New York Journal of Mathematics

New York J. Math. **28** (2022) 250–336.

On convergent Poincaré-Moser reduction for Levi degenerate embedded 5-dimensional CR manifolds

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ABSTRACT. Firstly, applying Lie's elementary theory for appropriate prolongations to jet spaces of orders 1 and 2, we show that any real-analytic 2-nondegenerate constant Levi rank 1 hypersurface $M^5 \subset \mathbb{C}^3$ carries *two* sorts of Cartan-Moser *chains*, that are of orders 1 and 2.

Secondly, integrating and straightening any given order 2 chain γ passing through any point $p \in M$ to be the v-axis in coordinates $(z, \zeta, w = u + i\,v)$ centered at p, without setting up the formal theory in advance, we show that there exists a *convergent* change of complex coordinates $(z, \zeta, w) \longmapsto (z', \zeta', w')$ fixing the origin in which γ is the v-axis and in which M has *Poincaré-Moser reduced equation* of a specific shape.

Thirdly, starting from an M having preliminary normalized equation

$$u \;=\; z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(5),$$

assigning weights $[z] := 1$, $[\zeta] := 0$, $[w] := 2$, we show that a normalizing biholomorphism exists and is *unique* when assumed of the form

$$\begin{aligned} z' \;&:=\; z + f_{\geqslant 2}(z,\zeta,w),\ \zeta' \;:=\; \zeta + g_{\geqslant 1}(z,\zeta,w), \qquad w' \;:=\; w + h_{\geqslant 3}(z,\zeta,w), \\ 0 \;&=\; f_w(0), \qquad\qquad\qquad\qquad\qquad\qquad 0 \;=\; \mathrm{Im}\, h_{ww}(0). \end{aligned}$$

The values at the origin of Pocchiola's two primary Cartan-type relative differential invariants are

$$W_0 \;=\; 4\,\overline{F_{3,0,0,2}(0)} \qquad\qquad \text{and} \qquad\qquad J_0 \;=\; 20\,F_{5,0,0,1}(0).$$

The proofs are detailed, accessible to non-experts. The computer-generated aspects (forthcoming) have been reduced to a minimum here.

CONTENTS

Received July 31, 2020.

2010 *Mathematics Subject Classification.* Primary: 32V40, 58K50, 32V35, 53A55, 53-08. Secondary: 58A15, 32A05, 53A07, 53B25, 22E05, 22E60, 58A30.

Key words and phrases. Normal forms, Power series method, Explicit differential invariants, Levi degenerate CR manifolds, Holomorphic mappings, Classification of hypersurfaces, Pocchiola's invariants.

ISSN 1076-9803/2022

1. Introduction

As explained in the survey introduction of [8], the appropriate local graphed model for 2-nondegenerate constant Levi rank 1 real analytic ($\mathcal{C}^\omega$) hypersurfaces $M^5 \subset \mathbb{C}^3$, generally graphed, in coordinates $\big(z, \zeta, w = u + i\, v\big)$ as

$$u \;=\; F\big(z, \zeta, \overline{z}, \overline{\zeta}, v\big),$$

is the so-called *Gaussier-Merker model*

$$u \;=\; \frac{z\overline{z} + \frac{1}{2}\,\overline{z}^2\zeta + \frac{1}{2}\,z^2\overline{\zeta}}{1 - \zeta\overline{\zeta}} \;=:\; m\big(z, \zeta, \overline{z}, \overline{\zeta}\big).$$

Fels-Kaup [14] showed that its (connected) intersection with $\{|\zeta| < 1\}$ is biholomorphic to a Zariski-open subset of the complex tube $S^2_{\mathrm{LC}} \times i\,\mathbb{R}^3$ over the real light cone $(\mathrm{Re}\, z_2)^2 - (\mathrm{Re}\, z_3)^2 = (\mathrm{Re}\, z_1)^2$. The light cone $S^2_{\mathrm{LC}} \subset \mathbb{R}^3$ is *the* maximally symmetric non-flat parabolic surface, characterized, according to [9], by the vanishing of certain two differential invariants.

By applying either Cartan's method of equivalence, or Tanaka's approach, several recent works ([22, 28, 29, 42, 35, 16]) have been devoted to construct absolute parallelisms, namely 10-dimensional $\{e\}$-structure bundles $P^{10} \longrightarrow M^5$ for such $M^5 \subset \mathbb{C}^3$, invariantly related to biholomorphic equivalences of such hypersurfaces.

By performing advanced electronic computations, Merker-Pocchiola [42, 35] found that only two primary curvature invariants exist, denoted W and I. These intensive computations have been redone manually by Foo-Merker in [16]. One obtains certain 'horizontal' (semi-basic) 1-forms $\{\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}\}$ with $\overline{\rho} = \rho$ together with four 'vertical' 1-forms $\pi^1, \pi^2, \overline{\pi}^1, \overline{\pi}^2$ which satisfy 'compact' structure equations of the form

$$\begin{aligned}
d\rho &= (\pi^1 + \overline{\pi}^1) \wedge \rho + i\,\kappa \wedge \overline{\kappa},\\
d\kappa &= \pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \overline{\kappa},\\
d\zeta &= (\pi^1 - \overline{\pi}^1) \wedge \zeta + i\,\pi^2 \wedge \kappa +\\
&\qquad + R\,\rho \wedge \zeta + i\,\frac{1}{\overline{\mathsf{c}}^3}\,\overline{J}_0\,\rho \wedge \overline{\kappa} + \frac{1}{\mathsf{c}}\,W_0\,\kappa \wedge \zeta,
\end{aligned}$$

conjugate structure equations for $d\overline{\kappa}$, $d\overline{\zeta}$ being easily deduced.

In Sections 20 and 24, we copy the expressions of the two primary relative differential invariants $W_0 : M \longrightarrow \mathbb{C}$ and $\overline{J}_0 : M \longrightarrow \mathbb{C}$, while R is a certain (useless) secondary invariant.

Theorem 1.1. [42, 35, 16] *Only two primary invariants, W_0 and J_0, occur for biholomorphic equivalences of 2-nondegenerate constant Levi rank 1 real analytic hypersurfaces $M^5 \subset \mathbb{C}^3$, and*

$$0 \equiv W_0 \equiv J_0 \qquad \Longleftrightarrow \qquad M \text{ is equivalent to the Gaussier-Merker model.}$$

Furthermore, when either $W_0 \neq 0$ or $J_0 \neq 0$, the equivalence problem reduces to a 5-dimensional $\{e\}$-structure on M^5, and every non-flat M^5 has CR automorphisms group of dimension $\leqslant 5$.

In this article, our motivation is to view again these relative CR differential invariants by putting the equation of such $M^5 \subset \mathbb{C}^3$ into normal form, like Chern-Moser did in [12]. Generally, the Poincaré-Moser normal form [12] provides a distinguished choice of local holomorphic coordinates for a hypersurface, in which its defining equation is approximated as far as possible by that

of the local model, for instance in $\mathbb{C}^{n+1} \ni (z_1, \dots, z_n, w = u + i\,v)$, a real hyperquadric

$$u \;=\; |z_1|^2 + \cdots + |z_p|^2 - |z_{p+1}|^2 - \cdots - |z_n|^2.$$

Usually, a biholomorphic transformation bringing a hypersurface to a normal form at the origin is defined up to composition with the automorphisms group of the model.

In [8], joint with Chen, we studied *rigid* $\mathcal{C}^\omega$ hypersurfaces $M^5 \subset \mathbb{C}^3$

$$u = F\big(z, \zeta, \overline{z}, \overline{\zeta}\big) = \sum_{a,b,c,d \geqslant 0} z^a \zeta^b \overline{z}^c \overline{\zeta}^d \, F_{a,b,c,d}, \quad F_{a,b,c,d} \in \mathbb{C}, \; \overline{F_{c,d,a,b}} = F_{a,b,c,d},$$

with graphing function F *independent of* v, which are everywhere 2-nondegenerate and of constant Levi rank 1, under the *rigid biholomorphisms group*, a group which consists of transformations of the form

$$(z, \zeta, w) \;\longmapsto\; \Big(f(z,\zeta),\, g(z,\zeta),\, \rho\, w + h(z,\zeta)\Big) =: \; \big(z', \zeta', w'\big),$$

having nonzero holomorphic Jacobian $f_z g_\zeta - f_\zeta g_z \neq 0$, with $\rho \in \mathbb{R}^*$. We established that every such rigid $M^5 \subset \mathbb{C}^3$ is *rigidly equivalent* to a 'perturbation' of the Gaussier-Merker model

$$\begin{aligned} u \;=\; & \frac{z\overline{z} + \frac{1}{2}\overline{z}^2 \zeta + \frac{1}{2} z^2 \overline{\zeta}}{1 - \zeta\overline{\zeta}} + 2\,\mathrm{Re}\Big\{F_{4,0,0,1}\, z^4 \overline{\zeta} + \mathrm{Re}\, F_{3,0,1,1}\, z^3 \overline{z}\overline{\zeta} + F_{3,0,0,2}\, z^3 \overline{\zeta}^2\Big\} \\ & + z^3 \overline{z}^3\, \mathrm{O}_{z,\overline{z}}(0) + 2\,\mathrm{Re}\, z^3 \overline{\zeta}\, \mathrm{O}_{z,\overline{z},\overline{\zeta}}(2) + \zeta\overline{\zeta}\, \mathrm{O}_{z,\overline{z}}(3)\, \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(1). \end{aligned}$$

Here, by writing $\mathrm{Re}\, F_{3,0,1,1}$, we mean that the (complex) coefficient $F_{3,0,1,1} \in \mathbb{C}$ has been normalized to be real.

Furthermore, writing

$$\begin{aligned} u \;=\; F\big(z,\zeta,\overline{z},\overline{\zeta}\big) \;&=\; m\big(z,\zeta,\overline{z},\overline{\zeta}\big) + G\big(z,\zeta,\overline{z},\overline{\zeta}\big) \\ &=\; m\big(z,\zeta,\overline{z},\overline{\zeta}\big) + \sum_{\substack{a,b,c,d \in \mathbb{N} \\ a+c \geqslant 3}} G_{a,b,c,d}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d, \end{aligned}$$

two such rigid $\mathcal{C}^\omega$ hypersurfaces $M^5 \subset \mathbb{C}^3$ and $M'^5 \subset \mathbb{C}'^3$, both brought into such a normal form, are rigidly biholomorphically equivalent if and only if there exist two constants $\rho \in \mathbb{R}^*_+$, $\varphi \in \mathbb{R}$, such that for all a, b, c, d

$$G_{a,b,c,d} \;=\; G'_{a,b,c,d}\, \rho^{\frac{a+c-2}{2}}\, e^{i\varphi(a+2b-c-2d)}.$$

This means that the normal form is defined only up to the 2-dimensional action of the *rigid* isotropy group of the origin

$$(z, \zeta, w) \;\longmapsto\; \big(\rho^{1/2}\, e^{i\varphi}\, z, \; e^{2i\varphi}\, \zeta, \; \rho\, w\big) \qquad (\rho \in \mathbb{R}^*_+, \; \varphi \in \mathbb{R}).$$

Before making public this normal form, in [19], we produced Cartan-type reduction to an $\{e\}$-structure for the equivalence problem, under *rigid* (local) biholomorphic transformations, of such rigid M^5 that are 2-nondegenerate of

constant Levi rank 1. We constructed an invariant 7-dimensional bundle $P^7 \longrightarrow M^5$ equipped with coordinates

$$\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v,\, \mathsf{c}, \overline{\mathsf{c}}\big),$$

with $\mathsf{c} \in \mathbb{C}$, together with of seven 1-forms generating T^*P^7, denoted

$$\{\rho,\, \kappa,\, \zeta,\, \overline{\kappa},\, \overline{\zeta},\, \alpha,\, \overline{\alpha}\} \qquad (\overline{\rho} \,=\, \rho),$$

which satisfy invariant structure equations of the form

$$\begin{aligned}
d\rho \,&=\, \big(\alpha + \overline{\alpha}\big) \wedge \rho + i\, \kappa \wedge \overline{\kappa},\\
d\kappa \,&=\, \alpha \wedge \kappa + \zeta \wedge \overline{\kappa},\\
d\zeta \,&=\, \big(\alpha - \overline{\alpha}\big) \wedge \zeta + \frac{1}{\mathsf{c}} I_0\, \kappa \wedge \zeta + \frac{1}{\mathsf{c}\overline{\mathsf{c}}} V_0\, \kappa \wedge \overline{\kappa},\\
d\alpha \,&=\, \zeta \wedge \overline{\zeta} - \frac{1}{\mathsf{c}} I_0\, \zeta \wedge \overline{\kappa} + \frac{1}{\mathsf{c}\overline{\mathsf{c}}} Q_0\, \kappa \wedge \overline{\kappa} + \frac{1}{\overline{\mathsf{c}}} \overline{I}_0\, \overline{\zeta} \wedge \kappa.
\end{aligned}$$

We refer to [8] for explicit expressions of the two primary invariants

$$I_0, V_0 :\quad M \,\longrightarrow\, \mathbb{C},$$

and of the secondary invariant $Q_0 :\, M \longrightarrow \mathbb{R}$, which is real. Once M is put into normal form as above, their values at the origin are

$$I_0 \,=\, 4\,\overline{F_{3,0,0,2}}, \qquad\qquad V_0 \,=\, -8\,\overline{F_{4,0,0,1}}, \qquad\qquad Q_0 \,=\, 4\,\mathrm{Re}\, F_{3,0,1,1}.$$

The goal of this article is to set up a rigorous *convergent* Poincaré-Moser normal form for any everywhere 2-nondegenerate constant Levi rank 1 general (nonrigid) $\mathcal{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$ under the *full* (not necessarily rigid) biholomorphisms group

$$(z, \zeta, w) \,\longmapsto\, \Big(f(z, \zeta, w),\, g(z, \zeta, w),\, h(z, \zeta, w)\Big).$$

Given such an $M^5 \subset \mathbb{C}^3$ with $0 \in M$, by examining terms of F up to order 4, it is elementary to find a holomorphic system of coordinates in which it is

$$u \,=\, F \,=\, z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2} z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(5).$$

Since the Gaussier-Merker model is invariant under the complex scalings

$$(z, \zeta, w) \,\longmapsto\, \big(\lambda\, z,\, \tfrac{\lambda}{\overline{\lambda}}\, \zeta,\, \lambda\overline{\lambda}\, w\big) \qquad (\lambda \,\in\, \mathbb{C}^*),$$

it is natural to assign the weights

$$[z] \,:=\, 1 \,=:\, [\overline{z}], \qquad\qquad [\zeta] \,:=\, 0 \,=:\, [\overline{\zeta}], \qquad\qquad [w] \,:=\, 2 \,=:\, [\overline{w}].$$

Then by $e_{\geqslant \nu}(z, \zeta, w)$, we will mean a holomorphic function near the origin all of whose monomials $z^a \zeta^b w^e$ are of weight $a + 2\,e \geqslant \nu$.

Theorem 1.2. [Main] *There exists a biholomorphism* $(z,\zeta,w) \longmapsto (z',\zeta',w')$ *fixing* 0 *which maps* $(M,0)$ *into* $(M',0)$ *of normalized equation (suppressing primes)*

$$\begin{aligned} u \;=\; & \frac{z\overline{z} + \frac{1}{2}\overline{z}^2\zeta + \frac{1}{2}z^2\overline{\zeta}}{1-\zeta\overline{\zeta}} \\ & + 2\,\mathrm{Re}\Big\{z^3\overline{\zeta}^2\,F_{3,0,0,2}(v) + \zeta\overline{\zeta}\,\big(3\,z^2\overline{z}\overline{\zeta}\,F_{3,0,0,2}(v)\big)\Big\} \\ & + 2\,\mathrm{Re}\Big\{z^5\overline{\zeta}\,F_{5,0,0,1}(v) + z^4\overline{\zeta}^2\,F_{4,0,0,2}(v) + z^3\overline{z}^2\overline{\zeta}\,F_{3,0,2,1}(v) \\ & \qquad\qquad + z^3\overline{z}\overline{\zeta}^2\,F_{3,0,1,2}(v) + z^3\overline{\zeta}^3\,F_{3,0,0,3}(v)\Big\} \\ & + z^3\overline{z}^3\,\mathrm{O}_{z,\overline{z}}(1) + \overline{z}^3\zeta\,\mathrm{O}_{z,\zeta,\overline{z}}(3) + z^3\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(3) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\overline{z}}(3)\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2). \end{aligned}$$

Furthermore, the map exists and is unique *if it is assumed to be of the form*

$$\begin{aligned} z' \;&:=\; z + f_{\geqslant 2}(z,\zeta,w) \quad \zeta' \;:=\; \zeta + g_{\geqslant 1}(z,\zeta,w), \qquad w' \;:=\; w + h_{\geqslant 3}(z,\zeta,w), \\ 0 \;&=\; f_w(0), \qquad\qquad 0 \;=\; \mathrm{Im}\, h_{ww}(0). \end{aligned}$$

Equivalently, writing

$$u \;=\; F \;=\; \sum_{a,b,c,d\geqslant 0} z^a\zeta^b\overline{z}^c\overline{\zeta}^d\,F_{a,b,c,d}(v),$$

the normal form is defined by the general *prenormalization conditions*

$$\begin{aligned} 0 \;&\equiv F_{a,b,0,0}(v) \equiv\; F_{0,0,c,d}(v), \\ 0 \;&\equiv F_{a,b,1,0}(v) \;\equiv\; F_{1,0,c,d}(v), \\ 0 \;&\equiv F_{a,b,2,0}(v) \;\equiv\; F_{2,0,c,d}(v), \end{aligned}$$

with the obvious two exceptions $F_{1,0,1,0}(v) \equiv 1$ and $F_{0,1,2,0}(v) \equiv \frac{1}{2} \equiv F_{2,0,0,1}(v)$, together with the *sporadic normalization conditions*, listed by increasing order 4, 5, 6

$$\begin{aligned} 0 \;&\equiv\; F_{3,0,0,1}(v) \;\equiv\; F_{0,1,3,0}(v), \\ 0 \;&\equiv\; F_{4,0,0,1}(v) \;\equiv\; F_{0,1,4,0}(v), \qquad & 0 \;&\equiv\; F_{3,0,1,1}(v) \;\equiv\; F_{1,1,3,0}(v), \\ 0 \;&\equiv\; F_{4,0,1,1}(v) \;\equiv\; F_{1,1,4,0}(v), \qquad & 0 \;&\equiv\; F_{3,0,3,0}(v). \end{aligned}$$

Without the above conditions $z' = z + f_{\geqslant 2}$, $\zeta' = \zeta + g_{\geqslant 1}$, $w' = w + h_{\geqslant 3}$ guaranteeing uniqueness, one can verify that a normalizing transformation is unique up to the right action of the 5-dimensional stability group of the

Gaussier-Merker model having the finite equations

$$
\begin{aligned}
z' &:= \lambda \frac{z + i\,\alpha\, z^2 + \big(i\,\alpha\,\zeta - i\,\overline{\alpha}\big)\, w}{1 + 2i\,\alpha\, z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,r\big)\, w},\\
\zeta' &:= \frac{\lambda}{\overline{\lambda}} \frac{\zeta + 2i\,\overline{\alpha}\, z - \big(\alpha\overline{\alpha} + i\,r\big)\, z^2 + \big(\overline{\alpha}^2 - i\,r\,\zeta - \alpha\overline{\alpha}\,\zeta\big)\, w}{1 + 2i\,\alpha\, z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,r\big)\, w},\\
w' &:= \lambda\overline{\lambda} \frac{w}{1 + 2i\,\alpha\, z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,r\big)\, w},
\end{aligned}
$$

where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$ are arbitrary.

Lastly, the values at the origin of Pocchiola's two primary Cartan-type relative differential invariants are

$$
W_0 \;=\; 4\,\overline{F_{3,0,0,2}(0)} \qquad\qquad \text{and} \qquad\qquad J_0 \;=\; 20\,F_{5,0,0,1}(0).
$$

However, Poincaré-Moser normal forms or Cartan-Tanaka reductions to $\{e\}$-structures are only a *preliminary* towards the understanding of the biholomorphic equivalence problem for embedded $\mathcal{C}^\omega$ CR submanifolds $M \subset \mathbb{C}^N$, quite far from any resolution, not even to be termed 'complete resolution'.

Indeed, focusing on CR geometry, we would like to indicate two 'defects' of Poincaré-Moser normal forms in comparison to Cartan-Tanaka principal bundles.

• Moser-type CR normal forms are in fact *incomplete* in the sense that their invariants are only *relative*, yet defined up to the action of a certain ambiguity (isotropy) group.

• Moser-type CR normal forms hold only at one point, hence are incapable to fully characterize flatness as Cartan's method *does*.

The main reason why Cartan's method is more powerful is that it embraces computations *at every point* of a given manifold. Objects manipulated by Cartan's thought are (often very complicated) rational differential expressions in partial derivatives of fundamental (graphing) functions. In comparison, objects manipulated by Moser's method are only plain Taylor coefficients, hence computations are *much more elementary*.

Fortunately, it is known that symmetries of a hypersurface can be read off from subsequently constructed *deeper* normal forms, not touched in the present paper, but forthcoming.

These comments conduct us to at least formulate and raise a certain number of questions showing that several mysteries remain.

Q① How to get rid of ambiguity in Moser CR-normal forms? What are the true (absolute) differential invariants? Can one retrieve Pocchiola's dimension drop $10 \downarrow 5$? Can one link Moser's punctual invariants with Cartan's invariants at every point?

Q② In all possibly existing branches, how to find a minimal set of generators for the differential algebra of absolute differential invariants? Using either Moser's or Cartan's method?

Q③ In each branch, what are the differential relations (syzygies) between differential invariants?

Q④ How to implement the determination of CR-homogeneous models beyond naive Taylor series manipulations at only one point? How to employ the theory of Lie? How to view Cartan's invariants in a Taylor series? (Recent results appear in [17].)

Q⑤ How to implement, from Moser's side of the bridge, any sub-branch assumption that requires that an ideal of differential invariants, or a collection of Taylor coefficients, vanish (identically)?

Acknowledgments. Zhangchi Chen provided the Maple figures of Sections 8 and 9.

This article provides an alternative approach to a result announced in the preprint [26], which appeared before on arxiv.org, and which was quite inspiring. But in contrast:

A① Analogs of Cartan-Moser chains will be 'discovered from scratch' by applying a method due to Lie, similarly as in [32].

A② Complete, rigorous, detailed technical arguments for the (most delicate) convergence part will be patiently provided.

A③ The standard order of [26]: '1 formal', '2 convergent', will be inverted, the main body of the paper building a convergent normal form, while leaving (easier) formal considerations to a uniqueness statement at the end.

2. $\mathfrak{C}_{2,1}$ hypersurfaces $M^5 \subset \mathbb{C}^3$

Our object of study is the collection of all real $\mathcal{C}^\omega$ hypersurfaces $M^5 \subset \mathbb{C}^3$ whose Levi form is of constant rank 1 at every point and that are everywhere 2-nondegenerate (*see* below), a *class* that we will denote as $\mathfrak{C}_{2,1}$.

Pick any point $p \in M$ and adapt affine holomorphic coordinates $\big(z, \zeta, w = u + i\,v\big) \in \mathbb{C}^3$ in which p is the origin, so that $T_0M \oplus \mathbb{R}_u = \mathbb{C}^3$. From any $\mathcal{C}^\omega$ real defining equation for M near p, the analytic implicit function theorem enables to solve for u as

$$u \,=\, F\big(z, \zeta, \overline{z}, \overline{\zeta}, v\big),$$

for some $\mathcal{C}^\omega$ graphing function F, the *core object* of our study. This F is expandable in converging power series as

$$F\big(z, \zeta, \overline{z}, \overline{\zeta}, v\big) \,=\, \sum_{a+b+c+d+e \geqslant 1} F_{a,b,c,d,e}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d\, v^e,$$

for some infinite collection of complex coefficients $F_{a,b,c,d,e} \in \mathbb{C}$. Then by conjugating only complex coefficients, *define*

$$\overline{F}(z,\zeta,\overline{z},\overline{\zeta},v) \ := \sum_{a+b+c+d+e\geqslant 1} \overline{F}_{a,b,c,d,e}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d v^e.$$

The reality $\overline{u} = u$ forces $\overline{F(z,\zeta,\overline{z},\overline{\zeta},v)} = F(z,\zeta,\overline{z},\overline{\zeta},v)$, that is

$$\overline{F}(\overline{z},\overline{\zeta},z,\zeta,v) \ \equiv \ F\big(z,\zeta,\overline{z},\overline{\zeta},v\big). \tag{2.1}$$

Applying $\frac{1}{a!}\partial_z^a \frac{1}{b!}\partial_\zeta^b \frac{1}{c!}\partial_{\overline{z}}^c \frac{1}{d!}\partial_{\overline{\zeta}}^d \frac{1}{e!}\partial_v^e$ at the origin $(0,0,0,0,0)$, we obtain the (known) condition on the $F_{a,b,c,d,e} \in \mathbb{C}$ which guarantees reality of the graphing function

$$\overline{F_{c,d,a,b,e}} \ = \ F_{a,b,c,d,e}.$$

Later, we will expand F in powers of $(z,\zeta,\overline{z},\overline{\zeta})$ only, by introducing

$$F\big(z,\zeta,\overline{z},\overline{\zeta},v\big) \ = \sum_{a,b,c,d} z^a \zeta^b \overline{z}^c \overline{\zeta}^d \sum_e F_{a,b,c,d,e}\, v^e \ =: \ \sum_{a,b,c,d} z^a \zeta^b \overline{z}^c \overline{\zeta}^d \, F_{a,b,c,d}(v).$$

The reality of F is then equivalent to

$$\overline{F_{c,d,a,b}(v)} \ = \ F_{a,b,c,d}(v). \tag{2.2}$$

In the literature [20, 21, 30, 15, 37, 22, 28, 29, 35, 34, 33, 16, 19], several equivalent definitions of the class $\mathfrak{C}_{2,1}$ exist. We propose a computational formulation of the two concepts of constant Levi rank 1 and of 2-nondegeneracy, already shown in [8] when M is *rigid*, namely when F is independent of v.

For this, we need the *complex graphed representation* of any $\mathcal{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$

$$w \ = \ Q\big(z,\zeta,\overline{z},\overline{\zeta},\overline{w}\big),$$

with a $\mathbb{C}$-valued analytic function Q which is obtained by solving for w in $\frac{w+\overline{w}}{2} = F\big(z,\zeta,\overline{z},\overline{\zeta},\frac{w-\overline{w}}{2i}\big)$, so that

$$\tfrac{1}{2}\, Q\big(z,\zeta,\overline{z},\overline{\zeta},\overline{w}\big) + \tfrac{1}{2}\,\overline{w} \ \equiv \ F\Big(z,\zeta,\overline{z},\overline{\zeta},\ \tfrac{1}{2i}\, Q\big(z,\zeta,\overline{z},\overline{\zeta},\overline{w}\big) - \tfrac{1}{2i}\,\overline{w}\Big).$$

Such an analytic function Q cannot be arbitrary, it must satisfy a compatibility condition obtained by replacing $\overline{w} := \overline{Q}$ in its last argument

$$w \ \equiv \ Q\Big(z,\zeta,\overline{z},\overline{\zeta},\ \overline{Q}(\overline{z},\overline{\zeta},z,\zeta,w)\Big).$$

3. Two invariant determinants

A local biholomorphism

$$(z,\zeta,w) \ \longmapsto \ \Big(f(z,\zeta,w),\, g(z,\zeta,w),\, h(z,\zeta,w)\Big) =: \ (z',\zeta',w'),$$

has nowhere vanishing holomorphic Jacobian determinant

$$0 \neq \begin{vmatrix} f_z & g_z & h_z \\ f_\zeta & g_\zeta & h_\zeta \\ f_w & g_w & h_w \end{vmatrix}.$$

Suppose that it makes a biholomorphism between two $\mathcal{C}^\omega$ hypersurfaces both represented by complex graphing functions

$$w = Q(z,\zeta,\overline{z},\overline{\zeta},\overline{w}) \qquad \text{and} \qquad w' = Q'(z',\zeta',\overline{z}',\overline{\zeta}',\overline{w}').$$

Plugging the three components of the biholomorphism in the target equation, we get the so-called *fundamental identity*

$$h(z,\zeta,w) = Q'\Big(f(z,\zeta,w),\, g(z,\zeta,w),\, \overline{f}(\overline{z},\overline{\zeta},\overline{w}),\, \overline{g}(\overline{z},\overline{\zeta},\overline{w}),\, \overline{h}(\overline{z},\overline{\zeta},\overline{w})\Big)\Big|_{w=Q(z,\zeta,\overline{z},\overline{\zeta},\overline{w})},$$

which holds identically in the ring of converging power series $\mathbb{C}\{z,\zeta,\overline{z},\overline{\zeta},\overline{w}\}$.

