma A.30] that for $x \in A \otimes C^*_r(\Gamma)$,

$$x = 0 \Leftrightarrow \text{ for all } t \in \Gamma,\ S_t(x) = 0.$$

Thus, every element $x \in A \otimes C^*_r(\Gamma)$ has a uniquely determined "Fourier series",

$$x \sim \sum_{t \in \Gamma} S_t(x) \otimes \delta_t.$$

Define continuous maps $\Phi_t : A \to A$ by

$$\Phi_t := S_t \circ \nu.$$

Then for every $a \in A$,

$$\nu(a) \sim \sum_{t\in\Gamma} \Phi_t(a) \otimes \delta_t. \tag{A.2}$$

While we will not need this fact here, the $*$-homomorphism property of ν and the series representation (A.2) implies that the "coefficient maps" $\{\Phi_t\}_{t\in\Gamma}$ behave much as Fourier coefficients do under convolution multiplication and adjoints: for every $t \in \Gamma$ and $a, b \in A$,

$$\Phi_t(ab) = \sum_{s\in\Gamma} \Phi_{ts^{-1}}(a)\Phi_s(b) \quad \text{and} \quad \Phi_t(a^*) = \Phi_{t^{-1}}(a)^*.$$

What we do require is that condition (ii) in the definition of coaction given above implies that for every $s, t \in \Gamma$ and $a \in A$,

$$\Phi_s(\Phi_t(a)) = \begin{cases} 0 & \text{if } s \neq t, \\ \Phi_t(a) & \text{when } s = t. \end{cases}$$

Define

$$A_t := \Phi_t(A).$$

Note that as S_e arises from the (faithful) trace on $C_r^*(\Gamma)$, $\Phi_e : A \to A_e$ is a faithful conditional expectation.

We will call an element $a \in A$ *homogeneous* if $a \in A_t$ for some $t \in \Gamma$. This gives us the Γ-grading and an analog of Lemma 3.4, which addresses the first two points of Paragraph A.1.

We now address the third and fourth items of Paragraph A.1. Assume we have a twist with a cocycle as in Section 5 but with Γ not necessarily abelian:

$$\begin{tikzcd} \mathbb{T} \times G^{(0)} \arrow[r] & \Sigma \arrow[r] \arrow[dr, "c_\Sigma"'] & G \arrow[d, "c_G"] \\ & & \Gamma. \end{tikzcd} \tag{A.3}$$

The proof of [10, Lemma 6.1] goes through without change to show there exists a coaction

$$\nu : C_r^*(\Sigma; G) \to C_r^*(\Sigma; G) \otimes C_r^*(\Gamma)$$

characterized by

$$\nu(f) = f \otimes \delta_t \quad \text{where } f \in C_c(G) \text{ and } \operatorname{supp}(f) \subseteq c_G^{-1}(t).$$

Note that for $f \in C_c(G)$ and $\operatorname{supp}(f) \subseteq c_G^{-1}(t)$,

$$\Phi_s(f) = S_s(f \otimes \delta_t) = f\langle \Lambda(\delta_t)\delta_e, \delta_s\rangle = f\langle \delta_t, \delta_s\rangle = f\delta_{s,t}.$$

Now for $f \in C_c(G)$, $f = \sum_{t\in\Gamma} f|_{c_G^{-1}(t)}$ so that the above computation shows that

$$\Phi_s(f) = f|_{c_G^{-1}(s)}.$$

By continuity of Φ_s and $j : C_r^*(\Sigma; G) \to C_0(\Sigma; G)$, we get that $j(\Phi_s(a)) = j(a)|_{c_G^{-1}(s)}$. As in Section 5, let $R = c_G^{-1}(e)$ and $\mathcal{P} = c_\Sigma^{-1}(e)$. Then Lemma 2.7

goes through without change, and if $a \in C_r^*(\Sigma;G)_e$ and $f_i \in C_r^*(\Sigma;G)$ is a net converging to a, by the above we have $\Phi_e(f_i) \to a$ and $\Phi_e(f_i) = f_i|_R \in C_c(R;\mathcal{P})$, so that $a \in C^*(R;\mathcal{P})$, giving Proposition 5.2.