By differentiating this identity (exercise!), one may express the invariancy of the Levi form as a relation between the two Levi determinants defined as

$$\begin{vmatrix} Q_{\overline{z}} & Q_{\overline{\zeta}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{\zeta}} & Q_{z\overline{w}} \\ Q_{\zeta\overline{z}} & Q_{\zeta\overline{\zeta}} & Q_{\zeta\overline{w}} \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} Q'_{\overline{z}'} & Q'_{\overline{\zeta}'} & Q'_{\overline{w}'} \\ Q'_{z'\overline{z}'} & Q'_{z'\overline{\zeta}'} & Q'_{z'\overline{w}'} \\ Q'_{\zeta'\overline{z}'} & Q'_{\zeta'\overline{\zeta}'} & Q'_{\zeta'\overline{w}'} \end{vmatrix}.$$

Indeed, abbreviate

$$\mathcal{L}_z := \frac{\partial}{\partial z} + Q_z(z,\zeta,\overline{z},\overline{\zeta},\overline{w})\,\frac{\partial}{\partial w} \qquad \text{and} \qquad \mathcal{L}_\zeta := \frac{\partial}{\partial \zeta} + Q_\zeta(z,\zeta,\overline{z},\overline{\zeta},\overline{w})\,\frac{\partial}{\partial w}.$$

Proposition 3.1. *Through any biholomorphism between real hypersurfaces* $\{w = Q\} \subset \mathbb{C}^3$ *and* $\{w' = Q'\} \subset \mathbb{C}'^3$*, one has*

$$\begin{vmatrix} Q'_{\overline{z}'} & Q'_{\overline{\zeta}'} & Q'_{\overline{w}'} \\ Q'_{z'\overline{z}'} & Q'_{z'\overline{\zeta}'} & Q'_{z'\overline{w}'} \\ Q'_{\zeta'\overline{z}'} & Q'_{\zeta'\overline{\zeta}'} & Q'_{\zeta'\overline{w}'} \end{vmatrix} = \frac{\begin{vmatrix} f_z & f_\zeta & f_w \\ g_z & g_\zeta & g_w \\ h_z & h_\zeta & h_w \end{vmatrix}^3}{\begin{vmatrix} \overline{f}_{\overline{z}} & \overline{f}_{\overline{\zeta}} & \overline{f}_{\overline{w}} \\ \overline{g}_{\overline{z}} & \overline{g}_{\overline{\zeta}} & \overline{g}_{\overline{w}} \\ \overline{h}_{\overline{z}} & \overline{h}_{\overline{\zeta}} & \overline{h}_{\overline{w}} \end{vmatrix}^1} \frac{1}{\begin{vmatrix} \mathcal{L}_z(f) & \mathcal{L}_\zeta(f) \\ \mathcal{L}_z(g) & \mathcal{L}_\zeta(g) \end{vmatrix}^4} \begin{vmatrix} Q_{\overline{z}} & Q_{\overline{\zeta}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{\zeta}} & Q_{z\overline{w}} \\ Q_{\zeta\overline{z}} & Q_{\zeta\overline{\zeta}} & Q_{\zeta\overline{w}} \end{vmatrix}.$$

Consequently, the property that the Levi form is of constant rank 1 is biholomorphically invariant. The 2-nondegeneracy property [34] then expresses as the nonvanishing of

$$\begin{vmatrix} Q_{\overline{z}} & Q_{\overline{\zeta}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{\zeta}} & Q_{z\overline{w}} \\ Q_{zz\overline{z}} & Q_{zz\overline{\zeta}} & Q_{zz\overline{w}} \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} Q'_{\overline{z}'} & Q'_{\overline{\zeta}'} & Q'_{\overline{w}'} \\ Q'_{z'\overline{z}'} & Q'_{z\overline{\zeta}'} & Q'_{z'\overline{w}'} \\ Q'_{z'z'\overline{z}'} & Q'_{z'z'\overline{\zeta}'} & Q'_{z'z'\overline{w}'} \end{vmatrix}.$$

Proposition 3.2. *When the Levi form is of constant rank* 1*, through any biholomorphism between real hypersurfaces* $\{w = Q\} \subset \mathbb{C}^3$ *and* $\{w' = Q'\} \subset \mathbb{C}'^3$*, one has*

$$\frac{\begin{vmatrix} Q'_{\overline{z}} & Q'_{\overline{\zeta}} & Q'_{\overline{w}} \\ Q'_{z'\overline{z}'} & Q'_{z'\overline{\zeta}'} & Q'_{z'\overline{w}'} \\ Q'_{z'z'\overline{z}'} & Q'_{z'z'\overline{\zeta}'} & Q'_{z'z'\overline{w}'} \end{vmatrix}}{\begin{vmatrix} Q_{\overline{z}} & Q_{\overline{\zeta}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{\zeta}} & Q_{z\overline{w}} \\ Q_{zz\overline{z}} & Q_{zz\overline{\zeta}} & Q_{zz\overline{w}} \end{vmatrix}} = \frac{\begin{vmatrix} f_z & f_\zeta & f_w \\ g_z & g_\zeta & g_w \\ h_z & h_\zeta & h_w \end{vmatrix}^3}{\begin{vmatrix} \overline{f}_{\overline{z}} & \overline{f}_{\overline{\zeta}} & \overline{f}_{\overline{w}} \\ \overline{g}_{\overline{z}} & \overline{g}_{\overline{\zeta}} & \overline{g}_{\overline{w}} \\ \overline{h}_{\overline{z}} & \overline{h}_{\overline{\zeta}} & \overline{h}_{\overline{w}} \end{vmatrix}^1} \cdot$$
$$\cdot \frac{\left(\mathcal{L}_\zeta(g) \begin{vmatrix} Q_{\overline{z}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{w}} \end{vmatrix} - \mathcal{L}_z(g) \begin{vmatrix} Q_{\overline{z}} & Q_{\overline{w}} \\ Q_{\zeta\overline{z}} & Q_{\zeta\overline{w}} \end{vmatrix} \right)^3}{\begin{vmatrix} \mathcal{L}_z(f) & \mathcal{L}_\zeta(f) \\ \mathcal{L}_z(g) & \mathcal{L}_\zeta(g) \end{vmatrix}^6 \begin{vmatrix} Q_{\overline{z}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{w}} \end{vmatrix}^3}.$$

The identities of these two propositions can be verified *e.g.* on a computer.

Recall that we denote the class of (local) hypersurfaces $M^5 \subset \mathbb{C}^3$ passing through the origin $0 \in M$ that are 2-nondegenerate and whose Levi form has constant rank 1 as $\mathfrak{C}_{2,1}$.

Repeatedly, we shall use the real expression of the *Levi determinant*

$$\mathsf{Levi}(F) := \begin{vmatrix} 0 & F_z & F_\zeta & -\frac{1}{2} + \frac{1}{2i} F_v \\ F_{\overline{z}} & F_{z\overline{z}} & F_{\zeta\overline{z}} & \frac{1}{2i} F_{\overline{z}v} \\ F_{\overline{\zeta}} & F_{z\overline{\zeta}} & F_{\zeta\overline{\zeta}} & \frac{1}{2i} F_{\overline{\zeta}v} \\ -\frac{1}{2} - \frac{1}{2i} F_v & -\frac{1}{2i} F_{zv} & -\frac{1}{2i} F_{\zeta v} & \frac{1}{4} F_{vv} \end{vmatrix}. \tag{3.1}$$

The next (known) statement applies to $\rho := -u + F$.

Lemma 3.3. [18] *If* $M^5 \subset \mathbb{C}^3$ *is implicitly defined by* $\rho(z, \zeta, w, \overline{z}, \overline{\zeta}, \overline{w}) = 0$ *with a* $\mathcal{C}^\omega$ *real function* $\rho = \overline{\rho}$ *satisfying* $\rho_w \neq 0$*, and if* $w = Q(z, \zeta, \overline{z}, \overline{\zeta}, \overline{w})$ *is its associated complex graphing function, then*

$$\begin{vmatrix} 0 & \rho_z & \rho_\zeta & \rho_w \\ \rho_{\overline{z}} & \rho_{z\overline{z}} & \rho_{\zeta\overline{z}} & \rho_{w\overline{z}} \\ \rho_{\overline{\zeta}} & \rho_{z\overline{\zeta}} & \rho_{\zeta\overline{\zeta}} & \rho_{w\overline{\zeta}} \\ \rho_{\overline{w}} & \rho_{z\overline{w}} & \rho_{\zeta\overline{w}} & \rho_{w\overline{w}} \end{vmatrix} = \rho_w^4 \begin{vmatrix} Q_{\overline{z}} & Q_{\overline{\zeta}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{\zeta}} & Q_{z\overline{w}} \\ Q_{\zeta\overline{z}} & Q_{\zeta\overline{\zeta}} & Q_{\zeta\overline{w}} \end{vmatrix}.$$

We leave as an exercise to find some invariant determinant expressed in terms of F which corresponds to the 2-nondegeneracy determinant of Proposition 3.2 in terms of Q.

4. Infinitesimal CR automorphisms

In the class $\mathfrak{C}_{2,1}$, the appropriate homogeneous model, named M_{LC}, was set up by Gaussier-Merker in [21] and Fels-Kaup in [14], *see* also [8]

$$M_{\mathsf{LC}}: \qquad u \,=\, \frac{z\overline{z} + \frac{1}{2}\, z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1-\zeta\overline{\zeta}} \;=:\; m\big(z,\zeta,\overline{z},\overline{\zeta}\big).$$

The letter m here stands for $m$odel.

The 10-dimensional simple Lie algebra of its infinitesimal CR automorphisms

$$\mathfrak{g} \,:=\, \mathfrak{aut}_{CR}\big(M_{\mathsf{LC}}\big) \,\cong\, \mathfrak{so}_{2,3}(\mathbb{R}),$$

has 10 natural generators $X_1, \dots, X_{10}$, which are $(1,0)$ vector fields in $\mathbb{C}^3$ having holomorphic coefficients with $X_\sigma + \overline{X}_\sigma$ tangent to M_{LC}.

It is natural to assign the following weights to variables and to vector fields

$$\begin{aligned} [z] \,&:=\, 1 \qquad & [\zeta] \,&:=\, 0, \qquad & [w] \,&:=\, 2, \\ \big[\partial_z\big] \,&:=\, -1, \qquad & \big[\partial_\zeta\big] \,&:=\, 0, \qquad & \big[\partial_w\big] \,&:=\, -2. \end{aligned} \tag{4.1}$$

The Lie algebra $\mathfrak{g} = \mathfrak{aut}_{CR}(M_{\mathsf{LC}})$ can be graded as

$$\mathfrak{g} \,=\, \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where, as shown in [21, 19]

$$\begin{aligned} \mathfrak{g}_{-2} \,&:=\, \mathsf{Span}\big\{i\,\partial_w\big\}, \\ \mathfrak{g}_{-1} \,&:=\, \mathsf{Span}\big\{(\zeta-1)\,\partial_z - 2z\,\partial_w, \quad (i+i\zeta)\,\partial_z - 2iz\,\partial_w\big\}, \end{aligned}$$

where $\mathfrak{g}_0 = \mathfrak{g}_0^{\mathsf{trans}} \oplus \mathfrak{g}_0^{\mathsf{iso}}$

$$\begin{aligned} \mathfrak{g}_0^{\mathsf{trans}} \,&:=\, \mathsf{Span}\Big\{z\zeta\,\partial_z + (\zeta^2-1)\,\partial_\zeta - z^2\,\partial_w, \quad iz\zeta\,\partial_z + (i+i\zeta^2)\,\partial_\zeta - iz^2\,\partial_w\Big\}, \\ \mathfrak{g}_0^{\mathsf{iso}} \,&:=\, \mathsf{Span}\big\{z\,\partial_z + 2w\,\partial_w, \quad iz\,\partial_z + 2i\zeta\,\partial_\zeta\big\}, \end{aligned}$$

while

$$\begin{aligned} \mathfrak{g}_1 \,&:=\, \mathsf{Span}\big\{\big(z^2 - \zeta w - w\big)\,\partial_z + \big(2z\zeta + 2z\big)\,\partial_\zeta + 2zw\,\partial_w, \\ &\qquad\qquad \big(-iz^2 + i\zeta w - iw\big)\,\partial_z + \big(-2iz\zeta + 2iz\big)\,\partial_\zeta - 2izw\,\partial_w\big\}, \\ \mathfrak{g}_2 \,&:=\, \mathsf{Span}\big\{izw\,\partial_z - iz^2\,\partial_\zeta + iw^2\,\partial_w\big\}. \end{aligned}$$

Calling these $X_1, \dots, X_{10}$ in order of appearance, the five $X_\sigma + \overline{X}_\sigma$ for $\sigma = 1,2,3,4,5$ span TM^5 while those for $\sigma = 6,7,8,9,10$ generate the isotropy subgroup of the origin.

In fact, we will use the alternative names for the 5 generators of the isotropy subroup

$$\begin{aligned}
\mathsf{D} &:= z\,\partial_z + 2w\,\partial_w,\\
\mathsf{R} &:= iz\,\partial_z + 2i\,\zeta\,\partial_\zeta,\\
\mathsf{I}_1 &:= \big(z^2 - \zeta w - w\big)\,\partial_z + \big(2\,z\zeta + 2\,z\big)\,\partial_\zeta + 2\,zw\,\partial_w,\\
\mathsf{I}_2 &:= \big(-i\,z^2 + i\,\zeta w - i\,w\big)\,\partial_z + \big(-2i\,z\zeta + 2i\,z\big)\,\partial_\zeta - 2i\,zw\,\partial_w,\\
\mathsf{J} &:= i\,zw\,\partial_z - i\,z^2\,\partial_\zeta + i\,w^2\,\partial_w,
\end{aligned}$$

having commutator table

	D	R	I_1	I_2	J
D	0	0	I_1	I_2	$2\,\mathsf{J}$
R	$*$	0	$-\mathsf{I}_2$	I_1	0
I_1	$*$	$*$	0	$4\,\mathsf{J}$	0
I_2	$*$	$*$	$*$	0	0
J	$*$	$*$	$*$	$*$	0

5. Fractional representation of the isotropy group

By integrating iterated flows of D, R, I_1, I_2, J, it can be shown (exercise) that the isotropy subgroup of the origin $0 \in M_{\mathsf{LC}}$ in the Gaussier-Merker model has the finite equations

$$\begin{aligned}
z' &:= \lambda\,\frac{z + i\,\alpha\,z^2 + \big(i\,\alpha\,\zeta - i\,\overline{\alpha}\big)\,w}{1 + 2i\,\alpha\,z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,r\big)\,w},\\
\zeta' &:= \frac{\lambda}{\overline{\lambda}}\,\frac{\zeta + 2i\,\overline{\alpha}\,z - \big(\alpha\overline{\alpha} + i\,r\big)\,z^2 + \big(\overline{\alpha}^2 - i\,r\,\zeta - \alpha\overline{\alpha}\,\zeta\big)\,w}{1 + 2i\,\alpha\,z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,r\big)\,w},\\
w' &:= \lambda\overline{\lambda}\,\frac{w}{1 + 2i\,\alpha\,z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,r\big)\,w},
\end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$ are arbitrary.

With the alternative weights

$$[z] := 1, \quad [y] := 1, \quad [w] := 2,$$

the Taylor expansions up to respective weighted orders 5, 4, 6, will soon be useful

$$\begin{aligned}
z' \;=\;& \lambda\,z\\
&- i\,\lambda\alpha\,z^2 - i\,\lambda\,\overline{\alpha}\,w\\
&- \lambda\alpha^2\,z^3 + \big(-3\,\lambda\alpha\overline{\alpha} + i\,\lambda r\big)\,zw + i\,\lambda\alpha\,\zeta w\\
&+ i\,\lambda\alpha^3\,z^4 + \big(6i\,\lambda\alpha^2\overline{\alpha} + 3\,\lambda\alpha r\big)\,z^2 w + \big(\lambda\overline{\alpha}r + i\,\lambda\overline{\alpha}^2\alpha\big)\,w^2 + 3\,\lambda\alpha^2\,z\zeta w\\
&+ \lambda\alpha^4\,z^5 + \big(-6i\,\alpha^2\lambda r + 10\,\lambda\alpha^3\overline{\alpha}\big)\,z^3 w - 6i\,\lambda\alpha^3\,z^2\zeta w\\
&\qquad + \big(5\,\lambda\alpha^2\overline{\alpha}^2 - 6i\,\lambda\alpha\overline{\alpha}r - \lambda r^2\big)\,zw^2 + \big(-2i\,\lambda\alpha^2\overline{\alpha} - \lambda\alpha r\big)\,\zeta w^2,
\end{aligned}$$

$$
\begin{aligned}
\zeta' \;=\; & 2i\,\frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}\,z + \frac{\lambda}{\overline{\lambda}}\,\zeta \\
& + \Big(- i\,\frac{\lambda}{\overline{\lambda}}\,r + 3\,\frac{\lambda}{\overline{\lambda}}\,\alpha\overline{\alpha}\Big)\,z^2 - 2i\,\frac{\lambda}{\overline{\lambda}}\,\alpha\,z\zeta + \frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}^2\,w \\
& + \Big(- 4i\,\frac{\lambda}{\overline{\lambda}}\,\alpha^2\overline{\alpha} - 2\,\frac{\lambda}{\overline{\lambda}}\,\alpha r\Big)\,z^3 - 3\,\frac{\lambda}{\overline{\lambda}}\,\alpha^2\,z^2\zeta + \Big(- 4i\,\frac{\lambda}{\overline{\lambda}}\,\alpha\overline{\alpha}^2 - 2\,\frac{\lambda}{\overline{\lambda}}\,\overline{\alpha} r\Big)\,zw - 2\,\frac{\lambda}{\overline{\lambda}}\,\alpha\overline{\alpha}\,\zeta w \\
& + \Big(- 5\,\frac{\lambda}{\overline{\lambda}}\,\alpha^3\overline{\alpha} + 3i\,\frac{\lambda}{\overline{\lambda}}\,\alpha^2 r\Big)\,z^4 + 4i\,\frac{\lambda}{\overline{\lambda}}\,\alpha^3\,z^3\zeta + \Big(- 10\,\frac{\lambda}{\overline{\lambda}}\,\alpha^2\overline{\alpha}^2 + 8i\,\frac{\lambda}{\overline{\lambda}}\,\alpha\overline{\alpha} r + \frac{\lambda}{\overline{\lambda}}\,r^2\Big)\,z^2 w \\
& \qquad + \Big(8i\,\frac{\lambda}{\overline{\lambda}}\,\alpha^2\overline{\alpha} + 2\,\frac{\lambda}{\overline{\lambda}}\,\alpha r\Big)\,z\zeta w + \frac{\lambda}{\overline{\lambda}}\,\alpha^2\,\zeta^2 w + \Big(i\,\frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}^2 r - \frac{\lambda}{\overline{\lambda}}\,\alpha\overline{\alpha}^3\Big)\,w^2, \\
w' \;=\; & 0 \\
& + \lambda\overline{\lambda}\,w \\
& - 2i\,\lambda\overline{\lambda}\,\alpha\,zw \\
& - 3\,\lambda\overline{\lambda}\alpha^2\,z^2 w + \big(i\,\lambda\overline{\lambda} r - \lambda\overline{\lambda}\alpha\overline{\alpha}\big)\,w^2 \\
& + 4i\,\lambda\overline{\lambda}\alpha^3\,z^3 + \big(4i\,\lambda\overline{\lambda}\alpha^2\overline{\alpha} + 4\,\lambda\overline{\lambda}\,\alpha r\big)\,zw^2 + \lambda\overline{\lambda}\alpha^2\,\zeta w^2 \\
& + 5\,\lambda\overline{\lambda}\alpha^4\,z^4 w + \big(10\,\lambda\overline{\lambda}\alpha^3\overline{\alpha} - 10i\,\lambda\overline{\lambda}\,\alpha^2 r\big)\,z^2 w^2 - 4i\,\lambda\overline{\lambda}\alpha^3\,z\zeta w^2 \\
& \qquad + \big(- \lambda\overline{\lambda} r^2 - 2i\,\lambda\overline{\lambda}\alpha\overline{\alpha} r + \lambda\overline{\lambda}\alpha^2\overline{\alpha}^2\big)\,w^3.
\end{aligned}
$$

6. Lie jet theory

To apply Lie's theory similarly as in [32], we must work with the five *intrinsic*, *real*, coordinates (x, y, s, t, v) on M^5, where

$$z \;=\; x + i\,y, \qquad\qquad \zeta \;=\; s + i\,t, \qquad\qquad w \;=\; u + i\,v.$$

As in [32], we consider parametrized local real $\mathcal{C}^\omega$ curves passing by the origin

$$\tau \;\longmapsto\; \big(x(\tau), y(\tau), s(\tau), t(\tau),\, \tau\big).$$

with $v(\tau) \equiv \tau$ guaranteeing that the curve is *not* CR-tangential. We then use the parameter-letter v instead of τ.

The eight independent coordinates corresponding to $\dot{x}(v)$, $\dot{y}(v)$, $\dot{s}(v)$, $\dot{t}(v)$, $\ddot{x}(v)$, $\ddot{y}(v)$, $\ddot{s}(v)$, $\ddot{t}(v)$ will be denoted

$$\big(v,\, x, y, s, t,\, x_1, y_1, s_1, t_1,\, x_2, y_2, s_2, t_2\big).$$

The first jet space is $J^1_{1,4} \equiv \mathbb{R}^{1+4+4}$, and the second jet space is $J^2_{1,4} \equiv \mathbb{R}^{1+4+4+4}$.

Any diffeomorphism $(v, x, y, s, t) \longmapsto (v', x', y', s', t')$ lifts to jet spaces of any order. Because the formulas rapidly become complicated [40, 30, 9], Lie linearized the action of diffeomorphisms.

As in [32], we will *apply* Lie's formulas. Start from a general vector field

$$
\begin{aligned}
\vec{\mathbf{v}} \;:=\; & \xi(v, x, y, s, t)\,\frac{\partial}{\partial v} + \varphi(v, x, y, s, t)\,\frac{\partial}{\partial x} + \psi(v, x, y, s, t)\,\frac{\partial}{\partial y} \\
& \qquad + \lambda(v, x, y, s, t)\,\frac{\partial}{\partial s} + \mu(v, x, y, s, t)\,\frac{\partial}{\partial t}.
\end{aligned}
$$

Introduce the *total differentiation operator*

$$\begin{aligned}\mathsf{D}_v \;:=\; &\frac{\partial}{\partial v} + x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + s_1\frac{\partial}{\partial s} + t_1\frac{\partial}{\partial t}\\ &+ x_2\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial y_1} + s_2\frac{\partial}{\partial s_1} + t_2\frac{\partial}{\partial t_1}\\ &+ x_3\frac{\partial}{\partial x_2} + y_3\frac{\partial}{\partial y_2} + s_3\frac{\partial}{\partial s_2} + t_3\frac{\partial}{\partial t_2}.\end{aligned}$$

Then the second prolongation of $\vec{\mathbf{v}}$

$$\begin{aligned}\vec{\mathbf{v}}^{(2)} \;=\; &\vec{\mathbf{v}} + \varphi_1\frac{\partial}{\partial x_1} + \psi_1\frac{\partial}{\partial y_1} + \lambda_1\frac{\partial}{\partial s_1} + \mu_1\frac{\partial}{\partial t_1}\\ &+ \varphi_2\frac{\partial}{\partial x_2} + \psi_2\frac{\partial}{\partial y_2} + \lambda_2\frac{\partial}{\partial s_2} + \mu_2\frac{\partial}{\partial t_2},\end{aligned}$$

has coefficients ([31, 27, 40, 30, 41, 9])

$$\begin{aligned}\varphi_1 &:= \mathsf{D}_v(\varphi - \xi\, x_1) + \xi\, x_2, \qquad \psi_1 := \mathsf{D}_v(\psi - \xi\, y_1) + \xi\, y_2,\\ \lambda_1 &:= \mathsf{D}_v(\lambda - \xi\, s_1) + \xi\, s_2, \qquad \mu_1 := \mathsf{D}_v(\mu - \xi\, t_1) + \xi\, t_2,\\ \varphi_2 &:= \mathsf{D}_v\mathsf{D}_v(\varphi - \xi\, x_1) + \xi\, x_3, \quad \psi_2 := \mathsf{D}_v\mathsf{D}_v(\psi - \xi\, y_1) + \xi\, y_3,\\ \lambda_2 &:= \mathsf{D}_v\mathsf{D}_v(\lambda - \xi\, s_1) + \xi\, s_3, \quad \mu_2 := \mathsf{D}_v\mathsf{D}_v(\mu - \xi\, t_1) + \xi\, t_3.\end{aligned}$$

7. Intrinsic isotropy automorphisms of the Gaussier-Merker model

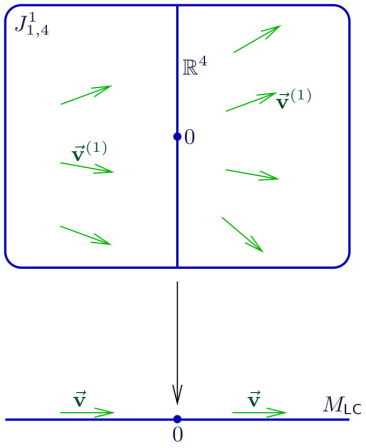
We want to apply Lie's prolongation formulas within the *first* jet space to our 5 vector fields $\mathsf{X} = \mathsf{D}, \mathsf{R}, \mathsf{I}_1, \mathsf{I}_2, \mathsf{J}$. But these *holomorphic* $(1,0)$ fields were *extrinsic*, defined in $\mathbb{C}^3$. We must therefore write up the five fields $\mathsf{X} + \overline{\mathsf{X}}$ in the *intrinsic* coordinates $(x, y, s, t, v) \in M^5_{\mathsf{LC}}$. By slight abuse, we keep the notation X instead of $\mathsf{X} + \overline{\mathsf{X}}$

$$\begin{aligned}\mathsf{D} \;&=\; x\,\partial_x + y\,\partial_y + 2v\,\partial_v,\\ \mathsf{R} \;&=\; -y\,\partial_x + x\,\partial_y - 2t\,\partial_s + 2s\,\partial_t,\\ \mathsf{I}_1 \;&=\; \Big[\Big(2\,x^2s^2 - 2\,y^2s^2 + 2\,y^2 + 2\,xyt + x^2t^2 - y^2t^2\\ &\qquad + 2\,xyst - tv + s^2tv + t^3v + 2\,x^2s\Big)\Big/\Big(-1 + s^2 + t^2\Big)\Big]\frac{\partial}{\partial x}\\ &\quad + \Big[\Big(-\,y^2t - x^2st - v + s^2v + t^2v - sv + s^3v + st^2v\\ &\qquad - x^2t - 4\,xyt^2 + y^2st + 2\,xy - 2\,xys^2\Big)\Big/\Big(1 - s^2 + t^2\Big)\Big]\frac{\partial}{\partial y}\\ &\qquad + \Big[2\,x - 2\,yt + 2\,xs\Big]\frac{\partial}{\partial s} + \Big[2\,y + 2\,ys + 2\,xt\Big]\frac{\partial}{\partial t}\\ &\quad + \Big[\Big(-\,4\,xy^2t - 2\,x^2ys - 2\,x^2y + 2\,y^3s - 2\,xv\\ &\qquad + 2\,xs^2v + 2\,xt^2v - 2\,y^3\Big)\Big/\Big(-1 + s^2 + t^2\Big)\Big]\frac{\partial}{\partial v},\end{aligned}$$

$$
\begin{aligned}
\mathtt{I}_2 \;=\; & \Big[\Big(-y^2t - x^2st - 4\,xyt^2 + y^2st - sv + s^3v \\
& + st^2v + 2\,xy - 2\,xys^2 + v - s^2v - t^2v - x^2t\Big)\Big/\Big(-1 + s^2 + t^2\Big)\Big]\frac{\partial}{\partial x} \\
& + \Big[\Big(-2\,x^2 + 2\,x^2s^2 + x^2t^2 - 2\,xyt - 2\,y^2s^2 - y^2t^2 \\
& \qquad + 2\,xyst - tv + s^2tv + t^3v + 2\,y^2s\Big)\Big/\Big(1 - s^2 + t^2\Big)\Big]\frac{\partial}{\partial y} \\
& + \Big[2\,xt - 2\,y + 2\,ys\Big]\frac{\partial}{\partial s} + \Big[-2\,xs + 2\,x + 2\,yt\Big]\frac{\partial}{\partial t} \\
& + \Big[\Big(-2\,xy^2s + 2\,xy^2 + 4\,x^2yt + 2\,x^3s \\
& \qquad + 2\,x^3 - 2\,yv + 2\,ys^2v + 2\,yt^2v\Big)\Big/\Big(-1 + s^2 + t^2\Big)\Big]\frac{\partial}{\partial v}, \\
\mathtt{J} \;=\; & \Big[\Big(-2\,xy^2t - x^2ys - x^2y + y^3s \\
& \qquad - xv + xs^2v + xt^2v - y^3\Big)\Big/\Big(-1 + s^2 + t^2\Big)\Big]\frac{\partial}{\partial x} \\
& + \Big[\Big(-xy^2s + xy^2 + 2\,x^2yt + x^3s \\
& \qquad + x^3 - yv + ys^2v + yt^2v\Big)\Big/\Big(1 - s^2 + t^2\Big)\Big]\frac{\partial}{\partial y} \\
& + \Big[2\,xy\Big]\frac{\partial}{\partial s} + \Big[-x^2 + y^2\Big]\frac{\partial}{\partial t} \\
& + \Big[\Big((v - s^2v - t^2v - x^2s - x^2 - 2\,xyt + y^2s - y^2)\cdot \\
& \cdot(-v + s^2v + t^2v - x^2 - x^2s - 2\,xyt + y^2s - y^2)\Big)\Big/\Big(\big(1 - s^2 - t^2\big)^2\Big)\Big]\frac{\partial}{\partial v}.
\end{aligned}
$$

8. Prolongation to the jet space of order 1

As said, we work above the origin $0 \in M_{\mathsf{LC}}$.

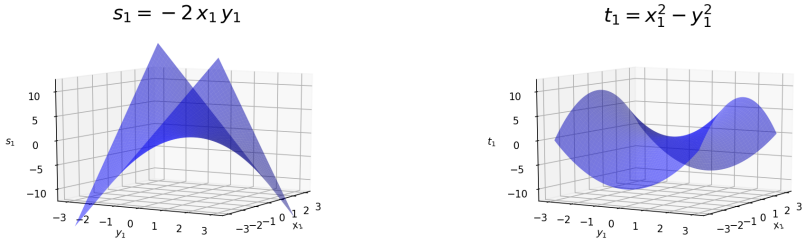


By Lie's theory, any vector field $\vec{\mathbf{v}}$ on the base M lifts as a vector field $\vec{\mathbf{v}}^{(1)}$ on the first jet space $J^1_{1,4} = \mathbb{R}^{1+4+4}$.

Because our five intrinsic vector fields D, R, I_1, I_2, J vanish at $v = x = y = s = t = 0$, their prolongations will automatically be tangent to the fiber $\{(0,0,0,0,0,x_1,y_1,s_1,t_1)\}$ above $(0,0,0,0,0)$ in the first jet space.

Lie's formulas yield the very simple values of these first prolongations above the origin $v = x = y = s = t = 0$

$$\begin{array}{cccccc} & \partial_{x_1} & \partial_{y_1} & \partial_{s_1} & \partial_{t_1} \\ \mathsf{D}^{(1)} & -x_1 & -y_1 & -2s_1 & -2t_1 \\ \mathsf{R}^{(1)} & -y_1 & x_1 & -2t_1 & 2s_1 \\ \mathsf{I}_1^{(1)} & 0 & -1 & 2x_1 & 2y_1 \\ \mathsf{I}_2^{(1)} & 1 & 0 & -2y_1 & 2y_1 \\ \mathsf{J}^{(1)} & 0 & 0 & 0 & 0 \end{array}$$



Observation 8.1. *On $\mathbb{R}^4 = \mathbb{R}^4_{x_1,y_1,s_1,t_1}$, there exists a unique 2-dimensional submanifold $\Sigma^1_0 \subset \mathbb{R}^4$, invariant under $\{\mathsf{D}^{(1)}, \mathsf{R}^{(1)}, \mathsf{I}_1^{(1)}, \mathsf{I}_2^{(1)}, \mathsf{J}^{(1)}\}$, algebraic, graphed as*

$$\begin{cases} s_1 &= -2\,x_1 y_1, \\ t_1 &= x_1^2 - y_1^2. \end{cases}$$

Moreover, the complement $\mathbb{R}^4 \setminus \Sigma^1_0$ is a unique (transitive) orbit under $\mathsf{D}^{(1)}$, $\mathsf{R}^{(1)}$, $\mathsf{I}_1^{(1)}$, $\mathsf{I}_2^{(1)}$, $\mathsf{J}^{(1)}$.