Theorem A.1. *Let A be a C^*-algebra and let D be an abelian C^*-algebra of A such that:*

- *there is a coaction ν of a discrete group Γ on A;*
- *D is Cartan in the algebra A_e; and*
- *$\overline{\operatorname{span}}\, N_h(A,D) = A$.*

Let A_c be the algebraic span of N_h. Then there exists a Γ-graded twist $\mathbb{T} \times X \hookrightarrow \Sigma \to G$ and a $*$-isomorphism $\Psi : A_c \to C_r^*(\Sigma;G)$ which induces an isomorphism $A \to C^*(\Sigma;G)$ taking D to $C_0(G^{(0)})$.

Proof. The arguments of Section 4 go through without change. $\square$

Remark A.2. *Compare the conditions of Theorem A.1 to the definition of Γ-Cartan (3.9), noting that by Lemma 3.10(1), the conditions on the normalizers coincide when Γ is abelian. While we have not checked details, we expect that if Γ is a (not necessarily abelian) discrete group, and the hypotheses of Theorem A.1 are weakened so that the condition $\overline{\operatorname{span} N_h(A,D)} = A$ is replaced with $\overline{\operatorname{span} N(A,D)} = A$, then it is still true that $\overline{N_h(A,D)} = A$. If this is the case, a Γ-Cartan pair could then be defined to be an inclusion of C^*-algebras $D \subseteq A$ satisfying the weakened hypotheses of Theorem A.1. All the results of this paper would be valid for this notion of Γ-Cartan pairs.*

The arguments of Sections 5 and 6 yield the following result for (possibly non-commutative) discrete groups Γ.

Theorem A.3. *Let $\Sigma \to G$ be a locally trivial twist, with a cocycle c into a discrete group Γ. Then $(C_r^*(\Sigma;G), C_0(G^{(0)}))$ is a Γ-Cartan pair. Let $\Sigma_1 \to G_1$ be the Γ-graded twist constructed from $(C_r^*(\Sigma;G), C_0(G^{(0)}))$. For each $\gamma \in \Sigma$, choose a homogeneous normalizer $n \in C_r^*(\Sigma, G)$ such that $n(\gamma) > 0$. Then the map*

$$\Upsilon_\Sigma : \Sigma \to \Sigma_1 \quad \textit{given by} \quad \gamma \mapsto [n, s(\gamma)]_{\Sigma_1}$$

descends to a well-defined isomorphism of twists such that the following diagram commutes.

$$\begin{tikzcd}
& \Sigma \arrow[rr, "q"] \arrow[dd, "\Upsilon_\Sigma"'] \arrow[dr, "c_\Sigma"] & & G \arrow[dl, "c_G"'] \arrow[dd, "\Upsilon_G"] \\
\mathbb{T} \times X \arrow[ur, "\iota"] \arrow[dr, "\iota_1"'] & & \Gamma & \\
& \Sigma_1 \arrow[rr, "q_1"'] \arrow[ur, "c_{\Sigma_1}"'] & & G_1 \arrow[ul, "c_{G_1}"]
\end{tikzcd}$$

References

[1] Abrams, Gene; Ara, Pere; Siles Molina, Mercedes. Leavitt path algebras. Lecture Notes in Mathematics, 2191. *Springer, London*, 2017. xiii+287 pp. ISBN: 978-1-4471-7343-4. MR3729290, Zbl 1393.16001, doi: 10.1007/978-1-4471-7344-1. 220