Proof. We can drop the fifth line of $\mathsf{J}^{(1)}$ containing only zeros. With a_1 and b_1 being parameters, any point of $\mathbb{R}^4$ can be written as (x_1, y_1, s_1, t_1) with

$$s_1 := -2\,x_1 y_1 + a_1, \qquad\qquad t_1 := x_1^2 - y_1^2 + b_1.$$

Then replacing s_1 and t_1

$$\begin{pmatrix} -x_1 & -y_1 & -2s_1 & -2t_1 \\ -y_1 & x_1 & -2t_1 & 2s_1 \\ 0 & -1 & 2x_1 & 2y_1 \\ 1 & 0 & -2y_1 & 2x_1 \end{pmatrix} \xrightarrow{\text{Gauss-pivot}} \begin{pmatrix} 0 & 0 & -2a_1 & -2b_1 \\ 0 & 0 & -2b_1 & 2a_1 \\ 0 & -1 & 2x_1 & 2y_1 \\ 1 & 0 & -2y_1 & 2x_1 \end{pmatrix}.$$

This matrix has determinant $-4a_1^2 - 4b_1^2$, hence is of rank 4 when $(a_1, b_1) \neq (0,0)$. In the corresponding locus, namely in $\mathbb{R}^4 \setminus \Sigma^1_0$, the five prolonged vector

fields $\mathsf{D}^{(1)}$, $\mathsf{R}^{(1)}$, $\mathsf{I}_1^{(1)}$, $\mathsf{I}_2^{(1)}$, $\mathsf{J}^{(1)}$ have everywhere rank 4, hence generate locally open orbits, so that $\mathbb{R}^4 \setminus \Sigma_0^1$ is a single orbit under their action.

When $a_1 = b_1 = 0$, the above matrix has rank 2. In this 2-dimensional graphed locus, the rank of $\mathsf{D}^{(1)}$, $\mathsf{R}^{(1)}$, $\mathsf{I}_1^{(1)}$, $\mathsf{I}_2^{(1)}$, $J^{(1)}$ is everywhere equal to 2, whence Σ_0^1 is a single orbit under their action. □

Thus, the model M_{LC} has an invariant cone

$$s_1 + i\,t_1 \,=\, i\,\big(x_1 + i\,y_1\big)^2,$$

namely a cone invariant under the action of $\mathsf{D}^{(1)}$, $\mathsf{R}^{(1)}$, $\mathsf{I}_1^{(1)}$, $\mathsf{I}_2^{(1)}$, $\mathsf{J}^{(1)}$. Soon, we will see that *every* $M^5 \subset \mathbb{C}^3$ in the class $\mathfrak{G}_{2,1}$ also possesses an invariant cone at *any* of its points $p \in M^5$.

9. Prolongation to the jet space of order 2

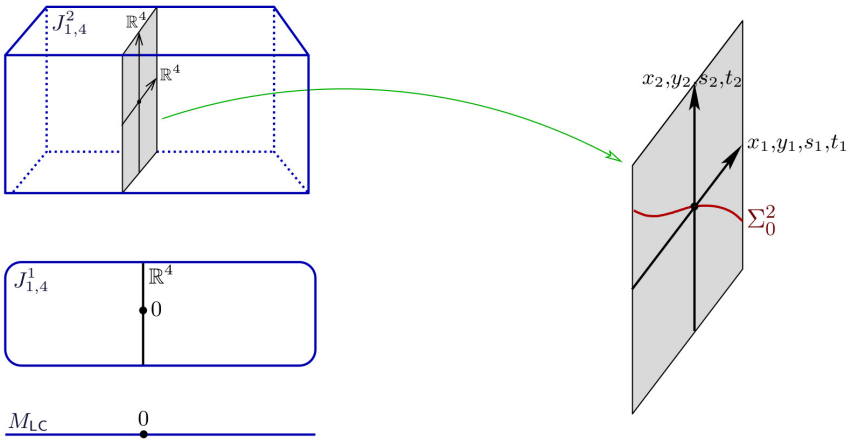
Next, we increment the jet order by one unit. The second order Lie prolongations $\mathsf{D}^{(2)}$, $\mathsf{R}^{(2)}$, $\mathsf{I}_1^{(2)}$, $\mathsf{I}_2^{(2)}$, $\mathsf{J}^{(2)}$ have the following coefficients above the origin, $v = x = y = s = t = 0$

	∂_{x_1}	∂_{y_1}	∂_{s_1}	∂_{t_1}	∂_{x_2}	∂_{y_2}	∂_{s_2}	∂_{t_2}
$\mathsf{D}^{(2)}$	$-x_1$	$-y_1$	$-2s_1$	$-2t_1$	$-3x_2$	$-3y_2$	$-4s_2$	$-4t_2$
$\mathsf{R}^{(2)}$	$-y_1$	x_1	$-2t_1$	$2s_1$	$-y_2$	x_2	$-2t_2$	$2s_2$
$\mathsf{I}_1^{(2)}$	0	-1	$2x_1$	$2y_1$	$2t_1 - 4x_1^2 - 4y_1^2$	$-2s_1$	$2x_2 - 4y_1t_1$	$2y_2 + 4x_1s_1$
$\mathsf{I}_2^{(2)}$	1	0	$-2y_1$	$2y_1$	$-2s_1$	$-2t_1 - 4x_1^2 - 4y_1^2$	$-2y_2 + 4x_1t_1$	$2x_2 - 4x_1s_1$
$\mathsf{J}^{(2)}$	0	0	0	0	0	0	$2s_1 + 4x_1y_1$	$2t_1 - 2x_1^2 + 2y_1^2$

Of course, we pull this matrix back to Σ_0^1, hence the last line becomes null. Keeping only the first 4 lines, and performing a Gauss pivot, we get

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 6x_1^2y_1 + 6y_1^3 - 3x_2 & -6x_1y_1^2 - 6x_1^3 - 3y_2 & \begin{matrix}-2x_2y_1 - 4y_1^4\\ -2x_1y_2 + 4x_1^4 - 4s_2\end{matrix} & \begin{matrix}-2y_1y_2 + 8x_1y_1^3\\ +2x_1x_2 + 8x_1^3y_1 - 4t_2\end{matrix} \\
0 & 0 & 0 & 0 & -2x_1^3 - 2x_1y_1^2 - y_2 & -2x_1^2y_1 - 2y_1^3 + x_2 & 2x_1x_2 - 2y_1y_2 - 2t_2 & 2x_1y_2 + 2y_1x_2 + 2s_2 \\
0 & -1 & 2x_1 & 2y_1 & -2x_1^2 - 6y_1^2 & 4x_1y_1 & 2x_2 - 4x_1^2y_1 + 4y_1^3 & 2y_2 - 8x_1y_1^2 \\
1 & 0 & -2y_1 & 2x_1 & 4x_1y_1 & -6x_1^2 - 2y_1^2 & -2y_2 + 4x_1^3 - 4x_1y_1^2 & 2x_2 + 8x_1^2y_1
\end{pmatrix}.$$

The upper 2×4 block, having 8 entries, then shows that x_2, y_2, s_2, t_2 can be uniquely and consistently defined in terms of x_1, y_1, so that they define an invariant surface under the action of $\mathsf{D}^{(2)}$, $\mathsf{R}^{(2)}$, $\mathsf{I}_1^{(2)}$, $\mathsf{I}_2^{(2)}$, $\mathsf{J}^{(2)}$.



Observation 9.1. *On*

$$\mathbb{R}^8 = \mathbb{R}^4_{x_1,y_1,s_1,t_1} \times \mathbb{R}^4_{x_2,y_2,s_2,t_2},$$

there exists a unique $\{\mathsf{D}^{(2)}, \mathsf{R}^{(2)}, \mathsf{I}_1^{(2)}, \mathsf{I}_2^{(2)}, \mathsf{J}^{(2)}\}$*-invariant 2-dimensional submanifold* $\Sigma^2_0 \subset \mathbb{R}^8$*, algebraic, graphed as*

$$\begin{cases} s_1 = -2\,x_1y_1, \\ t_1 = x_1^2 - y_1^2, \end{cases} \qquad \begin{cases} x_2 = 2\,x_1^2y_1 + 2\,y_1^3, \\ y_2 = -2\,x_1^3 - 2\,x_1y_1^2, \\ s_2 = -2\,y_1^4 + 2\,x_1^4, \\ t_2 = 4\,x_1^3y_1 + 4\,x_1y_1^3. \end{cases}$$

Moreover, the complement $\mathbb{R}^8 \backslash \Sigma^2_0$ *is a unique orbit under the transitive action of* $\mathsf{D}^{(2)}$, $\mathsf{R}^{(2)}$, $\mathsf{I}_1^{(2)}$, $\mathsf{I}_2^{(2)}$, $\mathsf{J}^{(2)}$.

$$\begin{array}{ccc} J^2_{1,4} & \supset & \Sigma^2_0 \\ \downarrow & & \downarrow \\ J^1_{1,4} & \supset & \Sigma^1_0 \\ \downarrow & & \downarrow \\ M & \ni & 0. \end{array}$$

Proof. As said, we pull everything back to Σ^1_0 having equations $s_1 = -2x_1y_1$, $t_1 = x_1^2 - y_1^2$. With a_2, b_2, c_2, d_2 being parameters, any point of $\mathbb{R}^4_{x_2,y_2,s_2,t_2}$ can be written as

$$\begin{cases} x_2 = 2\,x_1^2y_1 + 2\,y_1^3 + a_2, \\ y_2 = -2\,x_1^3 - 2\,x_1y_1^2 + b_2, \end{cases} \qquad \begin{cases} s_2 = -2\,y_1^4 + 2\,x_1^4 + c_2, \\ t_2 = 4\,x_1^3y_1 + 4\,x_1y_1^3 + d_2. \end{cases}$$

Replacing x_2, y_2 without replacing s_2, t_2, the upper right 2×4 block becomes

$$\begin{pmatrix} -3a_2 & -3b_2 & -8y_1^4+8x_1^4-4s_2-2y_1a_2-2x_1b_2 & 16x_1^3y_1+16x_1y_1^3-4t_2-2y_1b_2+2x_1a_2 \\ -b_2 & a_2 & 8x_1^3y_1+8x_1y_1^3-2t_2+2x_1a_2-2y_1b_2 & -4x_1^4+4y_1^4+2s_2+2x_1b_2+2y_1a_2 \end{pmatrix}.$$

Visibly, it is of rank 2 whenever $(a_2, b_2) \neq (0,0)$.

Thus, put in it $a_2 := 0$ and $b_2 := 0$

$$\begin{pmatrix} 0 & 0 & -8y_1^4+8x_1^4-4s_2 & 16x_1^3y_1+16x_1y_1^3-4t_2 \\ 0 & 0 & 8x_1^3y_1+8x_1y_1^3-2t_2 & -4x_1^4+4y_1^4+2s_2 \end{pmatrix},$$

and now replace s_2, t_2, to get

$$\begin{pmatrix} 0 & 0 & -4c_2 & -4d_2 \\ 0 & 0 & -2c_2 & 2d_2 \end{pmatrix},$$

a submatrix which has maximal rank 2 if and only if $(c_2, d_2) \neq (0,0)$. This concludes. □

We have therefore shown that, to every (fixed) 1-jet at the origin $0 \in M_{\mathsf{LC}}$ of the form

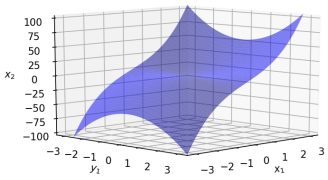
$$j_0^1 = (x_1, y_1, -2x_1y_1, x_1^2 - y_1^2)$$

is associated a unique second order jet at the origin

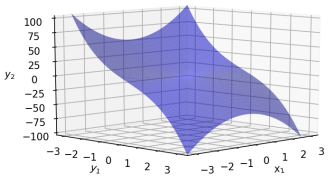
$$j_0^2 = \left(x_1, y_1, -2x_1y_1, x_1^2-y_1^2, 2x_1^2y_1+2y_1^3, -2x_1^3-2x_1y_1^2, -2y_1^4+2x_1^4, 4x_1^3y_1+4x_1y_1^3\right),$$

and since Σ_0^2 is invariant under the action of the stability group of the Gaussier-Merker model, this association is invariant.

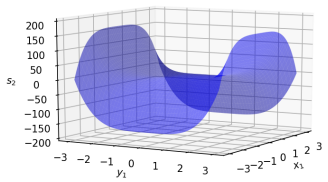
$x_2 = 2x_1^2y_1 + 2y^3$



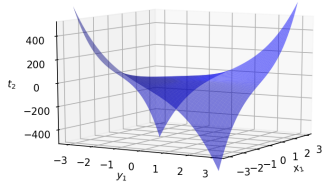
$y_2 = -2x_1^3 - 2x_1y_1^2$



$s_2 = -2y_1^4 + 2x_1^4$



$t_2 = 4x_1^3y_1 + 4x_1y_1^3$



Our next goal will be to transfer this invariancy property to *any* $M^5 \in \mathfrak{G}_{2,1}$. But subtleties will spice up our job.

10. Road map to convergent normal form

A certain *Lie-theoretic* construction of Cartan-Moser chains for Levi nondegenerate hypersurfaces $M^3 \subset \mathbb{C}^2$ was set up in [32] in order to be imitated when studying hypersurfaces $M^5 \subset \mathbb{C}^3$ in the class $\mathfrak{C}_{2,1}$, in the present memoir. However, we will encounter not only analogies, but also differences.

Recall that any Levi nondegenerate $M^3 \subset \mathbb{C}^2$, taken at any point $p \in M$, can be brought, in local coordinates $(z, w = u + iv)$ vanishing at p, to the preliminary normal form [32, Prp. 2.2]

$$v \;=\; z\overline{z} + \mathrm{O}(6),$$

where the remainder is *weighted* according to $[z] := 1, [w] := 2$. Furthermore, the *ambiguity* of such a punctual preliminary normalization, namely *any* map

$$z' \;=\; f_1 + f_2 + f_3 + f_4 + \mathrm{O}(5), \qquad\qquad w' \;=\; g_1 + g_2 + g_3 + g_4 + g_5 + \mathrm{O}(6),$$

which preserves this normalization, *i.e.* which sends $v = z\overline{z} + \mathrm{O}(6)$ to $v' = z'\overline{z}' + \mathrm{O}(6)$, can be shown to be necessarily of the form [32, Prp. 2.4]

$$\begin{aligned}
z' \;:=\; & \lambda\, z + 2i\lambda\overline{\alpha}\, z^2 + \big(-4\lambda\overline{\alpha}^2\big)\, z^3 + \big(-8i\lambda\overline{\alpha}^3\big)\, z^4 \\
& + \lambda\alpha\, w + \big(3i\lambda\alpha\overline{\alpha} + \lambda r\big)\, zw + \big(-8\lambda\alpha\overline{\alpha}^2 + 4i\overline{\alpha}\lambda r\big)\, z^2 w \\
& + \big(\lambda\alpha r + i\lambda\alpha^2\overline{\alpha}\big)\, w^2 + \mathrm{O}(5), \\
w' \;=\; & \lambda\overline{\lambda}\, w + 2i\lambda\overline{\lambda}\overline{\alpha}\, zw + \big(-4\lambda\overline{\lambda}\overline{\alpha}^2\big)\, z^2 w + \big(-8i\lambda\overline{\lambda}\overline{\alpha}^3\big)\, z^3 w \\
& + \big(i\lambda\overline{\lambda}\alpha\overline{\alpha} + \lambda\overline{\lambda} r\big)\, w^2 + \big(4i\lambda\overline{\lambda}\overline{\alpha} r - 4\lambda\overline{\lambda}\overline{\alpha}^2\alpha\big)\, zw^2 + \mathrm{O}(6),
\end{aligned}$$

and this form coincides exactly with the Taylor expansion, up to weighted orders 4, 5, of the general stability group of the *model* $\{v = z\overline{z}\} \longrightarrow \{v' = z'\overline{z}'\}$, which is well know to be

$$z' \;=\; \frac{\lambda\,(z + \alpha\, w)}{1 - 2i\overline{\alpha}\, z - (r + i\alpha\overline{\alpha})\, w}, \qquad\qquad w' \;=\; \frac{\lambda\overline{\lambda}\, w}{1 - 2i\overline{\alpha}\, z - (r + i\alpha\overline{\alpha})\, w},$$

with arbitrary $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$.

One could then figure out that precisely similar statements hold for $M^5 \in \mathfrak{C}_{2,1}$. However, some '*discrepancies*', which we will overcome, will occur. Indeed, let us briefly describe some differences, as a preliminary view on the technical road we will drive into the forest.

Taking the weights

$$[z] := 1, \quad [\zeta] := 1, \quad [w] := 2,$$

starting with $u = F(z, \zeta, \overline{z}, \overline{\zeta}, v)$ passing through the origin, by progressively normalizing the power series expansion of F, it is not difficult to show that any $M^5 \in \mathfrak{C}_{2,1}$ can be brought to the form

$$u \;=\; z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\, z^2\overline{\zeta} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(4).$$

As we know from Section 5, the isotropy group of the Gaussier-Merker model is *also* parametrized by 5 real constants $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$, and an expansion of the concerned fractional formulas was provided there.

However, one can verify (exercise) that the stability group of the above punctual normalization up to order 3 happens to be

$$\begin{aligned} z' &:= \lambda\, z + \Big(\frac{\delta}{\overline{\lambda}} - \tfrac{1}{2}\,\frac{\lambda^2}{\overline{\lambda}}\,\overline{\beta}\Big)\, z^2 - \tfrac{1}{2}\,\frac{\overline{\delta}}{\overline{\lambda}}\, w, \\ \zeta' &:= \frac{\lambda}{\overline{\lambda}}\,\zeta + \beta\, z, \\ w' &:= \lambda\overline{\lambda}\, w + \delta\, z w, \end{aligned}$$

with arbitrary $\lambda \in \mathbb{C}^*$, $\beta \in \mathbb{C}$, $\delta \in \mathbb{C}$. This looks different from the stability group of the model, shown in Section 5 and truncated to orders 2, 1, 3.

Next, it can be shown (and we will do it) that that any $M^5 \in \mathfrak{G}_{2,1}$ can be brought to the form

$$u \;=\; z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(5).$$

Lemma 20.1 will show that the stability of this equation reads as

$$\begin{aligned} z' &:= \lambda\, z - i\,\lambda\alpha\, z^2 - i\,\lambda\overline{\alpha}\, w - \frac{\lambda^2}{\overline{\lambda}}\,\overline{\beta}\, z^3 + \Big(i\,\lambda r - \tfrac{3}{2}\,\lambda\alpha\overline{\alpha} - \tfrac{1}{4}\,\frac{\lambda^2}{\overline{\lambda}}\,\overline{\varepsilon} - \tfrac{1}{4}\,\overline{\lambda}\varepsilon\Big)\, z w \\ &\qquad\qquad + i\,\lambda\alpha\,\zeta w, \\ \zeta' &:= \frac{\lambda}{\overline{\lambda}}\,\zeta + 2i\,\frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}\, z + \varepsilon\, z^2 - 2i\,\frac{\lambda}{\overline{\lambda}}\,\alpha\, z\zeta + \beta\, w, \\ w' &:= \lambda\overline{\lambda}\, w - 2i\,\lambda\overline{\lambda}\alpha\, z w - \big(2\,\lambda\overline{\lambda}\alpha^2 + \lambda^2\overline{\beta}\big)\, z^2 w + \big(-\lambda\overline{\lambda}\alpha\overline{\alpha} + i\,\lambda\overline{\lambda}\, r\big)\, w^2, \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$, $\beta \in \mathbb{C}$, $\varepsilon \in \mathbb{C}$ are arbitrary parameters. Thus, in comparison with the isotropy of the GM-model, shown in Section 5 and truncated to orders 3, 2, 4, there are two 'extra' complex parameters, namely β, ε.

Also, in Proposition 20.3 we will normalize, still at the origin only

$$\begin{aligned} u \;=\; z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} \\ + z^3\overline{\zeta}^2\, F_{3,0,0,2,0} + \overline{z}^3\zeta^2\,\overline{F_{3,0,0,2,0}} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(6), \end{aligned}$$

and in Lemma 20.4, we will see that the stability group of this normal form is

$$\begin{aligned} z' &:= \lambda\, z - i\,\lambda\alpha\, z^2 - i\,\lambda\overline{\alpha}\, w - \lambda\alpha^2\, z^3 \\ &\quad + \Big(i\,\lambda r - 3\,\lambda\alpha\overline{\alpha} + 2i\,\lambda\alpha\, F_{3,0,0,2,0} - 2i\,\lambda\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)\, z w + i\,\lambda\alpha\,\zeta w \\ &\quad + \Big(8i\,\lambda\alpha^2\overline{\alpha} + \tfrac{1}{2}\,\frac{\lambda^2}{\overline{\lambda}}\,\overline{\gamma} + 4\,\frac{\lambda}{\overline{\lambda}}\,\overline{\tau} + 4\,\lambda\alpha^2\, F_{3,0,0,2,0} - 8\,\lambda\alpha\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)\, z^2 w \\ &\quad + i\,\lambda\alpha^3\, z^4 + 3\,\lambda\alpha^2\, z\zeta w + \tau\, w^2, \end{aligned}$$

$$
\begin{aligned}
\zeta' \ :=\ & \frac{\lambda}{=}\,\zeta + 2i\,\frac{\lambda}{=}\,\overline{\alpha}\,z + \Big(3\,\frac{\lambda}{=}\,\alpha\overline{\alpha} - i\,\frac{\lambda}{=}\,r - 2i\,\frac{\lambda}{=}\,\alpha\,F_{3,0,0,2,0} + 6i\,\frac{\lambda}{=}\,\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)\,z^2 \\
& - 2i\,\frac{\lambda}{=}\,\alpha\,z\zeta + \frac{\lambda}{=}\,\overline{\alpha}^2\,w \\
& + \Big(2\,\frac{\lambda}{=}\,\alpha r - 4i\,\frac{\lambda}{=}\,\alpha^2\overline{\alpha} - 2\,\frac{\lambda^2}{\overline{=}}\,\overline{\gamma} - 8\,\frac{\lambda}{\overline{=}^2}\,\overline{\tau} + 12\,\frac{\lambda}{=}\,\alpha^2\,F_{3,0,0,2,0} \\
& \qquad\qquad + 4\,\frac{\lambda}{=}\,\alpha\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)\,z^3 - 3\,\frac{\lambda}{=}\,\alpha^2\,z^2\zeta + \gamma\,zw \\
& + \Big(-2\,\frac{\lambda}{=}\,\alpha\overline{\alpha} + 4i\,\frac{\lambda}{=}\,\alpha\,F_{3,0,0,2,0} - 4i\,\frac{\lambda}{=}\,\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)\,\zeta w,
\end{aligned}
$$

$$
\begin{aligned}
w' \ :=\ & \lambda\overline{\lambda}\,w - 2i\,\lambda\overline{\lambda}\alpha\,zw - 3\,\lambda\overline{\lambda}\alpha^2\,z^2w + \big(-\lambda\overline{\lambda}\alpha\overline{\alpha} + i\,\lambda\overline{\lambda}\,r\big)\,w^2 + 4i\,\lambda\overline{\lambda}\alpha^3\,z^3w \\
& + \Big(6i\,\lambda\overline{\lambda}\alpha^2\overline{\alpha} + 2\,\lambda\overline{\lambda}\alpha r + 2\,\lambda\overline{\tau} + 4\,\lambda\overline{\lambda}\alpha^2\,F_{3,0,0,2,0} - 4\,\lambda\overline{\lambda}\alpha\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)\,zw^2 \\
& + \lambda\overline{\lambda}\alpha^2\,\zeta w^2.
\end{aligned}
$$

where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\tau \in \mathbb{C}$ are arbitrary. Thus, there are again two 'extra' complex parameters, namely γ, τ.

To realize a Moser-like normal form for hypersurfaces $M^5 \in \mathfrak{C}_{2,1}$ and to define analogs of Cartan-Moser chains, we will therefore have to adapt a bit our ideas. Let us give a quick summary.

To start with, we will pick any curve $0 \in \gamma \subset M$ which is *CR-transversal* in the sense that $\dot{\gamma} \notin T^cM$. It is well known that one can always straighten it to be $\gamma = \{(0, 0, iv)\} \subset M$, the v-axis.

It is also well known that, after an appropriate biholomorphism, one can make the graphing function $F(z, \zeta, \overline{z}, \overline{\zeta}, v)$ to have *no* pluriharmonic terms, in the sense that $F(z, \zeta, 0, 0, v) \equiv 0$.

In Section 11 to 19, we will continue to *prenormalize* and even start to *normalize* F further, *without touching* γ, namely by always stabilizing $\{(0, 0, iv)\} \subset M$.

However, at some moment of the normalization process, exactly as what occurs [12, 23, 24] for Levi nondegenerate $M^3 \subset \mathbb{C}^2$, one is '*forced*' to perform additional normalizations which *bend* the v-axis, hence destroy what was preserved up to this point. This fact confirms that it was inappropriate to choose at the beginning any CR-transversal curve $0 \in \gamma \subset M$, '*at random*'.

It is at this crucial moment that the Cartan-Moser chains start to appear to eyes. By appropriately interpreting the algebraic or geometric normalization conditions that force to change the v-axis, one realizes that certain CR-transversal curves are *invariant* under biholomorphisms of $\mathbb{C}^2$. Our goal is to view something similar and new about $M^5 \in \mathfrak{C}_{2,1}$. We will do it.

The Lie-theoretical path taken in [32] consisted in normalizing the equation of M at only one point, only up to order 5, which is quite elementary, can be

done by hand or on a computer, and does not employ (at all) the implicit function theorem. In this memoir, we will conduct essentially the same method as in [32] but with two differences. Firstly, we will prenormalize the equation of M not only at 0 but all along the v-axis $\gamma \subset M$ (chosen at random) and reach Proposition 19.3, until we come to the point where chains start to appear to eyes. Then we will work only at 0, with power series expansions of orders 5, 6, 7, and 'discover' that the chains are the same as stated by Observations 8.1 and 9.1 for the Gaussier-Merker model, notwithstanding the presence of extra complex parameters.

Once chains are known, we will go back to the starting point, and choose the CR-transversal $\gamma \subset M$ to *be a chain*, then we will plainly apply all what was done for a random γ, and we will deduce that two normalizations of certain coefficients $F_{a,b,c,d}(v)$ realize themselves gratuitously thanks to chains, and lastly, we will obtain a complete Moser-like normal form.

To terminate our mathematical work and get some uniqueness property, we will work out the formal theory of the normal form only at the end of the paper.

11. Chain straightening and harmonic killing

Start with any $\mathfrak{G}_{2,1}$ hypersurface $M \subset \mathbb{C}^3$, passing by the origin $0 \in M$. Since $T^c_0 M \cong \mathbb{C}^2$, we can assume after a $\mathbb{C}$-linear transformation that $T^c_0 M = \mathbb{C}_z \times \mathbb{C}_\zeta \times \{0\}$, in coordinates $(z, \zeta, w) \in \mathbb{C}^3$.

The '*game*' is to transform M progressively into more and more *normalized* hypersurfaces. Each (partial) normalization step can represented by means of a biholomorphism fixing the origin as

$$\begin{array}{ccc} \mathbb{C}^3 \supset (M^5, 0) & \xrightarrow{\text{normalize}} & (M'^5, 0) \subset \mathbb{C}'^3, \\ (z, \zeta, w) & \longrightarrow & \big(f(z,\zeta,w),\, g(z,\zeta,w),\, h(z,\zeta,w)\big) \\ & & =: (z', \zeta', w'). \end{array}$$

Without loss of generality, both hypersurfaces will be assumed, with $w = u + iv$ and $w' = u' + iv'$, to be $\mathcal{C}^\omega$-graphed as

$$u = F(z, \zeta, \overline{z}, \overline{\zeta}, v) \qquad \text{and} \qquad u' = F'(z', \zeta', \overline{z}', \overline{\zeta}', v').$$

We may assume that $T^c_0 M = \{w = 0\}$ is left untouched, so that $T^c_{0'} M' = \{w' = 0\}$ too.

In fact, step by step, all previously achieved normalizations will be conserved while performing any further normalization. Once M has been partly normalized to some new M', we will erase primes to the obtained $M' =: M$, normalize once more, and so on.

Now, the hypothesis that the biholomorphism establishes a CR-diffeomorphism $M \xrightarrow{\sim} M'$, expresses as saying that $u' = F'$ when $u = F$, namely

$$\begin{aligned} 0 \ = \ & - \operatorname{Re} h(z,\zeta,w) \\ & + F'\Big(f(z,\zeta,w),\, g(z,\zeta,w),\, \overline{f}(\overline{z},\overline{\zeta},\overline{w}),\, \overline{g}(\overline{z},\overline{\zeta},\overline{w}), \\ & \qquad\qquad \operatorname{Im} h(z,\zeta,w)\Big)\Big|_{w=F(z,\zeta,\overline{z},\overline{\zeta},v)+iv}. \end{aligned}$$

Performing the indicated replacement $w = F + i\,v$ yields

Lemma 11.1. [Fundamental identity] *The map $(z',\zeta',w') = (f,g,h)$ sends $M = \{u = F\}$ to $M' = \{u' = F'\}$ if and only if*

$$\begin{aligned} 0 \ \equiv \ & -\tfrac{1}{2}\, h\big(z,\zeta,\, F(z,\zeta,\overline{z},\overline{\zeta},v) + iv\big) - \tfrac{1}{2}\, \overline{h}\big(\overline{z},\overline{\zeta},\, F(z,\zeta,\overline{z},\overline{\zeta},v) - iv\big) + \\ & + F'\Big(f\big(z,\zeta,\, F(z,\zeta,\overline{z},\overline{\zeta},v) + iv\big),\ g\big(z,\zeta,\, F(z,\zeta,\overline{z},\overline{\zeta},v) + iv\big), \\ & \qquad \overline{f}\big(\overline{z},\overline{\zeta},\, F(z,\zeta,\overline{z},\overline{\zeta},v) - iv\big),\ \overline{g}\big(\overline{z},\overline{\zeta},\, F(z,\zeta,\overline{z},\overline{\zeta},v) - iv\big), \\ & \qquad \tfrac{1}{2i}\, h\big(z,\zeta,\, F(z,\zeta,\overline{z},\overline{\zeta},v) + iv\big) - \tfrac{1}{2i}\, \overline{h}\big(\overline{z},\overline{\zeta},\, F(z,\zeta,\overline{z},\overline{\zeta},v) - iv\big)\Big), \end{aligned}$$

holds identically in $\mathbb{C}\{z,\zeta,\overline{z},\overline{\zeta},v\}$.

Although this equation looks complicated, it must be dealt with. Progressive normalizations will make it more tractable.

One of the first tasks is to annihilate all pluriharmonic monomials in (z,ζ) of the form $F_{a,b,0,0,e}\, z^a \zeta^b v^e$, and their conjugates as well. For completeness, we explain in details how to do this known normalization. We proceed in two steps.

As already explained in Section 10, a CR-transversal curve with $0 \in \gamma \subset M$ is now at first chosen 'at random', while a better choice will be made later, when the normalization process will reach a certain deeper point.

Lemma 11.2. *Let $\gamma : \mathbb{R} \longrightarrow M$ be any local $\mathcal{C}^\omega$ curve with $\gamma(0) = 0 \in M$ and $\dot\gamma(0) \notin T^c_0 M = \{w = 0\}$. Then there exists a biholomorphism $(z,\zeta,w) \longmapsto (z',w',\zeta')$ sending (stabilizing) $T^c_0 M = \{w = 0\}$ to $T^c_{0'} M' = \{w' = 0\}$ which sends γ to the curve $\gamma(t) = (0,0,it)$ straightened along the v-axis.*

Notice that the CR-transversal direction $\dot\gamma'(0) \in T_{0'}M' \big\backslash T^c_{0'} M'$ together with $T^c_{0'} M' = \{w' = 0\}$ implies $T_0 M' = \{u' = 0\}$.

Proof. Write the curve as

$$\gamma(t) \ = \ \big(\varphi(t),\, \psi(t),\, \chi(t)\big),$$

with some complex-valued analytic functions φ, ψ, χ. By assumption, $\dot\chi(0) \neq 0$. This guarantees invertibility of the *inverse* holomorphic change of coordinates

$$z \ := \ z' + \varphi\big(-iw'\big), \qquad\qquad \zeta \ := \ \zeta' + \psi\big(-iw'\big), \qquad\qquad w \ := \ \chi\big(-iw'\big).$$

Similarly, the target (transformed) curve can be written

$$\gamma'(t) \,=\, \big(\varphi'(t), \psi'(t), i\chi'(t)\big)$$

— note the i factor —, and the pointwise correspondence between curves writes as

$$\begin{aligned} \varphi(t) \,&\equiv\, \varphi'(t) + \varphi\big(-i(i\chi'(t))\big), \qquad & \psi(t) \,&\equiv\, \psi'(t) + \psi\big(-i(i\chi'(t))\big), \\ & & \chi(t) \,&\equiv\, \chi\big(-i(i\chi'(t))\big). \end{aligned}$$

This last identity yields $t \equiv \chi'(t)$ thanks to $0 \neq \dot{\chi}(0)$. Replacing then $\chi'(t) := t$ inside the first two identities concludes that $0 \equiv \varphi'(t) \equiv \psi'(t)$. □

Consequently, the graphing function of the transformed hypersurface writes, after erasing primes

$$M: \qquad u \,=\, F\big(z, \zeta, \overline{z}, \overline{\zeta}, v\big),$$

with $F = \mathrm{O}(2)$ and also $F(0,0,0,0,v) \equiv 0$. This last condition is technically needed for the next second elementary normalization.

Lemma 11.3. *Starting from $F = \mathrm{O}(2)$ with $F(0,0,0,0,v) \equiv 0$, there exists a biholomorphism of the form*

$$z' \,:=\, z, \qquad\qquad \zeta' \,:=\, \zeta, \qquad\qquad w' \,:=\, w + h(z,\zeta,w),$$

with $h = \mathrm{O}(2)$ and $h(0,0,w) \equiv 0$ which transforms $\{u = F\}$ to $\{u' = F'\}$ satisfying

$$0 \,\equiv\, F'\big(z', \zeta', 0, 0, v'\big) \,\equiv\, F'\big(0, 0, \overline{z}', \overline{\zeta}', v'\big).$$

The second vanishing identity is a consequence of the first by conjugation, thanks to (2.1). Equivalently, $F'_{a,b,0,0,e} = 0 = F'_{0,0,c,d,e}$ for all integer indices. Notice that $F'(0,0,0,0,v') \equiv 0$ still holds.

Proof. If such a biholomorphism exists, the identity of Lemma 11.1 shows that

$$\begin{aligned} 0 \,\equiv\, &-F(z,\zeta,\overline{z},\overline{\zeta},v) - \tfrac{1}{2}\, h\big(z,\zeta,\, F(z,\zeta,\overline{z},\overline{\zeta},v) + iv\big) \\ &\qquad\qquad - \tfrac{1}{2}\, \overline{h}\big(\overline{z},\overline{\zeta},\, F(z,\zeta,\overline{z},\overline{\zeta},v) - iv\big) + \\ &+ F'\Big(z,\zeta,\overline{z},\overline{\zeta},\, v + \tfrac{1}{2i}\, h\big(z,\zeta,\, F(z,\zeta,\overline{z},\overline{\zeta},v) + iv\big) \\ &\qquad\qquad - \tfrac{1}{2i}\, \overline{h}\big(\overline{z},\overline{\zeta},\, F(z,\zeta,\overline{z},\overline{\zeta},v) - iv\big)\Big). \end{aligned} \tag{11.1}$$

Our goal is to make $F'(z',\zeta',0,0,v) \equiv 0$.

If this vanishing identity would hold, putting $\overline{z} := 0 =: \overline{\zeta}$ in (11.1) we would deduce

$$\begin{aligned} 0 \,\equiv\, &-F(z,\zeta,0,0,v) - \tfrac{1}{2}\, h\big(z,\zeta,\, F(z,\zeta,0,0,v) + i\,v\big) \\ &\qquad\qquad - \tfrac{1}{2}\, \overline{h}\big(0,0,\, F(z,\zeta,0,0,v) - i\,v\big) + 0. \end{aligned} \tag{11.2}$$

We claim that such an identity can be employed in order to define $h(z,\zeta,w)$ uniquely, with the supplementary condition that the last term $-\frac{1}{2}\overline{h}$ of (11.2) is zero.

Indeed, thanks to $F = \mathrm{O}(2)$, we may apply the implicit function theorem to invert

$$F(z,\zeta,0,0,v) + i\,v =: \omega \qquad \Longleftrightarrow \qquad v = \mathrm{T}(z,\zeta,\omega) = -i\,\omega + \mathrm{O}(2).$$

Define therefore $h(z,\zeta,w)$ accordingly

$$0 \equiv -F\big(z,\zeta,\,\mathrm{T}(z,\zeta,\omega)\big) - \tfrac{1}{2}\,h(z,\zeta,\omega) - \tfrac{1}{2}\cdot 0.$$

Now, because $F(0,0,0,0,v) \equiv 0$ by hypothesis, it comes $0 \equiv h(0,0,\omega)$, just by putting $z := 0 =: \zeta$ in (11.2).

Consequently, the identity (11.2) is indeed realized with $-\frac{1}{2}\overline{h} = 0$. Finally, coming back to $(11.1)\big|_{\overline{z}=\overline{\zeta}=0}$, we get in conclusion what we want

$$0 \equiv 0 + F'\Big(z,\zeta,0,0,\,v + \tfrac{1}{2i}\,h\big(z,\zeta,\,F(z,\zeta,0,0,v) + i\,v\big) - 0\Big). \qquad \square$$

Thus, erasing primes, we have obtained the preliminary normalization

$$u = F = \sum_{\substack{a+b\geqslant 1\\ c+d\geqslant 1}} z^a\zeta^b\overline{z}^c\overline{\zeta}^d\,F_{a,b,c,d}(v) \quad \text{with} \quad F_{a,b,c,d}(v) := \sum_{e\geqslant 1} F_{a,b,c,d,e}\,v^e.$$

In the sequel, we shall perform normalizing biholomorphisms which stabilize this form.

12. Prenormalization: step I

To start with, let us expand

$$u = z\overline{z}\,F_{1,0,1,0}(v) + z\overline{\zeta}\,F_{1,0,0,1}(v) + \overline{z}\zeta\,F_{0,1,1,0}(v) + \zeta\overline{\zeta}\,F_{0,1,0,1}(v) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(3).$$

By assumption, the Levi matrix of F has rank 1 everywhere, hence in particular at the origin. We compute this matrix

$$\mathsf{Levi}(F) = \begin{pmatrix} 0 & \mathrm{O}(1) & \mathrm{O}(1) & -\frac{1}{2} + \mathrm{O}(2) \\ \mathrm{O}(1) & F_{1,0,1,0}(0) + \mathrm{O}(1) & F_{0,1,1,0}(0) + \mathrm{O}(1) & \mathrm{O}(1) \\ \mathrm{O}(1) & F_{1,0,0,1}(0) + \mathrm{O}(1) & F_{0,1,0,1}(0) + \mathrm{O}(1) & \mathrm{O}(1) \\ -\frac{1}{2} + \mathrm{O}(2) & \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(1) \end{pmatrix},$$

where $\mathrm{O}(\mathrm{N}) = \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(\mathrm{N})$ for any integer $\mathrm{N} \in \mathbb{N}$. Hence at the origin $(z,\zeta,\overline{z},\overline{\zeta},v) = (0,0,0,0,0)$ we have

$$1 = \mathsf{rank}\begin{pmatrix} F_{1,0,1,0}(0) & F_{0,1,1,0}(0) \\ F_{1,0,0,1}(0) & F_{0,1,0,1}(0) \end{pmatrix}.$$

After a $\mathbb{C}$-linear invertible transformation in the (z,ζ)-space, we can assume

$$1 = F_{1,0,1,0}(0) \qquad \text{and} \qquad 0 = F_{1,0,0,1}(0) = F_{0,1,1,0}(0) = F_{0,1,0,1}(0), \quad (12.1)$$

so that

$$u \;=\; z\overline{z} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(3).$$

Lemma 12.1. *There exists a biholomorphism of the form*

$$z' \;:=\; z\,\varphi(w), \qquad\qquad \zeta' \;:=\; \zeta, \qquad\qquad w' \;:=\; w,$$

which transforms $M = \{u = F\}$ into M' of equation

$$u' \;=\; z'\overline{z}' + \sum_{\substack{(a,b,c,d)\neq(1,0,1,0)\\ a+b\geqslant 1,\, c+d\geqslant 1}} z'^{a}\zeta'^{b}\overline{z}'^{c}\overline{w}'^{d}\, F'_{a,b,c,d}(v').$$

Proof. We write the source hypersurface as

$$u \;=\; F \;=\; z\overline{z}\,F_{1,0,1,0}(v) + \zeta\,(\cdots) + \overline{\zeta}\,(\cdots) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(3),$$

and similarly for the target

$$u' \;=\; F' \;=\; z'\overline{z}'\,F'_{1,0,1,0}(v') + \zeta'\,(\cdots) + \overline{\zeta}'\,(\cdots) + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(3).$$

Through any map of the form being considered, since $z' = z\,(\cdots)$ and $\zeta' = \zeta$, it is clear that the remainders correspond to one another

$$\zeta'\,(\cdots) \;=\; \zeta\,(\cdots), \qquad\qquad \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(3) \;=\; \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(3).$$

Since $u = u'$, the fundamental identity (11.1) writes

$$\begin{aligned} 0 \;\equiv\; & -F\big(z,\zeta,\overline{z},\overline{\zeta}\big) \\ & + F'\Big(z\,\varphi\big(F+iv\big),\ \zeta,\ \overline{z}\,\overline{\varphi}\big(F-iv\big),\ \overline{\zeta},\ v\Big), \end{aligned}$$

which implies, after taking account of the fact that remainders are the same and that $v = v'$

$$\begin{aligned} 0 \;\equiv\; & -z\overline{z}\,F_{1,0,1,0}(v) \\ & + z\overline{z}\,\varphi\big(F+iv\big)\,\overline{\varphi}\big(F-iv\big)\,F'_{1,0,1,0}(v) + \zeta\,(\cdots) + \overline{\zeta}\,(\cdots) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(3). \end{aligned}$$

Next, by Taylor expanding at $i\,v$, we get

$$\varphi\big(iv + F\big) \;=\; \varphi(iv) + F\,(\cdots) \;=\; \varphi(iv) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2),$$

and by inserting this above, we obtain

$$\begin{aligned} 0 \;\equiv\; & -z\overline{z}\,F_{1,0,1,0}(v) \\ & + z\overline{z}\,\varphi(iv)\,\overline{\varphi}(-iv)\,F'_{1,0,1,0}(v) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4) + \zeta\,(\cdots) + \overline{\zeta}\,(\cdots) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(3). \end{aligned}$$

Identifying the coefficients of $z\overline{z}$ yields

$$0 \;\equiv\; -F_{1,0,1,0}(v) + \varphi(iv)\,\overline{\varphi}(iv)\,F'_{1,0,1,0}(v).$$

We can normalize $F'_{1,0,1,0}(v) \equiv 1$ provided φ satisfies

$$\varphi(iv)\,\overline{\varphi}(-iv) \;\equiv\; F_{1,0,1,0}(v).$$

Observing that $\overline{F_{1,0,1,0}(v)} = F_{1,0,1,0}(v)$ by the reality condition (2.2), it suffices to set

$$\varphi(w) \,:=\, \sqrt{F_{1,0,1,0}(-\,i\,w)},$$

a function which is holomorphic thanks to $F_{1,0,1,0}(0) = 1$. □

So, erasing primes, we have obtained

$$u \,=\, z\overline{z} + \sum_{\substack{(a,b,c,d)\neq(1,0,1,0)\\ a+b\geqslant 1,\, c+d\geqslant 1}} z^a \zeta^b \overline{z}^c \overline{\zeta}^d \, F_{a,b,c,d}(v). \tag{12.2}$$

13. Dependent and independent jets

Now, the assumption of Levi degeneracy states as the vanishing identity

$$0 \,\equiv\, \mathsf{Levi}(F) \,:=\, \begin{vmatrix} 0 & F_z & F_\zeta & -\frac{1}{2} + \frac{1}{2i} F_v \\ F_{\overline{z}} & F_{z\overline{z}} & F_{\zeta\overline{z}} & \frac{1}{2i} F_{\overline{z}v} \\ F_{\overline{\zeta}} & F_{z\overline{\zeta}} & F_{\zeta\overline{\zeta}} & \frac{1}{2i} F_{\overline{\zeta}v} \\ -\frac{1}{2} - \frac{1}{2i} F_v & -\frac{1}{2i} F_{zv} & -\frac{1}{2i} F_{\zeta v} & \frac{1}{4} F_{vv} \end{vmatrix}.$$

But the Levi form is *not* assumed to be identically zero, it is assumed to be constantly of rank 1. With $F = z\overline{z} + \mathrm{O}(3)$ in (12.2), this assumption expresses as the nonvanishing of the minor

$$0 \,\neq\, \mathsf{Levi}_1(F) \,:=\, \begin{vmatrix} 0 & F_z & -\frac{1}{2} + \frac{1}{2i} F_v \\ F_{\overline{z}} & F_{z\overline{z}} & \frac{1}{2i} F_{\overline{z}v} \\ -\frac{1}{2} - \frac{1}{2i} F_v & -\frac{1}{2i} F_{zv} & \frac{1}{4} F_{vv} \end{vmatrix}.$$

Expanding $\mathsf{Levi}(F)$ along its third column gives

$$\begin{aligned} F_{\zeta\overline{\zeta}} \cdot \mathsf{Levi}_1(F) \,\equiv\, & -F_\zeta \begin{vmatrix} F_{\overline{z}} & F_{z\overline{z}} & \frac{1}{2i} F_{\overline{z}v} \\ F_{\overline{\zeta}} & F_{z\overline{\zeta}} & \frac{1}{2i} F_{\overline{\zeta}v} \\ -\frac{1}{2} - \frac{1}{2i} F_v & -\frac{1}{2i} F_{zv} & \frac{1}{4} F_{vv} \end{vmatrix} \\ & + F_{\zeta\overline{z}} \begin{vmatrix} 0 & F_z & -\frac{1}{2} + \frac{1}{2i} F_v \\ F_{\overline{\zeta}} & F_{z\overline{\zeta}} & \frac{1}{2i} F_{\overline{\zeta}v} \\ -\frac{1}{2} - \frac{1}{2i} F_v & -\frac{1}{2i} F_{zv} & \frac{1}{4} F_{vv} \end{vmatrix} \\ & - \tfrac{1}{2i} F_{\zeta v} \begin{vmatrix} 0 & F_z & -\frac{1}{2} + \frac{1}{2i} F_v \\ F_{\overline{z}} & F_{z\overline{z}} & \frac{1}{2i} F_{\overline{z}v} \\ F_{\overline{\zeta}} & F_{z\overline{\zeta}} & \frac{1}{2i} F_{\overline{\zeta}v} \end{vmatrix}. \end{aligned}$$

Expanding $\mathsf{Levi}_1(F)$ and dividing, we get a rational expression

$$F_{\zeta\bar{\zeta}} \equiv \frac{\mathcal{P}\big(F_z,\, F_\zeta,\, F_{\bar{z}},\, F_{\bar{\zeta}},\, F_v,\, F_{z\bar{z}},\, F_{z\bar{\zeta}},\, F_{\zeta\bar{z}},\, F_{zv},\, F_{\zeta v},\, F_{\bar{z}v},\, F_{\bar{\zeta}v},\, F_{vv}\big)}{F_{z\bar{z}} + F_v F_v F_{z\bar{z}} + i\, F_{\bar{z}} F_{zv} - i\, F_z F_{\bar{z}v} + F_z F_{\bar{z}} F_{vv} - F_v F_{\bar{z}} F_{zv} - F_z F_v F_{\bar{z}v}},$$

whose numerator $\mathcal{P}$ is a certain universal polynomial, not depending on F. By assumption, the denominator is nonvanishing (locally).

Differentiating this identity and successively performing appropriate replacements (exercise), we obtain

Proposition 13.1. *For all integers $a, b, c, d, e \in \mathbb{N}$ with $b \geqslant 1$ and $d \geqslant 1$, there exist a polynomial $\mathcal{P}_{a,b,c,d,e}$ and an exponent $\mathrm{N}_{a,b,c,d,e} \in \mathbb{N}_{\geqslant 1}$ such that*

$$\begin{aligned} F_{z^a \zeta^b \bar{z}^c \bar{\zeta}^d v^e} \equiv \frac{1}{\mathrm{Y}}\, \mathcal{P}_{a,b,c,d,e}\Big(& \big\{F_{z^{a'} \bar{z}^{c'} v^{e'}}\big\}_{a'+c'+e' \leqslant a+b+c+d+e}, \\ & \big\{F_{z^{a'} \zeta^{b'} \bar{z}^{c'} v^{e'}}\big\}^{b' \geqslant 1}_{a'+b'+c'+e' \leqslant a+b+c+d+e}, \\ & \big\{F_{z^{a'} \bar{z}^{c'} \bar{\zeta}^{d'} v^{e'}}\big\}^{d' \geqslant 1}_{a'+c'+d'+e' \leqslant a+b+c+d+e}\Big), \end{aligned}$$

with denominator

$$\mathrm{Y} := \big(F_{z\bar{z}} + F_v F_v F_{z\bar{z}} + i\, F_{\bar{z}} F_{zv} - i\, F_z F_{\bar{z}v} + F_z F_{\bar{z}} F_{vv} - F_v F_{\bar{z}} F_{zv} - F_z F_v F_{\bar{z}v}\big)^{\mathrm{N}_{a,b,c,d,e}}.$$

Accordingly, as in [9], we will term

$$\begin{aligned} \textit{Dependent derivatives} &:= \Big\{F_{z^a \zeta^b \bar{z}^c \bar{\zeta}^d v^e}\Big\}^{b \geqslant 1,\, d \geqslant 1}_{a,\, b,\, c,\, d,\, e \geqslant 0}, \\ \textit{Independent derivatives} &:= \big\{F_{z^a \bar{z}^c v^e}\big\}_{a,\, c,\, e \geqslant 0} \bigcup \big\{F_{z^a \zeta^b \bar{z}^c v^e}\big\}^{b \geqslant 1}_{a,\, c,\, e \geqslant 0} \\ &\qquad \bigcup \big\{F_{z^a \bar{z}^c \bar{\zeta}^d v^e}\big\}^{d \geqslant 1}_{a,\, c,\, e \geqslant 0}. \end{aligned}$$

At the origin when we will progressively normalize the power series F, any modification of the values of the *independent* derivatives of F at 0 will automatically transfer to the *dependent* derivatives of F at 0 *via* the formulas of Proposition 13.1. Thus, freedom of normalization concerns only *independent* derivatives

$$\tfrac{1}{a!}\, \tfrac{1}{b!}\, \tfrac{1}{c!}\, \tfrac{1}{d!}\, \tfrac{1}{e!}\, \partial_z^a\, \partial_\zeta^b\, \partial_{\bar{z}}^c\, \partial_{\bar{\zeta}}^d\, \partial_v^e\, F\big(0,0,0,0,0\big) \;=\; F_{a,b,c,d,e}, \quad (b+d \leqslant 1).$$

For this reason, we will often write

$$\begin{aligned} u &= F \\ &= z\bar{z} + \sum_{\substack{a+c \geqslant 3 \\ a \geqslant 1,\, c \geqslant 1}} z^a \bar{z}^c\, F_{a,0,c,0}(v) + \sum_{b \geqslant 1} z^a \zeta^b \bar{z}^c\, F_{a,b,c,0}(v) \\ &\qquad + \sum_{d \geqslant 1} z^a \bar{z}^c \bar{\zeta}^d\, F_{a,0,c,d}(v) + \zeta\bar{\zeta}\,(\cdots), \end{aligned}$$

pointing out that all terms behind $\zeta\overline{\zeta}\,(\cdots)$ are sorts of '*remainder terms*'. However, some information will be needed about these remainders anyway while normalizing the main independent derivatives. Indeed, regularly, we will come back to the Levi determinant (3.1).

14. Prenormalization: step II

Now, we come back to (12.2), which we rewrite by selecting monomials having $\overline{z}^1$ as single antiholomorphic component

$$\begin{aligned} u &= z\overline{z} + \sum_{\substack{a+b\geqslant 1\\(a,b)\neq(1,0)}} z^a\zeta^b\overline{z}^1\,F_{a,b,1,0}(v) + \sum_{\substack{a+b\geqslant 1\\c\geqslant 2}} z^a\zeta^b\overline{z}^c\,F_{a,b,c,0}(v) + \sum_{\substack{a+b\geqslant 1\\d\geqslant 1}} z^a\zeta^b\overline{z}^c\overline{\zeta}^d\,F_{a,b,c,d}(v)\\ &= \overline{z}\Big(z + \sum_{\substack{a+b\geqslant 1\\(a,b)\neq(1,0)}} z^a\zeta^b\,F_{a,b,1,0}(v)\Big) + \overline{z}^2\,(\cdots) + \overline{\zeta}\,(\cdots). \end{aligned}$$

Lemma 14.1. *There exists a biholomorphism of the form*

$$z' := z + \Lambda(z,\zeta,w) = z + \mathrm{O}_{z,\zeta,w}(2), \qquad \zeta' := \zeta, \qquad w' := w,$$

which transforms $M = \{u = F\}$ into M' of equation

$$u' = z'\overline{z}' + \overline{z}'^2\,(\cdots) + \overline{\zeta}'\,(\cdots).$$

Proof. Set

$$\Lambda(z,\zeta,w) := \sum_{\substack{a+b\geqslant 1\\(a,b)\neq(1,0)}} z^a\zeta^b\,F_{a,b,1,0}\big(-i\,w\big) = z^2\,(\cdots) + \zeta\,(\cdots).$$

Since $F_{0,1,1,0}(0) = 0$ by (12.1), we indeed have $\Lambda = \mathrm{O}_{z,\zeta,w}(2)$. Thus the equation of M writes

$$u = \overline{z}\,\big(z + \Lambda(z,\zeta,v)\big) + \overline{z}^2\,(\cdots) + \overline{\zeta}\,(\cdots).$$

Restricting $z' = z+\Lambda(z,\zeta,-iw)$ to M, Taylor expanding at (z,ζ,v), and using $0 \equiv F(z,\zeta,0,0,v)$ we obtain

$$\begin{aligned} z' = z + \Lambda\big(z,\,\zeta,\,v - iF\big) &= z + \Lambda(z,\zeta,v) + F\,(\cdots)\\ &= z + \Lambda(z,\zeta,v) + \overline{z}\,(\cdots) + \overline{\zeta}\,(\cdots), \end{aligned}$$

hence replacing $z + \Lambda(z,\zeta,v) = z' - \overline{z}(\cdots) - \overline{\zeta}(\cdots)$ and replacing $\zeta := \zeta'$

$$\begin{aligned} u' = u &= \overline{z}\Big(z' - \overline{z}\,(\cdots) - \overline{\zeta}\,(\cdots)\Big) + \overline{z}^2\,(\cdots) + \overline{\zeta}\,(\cdots)\\ &= \overline{z}\,z' + \overline{z}^2\,(\cdots) + \overline{\zeta}'\,(\cdots). \end{aligned}$$

Now, an inversion gives

$$\begin{aligned} z + \Lambda = z + z^2\,(\cdots) + \zeta\,(\cdots) = z' \iff z &= z' + z'^2\,(\cdots) + \zeta'\,(\cdots)\\ \Longrightarrow\ \overline{z}^2 &= \overline{z}'^2\,(\cdots) + \overline{\zeta}'\,(\cdots), \end{aligned}$$

which concludes

$$u' \,=\, z'\overline{z}' + \overline{z}'^{\,2}\,(\cdots) + \overline{\zeta}'\,(\cdots). \qquad \square$$

Erasing primes, and using the fact that the graphing function is real, we obtain

Corollary 14.2. *Any $\mathcal{C}^\omega$ hypersurface $0 \in M^5 \subset \mathbb{C}^3$ whose Levi form is of rank 1 at the origin can be brought to the form*

$$u \,=\, z\overline{z} + z^2\overline{z}^2\,(\cdots) + \overline{z}^2\zeta\,(\cdots) + z^2\overline{\zeta}\,(\cdots) + \zeta\overline{\zeta}\,(\cdots).$$

Next, as said, we need more information about the appearing dependent derivatives in the remainder $\zeta\overline{\zeta}(\cdots)$. We start to really use the assumption that the Levi form of $M \in \mathfrak{G}_{2,1}$ has constant rank 1.