[2] Ara, Pere; Bosa, Joan; Hazrat, Roozbeh; Sims, Aidan. Reconstruction of graded groupoids from graded Steinberg algebras. *Forum Math.* **29** (2017), no. 5, 1023–1037. MR3692025, Zbl 1373.22007, arXiv:1601.02872, doi: 10.1515/forum-2016-0072. 206, 207

[3] Baaj, Saad; Skandalis, Georges. C^*-algèbres de Hopf et théorie de Kasparov équivariante. *K-Theory* **2** (1989), no. 6, 683 –721. MR1010978, Zbl 0683.46048, doi: 10.1007/BF00538428. 247

[4] Barlak, Selçuk; Li, Xin. Cartan subalgebras and the UCT problem. *Adv. Math.* **316** (2017), 748–769. MR3672919, Zbl 1382.46048, doi: 10.1016/j.aim.2017.06.024. 207, 208, 246

[5] Brown, Jonathan; Clark, Lisa O.; Farthing, Cynthia; Sims, Aidan. Simplicity of algebras associated to étale groupoids, *Semigroup Forum* **88** (2014), no. 2, 433–452. MR3189105, Zbl 1304.46046, arXiv:1204.3127, doi: 10.1007/s00233-013-9546-z. 209

[6] Brown, Jonathan; Clark, Lisa O.; an Huef, Astrid. Diagonal-preserving ring $*$-isomorphisms of Leavitt path algebras. *J. Pure Appl. Algebra* **221** (2017), no. 10, 2458–2481. MR3646311, Zbl 1385.16027, doi: 10.1016/j.jpaa.2016.12.032. 206, 207

[7] Brownlowe, Nathan; Carlsen, Toke M.; Whittaker, Michael F. Graph algebras and orbit equivalence. *Ergodic Theory Dynam. Systems* **37** (2017), no. 2, 389–417. MR3614030, Zbl 1381.37014, doi: 10.1017/etds.2015.52. 206, 207

[8] Carlsen, Toke M. $*$-isomorphism of Leavitt path algebras over $\mathbb{Z}$. *Adv. Math.* **324** (2018), 326–335. MR3733888, Zbl 1431.16030, doi: 10.1016/j.aim.2017.11.018.

[9] Carlsen, Toke M.; Ruiz, Efren; Sims, Aidan Equivalence and stable isomorphism of groupoids, and diagonal-preserving stable isomorphisms of graph C^*-algebras and Leavitt path algebras. *Proc. Amer. Math. Soc.* **145** (2017), no. 4, 1581–1592. MR3601549, Zbl 1368.46042, doi: 10.1090/proc/13321.

[10] Carlsen, Toke M.; Ruiz, Efren; Sims, Aidan; Tomforde, Mark. Reconstruction of groupoids and C^*-rigidity of dynamical systems. Preprint, 2017. arXiv:1711.01052 207, 243, 248

[11] Clark, Lisa O.; Farthing, Cynthia; Sims, Aidan; Tomforde, Mark. A groupoid generalisation of Leavitt path algebras. *Semigroup Forum* **89** (2014), no. 3, 501–517. MR3274831, Zbl 1323.46033, doi: 10.1007/s00233-014-9594-z. 206

[12] Donsig, Allan P.; Pitts, David R. Coordinate systems and bounded isomorphisms. *J. Operator Theory* **59** (2008), no. 2, 359–416. MR2411050, Zbl 1199.46116. 228, 230

[13] Echterhoff, Siegfried; Kaliszewski, Steven P.; Quigg, John; Raeburn, Iain. A categorical approach to imprimitivity theorems for C^*-dynamical systems. *Mem. Amer. Math. Soc.* **180** (2006), no. 850, viii+169 pp. MR2203930, Zbl 1097.46042, doi: 10.1090/memo/0850. 246, 247

[14] Exel, Ruy. Amenability for Fell bundles. *J. Reine Angew. Math.* **492** (1997), 41–73. MR1488064, Zbl 0881.46046, doi: 10.1515/crll.1997.492.41. 219