Lemma 14.3. *Any $\mathcal{C}^\omega$ hypersurface $0 \in M^5 \subset \mathbb{C}^3$ whose Levi form is of* constant rank 1 *around the origin can be brought to the form*

$$u \,=\, z\overline{z} + z^2\overline{z}^2\,\mathrm{O}_{z,\overline{z}}(0) + \overline{z}^2\zeta\,\mathrm{O}_{z,\zeta,\overline{z}}(0) + z^2\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(0) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2).$$

Proof. Indeed, from the equation of Corollary 14.2, rewritten by emphasizing the remainder R, which is *real*, as

$$u \,=\, z\overline{z} + z^2\overline{z}^2\,(\cdots) + \overline{z}^2\zeta\,(\cdots) + z^2\overline{\zeta}\,(\cdots) + \zeta\overline{\zeta}\,R,$$

the Levi determinant (3.1) writes

$$0 \,\equiv\, \begin{vmatrix} 0 & \overline{z}+\mathrm{O}(2) & \mathrm{O}(1) & -\frac{1}{2}+\mathrm{O}(2) \\ z+\mathrm{O}(2) & 1+\mathrm{O}(2) & \mathrm{O}(1) & \mathrm{O}(2) \\ \mathrm{O}(1) & \mathrm{O}(1) & \big[\zeta\overline{\zeta}R\big]_{\zeta\overline{\zeta}} & \mathrm{O}(1) \\ -\frac{1}{2}+\mathrm{O}(2) & \mathrm{O}(2) & \mathrm{O}(1) & \mathrm{O}(2) \end{vmatrix},$$

where, for abbreviation, we denote shortly $\mathrm{O}(\mathrm{N})$ in the places of $\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(\mathrm{N})$, with $\mathrm{N} \in \mathbb{N}$. Expanding the determinant along its first column and computing modulo $\mathrm{O}(2)$, we get

$$\begin{aligned} 0 \,\equiv\, & -\big(z+\mathrm{O}(2)\big) \begin{vmatrix} \overline{z}+\mathrm{O}(2) & \mathrm{O}(1) & -\frac{1}{2}+\mathrm{O}(2) \\ \mathrm{O}(1) & \big[\zeta\overline{\zeta}R\big]_{\zeta\overline{\zeta}} & \mathrm{O}(1) \\ \mathrm{O}(2) & \mathrm{O}(1) & \mathrm{O}(2) \end{vmatrix} \\ & + \mathrm{O}(1) \begin{vmatrix} \overline{z}+\mathrm{O}(2) & \mathrm{O}(1) & -\frac{1}{2}+\mathrm{O}(2) \\ 1+\mathrm{O}(2) & \mathrm{O}(1) & \mathrm{O}(2) \\ \mathrm{O}(2) & \mathrm{O}(1) & \mathrm{O}(2) \end{vmatrix} \\ & - \big(-\tfrac{1}{2}+\mathrm{O}(2)\big) \begin{vmatrix} \overline{z}+\mathrm{O}(2) & \mathrm{O}(1) & -\frac{1}{2}+\mathrm{O}(2) \\ 1+\mathrm{O}(2) & \mathrm{O}(1) & \mathrm{O}(2) \\ \mathrm{O}(1) & \big[\zeta\overline{\zeta}R\big]_{\zeta\overline{\zeta}} & \mathrm{O}(1) \end{vmatrix} \\ \,=\, & \mathrm{O}(2) + \mathrm{O}(2) - \tfrac{1}{4}\big[\zeta\overline{\zeta}\,R\big]_{\zeta\overline{\zeta}} + \mathrm{O}(2), \end{aligned}$$

whence

$$R + \zeta R_\zeta + \overline{\zeta} R_{\overline{\zeta}} = \mathrm{O}(2).$$

Then certainly $R = \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(1)$. Since $\overline{R} = R$ is real

$$R = z A(v) + \zeta B(v) + \overline{z}\,\overline{A}(v) + \overline{\zeta}\,\overline{B}(v) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2),$$

and replacing R, R_ζ, R_z above yields $0 \equiv A(v) \equiv 2\,B(v)$, so $R = \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2)$. □

15. Expression of the assumption of 2-nondegeneracy at the origin

Consequently, abbreviating $\alpha := F_{2,0,0,1,0} \in \mathbb{C}$, we may show cubic terms

$$u = z\overline{z} + \alpha\, z^2\overline{\zeta} + \overline{\alpha}\,\overline{z}^2\zeta + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(4).$$

Writing $u = \frac{1}{2}w + \frac{1}{2}\overline{w}$, and solving for w, we get

$$w = Q\big(z,\zeta,\overline{z},\overline{\zeta},\overline{w}\big) = -\overline{w} + 2\,z\overline{z} + 2\,\alpha\, z^2\overline{\zeta} + 2\,\overline{\alpha}\,\overline{z}^2\zeta + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},\overline{w}}(4).$$

Inserting this in the 3×3 invariant determinant of Proposition 3.2, we get, with $\mathrm{O}(\mathrm{N})$ abbreviating $\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},\overline{w}}(\mathrm{N})$

$$0 \neq \begin{vmatrix} Q_{\overline{z}} & Q_{\overline{\zeta}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{\zeta}} & Q_{z\overline{w}} \\ Q_{zz\overline{z}} & Q_{zz\overline{\zeta}} & Q_{zz\overline{w}} \end{vmatrix} = \begin{vmatrix} 2z + \mathrm{O}(2) & 2\alpha z^2 + \mathrm{O}(3) & -1 + \mathrm{O}(3) \\ 2 + \mathrm{O}(2) & 4\alpha z + \mathrm{O}(2) & \mathrm{O}(2) \\ \mathrm{O}(1) & 4\alpha + \mathrm{O}(1) & \mathrm{O}(1) \end{vmatrix}.$$

Expanding along the last column and computing modulo $\mathrm{O}(1)$

$$0 \neq -8\,\alpha + \mathrm{O}(1).$$

So the assumption of 2-nondegeneracy at the origin means that $\alpha \neq 0$. After the dilation $\zeta \longmapsto \frac{1}{2\alpha}\,\zeta$, we obtain

$$u = z\overline{z} + \tfrac{1}{2}\, z^2\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(4).$$

16. Prenormalization: step III

Thus, we have obtained the partial normalization

$$\begin{aligned} u = {} & z\overline{z} + \overline{z}^2\zeta\, F_{0,1,2,0}(v) + z^2\overline{\zeta}\, F_{2,0,0,1}(v) + z^2\overline{z}^2\, \mathrm{O}_{z,\overline{z}}(0) \\ & + \overline{z}^2\zeta\, \mathrm{O}_{z,\zeta,\overline{z}}(1) + z^2\overline{\zeta}\, \mathrm{O}_{z,\overline{z},\overline{\zeta}}(1) + \zeta\overline{\zeta}\, \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2), \end{aligned}$$

with $F_{0,1,2,0}(0) = \frac{1}{2} = F_{2,0,0,1}(0)$.

Lemma 16.1. *There exists a biholomorphism of the form*

$$z' := z, \qquad \zeta' := \zeta\,\psi(w), \qquad w' := w,$$

with $\psi(0) \neq 0$*, which normalizes* $F'_{0,1,2,0}(v') \equiv \frac{1}{2} \equiv F'_{2,0,0,1}(v')$

$$\begin{aligned} u' \;=\; & z'\overline{z}' + \tfrac{1}{2}\,\overline{z}'^{\,2}\zeta' + \tfrac{1}{2}\,z'^{2}\overline{\zeta}' + z'^{2}\overline{z}'^{\,2}\,\mathrm{O}_{z',\overline{z}'}(0) \\ & + \overline{z}'^{\,2}\zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(1) + z'^{2}\overline{\zeta}'\,\mathrm{O}_{z',\overline{z}',\overline{\zeta}'}(1) + \zeta'\overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(2). \end{aligned}$$

Proof. It is obvious that $\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(\mathrm{N}) = \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(\mathrm{N})$.

From the source equation

$$u \;=\; z\overline{z} + \overline{z}^{2}\zeta\,F_{0,1,2,0}(v) + z^{2}\overline{\zeta}\,F_{2,0,0,1}(v) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4),$$

with $F_{0,1,2,0}(0) = \frac{1}{2} = F_{2,0,0,1}(0)$, the target equation will be of a similar form

$$u' \;=\; z'\overline{z}' + \overline{z}'^{\,2}\zeta'\,F'_{0,1,2,0}(v') + z'^{2}\overline{\zeta}'\,F'_{2,0,0,1}(v') + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(4).$$

Since $u = F$ and $u = u' = F'$, the fundamental equation writes

$$\begin{aligned} 0 \;\equiv\; & -F\big(z,\zeta,\overline{z},\overline{\zeta},v\big) \\ & + F'\Big(z,\,\zeta\,\psi\big(F+iv\big),\;\overline{z},\;\zeta\,\overline{\psi}\big(F-iv\big),\;v\Big), \end{aligned}$$

that is

$$\begin{aligned} 0 \;\equiv\; & -z\overline{z} - \overline{z}^{2}\zeta\,F_{0,1,2,0}(v) - z^{2}\overline{\zeta}\,F_{2,0,0,1}(v) - \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4) \\ & + z\overline{z} + \overline{z}^{2}\zeta\,\psi\big(F+iv\big)\,F'_{0,1,2,0}(v) + z^{2}\overline{\zeta}\,\overline{\psi}\big(F-iv\big)\,F'_{2,0,0,1}(v) \\ & \qquad + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4). \end{aligned}$$

Next, by Taylor expanding at iv

$$\psi\big(F+iv\big) \;=\; \psi(iv) + F\,\big(\,\cdots\big) \;=\; \psi(iv) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2),$$

we get

$$\begin{aligned} 0 \;\equiv\; & -\overline{z}^{2}\zeta\,\Big(F_{0,1,2,0}(v) - \psi(iv)\,F'_{0,1,2,0}(v)\Big) \\ & - z^{2}\overline{\zeta}\,\Big(F_{2,0,0,1}(v) - \overline{\psi}\big(-iv\big)\,F'_{2,0,0,1}(v)\Big) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4). \end{aligned}$$

Thus, to normalize $F'_{0,1,2,0}(v) \equiv \frac{1}{2} \equiv F'_{2,0,0,1}(v)$, it suffices to set

$$\psi(w) \;:=\; 2\,F_{0,1,2,0}\big(-i\,w\big). \qquad \square$$

So erasing primes, we have normalized

$$\begin{aligned} u \;=\; & z\overline{z} + \tfrac{1}{2}\,\overline{z}^{2}\zeta + \tfrac{1}{2}\,z^{2}\overline{\zeta} + z^{2}\overline{z}^{2}\,F_{2,0,2,0}(v) + z^{2}\overline{z}^{2}\,\mathrm{O}_{z,\overline{z}}(1) \\ & + \overline{z}^{2}\zeta\,\mathrm{O}_{z,\zeta,\overline{\zeta}}(1) + z^{2}\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(1) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2). \quad (16.1) \end{aligned}$$

Our next goal is to eliminate $F_{2,0,2,0}(v)$.

Lemma 16.2. *There exists a biholomorphism of the form*

$$z' \,:=\, z, \qquad\qquad \zeta' \,:=\, \zeta + z^2\,\psi(w), \qquad\qquad w' \,:=\, w,$$

which normalizes $F'_{2,0,2,0}(v') \equiv 0$

$$\begin{aligned} u' \,=\, z'\overline{z}' + \tfrac{1}{2}\,\overline{z}'^{\,2}\zeta' + \tfrac{1}{2}\,z'^{2}\overline{\zeta}' + z'^{2}\overline{z}'^{\,2}\,\mathrm{O}_{z',\overline{z}'}(1) \\ + \overline{z}'^{\,2}\zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(1) + z'^{2}\overline{\zeta}'\,\mathrm{O}_{z',\overline{z}',\overline{\zeta}'}(1) + \zeta'\overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(2). \end{aligned}$$

Proof. In (16.1), extract the real term $F_{2,0,2,0}(v)$ and split it

$$\begin{aligned} u \,=\, z\overline{z} + \tfrac{1}{2}\,\overline{z}^{2}\big(\zeta + z^2\,F_{2,0,2,0}(v)\big) + \tfrac{1}{2}\,z^2\,\big(\overline{\zeta} + \overline{z}^{2}\,F_{2,0,2,0}(v)\big) + z^2\overline{z}^{2}\,\mathrm{O}_{z,\overline{z}}(1) \\ + \overline{z}^{2}\zeta\,\mathrm{O}_{z,\zeta,\overline{z}}(1) + z^2\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(1) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2). \end{aligned} \tag{16.2}$$

We claim that the biholomorphism which works is

$$z' \,:=\, z, \qquad\qquad \zeta' \,:=\, \zeta + z^2\,F_{2,0,2,0}\big(-i\,w\big), \qquad\qquad w' \,:=\, w.$$

The inverse is

$$\zeta \,=\, \zeta' - z'^{2}\,F_{2,0,2,0}\big(-i\,w'\big) \,=\, \zeta' + z'^{2}\,\big(\cdots\big).$$

We verify first that all remainders correspond to one another

$$\begin{aligned} z^2\overline{z}^{2}\,\mathrm{O}_{z,\overline{z}}(1) \,&=\, z'^{2}\overline{z}'^{\,2}\,\mathrm{O}_{z',\overline{z}'}(1), \\ \overline{z}^{2}\zeta\,\mathrm{O}_{z,\zeta,\overline{z}}(1) \,&=\, \overline{z}'^{\,2}\,\big(\zeta' + z'^{2}\,(\cdots)\big)\,\mathrm{O}_{z',\zeta',\overline{z}'}(1) \\ &=\, \overline{z}'^{\,2}\zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(1) + z'^{2}\overline{z}'^{\,2}\,\big[\mathrm{O}_{z',\overline{z}'}(1) + \zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(0)\big] \\ &=\, \overline{z}'^{\,2}\zeta'\,\mathrm{O}_{z',\zeta,\overline{z}'}(1) + z'^{2}\overline{z}'^{\,2}\,\mathrm{O}_{z',\overline{z}'}(1), \\ \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2) \,&=\, \big(\zeta' + z'^{2}\,(\cdots)\big)\,\big(\overline{\zeta}' + \overline{z}'^{\,2}\,(\cdots)\big)\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(2) \\ &=\, \zeta'\overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(2) + \zeta'\overline{z}'^{\,2}\,\big[\mathrm{O}_{z',\zeta',\overline{z}'}(2) + \overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(1)\big] \\ &\quad + \overline{\zeta}'\,z'^{2}\,\big[\mathrm{O}_{z',\overline{z}',\overline{\zeta}'}(2) + \zeta'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(1)\big] \\ &\quad + z'^{2}\overline{z}'^{\,2}\,\big[\mathrm{O}_{z',\overline{z}'}(2) + \zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(1) + \overline{\zeta}'\,\mathrm{O}_{z',\overline{z}',\overline{\zeta}'}(1) + \zeta'\overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(0)\big] \\ &=\, z'^{2}\overline{z}'^{\,2}\,\mathrm{O}_{z',\overline{z}'}(1) + \overline{z}'^{\,2}\zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(1) + z'^{2}\overline{\zeta}'\,\mathrm{O}_{z',\overline{z}',\overline{\zeta}'}(1) + \zeta'\overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(2). \end{aligned}$$

Next, using $0 \equiv F\big(0,0,\overline{z},\overline{\zeta},v\big)$, and Taylor expanding at v', we can write

$$\begin{aligned} \zeta \,&=\, \zeta' - z'^{2}\,F_{2,0,2,0}\big(v' - iF\big) \\ &=\, \zeta' - z'^{2}\,F_{2,0,2,0}(v') - z'^{2}\,F\,\big(\cdots\big) \\ &=\, \zeta' - z'^{2}\,F_{2,0,2,0}(v') - z'^{2}\,\big[z\,(\cdots) + \zeta\,(\cdots)\big] \\ &=\, \zeta' - z'^{2}\,F_{2,0,2,0}(v') - z'^{2}\,\big[z'\,(\cdots) + \zeta'(\cdots)\big]. \end{aligned}$$

Lastly, replacing $z, \zeta, \overline{z}, \overline{\zeta}, u, v$ in terms of $z', \zeta', \overline{z}', \overline{\zeta}', u', v'$ in (16.2), we obtain what was asserted

$$\begin{aligned}
u' \;=\; & z'\overline{z}' + \tfrac{1}{2}\overline{z}'^{2}\Big(\zeta' - \underline{z'^{2}\,F_{2,0,2,0}(v')}_{\circ} - z'^{3}\,(\cdots) - z'^{2}\zeta'\,(\cdots) + \underline{z'^{2}\,F_{2,0,2,0}(v')}_{\circ}\Big) \\
& + \tfrac{1}{2}z'^{2}\Big(\overline{\zeta}' - \underline{\overline{z}'^{2}\,F_{2,0,2,0}(v')}_{\circ\circ} - \overline{z}'^{3}\,(\cdots) - \overline{z}'^{2}\overline{\zeta}'\,(\cdots) + \underline{\overline{z}'^{2}\,F_{2,0,2,0}(v')}_{\circ\circ}\Big) \\
& + z'^{2}\overline{z}'^{2}\,\mathrm{O}_{z',\overline{z}'}(1) + \overline{z}'^{2}\zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(1) + z'^{2}\overline{\zeta}'\,\mathrm{O}_{z',\overline{z}',\overline{\zeta}'}(1) + \zeta'\overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(2) \\
=\; & z'\overline{z}' + \tfrac{1}{2}\overline{z}'^{2}\zeta' + \tfrac{1}{2}z'^{2}\overline{\zeta}' \\
& + z'^{2}\overline{z}'^{2}\,\mathrm{O}_{z',\overline{z}'}(1) + \overline{z}'^{2}\zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(1) + z'^{2}\overline{\zeta}'\,\mathrm{O}_{z',\overline{z}',\overline{\zeta}'}(1) + \zeta'\overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(2). \qquad \square
\end{aligned}$$

Thus, dropping primes, we have reached the following normalization, where we show all monomials in F which have $\overline{z}^{2}$ as only antiholomorphic part

$$\begin{aligned}
u \;=\; & z\overline{z} + \tfrac{1}{2}\overline{z}^{2}\zeta + \tfrac{1}{2}z^{2}\overline{\zeta} + \sum_{\substack{a+c\geqslant 5\\ a\geqslant 2,\,c\geqslant 2}} z^{a}\overline{z}^{c}\,F_{a,0,c,0}(v) + \sum_{\substack{a+b+c\geqslant 4\\ b\geqslant 1,\,c\geqslant 2}} z^{a}\zeta^{b}\overline{z}^{c}\,F_{a,b,c,0}(v) \\
& + \sum_{\substack{a+c+d\geqslant 4\\ a\geqslant 2,\,d\geqslant 1}} z^{a}\overline{z}^{c}\overline{\zeta}^{d}\,F_{a,0,c,d}(v) + \sum_{\substack{a+b+c+d\geqslant 4\\ b\geqslant 1,\,d\geqslant 1}} z^{a}\zeta^{b}\overline{z}^{c}\overline{\zeta}^{d}\,F_{a,b,c,d}(v).
\end{aligned}$$

Now, we will work modulo $\overline{z}^{3}\,(\cdots) + \overline{\zeta}\,(\cdots)$, so the last two sums above disappear and many terms in the first two sums as well, so that it remains

$$\begin{aligned}
u \;=\; & z\overline{z} + \tfrac{1}{2}\overline{z}^{2}\Big[\zeta + 2\sum_{a\geqslant 3} z^{a}\,F_{a,0,2,0}(v) + 2\sum_{\substack{a+b\geqslant 2\\ b\geqslant 1}} z^{a}\zeta^{b}\,F_{a,b,2,0}(v)\Big] \\
& + \overline{z}^{3}\,(\cdots) + \overline{\zeta}\,(\cdots). \qquad (16.3)
\end{aligned}$$

Lemma 16.3. *The biholomorphism*

$$\begin{aligned}
z' \;&:=\; z, \\
\zeta' \;&:=\; \zeta + 2\sum_{a\geqslant 3} z^{a}\,F_{a,0,2,0}\big(-i\,w\big) + 2\sum_{\substack{a+b\geqslant 2\\ b\geqslant 1}} z^{a}\zeta^{b}\,F_{a,b,2,0}\big(-i\,w\big), \\
w' \;&:=\; w,
\end{aligned}$$

transforms M into M' of equation

$$u' \;=\; z'\overline{z}' + \tfrac{1}{2}\overline{z}'^{2}\zeta' + \tfrac{1}{2}z'^{2}\overline{\zeta}' + \overline{z}'^{3}\,(\cdots) + \overline{\zeta}'\,(\cdots).$$

Proof. As in [19], we write

$$\zeta' \;:=\; \zeta + \tau(z,w) + \zeta\,\omega(z,\zeta,w),$$

where

$$\tau \;=\; z^{3}\,(\cdots) \qquad \text{and} \qquad \omega \;=\; \mathrm{O}_{z,\zeta,w}(1).$$

The inverse is certainly of the form $\zeta = \zeta' + \mathrm{O}_{z',\zeta',w'}(2)$, hence

$$\zeta \;=\; \zeta' + \tau'(z',w') + \zeta'\,\omega'(z',\zeta',w'),$$

with $\tau' = \mathrm{O}_{z',w'}(2)$ and $\omega' = \mathrm{O}_{z',\zeta',w'}(1)$. We claim that $\tau' = {z'}^3(\cdots)$.

Indeed, replacing $\zeta' = \tau(z,w) + \zeta\,[1+\omega(z,\zeta,\omega)]$ into $\zeta = \tau'(z',w') + \zeta'\,[1+\omega'(z',\zeta',w')]$, the following identity must hold in $\mathbb{C}\{z,\zeta,w\}$

$$\begin{aligned}\zeta \;\equiv\; \tau'(z,w) &+ \big(\tau(z,w) + \zeta\,[1+\omega(z,\zeta,w)]\big)\cdot \\ &\cdot\Big[1+\omega'\big(z,\ \tau(z,w)+\zeta\,[1+\omega(z,\zeta,w)],\ w\big)\Big].\end{aligned}$$

Putting $\zeta := 0$, it comes

$$0 \;\equiv\; \tau'(z,w) + \tau(z,w)\big[1+\mathrm{O}_{z,w}(1)\big] \;\equiv\; \tau'(z,w) + z^3(\cdots)\big[1+\mathrm{O}_{z,w}(1)\big].$$

Thus $\zeta = \zeta'(\cdots) + {z'}^3(\cdots)$, which enables us to verify that remainders correspond as follows

$$\begin{aligned}\overline{\zeta}\,(\cdots) &= \overline{\zeta}'\,(\cdots) + \overline{z}'^3\,(\cdots),\\ \overline{z}^3\,(\cdots) &= \overline{z}'^3\,(\cdots).\end{aligned}$$

Next, using $0 \equiv F(z,\zeta,0,0,0)$, so that $F = \overline{z}(\cdots) + \overline{\zeta}(\cdots) = \overline{z}'(\cdots) + \overline{\zeta}'(\cdots)$, we have

$$\begin{aligned}\zeta' &= \zeta + 2\sum_{a\geqslant 3} z^a F_{a,0,2,0}\big(v-iF\big) + 2\sum_{\substack{a+b\geqslant 2\\ b\geqslant 1}} z^a\zeta^b F_{a,b,2,0}\big(v-iF\big)\\ &= \zeta + 2\sum_{a\geqslant 3} z^a F_{a,0,2,0}(v) + F\,(\cdots) + 2\sum_{\substack{a+b\geqslant 2\\ b\geqslant 1}} z^a\zeta^b F_{a,b,2,0}(v) + F\,(\cdots).\end{aligned}$$

Lastly, coming back to (16.3), we conclude

$$\begin{aligned}u' \;=\; u \;&=\; z'\overline{z}' + \tfrac{1}{2}\overline{z}'^2\big[\zeta' - \overline{\zeta}\,(\cdots) - \overline{z}\,(\cdots)\big] + \overline{z}'^3\,(\cdots) + \overline{\zeta}'\,(\cdots)\\ &=\; z'\overline{z}' + \tfrac{1}{2}\overline{z}'^2\zeta' + \overline{z}'^3\,(\cdots) + \overline{\zeta}'\,(\cdots).\end{aligned}$$

□

Erasing primes, and using the fact that the graphing function is real, we obtain

$$u \;=\; z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z^3\overline{z}^3\,(\cdots) + \overline{z}^3\zeta\,(\cdots) + z^3\overline{\zeta}\,(\cdots) + \zeta\overline{\zeta}\,(\cdots).$$

It remains only to analyze the dependent-derivatives remainder $\zeta\overline{\zeta}(\cdots)$. For this, we must extract the single 4^{th} order monomial $z\overline{z}\zeta\overline{\zeta}$ in the GM-model $m(z,\zeta,\overline{z},\overline{\zeta})$. Then we realize that behind $\zeta\overline{\zeta}(\cdots)$, there must be order 3 terms only.

Proposition 16.4. [Prenormalization] *Any hypersurface $M^5 \in \mathfrak{C}_{2,1}$ can be brought to the prenormal form*

$$\begin{aligned}u \;=\; z\overline{z} &+ \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta}\\ &+ z^3\overline{z}^3\,\mathrm{O}_{z,\overline{z}}(0) + \overline{z}^3\zeta\,\mathrm{O}_{z,\zeta,\overline{z}}(0) + z^3\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(0) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(3).\end{aligned}$$

Proof. We write

$$u = \overline{z}z + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + z^3\overline{z}^3\,(\cdots) + \overline{z}^3\zeta\,(\cdots) + z^3\overline{\zeta}\,(\cdots) + \zeta\overline{\zeta}\,R.$$

From Lemma 14.3, we already know that $R = \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2)$.

To get more, we look at the Levi determinant

$$0 \equiv \begin{vmatrix} 0 & \overline{z} + z\overline{\zeta} + \mathrm{O}(3) & \frac{1}{2}\overline{z}^2 + \mathrm{O}(3) & -\frac{1}{2} + \mathrm{O}(4) \\ z + \overline{z}\zeta + \mathrm{O}(3) & 1 + \mathrm{O}(2) & \overline{z} + \mathrm{O}(2) & \mathrm{O}(3) \\ \frac{1}{2}z^2 + \mathrm{O}(3) & z + \mathrm{O}(2) & z\overline{z} + \big[\zeta\overline{\zeta}R\big]_{\zeta\overline{\zeta}} & \mathrm{O}(3) \\ -\frac{1}{2} + \mathrm{O}(4) & \mathrm{O}(3) & \mathrm{O}(3) & \mathrm{O}(4) \end{vmatrix}.$$

Computing modulo $\mathrm{O}(3)$, so that the entries $(2,4)$, $(3,4)$, $(4,2)$, $(4,3)$, $(4,4)$ are 'zero', we get

$$0 \equiv -\big(-\tfrac{1}{2}\big)\big(-\tfrac{1}{2}\big) \begin{vmatrix} 1 + \mathrm{O}(2) & \overline{z} + \mathrm{O}(2) \\ z + \mathrm{O}(2) & z\overline{z} + \big[\zeta\overline{\zeta}R\big]_{\zeta\overline{\zeta}} \end{vmatrix} + \mathrm{O}(3),$$

that is

$$\big[\zeta\overline{\zeta}R\big]_{\zeta\overline{\zeta}} \equiv \mathrm{O}(3).$$

Thanks to the already known $R = \mathrm{O}(2)$

$$\begin{aligned} R \;=\; & A\,zz + B\,z\zeta + C\,z\overline{z} + D\,z\overline{\zeta} + E\,\zeta\zeta + \overline{D}\,\zeta\overline{z} \\ & + G\,\zeta\overline{\zeta} + \overline{A}\,\overline{z}z + \overline{B}\,\overline{z}\overline{\zeta} + \overline{E}\overline{\zeta\zeta} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(3), \end{aligned}$$

with both $\overline{C} = C$ and $\overline{G} = G$ real, hence

$$\begin{aligned} \mathrm{O}(3) \;\equiv\; & R + \zeta\,R_{\zeta} + \overline{\zeta}\,R_{\overline{\zeta}} + \zeta\overline{\zeta}\,R_{\zeta\overline{\zeta}} \\ \equiv\; & A\,zz + 2B\,z\zeta + \big(\overline{A} + C\big)\,z\overline{z} + 2D\,z\overline{\zeta} + 3E\,\zeta\zeta + 2\overline{D}\,\zeta\overline{z} \\ & + 4G\,\zeta\overline{\zeta} + 2\overline{B}\,\overline{z}\overline{\zeta} + 3\overline{E}\,\overline{\zeta\zeta}, \end{aligned}$$

and this forces $A = B = C = D = E = G = 0$, whence $R = \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(3)$. □

17. Normalization $F_{3,0,0,1}(v) = 0$

Now, we specify the unique term of order 4 in $(z, \zeta, \overline{z}, \overline{\zeta})$

$$u \;=\; z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + z^3\overline{\zeta}\,F_{3,0,0,1}(v) + \overline{z}^3\zeta\,\overline{F_{3,0,0,1}(v)} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(5).$$

Abbreviate

$$\varphi(v) \;:=\; F_{3,0,0,1}(v).$$

Lemma 17.1. *The biholomorphism*

$$\begin{aligned} z' \;&:=\; z + z^2\,\varphi(-iw) + 2\,z^3\,\varphi(-iw)\,\varphi(-iw), \\ \zeta' \;&:=\; \zeta - 2\,z\,\overline{\varphi}(-iw) + 4\,z\zeta\,\varphi(-iw) - 5\,z^2\,\varphi(-iw)\,\overline{\varphi}(-iw), \\ w' \;&:=\; w, \end{aligned}$$

transforms M into M' of equation

$$u' \,=\, z'\overline{z}' + \tfrac{1}{2}\,\overline{z}'^{\,2}\zeta' + \tfrac{1}{2}\,z'^{2}\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}' + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(5).$$

Proof. On restriction to M where $-iw = v - iF$

$$\begin{aligned} z' \,&:=\, z + z^2\,\varphi\big(v-iF\big) + 2\,z^3\,\varphi\big(v-iF\big)\,\varphi\big(v-iF\big), \\ \zeta' \,&:=\, \zeta - 2\,z\,\overline{\varphi}\big(v-iF\big) + 4\,z\zeta\,\varphi\big(v-iF\big) - 5\,z^2\,\varphi\big(v-iF\big)\,\overline{\varphi}\big(v-iF\big), \end{aligned}$$

hence Taylor expanding at v and using $F = \mathrm{O}(2)$

$$\begin{aligned} z' \,&=\, z + z^2\,\varphi(v) + 2\,z^3\,\varphi(v)\,\varphi(v) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4), \\ \zeta' \,&=\, \zeta - 2\,z\,\overline{\varphi}(v) + 4\,z\zeta\,\varphi(v) - 5\,z^2\,\varphi(v)\,\overline{\varphi}(v) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(3). \end{aligned}$$

An expansion concludes

$$\begin{aligned} &z'\overline{z}' + \tfrac{1}{2}\,\overline{z}'^{\,2}\zeta' + \tfrac{1}{2}\,z'^{2}\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}' + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(5) \,= \\ &= \Big(z + z^2\,\varphi(v) + 2\,z^3\,\varphi(v)\,\varphi(v)\Big)\Big(\overline{z} + \overline{z}^{\,2}\,\overline{\varphi}(v) + 2\,\overline{z}^{\,3}\,\overline{\varphi}(v)\,\overline{\varphi}(v)\Big) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(5) \\ &\quad + \tfrac{1}{2}\Big(\overline{z} + \overline{z}^{\,2}\,\overline{\varphi}(v)\Big)^2\Big(\zeta - 2\,z\,\overline{\varphi}(v) + 4\,z\zeta\,\varphi(v) - 5\,z^2\,\varphi(v)\,\overline{\varphi}(v)\Big) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(5) \\ &\quad + \tfrac{1}{2}\Big(z + z^2\,\varphi(v)\Big)^2\Big(\overline{\zeta} - 2\,\overline{z}\,\varphi(v) + 4\,\overline{z}\overline{\zeta}\,\overline{\varphi}(v) - 5\,\overline{z}^{\,2}\,\overline{\varphi}(v)\,\varphi(v)\Big) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(5) \\ &\quad + z\overline{z}\,\big(\zeta - 2\,z\,\overline{\varphi}(v)\big)\big(\overline{\zeta} - 2\,\overline{z}\,\varphi(v)\big) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(5) \\ &= z\overline{z} + \tfrac{1}{2}\,\overline{z}^{\,2}\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + z^3\overline{\zeta}\,\varphi(v) + \overline{z}^{\,3}\zeta\,\overline{\varphi}(v) + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(5). \end{aligned}$$

□

After this, although $F_{a,b,0,0}(v) \equiv 0$ for all (a,b), it is not necessarily still true that prenormalization holds

$$\begin{aligned} 0 \,&\overset{?}{\equiv}\, F_{a,b,1,0}(v) \qquad\qquad (\forall\,(a,b) \,\neq\, (1,0)), \\ 0 \,&\overset{?}{\equiv}\, F_{a,b,2,0}(v) \qquad\qquad (\forall\,(a,b) \,\neq\, (0,1)). \end{aligned}$$

18. Repetition of prenormalization

Fortunately, we can repeat the prenormalization. Indeed, let us write

$$u \,=\, z\overline{z} + \tfrac{1}{2}\,\overline{z}^{\,2}\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \sum_{\substack{a+b+c+d\geqslant 5 \\ a+b\geqslant 1,\,c+d\geqslant 1}} z^a\zeta^b\overline{z}^{\,c}\overline{\zeta}^{\,d}\,F_{a,b,c,d}(v).$$

We will perform two biholomorphisms of the form

$$z' \,:=\, z + \mathrm{O}_{z,\zeta}(4), \qquad\qquad \zeta' \,:=\, \zeta + \mathrm{O}_{z,\zeta}(3), \qquad\qquad w' \,=\, w,$$

so that normalizations of terms up to order 4 included will be stabilized and preserved.

Starting from

$$u \,=\, \overline{z}\Big(z + \sum_{a+b\geqslant 4} z^a\zeta^b\, F_{a,b,1,0}(v)\Big) + \overline{z}^{\,2}\,(\cdots) + \overline{\zeta}\,(\cdots),$$

we perform the following first biholomorphism, with $z' := z+\mathrm{O}_{z,\zeta}(4)$, $\zeta' := \zeta$, $w' := w$, which we restrict to M, using $F = \overline{z}(\cdots) + \overline{\zeta}(\cdots)$

$$\begin{aligned} z' \,&:=\, z + \sum_{a+b\geqslant 4} z^a\zeta^b\, F_{a,b,1,0}(-iw) \\ &= z + \sum_{a+b\geqslant 4} z^a\zeta^b \Big[F_{a,b,1,0}(v) + F\,(\cdots)\Big] \\ &= z + \sum_{a+b\geqslant 4} z^a\zeta^b\, F_{a,b,1,0}(v) + \overline{z}\,(\cdots) + \overline{\zeta}\,(\cdots), \end{aligned}$$

hence

$$z' - \overline{z}'\,(\cdots) - \overline{\zeta}'\,(\cdots) \,=\, z + \sum_{a+b\geqslant 4} z^a\zeta^b\, F_{a,b,1,0}(v),$$

so we can replace, using $z' \,=\, z + z^4(\cdots) + \zeta(\cdots)$ which gives by inversion $z = z' + {z'}^4(\cdots) + \zeta'(\cdots)$

$$\begin{aligned} u' \,=\, u \,&=\, \big(\overline{z}' + \overline{z}'^{\,4}\,(\cdots) + \overline{\zeta}'\,(\cdots)\big)\big(z' - \overline{z}'\,(\cdots) - \overline{\zeta}\,(\cdots)\big) \\ &\qquad\qquad + \overline{z}'^{\,2}\,(\cdots) + \overline{\zeta}'\,(\cdots) \\ &= \overline{z}'\,z' + \overline{z}'^{\,2}\,(\cdots) + \overline{\zeta}'\,(\cdots). \end{aligned}$$

Next, erase primes, specify terms having $\overline{z}^{\,2}$ as only antiholomorphic part

$$u \,=\, z\overline{z} + \tfrac{1}{2}\Big(\zeta + 2\sum_{a+b\geqslant 3} z^a\zeta^b\, F_{a,b,2,0}(v)\Big)\overline{z}^{\,2} + \overline{z}^{\,3}\,(\cdots) + \overline{\zeta}\,(\cdots),$$

and perform the second biholomorphism

$$z' \,:=\, z, \qquad\qquad \zeta' \,:=\, \zeta + 2\sum_{a+b\geqslant 3} z^a\zeta^b\, F_{a,b,2,0}(-iw), \qquad\qquad w' \,:=\, w.$$

Since $-iw = v - iF$ on M, using $F = \overline{z}(\cdots) + \overline{\zeta}(\cdots)$, we have

$$\begin{aligned} \zeta' \,&=\, \zeta + 2\sum_{a+b\geqslant 3} z^a\zeta^b\, F_{a,b,2,0}(v - iF) \\ &= \zeta + 2\sum_{a+b\geqslant 3} z^a\zeta^b\, F_{a,b,2,0}(v) + \overline{z}\,(\cdots) + \overline{\zeta}\,(\cdots), \end{aligned}$$

hence after an inversion

$$\zeta' - \overline{z}'\,(\cdots) - \overline{\zeta}'\,(\cdots) \,=\, \zeta + 2\sum_{a+b\geqslant 3} z^a\zeta^b\, F_{a,b,2,0}(v).$$

So using $\zeta' = \zeta + z^3(\cdots) + \zeta\,\mathrm{O}_{z,\zeta}(2)$ which gives after inversion $\zeta = \zeta' + z'^3(\cdots) + \zeta'\mathrm{O}_{z',\zeta'}(2)$, and observing that remainders correspond to one another, we can replace

$$\begin{aligned} u' \,=\, u \,&=\, z'\overline{z}' + \tfrac{1}{2}\overline{z}'^2\big(\zeta' - \overline{z}'(\cdots) - \overline{\zeta}'(\cdots)\big) + \overline{z}'^3\big(\cdots\big) + \overline{\zeta}'\big(\cdots\big) \\ &=\, z'\overline{z}' + \tfrac{1}{2}\overline{z}'^2\zeta' + \overline{z}'^3\big(\cdots\big) + \overline{\zeta}'\big(\cdots\big). \end{aligned}$$

Since terms are unchanged up to order 5, and since the right-hand side is real, we have reached

$$\begin{aligned} u' \,=\, z'\overline{z}' + \tfrac{1}{2}\overline{z}'^2\zeta' + \tfrac{1}{2}z'^2\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}' + z'^3\overline{z}'^3\,\mathrm{O}_{z',\overline{z}'}(0) + \overline{z}'^3\zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(1) \\ + z'^3\overline{\zeta}'\,\mathrm{O}_{z',\overline{z}',\overline{\zeta}'}(1) + \zeta'\overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(3). \end{aligned}$$

Lemma 18.1. *Starting from*

$$u \,=\, z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \sum_{\substack{a+b+c+d\geqslant 5\\ a+b\geqslant 1,\,c+d\geqslant 1}} z^a\zeta^b\overline{z}^c\overline{\zeta}^d\,F_{a,b,c,d}(v),$$

there exists a biholomorphism of the form

$$z' \,=\, z + \mathrm{O}_{z,\zeta}(4), \qquad\qquad \zeta' \,=\, \zeta + \mathrm{O}_{z,\zeta}(3), \qquad\qquad w' \,:=\, w,$$

which transforms M into M' of equation

$$\begin{aligned} u' \,=\, z'\overline{z}' + \tfrac{1}{2}\overline{z}'^2\zeta' + \tfrac{1}{2}z'^2\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}' + z'^3\overline{z}'^3\,\mathrm{O}_{z',\overline{z}'}(0) + \overline{z}'^3\zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(1) \\ + z'^3\overline{\zeta}'\,\mathrm{O}_{z',\overline{z}',\overline{\zeta}'}(1) + \zeta'\overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(4). \end{aligned}$$

Proof. The only modification is the information about the dependent jets remainder being an O(4) after $\zeta'\overline{\zeta}'$, which improves the previous O(3). The proof consists in examining the Levi determinant, and proceeds similarly as at the end of the proof of Proposition 16.4. □

19. Normalization $F_{3,0,1,1}(v) = 0$

Including order 5 terms from $z^3\overline{\zeta}\mathrm{O}_{z,\overline{z},\overline{\zeta}}(1)$, three new terms appear

$$\begin{aligned} u \,=\, z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}z^2\overline{\zeta}\zeta\overline{\zeta} \\ + 2\,\mathrm{Re}\Big\{z^3\overline{z}\overline{\zeta}\,F_{3,0,1,1}(v) + z^4\overline{\zeta}\,F_{4,0,0,1}(v) + z^3\overline{\zeta}^2\,F_{3,0,0,2}(v)\Big\} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(6), \end{aligned} \tag{19.1}$$

and we gather all remainder terms as an O(6).

Lemma 19.1. *There exists a biholomorphism of the form*

$$z' := z, \qquad \zeta' := \zeta + i\,\varphi(-iw)\,z^2, \qquad w' := w,$$

with $\varphi(v) \in \mathbb{R}$ for $v \in \mathbb{R}$, which normalizes

$$\operatorname{Im} F'_{3,0,1,1}(v') \equiv 0.$$

Proof. On restriction to M, the inverse writes

$$\begin{aligned}
\zeta &= \zeta' - i\,\varphi(-iw)\,z'^2 \\
&= \zeta' - i\,\varphi\big(v - iF\big)\,z'^2 \\
&= \zeta' - i\,\varphi(v)\,z'^2 + z'^2\,F\big(\cdots\big) \\
&= \zeta' - i\,\varphi(v')\,z'^2 + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(4).
\end{aligned}$$

So we insert in (19.1) and we conclude

$$\begin{aligned}
u' &= u \\
&= z'\overline{z}' + \tfrac{1}{2}\,\overline{z}'^2\,\big(\zeta' - i\,\varphi(v')\,z'^2 + \mathrm{O}(4)\big) + \tfrac{1}{2}\,z'^2\,\big(\overline{\zeta}' + i\,\overline{\varphi}(v')\,\overline{z}'^2 + \mathrm{O}(4)\big) \\
&\quad + z'\overline{z}'\,\big(\zeta' - i\,\varphi(v')\,z'^2\big)\,\big(\overline{\zeta}' + i\,\overline{\varphi}(v)\,\overline{z}'^2\big) + \tfrac{1}{2}\,\overline{z}'^2\,\zeta'\zeta'\overline{\zeta}' + \tfrac{1}{2}\,z'^2\overline{\zeta}'\,\zeta'\overline{\zeta}' \\
&\quad + 2\,\mathrm{Re}\Big\{z'^3\overline{z}'\overline{\zeta}'\,F_{3,0,1,1}(v') + z'^4\overline{\zeta}'\,F_{4,0,0,1}(v') + z'^3\overline{\zeta}'^2\,F_{3,0,0,2}(v')\Big\} + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(6) \\
&= z'\overline{z}' + \tfrac{1}{2}\,\overline{z}'^2\zeta' + \tfrac{1}{2}\,z'^2\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}' + \tfrac{1}{2}\,\overline{z}'^2\zeta'\zeta'\overline{\zeta}' + \tfrac{1}{2}\,z'^2\overline{\zeta}'\,\zeta'\overline{\zeta}' \\
&\quad + z'^2\overline{z}'^2\,\big[-\tfrac{i}{2}\,\varphi(v') + \tfrac{i}{2}\,\overline{\varphi}(v')\big] \\
&\quad + 2\,\mathrm{Re}\Big\{z'^3\overline{z}'\overline{\zeta}'\,\big[F_{3,0,1,1}(v') - i\,\varphi(v')\big] + z'^4\overline{\zeta}'\,F_{4,0,0,1}(v') + z'^3\overline{\zeta}'^2\,F_{3,0,0,2}(v')\Big\} \\
&\quad + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(6).
\end{aligned}$$

□

Breaking routine, we do not erase primes.

Lemma 19.2. *There exists a biholomorphism whose inverse is of the form*

$$z' := z\,e^{i\varphi(-iw)}, \qquad \zeta' := \zeta\,e^{2i\varphi(-iw)} + \psi(-iw)\,z^2, \qquad w' := w,$$

with $\varphi(v) \in \mathbb{R}$ for $v \in \mathbb{R}$, which normalizes $u' = F'$ above to $u = F$ of the same shape, but with

$$\operatorname{Re} F'_{3,0,1,1}(v) \equiv 0.$$

Proof. Start with

$$\begin{aligned}
z'\overline{z}' &= z\overline{z}\,e^{i[\varphi(v-iF)-\overline{\varphi}(v+iF)]} \\
&= z\overline{z}\,e^{i[\varphi(v)+\varphi_v(v)(-iF)+F^2(\cdots)-\overline{\varphi}(v)-\overline{\varphi}_v(v)(iF)-F^2(\cdots)]} \\
&= z\overline{z}\,e^{2\varphi_v(v)\,F+F^2(\cdots)} \\
&= z\overline{z}\,\big(1 + 2\,\varphi_v(v)\,F + \mathrm{O}(4)\big) \\
&= z\overline{z} + 2\,\varphi_v(v)\,z^2\overline{z}^2 + \varphi_v(v)\,z\zeta\overline{z}^3 + \varphi_v(v)\,z^3\overline{z}\overline{\zeta} + \mathrm{O}(6).
\end{aligned}$$

Next

$$\begin{aligned}
\operatorname{Re}\big(\overline{z}'^{\,2}\zeta'\big) &= \operatorname{Re}\Big(\overline{z}^2\, e^{-2i\overline{\varphi}(i\overline{w})}\big[\zeta\, e^{2i\varphi(-iw)} + \psi(-iw)\, z^2\big]\Big) \\
&= \operatorname{Re}\Big(\overline{z}^2\zeta\, e^{2i[-\overline{\varphi}(v+iF)+\varphi(v-iF)]} + z^2\overline{z}^2\, e^{-2i\overline{\varphi}(v+iF)}\,\psi(v-iF)\Big) \\
&= \operatorname{Re}\Big(\overline{z}^2\zeta\, e^{2i[-\overline{\varphi}(v)-\overline{\varphi}_v(v)(iF)-F^2(\cdots)+\varphi(v)+\varphi_v(v)(-iF)+F^2(\cdots)]}\Big) \\
&\qquad\qquad + z^2\overline{z}^2\,\psi(v) + \mathrm{O}(6) \\
&= \operatorname{Re}\Big(\overline{z}^2\zeta\, e^{2i[\overline{\varphi}_v(v)+\varphi_v(v)]\,F+F^2(\cdots)}\Big) + z^2\overline{z}^2\,\psi(v) + \mathrm{O}(6) \\
&= \operatorname{Re}\Big(\overline{z}^2\zeta\,\big[1+4\,\varphi_v(v)\,\big(z\overline{z}+\mathrm{O}(3)\big)+\mathrm{O}(4)\big]\Big) + z^2\overline{z}^2\,\psi(v) + \mathrm{O}(6) \\
&= \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z^2\overline{z}^2\,\psi(v) + 2\,\varphi_v(v)\,\overline{z}^3 z\zeta + 2\,\varphi_v(v)\,z^3\overline{z}\overline{\zeta} + \mathrm{O}(6).
\end{aligned}$$

Lastly

$$\begin{aligned}
z'\overline{z}'\zeta'\overline{\zeta}' &= \big(z\overline{z}+\mathrm{O}(4)\big)\,\big(\zeta\, e^{2i\varphi(v)+F(\cdots)} + (\psi(v)+F(\cdots))\,z^2\big)\cdot \\
&\qquad\qquad \cdot\big(\overline{\zeta}\, e^{-2i\overline{\varphi}(v)+F(\cdots)} + (\overline{\psi}(v)+F(\cdots))\,\overline{z}^2\big) \\
&= z\overline{z}\,\big(\zeta+\psi(v)\,z^2\big)\,\big(\overline{\zeta}+\overline{\psi}(v)\,\overline{z}^2\big) + \mathrm{O}(6) \\
&= z\overline{z}\zeta\overline{\zeta} + z\overline{z}\zeta\overline{z}^2\,\overline{\psi}(v) + z\overline{z}\,\psi(v)\,z^2\overline{\zeta} + \mathrm{O}(6).
\end{aligned}$$

Summing, we conclude by taking $\psi(v) := -2\,\varphi_v(v)$ and by taking $\varphi_v(v) := -\operatorname{Re} F'_{3,0,1,1}(v)$

$$\begin{aligned}
u' &= z'\overline{z}' + \tfrac{1}{2}\,\overline{z}'^{\,2}\zeta' + \tfrac{1}{2}\,z'^{\,2}\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}' + \tfrac{1}{2}\,\overline{z}'^{\,2}\zeta'\zeta'\overline{\zeta}' + \tfrac{1}{2}\,z'^{\,2}\overline{\zeta}'\zeta'\overline{\zeta}' \\
&\quad + 2\operatorname{Re}\Big\{F'_{4,0,0,1}(v')\,z'^{\,3}\overline{z}'\overline{\zeta}' + F'_{3,0,0,1}z'^{\,3}\overline{z}'\overline{\zeta}' + F'_{3,0,0,2}(v')\,z'^{\,3}\overline{\zeta}'^{\,2}\Big\} + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(6) \\
&= z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} \\
&\quad + z^2\overline{z}^2\,\big[2\,\varphi_v(v)+\psi(v)\big] \\
&\quad + 2\operatorname{Re}\Big\{2\,\varphi_v(v)+\psi(v)+\varphi_v(v)+F'_{3,0,1,1}(v)\Big\} \\
&\quad + 2\operatorname{Re}\Big\{F'_{4,0,0,1}(v)\,z^4\overline{\zeta} + F'_{3,0,0,2}(v)\,z^3\overline{\zeta}^2\Big\} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(6). \qquad \square
\end{aligned}$$

Proposition 19.3. *For every hypersurface $M^5 \in \mathfrak{C}_{2,1}$, at any point $p \in M$, given any CR-transversal curve $p \in \gamma \subset M$, there exist holomorphic coordinates $(z,\zeta,w) \in \mathbb{C}^3$ vanishing at p in which γ is the v-axis and in which M has equation*

$$\begin{aligned}
u &= z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} \\
&\quad + z^3\overline{z}^3\,\mathrm{O}_{z,\overline{z}}(0) \\
&\quad + 2\operatorname{Re}\Big\{0 + z^4\overline{\zeta}\,F_{4,0,0,1}(v) + z^3\overline{\zeta}^2\,F_{3,0,0,2}(v)\Big\} \\
&\quad + \overline{z}^3\zeta\,\mathrm{O}_{z,\zeta,\overline{\zeta}}(2) + z^3\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(2) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4).
\end{aligned}$$

Proof. The annihilation of $F_{3,0,1,1}(v) \equiv 0$ has been performed above. After that, it is necessary to repeat prenormalization, as was done in Section 18, and this does not perturb the normalizations done up to order 5 in $(z, \zeta, \overline{z}, \overline{\zeta})$.

Lastly, it remains to justify the vanishing order 4 of the dependent-derivatives remainder $\zeta\overline{\zeta}(\cdots)$. This can be done by examining the Levi determinant (3.1), similarly as was done in *e.g.* the proof of Proposition 16.4. □

20. Normalizations at the origin

Now, we work at the origin. Expanding now in terms of all five variables $(z, \zeta, \overline{z}, \overline{\zeta}, v)$, and working modulo *weighted* order 6 terms, for the weights $[z] = 1$, $[\zeta] = 1$, $[w] = 2$, we have obtained

$$\begin{aligned} u \;=\; & z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}z^2\overline{\zeta}\zeta\overline{\zeta} \\ & + z^4\overline{\zeta}\,F_{4,0,0,1,0} + \overline{z}^4\zeta\,\overline{F_{4,0,0,1,0}} + z^3\overline{\zeta}^2\,F_{3,0,0,2,0} + \overline{z}^3\zeta^2\,\overline{F_{3,0,0,2,0}} \\ & + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(6). \end{aligned}$$

To normalize further, we can assume that the target hypersurface has already been normalized in the same way

$$\begin{aligned} u' \;=\; & z'\overline{z}' + \tfrac{1}{2}\overline{z}'^2\zeta' + \tfrac{1}{2}z'^2\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}' + \tfrac{1}{2}\overline{z}'^2\zeta'\zeta'\overline{\zeta}' + \tfrac{1}{2}z'^2\overline{\zeta}'\zeta'\overline{\zeta}' \\ & + z'^4\overline{\zeta}'\,F'_{4,0,0,1,0} + \overline{z}'^4\zeta'\,\overline{F'_{4,0,0,1,0}} + z'^3\overline{\zeta}'^2\,F'_{3,0,0,2,0} + \overline{z}'^3\zeta'^2\,\overline{F'_{3,0,0,2,0}} \\ & + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}',v'}(6). \end{aligned}$$

But then, it is necessary to stabilize the normalization obtained up to order 4. With the help of a computer, one can prove the following

Lemma 20.1. *Any biholomorphic map of the form*

$$z' \;:=\; f_1 + f_2 + f_3, \qquad \zeta' \;:=\; g_1 + g_2, \qquad w' \;:=\; h_1 + h_2 + h_3 + h_4,$$

where f_1, f_2, f_3, g_1, g_2, h_1, h_2, h_3, h_4 are weighted homogeneous polynomials in (z, ζ, w) of degrees equal to their indices, which stabilizes the normalization up to order 4

$$\begin{aligned} & z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(5) \qquad \longrightarrow \\ & \longrightarrow \qquad z'\overline{z}' + \tfrac{1}{2}\overline{z}'^2\zeta' + \tfrac{1}{2}z'^2\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}' + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}',v'}(5) \end{aligned}$$

is of the form

$$\begin{aligned}
z' &:= \lambda z - i\lambda\alpha z^2 - i\lambda\overline{\alpha} w - \frac{\lambda^2}{\overline{\lambda}}\overline{\beta} z^3 \\
&\qquad + \Big(i\lambda r - \tfrac{3}{2}\lambda\alpha\overline{\alpha} - \tfrac{1}{4}\frac{\lambda^2}{\overline{\lambda}}\overline{\varepsilon} - \tfrac{1}{4}\overline{\lambda}\varepsilon\Big) zw + i\lambda\alpha\zeta w, \\
\zeta' &:= \frac{\lambda}{\overline{\lambda}}\zeta + 2i\frac{\lambda}{\overline{\lambda}}\overline{\alpha} z + \varepsilon z^2 - 2i\frac{\lambda}{\overline{\lambda}}\alpha z\zeta + \beta w, \\
w' &:= \lambda\overline{\lambda} w - 2i\lambda\overline{\lambda}\alpha zw - \big(2\lambda\overline{\lambda}\alpha^2 + \lambda^2\overline{\beta}\big) z^2 w + \big(-\lambda\overline{\lambda}\alpha\overline{\alpha} + i\lambda\overline{\lambda} r\big) w^2,
\end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$, $\beta \in \mathbb{C}$, $\varepsilon \in \mathbb{C}$ *are arbitrary parameters.*

Compared to the expansions to orders 3, 2, 4 of the components of the isotropy group of the Gaussier-Merker model shown in Section 5, *two new parameters appear*, namely β and ε. This causes little trouble to define *chains* for $M^5 \in \mathfrak{C}_{2,1}$, analogous to the Cartan-Moser chains for Levi nondegenerate $M^3 \subset \mathbb{C}^2$ redefined in [32], because the linearization of the above collection of maps (in fact a group) is

$$\begin{aligned}
z' &:= \lambda z - i\lambda\overline{\alpha} w, \\
\zeta' &:= \frac{\lambda}{\overline{\lambda}}\zeta + 2i\frac{\lambda}{\overline{\lambda}}\overline{\alpha} z + \beta w, \\
w' &:= \lambda\overline{\lambda} w,
\end{aligned}$$

and this action, parametrized by 6 variables $\lambda, \overline{\lambda}, \alpha, \overline{\alpha}, \beta, \overline{\beta}$, is *transitive* on 1-jets at the origin (exercise), contrary to the linearization of the action of the isotropy group of the Gaussier-Merker model

$$\begin{aligned}
z' &:= \lambda z - i\lambda\overline{\alpha} w, \\
\zeta' &:= \frac{\lambda}{\overline{\lambda}}\zeta + 2i\frac{\lambda}{\overline{\lambda}}\overline{\alpha} z + \frac{\lambda}{\overline{\lambda}}\overline{\alpha}^2 w, \\
w' &:= \lambda\overline{\lambda} w,
\end{aligned}$$

in which $\beta = \frac{\lambda}{\overline{\lambda}}\overline{\alpha}^2$ is a *dependent* parameter. This is why we obtained an invariant submanifold Σ^1_0 in Observation 8.1.

To resolve this little discrepancy, we must normalize to higher order at the origin.

So to normalize further, we will employ maps of the form

$$\begin{aligned}
z' &:= \lambda\, z - i\,\lambda\alpha\, z^2 - i\,\lambda\overline{\alpha}\, w - \frac{\lambda^2}{\overline{\lambda}}\,\overline{\beta}\, z^3 \\
&\quad + \Big(i\,\lambda r - \tfrac{3}{2}\,\lambda\alpha\overline{\alpha} - \tfrac{1}{4}\,\frac{\lambda^2}{\overline{\lambda}}\,\overline{\varepsilon} - \tfrac{1}{4}\,\overline{\lambda}\varepsilon\Big)\, zw + i\,\lambda\alpha\,\zeta w \\
&\quad + \sum_{a+b+2e=4} f_{a,b,e}\, z^a \zeta^b w^e, \\
\zeta' &:= \frac{\lambda}{\overline{\lambda}}\,\zeta + 2i\,\frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}\, z + \varepsilon\, z^2 - 2i\,\frac{\lambda}{\overline{\lambda}}\,\alpha\, z\zeta + \beta\, w \\
&\quad + \sum_{a+b+2e=3} g_{a,b,e}\, z^a \zeta^b w^e, \\
w' &:= \lambda\overline{\lambda}\, w - 2i\,\lambda\overline{\lambda}\alpha\, zw - \big(2\,\lambda\overline{\lambda}\alpha^2 + \lambda^2\overline{\beta}\big)\, z^2 w + \big(-\lambda\overline{\lambda}\alpha\overline{\alpha} + i\,\lambda\overline{\lambda}\, r\big)\, w^2 \\
&\quad + \sum_{a+b+2e=5} h_{a,b,e}\, z^a \zeta^b w^e.
\end{aligned}$$

Still on a computer, we verify

Assertion 20.2. *Whatever map is chosen, one has*

$$F'_{3,0,0,2,0} \;=\; \frac{1}{\overline{\lambda}}\, F_{3,0,0,2,0}.$$

Furthermore, the map

$$\begin{aligned}
z' &:= z + 2\, F_{4,0,0,1,0}\, z^3 - 2\, F_{4,0,0,1,0}\, z\zeta w, \\
\zeta' &:= \zeta - 2\,\overline{F_{4,0,0,1,0}}\, w + 10\, z^2\zeta\, F_{4,0,0,1,0}, \\
w' &:= w + 2\, z^2 w\, F_{4,0,0,1,0},
\end{aligned}$$

normalizes $F'_{4,0,0,1,0} := 0$ (exercise). What we have proved so far deserved to be stated as a

Proposition 20.3. *At every point $p \in M^5$ of a hypersurface $M^5 \subset \mathbb{C}^3$ in the class $\mathfrak{G}_{2,1}$, there exist holomorphic coordinates $(z, \zeta, w) \in \mathbb{C}^3$ centered at $p = (z_p, \zeta_p, w_p) = (0, 0, 0)$ in which M has equation*

$$\begin{aligned}
u \;=\; & z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\, z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\, z^2\overline{\zeta}\zeta\overline{\zeta} \\
& + z^3\overline{\zeta}^2\, F_{3,0,0,2,0} + \overline{z}^3\zeta^2\,\overline{F_{3,0,0,2,0}} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(6).
\end{aligned}$$

By applying the technique of Chen-Foo-Merker-Ta [8, Sections 9, 10], one can realize, after rather hard computations, that there corresponds to this Taylor coefficient $F_{3,0,0,2,0}$, the relative invariant W_0 of Pocchiola, presented in [42, 35,

16]

$$W_0 := -\frac{1}{3}\frac{\mathcal{K}(\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(k)))}{\overline{\mathcal{L}}_1(k)^2} + \frac{1}{3}\frac{\mathcal{K}(\overline{\mathcal{L}}_1(k))\,\overline{\mathcal{L}}_1(\overline{\mathcal{L}}_1(k))}{\overline{\mathcal{L}}_1(k)^3} + \\ + \frac{2}{3}\frac{\mathcal{L}_1(\mathcal{L}_1(\overline{k}))}{\mathcal{L}_1(\overline{k})} + \frac{2}{3}\frac{\mathcal{L}_1(\overline{\mathcal{L}}_1(k))}{\overline{\mathcal{L}}_1(k)} + \frac{i}{3}\frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)},$$

Much more simply, by plugging this normalized F into this formula, we obtain its value *only at one point*, namely at the origin

$$W_0 = 4\overline{F_{3,0,0,2,0}}.$$

Next, we determine the isotropy of this normalization.

Lemma 20.4. *Any biholomorphic map of the form*

$$z' := f_1 + f_2 + f_3 + f_4, \quad \zeta' := g_1 + g_2 + g_3, \quad w' := h_1 + h_2 + h_3 + h_4 + h_5,$$

where f_1, f_2, f_3, f_4, g_1, g_2, g_3, h_1, h_2, h_3, h_4, h_5, are weighted homogeneous polynomials in (z, ζ, w) of degrees equal to their indices, which stabilizes the normalization up to order 5 included

$$z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}z^2\overline{\zeta}\zeta\overline{\zeta} \\ + F_{3,0,0,2,0}\,z^3\overline{\zeta}^2 + \overline{F_{3,0,0,2,0}}\,\overline{z}^3\zeta^2 + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(6) \quad \longrightarrow$$

$$\longrightarrow \quad z'\overline{z}' + \tfrac{1}{2}\overline{z}'^2\zeta' + \tfrac{1}{2}z'^2\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}' + \tfrac{1}{2}\overline{z}'^2\zeta'\zeta'\overline{\zeta}' + \tfrac{1}{2}z'^2\overline{\zeta}'\zeta'\overline{\zeta}' \\ + F'_{3,0,0,2,0}\,z'^3\overline{\zeta}'^2 + \overline{F'_{3,0,0,2,0}}\,\overline{z}'^3\zeta'^2 + \mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}',v'}(6),$$

is of the form

$$\begin{aligned}
z' :=\ & \lambda z - i\lambda\alpha z^2 - i\lambda\overline{\alpha}w - \lambda\alpha^2 z^3 \\
& + \Big(i\lambda r - 3\lambda\alpha\overline{\alpha} + 2i\lambda\alpha F_{3,0,0,2,0} - 2i\lambda\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)zw + i\lambda\alpha\zeta w \\
& + \Big(8i\lambda\alpha^2\overline{\alpha} + \tfrac{1}{2}\frac{\lambda^2}{\overline{\lambda}}\overline{\gamma} + 4\frac{\lambda}{\overline{\lambda}}\overline{\tau} + 4\lambda\alpha^2 F_{3,0,0,2,0} - 8\lambda\alpha\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)z^2 w \\
& + i\lambda\alpha^3 z^4 + 3\lambda\alpha^2 z\zeta w + \tau w^2,
\end{aligned}$$

$$\begin{aligned}
\zeta' :=\ & \frac{\lambda}{\overline{\lambda}}\zeta + 2i\frac{\lambda}{\overline{\lambda}}\overline{\alpha}z + \Big(3\frac{\lambda}{\overline{\lambda}}\alpha\overline{\alpha} - i\frac{\lambda}{\overline{\lambda}}r - 2i\frac{\lambda}{\overline{\lambda}}\alpha F_{3,0,0,2,0} + 6i\frac{\lambda}{\overline{\lambda}}\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)z^2 \\
& - 2i\frac{\lambda}{\overline{\lambda}}\alpha z\zeta + \frac{\lambda}{\overline{\lambda}}\overline{\alpha}^2 w \\
& + \Big(2\frac{\lambda}{\overline{\lambda}}\alpha r - 4i\frac{\lambda}{\overline{\lambda}}\alpha^2\overline{\alpha} - 2\frac{\lambda^2}{\overline{\lambda}^2}\overline{\gamma} - 8\frac{\lambda}{\overline{\lambda}^2}\overline{\tau} + 12\frac{\lambda}{\overline{\lambda}}\alpha^2 F_{3,0,0,2,0} + 4\frac{\lambda}{\overline{\lambda}}\alpha\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)z^3 \\
& - 3\frac{\lambda}{\overline{\lambda}}\alpha^2 z^2\zeta + \gamma zw \\
& + \Big(-2\frac{\lambda}{\overline{\lambda}}\alpha\overline{\alpha} + 4i\frac{\lambda}{\overline{\lambda}}\alpha F_{3,0,0,2,0} - 4i\frac{\lambda}{\overline{\lambda}}\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)\zeta w,
\end{aligned}$$

$$\begin{aligned} w' \ :=\ & \lambda\overline{\lambda}\, w - 2i\,\lambda\overline{\lambda}\alpha\, zw - 3\,\lambda\overline{\lambda}\alpha^2\, z^2 w + \big(-\lambda\overline{\lambda}\alpha\overline{\alpha} + i\,\lambda\overline{\lambda}\, r\big)\, w^2 + 4i\,\lambda\overline{\lambda}\alpha^3\, z^3 w \\ & + \Big(6i\,\lambda\overline{\lambda}\alpha^2\overline{\alpha} + 2\,\lambda\overline{\lambda}\alpha r + 2\,\lambda\overline{\tau} + 4\,\lambda\overline{\lambda}\alpha^2\, F_{3,0,0,2,0} - 4\,\lambda\overline{\lambda}\alpha\overline{\alpha}\,\overline{F_{3,0,0,2,0}}\Big)\, zw^2 \\ & + \lambda\overline{\lambda}\alpha^2\,\zeta w^2. \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\tau \in \mathbb{C}$ are arbitrary parameters.