[15] Exel, Ruy. Inverse semigroups and combinatorial C^*-algebras. *Bull. Braz. Math. Soc. (N.S.)* **39** (2008), no. 2, 191–313. MR2419901, Zbl 1173.46035, doi: 10.1007/s00574-008-0080-7. 213

[16] Exel, Ruy. Partial dynamical systems, Fell bundles and applications. Mathematical Surveys and Monographs, 224. *American Mathematical Society, Providence, RI,* 2017. vi+321 pp. ISBN: 978-1-4704-3785-5. MR3699795, Zbl 1405.46003, doi: 10.1090/surv/224. 219

[17] Feldman, Jacob; Moore, Calvin C. Ergodic equivalence relations, cohomology and von Neumann algebras. II. *Trans. Amer. Math. Soc.* **234** (1977), 325–359. MR0578730 , Zbl 0369.22010, doi: 10.2307/1997925. 205, 206

[18] Kumjian, Alex. On C^*-diagonals. *Canad. J. Math.* **38** (1986), no. 4, 969–1008. MR854149, Zbl 0627.46071, doi: 10.4153/CJM-1986-048-0. 206, 223, 224

[19] Kumjian, Alex; Pask, David. Higher rank graph C^*-algebras. *New York J. Math.* **6** (2000), 1–20. MR1745529, Zbl 0946.46044. 245

[20] Kumjian, Alex; Pask, David; Raeburn, Iain; Renault, Jean. Graphs, groupoids, and Cuntz–Krieger algebras. *J. Funct. Anal.* **144** (1997), no. 2, 505–541. MR1432596, Zbl 0929.46055, doi: 10.1006/jfan.1996.3001. 245

[21] Kumjian, Alex; Pask, David; Sims, Aidan. Homology for higher-rank graphs and twisted C^*-algebras. *J. Funct. Anal.* **263** (2012), no. 6, 1539–1574. MR2948223, Zbl 1253.55006, arXiv:1110.1433, doi: 10.1016/j.jfa.2012.05.023. 208, 243

[22] Kumjian, Alex; Pask, David; Sims, Aidan. On twisted higher-rank graph C^*-algebras. *Trans. Amer. Math. Soc.* **367** (2015), no. 7, 5177–5216. MR3335414, Zbl 1329.46050, doi: 10.1090/S0002-9947-2015-06209-6. 208, 243, 244, 245, 246

[23] Matsumoto, Kengo. Classification of Cuntz–Krieger algebras by orbit equivalence of topological Markov shifts. *Proc. Amer. Math. Soc.* **141** (2013), no. 7, 2329–2342. MR3043014, Zbl 1274.46129, doi: 10.1090/S0002-9939-2013-11519-4. 206

[24] Matsumoto, Kengo; Matui, Hiroki. Continuous orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras. *Kyoto J. Math.* **54** (2014), no. 4, 863–877. MR3276420, Zbl 1320.46055, doi: 10.1215/21562261-2801849. 206

[25] Matui, Hiroki. Homology and topological full groups of étale groupoids on totally disconnected spaces. *Proc. Lond. Math. Soc. (3)* **104** (2012), no. 1, 27–56. MR2876963, Zbl 1325.19001, doi: 10.1112/plms/pdr029. 206

[26] Pitts, David R. Structure for regular inclusions. Preprint, 2012. arXiv:1202.6413v2. 223, 230, 231

[27] Pitts, David R. Structure for regular inclusions. I. *J. Operator Theory* **78** (2017), no. 2, 357–416. MR3725511, Zbl 1424.46078. 224, 225, 228

[28] Pitts, David R. Structure for regular inclusions. II: Cartan envelopes, pseudo-expectations and twists. Preprint, 2020. arXiv:2006.00834 223, 230, 231