In comparison to the normalization up to order 4, observe that the previous two supplementary parameters *have now been normalized*

$$\begin{aligned} \beta \ :=\ & \frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}^2, \\ \varepsilon \ :=\ & -2i\,\frac{\lambda}{\overline{\lambda}}\,\alpha\, F_{3,0,0,2,0} + 6i\,\frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}\,\overline{F_{3,0,0,2,0}} + 3\,\frac{\lambda}{\overline{\lambda}}\,\alpha\overline{\alpha} - i\,\frac{\lambda}{\overline{\lambda}}\, r. \end{aligned}$$

With this, the *linearized* isotropy has become the *same* as the one of the GM-model written above

$$\begin{aligned} z' \ :=\ & \lambda\, z - i\,\lambda\overline{\alpha}\, w, \\ \zeta' \ :=\ & \frac{\lambda}{\overline{\lambda}}\,\zeta + 2i\,\frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}\, z + \frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}^2\, w, \\ w' \ :=\ & \lambda\overline{\lambda}\, w. \end{aligned} \tag{20.1}$$

This key fact will enable us to define, at *every* point of *any* $\mathfrak{G}_{2,1}$ hypersurface $M^5 \subset \mathbb{C}^3$, a CR-invariant 1-jet locus $\Sigma^1_p \subset J^1_{M,p}$ in the bundle of CR-transversal 1-jets of $\mathcal{C}^\omega$ curves $\gamma \subset M$.

We will follow the guide [32], which was prepared in advance on this purpose.

21. Point translations of $\mathcal{C}^\omega$ hypersurfaces $M^5 \subset \mathbb{C}^3$

Consider as before a local $\mathcal{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$ which is 2-nondegenerate and of constant Levi rank 1, namely belongs to the class $\mathfrak{G}_{2,1}$.

In coordinates $(z,\zeta,w) = (x+iy,\, s+it,\, u+iv)$, assume that M is locally graphed as $u = F(z,\zeta,\overline{z},\overline{\zeta},v)$. At all points $p = (z_p,\zeta_p,w_p) \in M$ with $u_p = F\big(z_p,\zeta_p,\overline{z}_p,\overline{\zeta}_p,v_p\big)$, let us expand up to weighted order 5

$$\begin{aligned} u \ =\ F\big(z,\zeta,\overline{z},\overline{\zeta},v\big) \ =\ \sum_{a+b+c+d+2e\leqslant 5} & \frac{(z-z_p)^a}{a!}\,\frac{(\zeta-\zeta_p)^b}{b!}\,\frac{(\overline{z}-\overline{z}_p)^c}{c!}\,\frac{(\overline{\zeta}-\overline{\zeta}_p)^d}{d!}\,\frac{(v-v_p)^e}{e!}\cdot \\ & \cdot F_{z^a\zeta^b\overline{z}^c\overline{\zeta}^d v^e}\big(z_p,\zeta_p,\overline{z}_p,\overline{\zeta}_p,v_p\big) + \mathrm{O}(6), \end{aligned}$$

subtract $u-u_p$, translate coordinates $z := z-z_p$, $\zeta := \zeta-\zeta_p$, $w := w-w_p$, and get a family of hypersurfaces $M^p \subset \mathbb{C}^3$, parametrized by $p \in M$ and passing through the origin

$$u \ =\ F^p\big(z,\zeta,\overline{z},\overline{\zeta},v\big) \ =\ \sum_{1\leqslant a+b+c+d+2e\leqslant 5} z^a\zeta^b\overline{z}^c\overline{\zeta}^d v^e\, F^p_{a,b,c,d,e} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(6),$$

namely with $F^p(0,0,0,0,0)=0$, whose graphing function has coefficients

$$F^p_{a,b,c,d,e} := \tfrac{1}{a!}\tfrac{1}{b!}\tfrac{1}{c!}\tfrac{1}{d!}\tfrac{1}{e!}F_{z^a\zeta^b\overline{z}^c\overline{\zeta}^d v^e}\big(z_p,\zeta_p,\overline{z}_p,\overline{\zeta}_p,v_p\big),$$

analytically parametrized by $p\in M$. Thanks to this, working at *only one* point, namely at the origin, we will treat *all* points $p\in M$.

Question 21.1. *Are there analogs, on hypersurfaces $M^5\in\mathfrak{C}_{2,1}$, of Cartan-Moser chains* [6, 5, 4, 12, 23, 10, 24, 25, 13, 39, 2, 7, 1, 3, 38, 11, 32] *for Levi nondegenerate hypersurfaces $M^3\subset\mathbb{C}^2$?*

Thanks to Lemma 20.4, we will construct, at each point $p\in M$, an invariant surface in the bundle of 1-jets of CR-transversal curves in M. So there will be an important difference with Cartan-Moser chains for Levi nondegenerate $M^3\subset\mathbb{C}^2$: the phenomenon that there exists a CR-transversal invariant object which is of *order* 1.

To view this object, similarly as in [32], we need to introduce bundles J^1_M and J^2_M of 1-jets and 2-jets of CR-transversal curves $\gamma:\mathbb{R}\longrightarrow M$ with $\dot\gamma\not\in T^c_\gamma M$ nowhere complex-tangential.

22. CR-invariant 2-codimensional submanifold $\Sigma^1\subset J^1_M\cong M^5\times\mathbb{R}^4$

In local coordinates for which M is locally graphed as $u=F(z,\zeta,\overline{z},\overline{\zeta},v)$, at any point $p\in M$, the CR-transversal curves can be parametrized as

$$v\longmapsto\big(x(v),\,y(v),\,s(v),\,t(v),\,v\big)\in\mathbb{R}^5_{x,y,z,t,v},$$

with $\gamma(0)=p=(x_p,y_p,s_p,t_p,v_p)$.

The $4+4=8$ independent coordinates corresponding to the 1^{st} derivatives $\big(\dot x(v),\dot y(v),\dot s(v),\dot t(v)\big)$ and to the 2^{nd} derivatives $\big(\ddot x(v),\ddot y(v),\ddot s(v),\ddot t(v)\big)$ will be denoted as follows

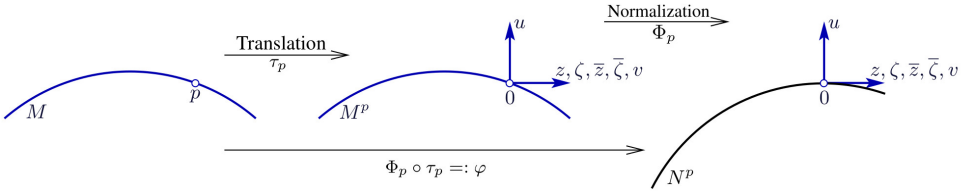
$$J^1_M := \big\{\big(x_p,y_p,s_p,t_p,v_p,\,x^1_p,y^1_p,s^1_p,t^1_p,v^1_p\big)\big\} = \mathbb{R}^{5+4},$$
$$J^2_M := \big\{\big(x_p,y_p,s_p,t_p,v_p,\,x^1_p,y^1_p,s^1_p,t^1_p,v^1_p,\,x^2_p,y^2_p,s^2_p,t^2_p,v^2_p\big)\big\} = \mathbb{R}^{5+4+4}.$$

Now, denote the translation map as

$$\tau_p:\quad (z,\zeta,w)\longrightarrow\big(z-z_p,\;\zeta-\zeta_p,\;w-w_p\big)=:(z,\zeta,w),$$

so that

$$\tau_p\big(M,p\big)=:\big(M^p,0\big).$$



Also, let the punctual (at the origin) normalization map constructed up to now, by Proposition 20.3, be denoted by

$$\begin{aligned}\Phi_p : \quad (M^p,0) &= \Big\{u = \sum_{1\leqslant a+b+c+d+2e\leqslant 5} F^p_{a,b,c,d,e}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d v^e + \mathrm{O}(6)\Big\} \\ \longrightarrow \quad (N^p,0) &= \Big\{u = z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} \\ &\quad + \tfrac{1}{2}\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}z^2\overline{\zeta}\zeta\overline{\zeta} \\ &\quad + z^3\overline{\zeta}^2 F^p_{3,0,0,2,0} + \overline{z}^3\zeta^2\,\overline{F^p_{3,0,0,2,0}} \\ &\quad + \mathrm{O}(6)\Big\}.\end{aligned}$$

According to the constructions done in Sections 11 to 19 and according to Proposition 20.3, we know that Φ_p depends analytically on p.

Abbreviate

$$\varphi := \Phi_p \circ \tau_p,$$

and consider the diagram

$$\begin{array}{ccc} J^1_{M,p} & \xrightarrow{\varphi^{(1)}} & J^1_{N^p,0} \\ \downarrow & & \downarrow \\ (M,p) & \xrightarrow[\varphi]{} & (N^p,0). \end{array}$$

As in Observation 8.1, in the 1-jet fiber above $0 \in N^p$, introduce the surface

$$\Sigma^1_0 := \big\{(x_1,y_1,s_1,t_1) \in J^1_{N^p,0} : \ s_1 = -2\,x_1 y_1, \ \ t_1 = x_1^2 - y_1^2\big\}.$$

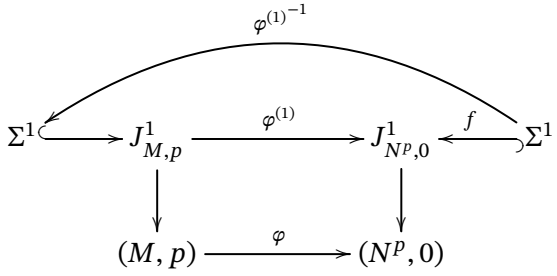
Using the first prolongation $\varphi^{(1)}$, define the 2-dimensional submanifold of $J^1_{M,p}$

$$\Sigma^1_p := {\varphi^{(1)}}^{-1}\big(\Sigma^1_0\big).$$

Since $\varphi^{(1)}$ is a diffeomorphism $J^1_{M,p} \xrightarrow{\sim} J^1_{N^p,0}$, this Σ^1_p is also graphed, say of the form

$$s^1_p = A\big(x^1_p, y^1_p\big), \qquad\qquad t^1_p = B\big(x^1_p, y^1_p\big),$$

with two $\mathcal{C}^\omega$ functions A, B which depend on p, and depend *also* a priori on the normalizing map φ.



The union

$$\bigcup_{p\in M} \Sigma^1_p \;=:\; \Sigma^1 \;\subset\; J^1_M$$

is a $\mathcal{C}^\omega$ submanifold of dimension $5+2$ within J^1_M which has dimension $5+4$.

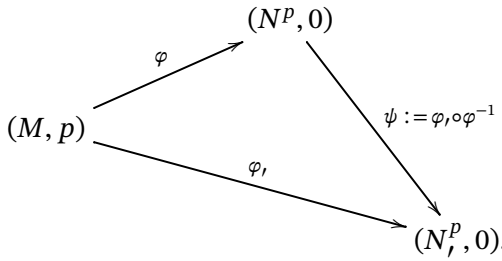
Assertion 22.1. *This graphed surface*

$$\Sigma^1_p \subset J^1_{M,p} \cong \mathbb{R}^4$$

is independent *of the map $\varphi = \Phi_p \circ \tau_p$ normalizing the initial hypersurface M of equation $u = F(z,\zeta,\overline{z},\overline{\zeta},v)$ near any of its points $p \in M$, to*

$$\begin{aligned} u \;=\; z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}z^2\overline{\zeta}\zeta\overline{\zeta} \\ + z^3\overline{\zeta}^2 F^p_{3,0,0,2,0} + \overline{z}^3\zeta^2\overline{F^p_{3,0,0,2,0}} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(6). \end{aligned}$$

Proof. Suppose another such normalizing map is given



By Lemma 20.4 which holds for maps stabilizing the origin, ψ has linear terms exactly equal to the linear terms of the isotropy group of the GM-model, for which we already know, thanks to Observation 8.1, that

$$\psi^{(1)}\big(\Sigma^1_0\big) \;=\; \Sigma'^1_0.$$

Hence in conclusion

$$\begin{aligned} \Sigma'^1_p \;=\; \varphi_\prime^{(1)^{-1}}\big(\Sigma'^1_0\big) \;=\; \varphi_\prime^{(1)^{-1}}\Big(\psi^{(1)}\big(\Sigma^1_0\big)\Big) \;=\; \varphi_\prime^{(1)^{-1}}\Big(\big(\varphi_\prime\circ\varphi^{-1}\big)^{(1)}\big(\Sigma^1_0\big)\Big) \\ = \;\varphi^{(1)^{-1}}\big(\Sigma^1_0\big) \;=\; \Sigma^1_p. \end{aligned}$$

□

So at each point $p \in M$, there exists a CR-invariant, or biholomorphically invariant, surface $\Sigma^1_p \subset J^1_{M,p}$. Therefore, it is natural to select only CR-transversal curves $\gamma :\, \mathbb{R} \longrightarrow M$, $\gamma(0) = p$, such that $\dot\gamma(\tau) \in \Sigma^1_{\gamma(\tau)}$ for every $\tau \in \mathbb{R}$.

But the 'discovery' of this CR-invariant submanifold $\Sigma^1_M \subset J^1_M$ does not suffice, because the linear action

$$\begin{aligned} z' \;&:=\; \lambda\, z - i\,\lambda\overline{\alpha}\, w, \\ \zeta' \;&:=\; \frac{\lambda}{\overline{\lambda}}\,\zeta + 2i\,\frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}\, z + \frac{\lambda}{\overline{\lambda}}\,\overline{\alpha}^2\, w, \\ w' \;&:=\; \lambda\overline{\lambda}\, w, \end{aligned}$$

happens to be *transitive* on the invariant surface $\Sigma^1_0 \subset \mathbb{R}^4$ of 1-jets, according to the fact that the prolonged symmetry vector fields $\mathsf{D}^{(1)}$, $\mathsf{R}^{(1)}$, $\mathsf{I}^{(1)}_1$, $\mathsf{I}^{(1)}_2$, $\mathsf{J}^{(1)}$, shown in Section 8, are of rank $2 = \dim\, \Sigma^1_0$ everywhere.

Remind from [6, 5, 24, 32] that Cartan-Moser chains were strictly of *second order*. Hence, we need to explore deeper, and to normalize further, still at $0 \in M^p$. We will realize that to each 1-jet $j^1_p \in \Sigma^1_p$, there is associated a unique invariant 2-jet $j^2_p = j^2_p(j^1_p)$, as we already saw when studying the GM-model in Section 9.

23. Order 1 chains in $\mathfrak{C}_{2,1}$ hypersurfaces $M^5 \subset \mathbb{C}^3$

So far, at the origin, we have constructed a normalizing map Φ_p, composed with a translation map τ_p

$$\varphi: \quad (M,p) \xrightarrow{\ \tau_p\ } (M^p,0) \xrightarrow{\ \Phi_p\ } (N^p,0),$$

which brings (M,p) to $(N^p,0)$ at the origin of equation fully normalized up to order 5 included

$$\begin{aligned} u \,=\, z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} \\ + 2\,\mathrm{Re}\Big\{0 + 0 + z^3\overline{\zeta}^2\,F^p_{3,0,0,2,0}\Big\} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(6), \end{aligned}$$

namely with $0 = F^p_{3,0,1,1,0} = F^p_{4,0,0,1,0}$, knowing that $F^p_{3,0,0,2,0}$ is a relative invariant.

The differential φ_* establishes isomorphisms

$$\begin{aligned} T_pM &\xrightarrow{\ \sim\ } T_0N^p, \\ T^c_pM &\xrightarrow{\ \sim\ } T^c_0N^p, \\ K^c_pM &\xrightarrow{\ \sim\ } K^c_0N^p, \end{aligned}$$

where $K^cM \subset T^cM$ is the Levi-kernel subbundle [37]. It follows that φ_* establishes an isomorphism between the 3-dimensional real quotient bundles

$$T_pM\Big/\big(T^c_pM/K^c_pM\big) \xrightarrow{\ \sim\ } T_0N^p\Big/\big(T^c_0N^p/K^c_0N^p\big).$$

By definition, on these bundles T^c/K^c, the Levi form of M is nondegenerate, of maximal possible rank 1.

In a neighborhood of some reference point $p_0 \in M$, we can take coordinates (z,w,ζ) with $z = x+iy$, $\zeta = s+it$, $w = u+iv$, so that M is locally graphed as $u = F(z,\zeta,\overline{z},\overline{\zeta},v)$, with $(v,x,y,s,t) \in M^5$ being intrinsic coordinates, so that the Levi form of M is *nonzero near* p_0 along the intrinsic $(1,0)$ vector field

$$\mathcal{L} \,:=\, \frac{\partial}{\partial z} - i\,\frac{F_z}{1+i\,F_v}\,\frac{\partial}{\partial v}.$$

We will let $p \sim p_0$ vary in a neighborhood of p_0.

Taking jet coordinates (x_1, y_1, s_1, t_1) near p_0 so that

$$J^1_M \,=\, \big\{(v, x, y, s, t,\, x_1, y_1, s_1, t_1)\big\},$$

it follows from the above isomorphisms and from the definition of $\Sigma^1_0 \subset J^1_{M^p,0}$ that $\Sigma^1 \subset J^1_M$ is locally defined near p_0 as a graph

$$s_1 \,=\, A\big(v, x, y, s, t,\, x_1, y_1\big), \qquad\qquad t_1 \,=\, B\big(v, x, y, s, t,\, x_1, y_1\big),$$

in terms of certain two $\mathcal{C}^\omega$ functions A, B, which vanish for $x_1 = y_1 = 0$. In this respect, the first two coordinates (x^1_p, y^1_p) of a 1-jet j^1_p at some point $p = (v_p, x_p, y_p, s_p, t_p) \in M$ near p_0 should be thought of as being *horizontal*, and the last two coordinates (s^1_p, t^1_p) as being *vertical*.

An alternative presentation of CR-invariant CR-transversal 1-jets on hypersurfaces $M^5 \subset \mathbb{C}^3$ will be useful in a moment.

Definition 23.1. A 1-jet $j^1_p \in J^1_{M,p}$ is said to be the jet of an *order* 1 *chain* at a point $p \in M$, or to *belong* to the invariant surface $\Sigma^1_p \subset J^1_{M,p}$, if, given any punctual normalizing map from (M, p) to $(N^p, 0)$ up to order 5 as in Proposition 20.3

$$\begin{aligned} u \,=\, z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} \\ + 2\,\mathrm{Re}\Big\{0 + 0 + z^3\overline{\zeta}^2\,F^p_{3,0,0,2,0}\Big\} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(6), \end{aligned}$$

which sends j^1_p to a 1-jet at $0 \in N^p$ having *vanishing* horizontal part

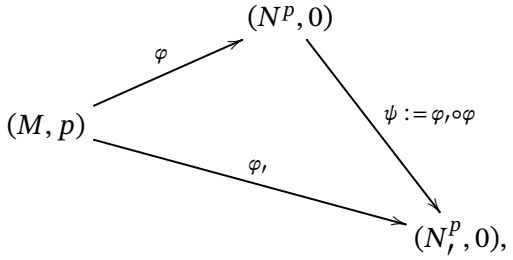
$$\varphi^{(1)}\big(j^1_p\big) \,=\, \big(0, 0,\, s^1_0, t^1_0\big),$$

then in fact j^1_p is the inverse image of the *flat* 1-*jet* at the origin

$$j^1_p \,=\, {\varphi^{(1)}}^{-1}\big(0, 0,\, 0, 0\big),$$

or equivalently $s^1_0 = t^1_0 = 0$.

This definition does not depend on the normalizing map Φ_p in $\varphi = \tau_p \circ \Phi_p$, because if another Φ'_p is chosen, which leads to the diagram



with $(N^p_,, 0)$ having an equation similar to the one of $(N^p, 0)$ above, then the ambiguity map $\psi := \varphi_, \circ \varphi$ should stabilize the flat 1-jet, and for this to hold, we already know from the formulas (20.1) that this forces $\alpha = 0$.

We will now employ this definition in two ways. It is clear that the graphed equations of $\Sigma^1 \subset J^1_M$ lead to a system of two first-order ordinary differential equations

$$\dot{s} = A\big(v, x, y, s, t, \dot{x}, \dot{y}\big), \qquad\qquad \dot{t} = B\big(v, x, y, s, t, \dot{x}, \dot{y}\big),$$

the time parameter being v. For any choice of any two functions $(x(v), y(v))$ with $(x(0), y(0)) = (x_p, y_p)$, with $(\dot{x}(0), \dot{y}(0)) \neq (0, 0)$, and with $(s(0), t(0)) = (s_p, t_p)$, there exists a unique local $\mathcal{C}^\omega$ solution to this system passing through p at 'time' $v = 0$, which is a CR-transversal curve having tangents in Σ^1_M.

Terminology 23.2. Such a curve will be called an *order* 1 *chain*.

Later, when passing to *order* 2 *chains*, we will see that the large freedom in the choice of arbitrary functions $(x(v), y(v))$ will drop.

Once order 1 chains are known, it is natural to restart the whole process of prenormalization and of partial normalization which begun in Section 11, by assuming that the CR-transversal curve $p \in \gamma \subset M$ (not anymore chosen at random) *is* an order 1 chain.

Then, coming back to Proposition 19.3, but viewed at the origin up to order 6 in *all* variables $(z, \zeta, \overline{z}, \overline{\zeta}, v)$, we remember that we have constructed a normalizing map Φ_p, composed with a translation map τ_p

$$\varphi : \quad (M, p) \xrightarrow{\quad\tau_p\quad} (M^p, 0) \xrightarrow{\quad\Phi_p\quad} (N^p, 0),$$

which brings (M, p) to $(N^p, 0)$ at the origin of equation

$$\begin{aligned} u \;=\; & z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} \\ & + 2\,\mathrm{Re}\Big\{0 + z^4\overline{\zeta}\,F^p_{4,0,0,1,0} + z^3\overline{\zeta}^2\,F^p_{3,0,0,2,0}\Big\} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(6), \end{aligned}$$

without changing the CR-transversal curve $0 \in \gamma \subset M$ being the v-axis, hence having *flat* 1-jet at the origin.

Assertion 23.3. *Then* $F^p_{4,0,0,1,0} = 0$ *holds automatically, without having the needs to perform any further biholomorphism.*

Proof. Indeed, we already know that one can continue to normalize and make $F^p_{4,0,0,1,0} = 0$ by means of the map

$$\begin{aligned} z' &:= z + 2\,F^p_{4,0,0,1,0}\,z^3 - 2\,F^p_{4,0,0,1,0}\,z\zeta w, \\ \zeta' &:= \zeta - 2\,\overline{F^p_{4,0,0,1,0}}\,w + 10\,z^2\zeta\,F^p_{4,0,0,1,0}, \\ w' &:= w + 2\,z^2 w\,F^p_{4,0,0,1,0}, \end{aligned}$$

which we may call $\Psi : (N^p, 0) \longrightarrow (N^p_\prime, 0)$. We then reason as in [32, 9.5].

If $F^p_{4,0,0,1,0} \neq 0$ would be nonzero, due to the presence in ζ' of the linear term $2\,\overline{F^p_{4,0,0,1,0}}\,w$, this map Ψ would *not* stabilize the flat order 1 jet $j^1_0 = (0, 0, 0, 0)$,

and so, this would contradict Definition 23.1 applied to $(M, p) := (N^p, 0)$, to $\varphi := \Psi$, and to $(N^p, 0) := (N'^p, 0)$. □

Lastly, coming again back to Proposition 19.3, we remember that we have constructed a normalizing map which brings M near $0 \in M$ to the equation

$$\begin{aligned} u \;=\; & z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} \\ & + z^3\overline{z}^3\,\mathrm{O}_{z,\overline{z}}(0) \\ & + 2\,\mathrm{Re}\Big\{0 + z^4\overline{\zeta}\,F_{4,0,0,1}(v) + z^3\overline{\zeta}^2\,F_{3,0,0,2}(v)\Big\} \\ & + \overline{z}^3\zeta\,\mathrm{O}_{z,\zeta,\overline{z}}(2) + z^3\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(2) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4), \end{aligned}$$

without changing any starting CR-transversal curve $0 \in \gamma \subset M$. We now realize that $F_{4,0,0,1}(v) \equiv 0$ vanishes for free.

Proposition 23.4. *For every hypersurface $M^5 \in \mathfrak{C}_{2,1}$, at any point $p \in M$, given any CR-transversal curve $p \in \gamma \subset M$ which is an order 1 chain, there exist holomorphic coordinates $(z, \zeta, w) \in \mathbb{C}^3$ vanishing at p in which γ is the v-axis and in which M has equation*

$$\begin{aligned} u \;=\; & z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} \\ & + z^3\overline{z}^3\,\mathrm{O}_{z,\overline{z}}(0) \\ & + 2\,\mathrm{Re}\Big\{0 + 0 + z^3\overline{\zeta}^2\,F_{3,0,0,2}(v)\Big\} \\ & + \overline{z}^3\zeta\,\mathrm{O}_{z,\zeta,\overline{z}}(2) + z^3\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(2) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4). \end{aligned}$$

Proof. What was done an instant ago by Assertion 23.3 at the origin $(z, \zeta, w) = (0, 0, 0)$ applies in fact at *every* point $(0, 0, iv)$ along the v-axis, thanks to the fact that the (pre)normalizations of Propositions 16.4 and 19.3 were achieved *all along* the v-axis. □

Because we know the existence of a CR-invariant surface $\Sigma^1_p \subset J^1_{M,p}$ on which the isotropy is transitive, we will assume that, starting with any fixed 1-jet $j^1_p \in \Sigma^1_p$, the partial normalization map performed up to now sends j^1_p to the *flat* 1-jet at $0 \in M^p$, namely to $j^1_0 = (0, 0, 0, 0)$. We will assume that subsequent normalizations *stabilize* this invariant flat 1-jet. For this, at the very beginning, we have to assume that the CR-transversal curve used in Section 11, whose choice was left free, has 1-jet at the origin 0 equal to the flat 1-jet. By surveying all normalizations done up to now, one realizes that the v-axis was always stabilized, contained in M, hence the flat 1-jet was always preserved (implicitly).