[29] Raeburn, Iain. Graph algebras. CBMS Regional Conference Series in Mathematics, 103. *Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI,* 2005. vi+113 pp. ISBN: 0-8218-3660-9. MR2135030, Zbl 1079.46002, doi: 10.1090/cbms/103. 245

[30] Raeburn, Iain. On graded C^*-algebras. *Bull. Aust. Math. Soc.* **97** (2018), no. 1, 127–132. MR3744874, Zbl 1393.46041, doi: 10.1017/S0004972717000776. 206, 221, 223

[31] Renault, Jean. A groupoid approach to C^*-algebras. Lecture Notes in Mathematics, 793. *Springer, Berlin,* 1980. ii+160 pp. ISBN: 3-540-09977-8. MR584266, Zbl 0433.46049, doi: 10.1007/BFb0091072. 214, 215, 245, 246

[32] Renault, Jean. Représentation des produits croisés d'algèbres de groupoïdes. *J. Operator Theory* **18** (1987), no. 1, 67–97. MR0912813, Zbl 0659.46058. 239

[33] Renault, Jean. Cartan subalgebras in C^*-algebras. *Irish Math. Soc. Bull.* **61** (2008), 29–63. MR2460017, Zbl 1175.46050, arXiv:0803.2284. 206, 209, 215, 223, 224, 226, 238, 240

[34] Sims, Aidan. Étale groupoids and their C^*-algebras. Preprint, 2018. arXiv:1710.10897v2. 209

[35] Spielberg, Jack. Groupoids and C^*-algebras for categories of paths. *Trans. Amer. Math. Soc.* **366** (2014), no. 11, 5771–5819. MR3256184, Zbl 1348.46056, arXiv:1111.6924, doi: 10.1090/S0002-9947-2014-06008-X.

[36] Strătilă, Şerban; Voiculescu, Dan. Representations of AF-algebras and of the group $U(\infty)$. Lecture Notes in Mathematics, 486. *Springer-Verlag, Berlin-New York,* 1975. viii+169 pp. MR0458188, Zbl 0318.46069, doi: 10.1007/BFb0082276. 245

[37] Steinberg, Benjamin. A groupoid approach to discrete inverse semigroup algebras. *Adv. Math.* **223** (2010), no. 2, 689–727. MR2565546, Zbl 1188.22003, doi: 10.1016/j.aim.2009.09.001. 206

[38] Tikuisis, Aaron; White, Stuart; Winter, Wilhelm. Quasidiagonality of nuclear C*-algebras. *Ann. of Math. (2)* **185** (2017), no. 1, 229–284. MR3583354, Zbl 1367.46044, arXiv:1509.08318, doi: 10.4007/annals.2017.185.1.4. 207

[39] Tomforde, Mark. Uniqueness theorems and ideal structure for Leavitt path algebras *J. Algebra* **318** (2007), no. 1, 270–299. MR2363133, Zbl 1132.46035, arXiv:math/0612628, doi: 10.1016/j.jalgebra.2007.01.031. 206

[40] Tomiyama, Jun. Applications of Fubini type theorem to the tensor products of C^*-algebras. *Tohoku Math. J. (2)* **19** (1967), 213–226. MR0218906, Zbl 0166.11401, doi: 10.2748/tmj/1178243318. 247

[41] Willard, Stephen. General topology. *Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont.,* 1970. xii+369 pp. MR0264581, Zbl 0205.26601. 210

(J.H. Brown) Department of Mathematics, University of Dayton, 300 College Park Dayton, OH 45469-2316, U.S.A.
jonathan.henry.brown@gmail.com

(A.H. Fuller) Department of Mathematics, Ohio University, Athens, OH 45701, U.S.A.
fullera@ohio.edu

(D.R. Pitts) Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130, U.S.A.
dpitts2@unl.edu

(S. A. Reznikoff) Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, U.S.A.
sarahrez@math.ksu.edu

This paper is available via http://nyjm.albany.edu/j/2021/27-7.html.