Preserving the flat 1-jet at 0 corresponds to making $\alpha := 0$ in the formulas of Section 9 and of Lemma 20.4. We state this explicitly as a

Corollary 23.5. *The biholomorphic maps of Lemma* 20.4 *which stabilize punctual normalizations of* $(M^p, 0)$ *at the origin up to order* 5 and *which stabilize* also *the flat* 1*-jet* $j^1_0 = (0,0,0,0) \in \Sigma^1_0$ *read, with* $\alpha := 0$ *and* $\theta := \gamma$, *as*

$$\begin{aligned}
z' &:= \lambda\, z + i\,\lambda\, r\, zw + \Big(\frac{1}{2}\,\frac{\lambda^2}{\overline{\lambda}}\,\overline{\theta} + 4\,\frac{\lambda}{\overline{\lambda}}\,\overline{\tau}\Big)\, z^2 w + \tau\, w^2,\\
\zeta' &:= \frac{\lambda}{\overline{\lambda}}\,\zeta - i\,\frac{\lambda}{\overline{\lambda}}\, r\, z^2 + \Big(-2\,\frac{\lambda^2}{\overline{\lambda}^2}\,\overline{\theta} - 8\,\frac{\lambda}{\overline{\lambda}^2}\,\overline{\tau}\Big)\, z^3 + \theta\, zw,\\
w' &:= \lambda\overline{\lambda}\, w + i\,\lambda\overline{\lambda}\, r\, w^2 + 2\,\lambda\overline{\tau}\, zw^2.
\end{aligned}$$

24. End of point normalization of $\mathcal{C}^\omega$ hypersurfaces $M^5 \subset \mathbb{C}^3$

Thus, we have to look at 6^{th} order terms in the currently normalized equation of $(M^p, 0)$, which, taking account of the vanishing of the Levi determinant, are of the form (exercise)

$$\begin{aligned}
u \;=\;& z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} + z\overline{z}\zeta\overline{\zeta}\zeta\overline{\zeta}\\
&+ F^p_{3,0,0,2,0}\, z^3\overline{\zeta}^2 + \overline{F^p_{3,0,0,2,0}}\,\overline{z}^3\zeta^2 + \zeta\overline{\zeta}\,\Big(3\,F^p_{3,0,0,2,0}\, z^2\overline{z}\overline{\zeta} + 3\,\overline{F^p_{3,0,0,2,0}}\, z\zeta\overline{z}^2\Big)\\
&+ z^3\overline{z}^3\, F^p_{3,0,3,0,0}\\
&+ 2\,\mathrm{Re}\Big\{z^5\overline{\zeta}\, F^p_{5,0,0,1,0} + z^4\overline{z}\overline{\zeta}\, F^p_{4,0,1,1,0} + z^4\overline{\zeta}^2\, F^p_{4,0,0,2,0}\\
&\qquad + z^3\overline{z}^2\overline{\zeta}\, F^p_{3,0,2,1,0} + z^3\overline{z}\overline{\zeta}^2\, F^p_{3,0,1,2,0}\\
&\qquad\qquad + z^3\overline{\zeta}^3\, F^p_{3,0,0,3,0}\Big\} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(7).
\end{aligned}$$

To normalize further order 6 terms, it is natural to assume that the normalizations up to order 5 included are stabilized, *and also* that the flat 1-jet at the origin is stabilized as well. Thus we will employ maps of the form

$$\begin{aligned}
z' &:= \lambda\, z + i\,\lambda\, r\, zw + \Big(\tfrac{1}{2}\,\frac{\lambda^2}{\overline{\lambda}}\,\overline{\theta} + 4\,\frac{\lambda}{\overline{\lambda}}\,\overline{\tau}\Big)\, z^2 w + \tau\, w^2 + \sum_{a+b+2e=5} f_{a,b,e}\, z^a\zeta^b w^e,\\
\zeta' &:= \frac{\lambda}{\overline{\lambda}}\,\zeta - i\,\frac{\lambda}{\overline{\lambda}}\, r\, z^2 + \Big(-2\,\frac{\lambda^2}{\overline{\lambda}^2}\,\overline{\theta} - 8\,\frac{\lambda}{\overline{\lambda}^2}\,\overline{\tau}\Big)\, z^3 + \theta\, zw + \sum_{a+b+2e=4} g_{a,b,e}\, z^a\zeta^b w^e,\\
w' &:= \lambda\overline{\lambda}\, w + i\,\lambda\overline{\lambda}\, r\, w^2 + 2\,\lambda\overline{\tau}\, zw^2 + \sum_{a+b+2e=6} h_{a,b,e}\, z^a\zeta^b w^e.
\end{aligned}$$

Lemma 24.1. *One can annihilate*

$$F^p_{3,0,3,0,0} \;=\; 0 \quad \textit{and} \quad \Big(\textit{either} \quad F^p_{4,0,1,1,0} \;=\; 0 \quad \textit{or} \quad F^p_{3,0,2,1,0} \;=\; 0\Big).$$

Proof. By hand or on a computer, one verifies that the map

$$\begin{aligned}
z' &:= z + \tfrac{3}{4}\, F^p_{3,0,3,0,0}\, zw^2,\\
\zeta' &:= \zeta,\\
w' &:= w + \big(\tfrac{1}{4}\, F^p_{3,0,3,0,0} + \overline{F^p_{3,0,3,0,0}}\big)\, w^3,
\end{aligned}$$

makes $F^{p\,\prime}_{3,0,3,0,0} = 0$. It is visible (eyes exercise) that this map stabilizes the flat 1-jet $j^1_0 = (0,0,0,0)$.

Next, assuming that $F^{p}_{3,0,3,0,0} = 0 = F^{p\,\prime}_{3,0,3,0,0}$, the map parametrized by $\tau \in \mathbb{C}$

$$\begin{aligned} z' &:= z + 2\,\overline{\tau}\,z^2 w + \tau\,w^2 - \overline{\tau}\,\zeta w^2, \\ \zeta' &:= \zeta - 4\,\tau\,zw + 4\,\overline{\tau}\,z\zeta w, \\ w' &:= w + 2\,\overline{\tau}\,zw^2, \end{aligned}$$

also stabilizes the flat 1-jet $j^1_0 = (0,0,0,0)$, and it transforms as follows the six remaining coefficients

$$\begin{aligned} F^{p\,\prime}_{5,0,0,1,0} &= F^{p}_{5,0,0,1,0}, \quad F^{p\,\prime}_{4,0,1,1,0} = F^{p}_{4,0,1,1,0} - 2\,\overline{\tau}, \quad F^{p\,\prime}_{4,0,0,2,0} = F^{p}_{4,0,0,2,0}, \\ & \qquad F^{p\,\prime}_{3,0,2,1,0} = F^{p}_{3,0,2,1,0} + 2\,\tau, \quad F^{p\,\prime}_{3,0,1,2,0} = F^{p}_{3,0,1,2,0}, \\ & \qquad\qquad\qquad\qquad F^{p\,\prime}_{3,0,0,3,0} = F^{p}_{3,0,0,3,0}. \end{aligned}$$

So one of the two mentioned coefficients can be normalized. □

A choice must be made. We then determine the stability group for both choices of normalizations, again with the constraint of stabilizing the flat 1-jet j^1_0. Both choices lead to the same stability group (exercise on a computer).

Lemma 24.2. *Any biholomorphic map of the form*

$$\begin{aligned} z' &:= f_1 + f_2 + f_3 + f_4 + f_5, \quad \zeta' := g_1 + g_2 + g_3 + g_4, \\ & \qquad\qquad\qquad\quad w' := h_1 + h_2 + h_3 + h_4 + h_5 + h_6, \end{aligned}$$

where f_1, f_2, f_3, f_4, f_5, g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4, h_5, h_6, are weighted homogeneous, which stabilizes the normalization up to order 6 *included*

$$\begin{aligned} u \,=\, & z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} + z\overline{z}\zeta\overline{\zeta}\zeta\overline{\zeta} \\ & + F^{p}_{3,0,0,2,0}\,z^3\overline{\zeta}^2 + \overline{F^{p}_{3,0,0,2,0}}\,\overline{z}^3\zeta^2 + \zeta\overline{\zeta}\left(3\,F^{p}_{3,0,0,2,0}\,z^2\overline{z}\overline{\zeta} + 3\,\overline{F^{p}_{3,0,0,2,0}}\,z\zeta\overline{z}^2\right) \\ & + 0 + 2\,\mathrm{Re}\Big\{z^5\overline{\zeta}\,F^{p}_{5,0,0,1,0} + \quad 0 \quad + z^4\overline{\zeta}^2\,F^{p}_{4,0,0,2,0} \\ & \qquad\qquad + z^3\overline{z}^2\zeta\,F^{p}_{3,0,2,1,0} + z^3\overline{z}\overline{\zeta}^2\,F^{p}_{3,0,1,2,0} \\ & \qquad\qquad\qquad + z^3\overline{\zeta}^3\,F^{p}_{3,0,0,3,0}\Big\} \\ & \qquad\qquad\qquad + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(7), \end{aligned}$$

and which stabilizes the flat 1*-jet at the origin, is of the form*

$$\begin{aligned} z' &:= \lambda\,z + i\,\lambda\,r\,zw + 2\,\frac{\lambda^2}{\overline{\lambda}}\,\overline{\chi}\,z^3 w + \psi\,zw^2, \\ \zeta' &:= \frac{\lambda}{\overline{\lambda}}\,\zeta - i\,\frac{\lambda}{\overline{\lambda}}\,r\,z^2 - 4\,\frac{\lambda^2}{\overline{\lambda}^2}\,\overline{\chi}\,z^4 + \Big(-\frac{8}{3}\,\frac{\psi}{\overline{\lambda}} + \frac{4}{3}\,\frac{\lambda}{\overline{\lambda}^2}\,\overline{\psi} - \frac{1}{3}\,\frac{\lambda}{\overline{\lambda}}\,r^2\Big)\,z^2 w + \chi\,w^2, \\ w' &:= \lambda\overline{\lambda}\,w + i\,\lambda\overline{\lambda}\,r\,w^2 + \lambda^2\overline{\chi}\,z^2 w^2 + \Big(-\frac{1}{3}\,\lambda\overline{\lambda}\,r^2 + \frac{1}{3}\,\overline{\lambda}\psi + \frac{1}{3}\,\lambda\overline{\psi}\Big)\,w^3, \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $r \in \mathbb{R}$, $\psi \in \mathbb{C}$, $\chi \in \mathbb{C}$ are arbitrary parameters.

Furthermore, with this map, if one stabilizes the normalization $F_{4,0,1,1,0} = 0 = F^{p\prime}_{4,0,1,1,0}$, the other coefficients transform as

$$\begin{aligned}
F^{p\prime}_{5,0,0,1,0} &= \tfrac{1}{\lambda^3} F^p_{5,0,0,1,0}, & 0 &= 0, & F^{p\prime}_{4,0,0,2,0} &= \tfrac{1}{\lambda\overline{\lambda}} F^p_{4,0,0,2,0},\\
F^{p\prime}_{3,0,2,1,0} &= \tfrac{1}{\lambda\overline{\lambda}^2} F^p_{3,0,2,1,0} - 2i\,\tfrac{1}{\lambda\overline{\lambda}^2} F^p_{3,0,0,2,0}, & & & F^{p\prime}_{3,0,1,2,0} &= \tfrac{1}{\overline{\lambda}^2} F^p_{3,0,1,2,0},\\
& & & & F^{p\prime}_{3,0,0,3,0} &= \tfrac{\lambda}{\overline{\lambda}^2} F^p_{3,0,0,3,0},
\end{aligned}$$

while if one stabilizes the normalization $F_{3,0,2,1,0} = 0 = F^{p\prime}_{3,0,2,1,0}$, the other coefficients transform as

$$\begin{aligned}
F^{p\prime}_{5,0,0,1,0} &= \tfrac{1}{\lambda^3} F^p_{5,0,0,1,0}, & F^{p\prime}_{4,0,1,1,0} &= \tfrac{1}{\lambda^2\overline{\lambda}^2} F^p_{4,0,1,1,0} - 2\overline{\tau}, & F^{p\prime}_{4,0,0,2,0} &= \tfrac{1}{\lambda\overline{\lambda}} F^p_{4,0,0,2,0},\\
& & 0 &= 2i\,\lambda\overline{\lambda}\, r F^p_{3,0,0,2,0}, & F^{p\prime}_{3,0,1,2,0} &= \tfrac{1}{\overline{\lambda}^2} F^p_{3,0,1,2,0},\\
& & & & F^{p\prime}_{3,0,0,3,0} &= \tfrac{\lambda}{\overline{\lambda}^2} F^p_{3,0,0,3,0}.
\end{aligned}$$

This second choice happens to be less natural than the first one, because it forces to discuss the dichotomy branching

$$F^p_{3,0,0,2,0} \;\begin{cases} \nearrow\; F^p_{3,0,0,2,0} = 0, \\ \searrow\; F^p_{3,0,0,2,0} \neq 0, \end{cases}$$

and when $F^p_{3,0,0,2,0} \neq 0$, it leads to normalize the parameter r, which belongs to the isotropy of the GM-model, and such a normalization is too early to be done.

Therefore, we choose the normalization $F^{p\prime}_{4,0,1,1,0} = 0$.

By applying the technique of Chen-Foo-Merker-Ta [8, Sections 9, 10], one can realize, after rather hard computations, that there corresponds to the Taylor coefficient $F_{5,0,0,1,0}$, the relative invariant J_0 of Pocchiola, presented in [42, 35, 16]

$$\begin{aligned}
\overline{J}_0 \;:=\; & \frac{1}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)\big)}{\overline{\mathcal{L}}_1(k)} - \frac{5}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^2}\\
& - \frac{1}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)}\,\overline{P} + \frac{20}{27}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^3}{\overline{\mathcal{L}}_1(k)^3} + \frac{5}{18}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2}\,\overline{P}\\
& + \frac{1}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{P}\big)}{\overline{\mathcal{L}}_1(k)} - \frac{1}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\overline{P}\,\overline{P}\\
& - \frac{1}{6}\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{P}\big)\big) + \frac{1}{3}\,\overline{\mathcal{L}}_1\big(\overline{P}\big)\overline{P} - \frac{2}{27}\,\overline{P}\,\overline{P}\,\overline{P}.
\end{aligned}$$

Much more simply, by plugging this normalized F into this formula, we obtain its value *only at one point*, namely at the origin

$$\overline{J}_0 \;=\; 20\,\overline{F_{5,0,0,1,0}}.$$

25. Order 2 chains in $\mathfrak{C}_{2,1}$ hypersurfaces $M^5 \subset \mathbb{C}^3$

In Lemma 24.2, the presence of the free parameter $\chi \in \mathbb{C}$ in the last term $\chi\, w^2$, of order 4, of $\zeta' = \frac{\lambda}{\overline{\lambda}}\,\zeta + \cdots + \chi\, w^2$, shows that the flat second jet $j^2_0 = (0,0,0,0,\ 0,0,0,0)$ is *not* invariant by transformations which stabilize the normalizations achieved up to now at order 6.

To define chains as in Definition 8.4 of [32], we need then to explore a bit further the normalizations.

As we already know thanks to Proposition 16.4, it is possible, by some punctual normalization, to also make, at order 7

$$\begin{aligned}
0 \;&=\; F^p_{a,b,0,0,e} && (a+b+2e=7),\\
0 \;&=\; F^p_{a,b,1,0,e} && (a+b+2e=6),\\
0 \;&=\; F^p_{a,b,2,0,e} && (a+b+2e=5).
\end{aligned}$$

Once these normalizations are done, the condition that they are preserved forces $\chi = 0$ (exercise).

We therefore come to maps which express the 'ambiguity' of punctual normalizations being of the form

$$\begin{aligned}
z' \;&:=\; \lambda\, z + i\,\lambda\, r\, zw + \psi\, zw^2,\\
\zeta' \;&:=\; \frac{\lambda}{\overline{\lambda}}\,\zeta - i\,\frac{\lambda}{\overline{\lambda}}\,r\,z^2 + \Big(-\tfrac{8}{3}\,\frac{\psi}{\overline{\lambda}} + \tfrac{4}{3}\,\frac{\lambda}{\overline{\lambda}^2}\,\overline{\psi} - \tfrac{1}{3}\,\frac{\lambda}{\overline{\lambda}}\,r^2\Big)\,z^2w,\\
w' \;&:=\; \lambda\overline{\lambda}\,w + i\,\lambda\overline{\lambda}\,r\,w^2 + \Big(-\tfrac{1}{3}\,\lambda\overline{\lambda}\,r^2 + \tfrac{1}{3}\,\overline{\lambda}\psi + \tfrac{1}{3}\,\lambda\overline{\psi}\Big)\,w^3.
\end{aligned}$$

Then such maps have the property that they send curves $\mathbb{R}^1_v \longrightarrow \mathbb{R}^4_{x,y,s,t}$ of the form

$$x \;=\; \mathrm{O}_v(2), \qquad y \;=\; \mathrm{O}_v(2), \qquad s \;=\; \mathrm{O}_v(2), \qquad t \;=\; \mathrm{O}_v(2),$$

to curves of the similar form

$$x' \;=\; \mathrm{O}_{v'}(2), \qquad y \;=\; \mathrm{O}_{v'}(2), \qquad s \;=\; \mathrm{O}_{v'}(2), \qquad t \;=\; \mathrm{O}_{v'}(2),$$

hence they *stabilize the flat* 2*-jet* $j^2_0 = (0,0,0,0,\ 0,0,0,0)$.

In conclusion, we have reached a point at which we can state an analog of Definition 8.4 in [32].

Definition 25.1. Given a hypersurface $M^5 \subset \mathbb{C}^3$ in the class $\mathfrak{C}_{2,1}$, a point $p \in M$, a 1-jet $j^1_p \in \Sigma^1_p$ at p, given the translation map $\tau_p : (M,p) \longrightarrow (M^p, 0)$,

and using *any* normalizing map $\Phi_p : M^p \longrightarrow N^p$ which sends $(M^p, 0)$ to a hypersurface $(N^p, 0)$ of equation

$$\begin{aligned} z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}z^2\overline{\zeta}\zeta\overline{\zeta} \\ + 2\,\mathrm{Re}\Big\{0 + 0 + F^p_{3,0,0,2,0}\,z^3\overline{\zeta}^2 + \zeta\overline{\zeta}\,\big(3\,z^2\overline{z}\overline{\zeta}\,F^p_{3,0,0,2,0}\big)\Big\} \\ + 0 + 2\,\mathrm{Re}\Big\{z^5\overline{\zeta}\,F^p_{5,0,0,1,0} + \quad 0 \quad + z^4\overline{\zeta}^2\,F^p_{4,0,0,2,0} \\ + z^3\overline{z}^2\overline{\zeta}\,F^p_{3,0,2,1,0} + z^3\overline{z}\overline{\zeta}^2\,F^p_{3,0,1,2,0} \\ + z^3\overline{\zeta}^3\,F^p_{3,0,0,3,0}\Big\} \\ + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(7), \end{aligned}$$

with in addition

$$\begin{aligned} 0 &= F^p_{a,b,0,0,e} \qquad (a+b+2e=7), \\ 0 &= F^p_{a,b,1,0,e} \qquad (a+b+2e=6), \\ 0 &= F^p_{a,b,2,0,e} \qquad (a+b+2e=5), \end{aligned}$$

and which *also sends* j^1_p to the flat 1-jet $j^1_0 = (0,0,0,0)$ at $0 \in N^p$, assign the 2-jet j^2_p of the *chain* at $p \in M$ associated with j^1_p to be the inverse image of the *flat* 2-jet at $0 \in N^p$

$$j^2_p := \big(\Phi_p \circ \tau_p\big)^{(2)^{-1}}\big(0,0,0,0,\ 0,0,0,0\big).$$

Thanks to the preceding reasonings, the result j^2_p is independent of the normalizing map $\Phi_p \circ \tau_p$ satisfying $\big(\Phi_p \circ \tau_p\big)^{(1)}(j^1_p) = (0,0,0,0)$, the flat 1-jet at $0 \in N^p$.

Furthermore, there are $\mathcal{C}^\omega$ functions A, B, C, D, E, F, which can be made explicit in terms of $\big\{F_{a,b,c,d,e}\big\}_{1 \leqslant a+b+c+d+2e \leqslant 6}$, such that equations of chains are, with time parameter v

$$\begin{aligned} \dot{s} &= A\big(v,x,y,s,t,\dot{x},\dot{y}\big), & \ddot{x} &= C\big(v,x,y,s,t,\dot{x},\dot{y}\big), \\ \dot{t} &= B\big(v,x,y,s,t,\dot{x},\dot{y}\big), & \ddot{y} &= D\big(v,x,y,s,t,\dot{x},\dot{y}\big), \\ & & \ddot{s} &= E\big(v,x,y,s,t,\dot{x},\dot{y}\big), \\ & & \ddot{t} &= F\big(v,x,y,s,t,\dot{x},\dot{y}\big). \end{aligned}$$

Integrability follows from the fact that Σ^2_0 *is* a surface.

After that order 2 chains are known, it is natural to restart once more the whole process of prenormalization and of partial normalization which begun in Section 11, by assuming that the CR-transversal curve $p \in \gamma \subset M$ (not anymore chosen at random) *is* an order 2 chain. In fact, to have a second order chain at a point $p \in M$, it suffices to prescribe two real constants, the initial values $\dot{x}(0)$, $\dot{y}(0)$.

Then, coming back to Proposition 23.4, but viewed at the origin up to order 6 in *all* variables $(z,\zeta,\overline{z},\overline{\zeta},v)$, we remember that we have constructed a normalizing map Φ_p, composed with a translation map τ_p

$$\varphi:\quad (M,p)\ \xrightarrow{\quad\tau_p\quad}\ (M^p,0)\ \xrightarrow{\quad\Phi_p\quad}\ (N^p,0),$$

which brings (M,p) to $(N^p,0)$ at the origin of equation

$$\begin{aligned}
u\ =\ & z\overline{z}+\tfrac{1}{2}\overline{z}^2\zeta+\tfrac{1}{2}z^2\overline{\zeta}+z\overline{z}\zeta\overline{\zeta}+\tfrac{1}{2}\overline{z}^2\zeta\zeta\overline{\zeta}+\tfrac{1}{2}z^2\overline{\zeta}\zeta\overline{\zeta}+z\overline{z}\zeta\overline{\zeta}\zeta\overline{\zeta}\\
&+z^3\overline{z}^3F^p_{3,0,3,0,0}+z^3\overline{z}^3\,\mathrm{O}_{z,\overline{z},v}(1)\\
&+2\,\mathrm{Re}\Big\{0+0+z^3\overline{\zeta}^2F^p_{3,0,0,2,0}+\zeta\overline{\zeta}\left(3\,z^2\overline{z}\overline{\zeta}F^p_{3,0,0,2,0}\right)\Big\}\\
&+2\,\mathrm{Re}\Big\{z^5\overline{\zeta}F^p_{5,0,0,1,0}+z^4\overline{z}\overline{\zeta}F^p_{4,0,1,1,0}+z^4\overline{\zeta}^2F^p_{4,0,0,2,0}\\
&\qquad+z^3\overline{z}^2\overline{\zeta}F^p_{3,0,2,1,0}+z^3\overline{z}\overline{\zeta}^2F^p_{3,0,1,2,0}\\
&\qquad+z^3\overline{\zeta}^3F^p_{3,0,0,3,0}\Big\}\\
&\qquad+\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(7),
\end{aligned}$$

without changing the CR-transversal curve $0\in\gamma\subset M$ being the v-axis, hence having *flat* 1-jet at the origin.

Assertion 25.2. *Then* $F^p_{4,0,1,1,0}=0$ *holds automatically, without having the needs to perform any further biholomorphism.*

Proof. Indeed, from the proof of Lemma 24.1 we already know that with the choice

$$\tau\ :=\ \frac{1}{2}\overline{F^p_{4,0,1,1,0}},$$

one can continue to normalize and make $F^{p\prime}_{4,0,1,1,0}=0$ by means of the map

$$\begin{aligned}
z'\ &:=\ z+F^p_{4,0,1,1,0}\,z^2w-\tfrac{1}{2}F^p_{4,0,1,1,0}\,\zeta w^2+\tfrac{1}{2}\overline{F^p_{4,0,1,1,0}}\,w^2,\\
\zeta'\ &:=\ \zeta-2\overline{F^p_{4,0,1,1,0}}\,zw+2F^p_{4,0,1,1,0}\,\zeta w^2,\\
w'\ &:=\ w+F^p_{4,0,1,1,0}\,zw^2,
\end{aligned}$$

which we may call $\Psi:\,(N^p,0)\longrightarrow(N'^p,0)$. We then reason as in [32, 9.5]

If $F^p_{4,0,1,1,0}\neq 0$ would be nonzero, due to the presence in z' of the quadratic term $\frac{1}{2}\overline{F^p_{4,0,1,1,0}}\,w^2$, this map Ψ would *not* stabilize the flat order 2 jet $j^2_0=(0,0,0,0,\ 0,0,0,0)$, and so, this would contradict Definition 25.1 applied to $(M,p):=(N^p,0)$, to $\varphi:=\Psi$, and to $(N^p,0):=(N'^p,0)$. □

26. Moser-like normal form for $\mathfrak{C}_{2,1}$ hypersurfaces $M^5 \subset \mathbb{C}^3$

Lastly, coming again back to Proposition 19.3, all what precedes showed that, *without changing* any starting order 2 chain $0 \in \gamma \subset M$ to be straightened to be the v-axis, we have constructed a normalizing map $(M, 0) \longrightarrow (N, 0)$ so that, in the equation of N, we may (at last!) let appear all the terms of order 6 in $(z, \zeta, \overline{z}, \overline{\zeta})$

$$
\begin{aligned}
u \;=\; & z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}z^2\overline{\zeta}\zeta\overline{\zeta} + z\overline{z}\zeta\overline{\zeta}\zeta\overline{\zeta} \\
& + z^3\overline{z}^3 F_{3,0,3,0}(v) + z^3\overline{z}^3 \mathrm{O}_{z,\overline{z}}(1) \\
& + 2\,\mathrm{Re}\Big\{0 + 0 + z^3\overline{\zeta}^2 F_{3,0,0,2}(v) + \zeta\overline{\zeta}\,\big(3\,z^2\overline{z}\overline{\zeta}\,F_{3,0,0,2}(v)\big)\Big\} \\
& + 2\,\mathrm{Re}\Big\{z^5\overline{\zeta}\,F_{5,0,0,1}(v) + z^4\overline{z}\overline{\zeta}\,\underline{F_{4,0,1,1}(v)}_{\circ} + z^4\overline{\zeta}^2 F_{4,0,0,2}(v) \\
& \qquad + z^3\overline{z}^2\overline{\zeta}\,F_{3,0,2,1}(v) + z^3\overline{z}\overline{\zeta}^2 F_{3,0,1,2}(v) \\
& \qquad\qquad + z^3\overline{\zeta}^3 F_{3,0,0,3}(v)\Big\} \\
& + \overline{z}^3\zeta\,\mathrm{O}_{z,\zeta,\overline{z}}(3) + z^3\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(3) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(5).
\end{aligned}
$$

Assertion 26.1. *The function $F_{4,0,1,1}(v) \equiv 0$ vanishes for free.*

Proof. What was done an instant ago by Assertion 25.2 at the origin $(z, \zeta, w) = (0, 0, 0)$ applies in fact at *every* point $(0, 0, iv)$ along the v-axis, thanks to the fact that the above graphed equation is the same *all along* the v-axis. □

Proposition 26.2. *There exists a biholomorphism of the form*

$$
z' \;:=\; z\,\varphi(-iw), \qquad\qquad \zeta' \;:=\; \zeta + \chi(-iw)\,z^2, \qquad\qquad w' \;:=\; i\,\psi(-iw),
$$

with $\psi(v) \in \mathbb{R}$ for $v \in \mathbb{R}$, which normalizes in addition $F'_{3,0,3,0}(v') \equiv 0$.

Proof. Left to the reader. Hint: imitate [32, Lm. 12.4]. □

In summary, we can state

Theorem 26.3. [Existence of normal form] *For every 2-nondegenerate hypersurface $M^5 \in \mathfrak{C}_{2,1}$ whose Levi form has constant rank 1, at any point $p \in M$, given any order 2 CR-transversal chain $p \in \gamma \subset M$, there exist holomorphic coordinates $(z, \zeta, w) \in \mathbb{C}^3$ vanishing at p in which γ is the v-axis and in which M*

has normalized equation

$$\begin{aligned} u \;=\; & \tfrac{z\overline{z}+\frac{1}{2}\overline{z}^2\zeta+z^2\overline{\zeta}}{1-\zeta\overline{\zeta}} \\ & + z^3\overline{z}^3\,\mathrm{O}_{z,\overline{z}}(1) + 2\,\mathrm{Re}\Big\{z^3\overline{\zeta}^2\,F_{3,0,0,2}(v) + \zeta\overline{\zeta}\,\big(3\,z^2\overline{z}\overline{\zeta}\,F_{3,0,0,2}(v)\big)\Big\} \\ & + 2\,\mathrm{Re}\Big\{z^5\overline{\zeta}\,F_{5,0,0,1}(v) + z^4\overline{\zeta}^2\,F_{4,0,0,2}(v) \\ & \qquad + z^3\overline{z}^2\overline{\zeta}\,F_{3,0,2,1}(v) + z^3\overline{z}\overline{\zeta}^2\,F_{3,0,1,2}(v) + z^3\overline{\zeta}^3\,F_{3,0,0,3}(v)\Big\} \\ & + \overline{z}^3\zeta\,\mathrm{O}_{z,\zeta,\overline{z}}(3) + z^3\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(3) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(5). \end{aligned}$$

27. Consequence of prenormalization on dependent jets

After the prenormalization Proposition 16.4, we know that we have

$$\begin{aligned} u \;=\; F \;&=\; m+G \\ &=\; m + z^3\overline{z}^3\,\mathrm{O}_{z,\overline{z}}(0) + z^3\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z}}(0) + \overline{z}^3\zeta\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(0) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(3). \end{aligned}$$

The next statement shows that the dependent-jets remainder is *in addition* an $\mathrm{O}_{z,\overline{z}}(3)$.

Proposition 27.1. *In prenormalized coordinates,* $G = \mathrm{O}_{z,\overline{z}}(3)$.

This writing means here that G is of order 3 in $(z,\overline{z})$, with coefficients being arbitrary functions of $(z,\zeta,\overline{z},\overline{\zeta},v)$, namely that

$$G \;=\; z^3\,(\cdots) + z^2\overline{z}\,(\cdots) + z\overline{z}^2\,(\cdots) + \overline{z}^3\,(\cdots).$$

Proof. Since the coordinates are prenormalized, we have at least

$$u \;=\; z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4) \;=\; m+G.$$

Thus if we write

$$G \;=\; \sum_{\kappa\geqslant 2}\;\sum_{a+b+c+d=\kappa} G_{a,b,c,d}(v)\,z^a\zeta^b\overline{z}^c\overline{\zeta}^d \;=:\; \sum_{\kappa\geqslant 2} G^\kappa(v).$$

we have $0 = G^2 = G^3$, which are certainly both $\mathrm{O}_{z,\overline{z}}(3)$.

The proof will consist in examining, order by order, the Levi determinant for $F = m + G$

$$\begin{vmatrix} 0 & F_z & F_\zeta & -\frac{1}{2}+\frac{1}{2i}F_v \\ F_{\overline{z}} & F_{z\overline{z}} & F_{\zeta\overline{z}} & \frac{1}{2i}F_{\overline{z}v} \\ F_{\overline{\zeta}} & F_{z\overline{\zeta}} & F_{\zeta\overline{\zeta}} & \frac{1}{2i}F_{\overline{\zeta}v} \\ -\frac{1}{2}-\frac{1}{2i}F_v & -\frac{1}{2i}F_{zv} & -\frac{1}{2i}F_{\zeta v} & \frac{1}{4}F_{vv} \end{vmatrix}.$$

Reasoning by induction, assume, for some $\kappa \geqslant 4$, that $G^2, G^3, \dots, G^{\kappa-1}$ are all $\mathrm{O}_{z,\overline{z}}(3)$. For all $2 \leqslant \ell \leqslant \kappa - 1$, it then follows that

$$\begin{array}{cccc}
G^\ell_z = \mathrm{O}_{z,\overline{z}}(2), & G^\ell_\zeta = \mathrm{O}_{z,\overline{z}}(3), & G^\ell_v = \mathrm{O}_{z,\overline{z}}(3), \\
G^\ell_{\overline{z}} = \mathrm{O}_{z,\overline{z}}(2), & G^\ell_{z\overline{z}} = \mathrm{O}_{z,\overline{z}}(1), & G^\ell_{\zeta\overline{z}} = \mathrm{O}_{z,\overline{z}}(2), & G^\ell_{\overline{z}v} = \mathrm{O}_{z,\overline{z}}(2), \\
G^\ell_{\overline{\zeta}} = \mathrm{O}_{z,\overline{z}}(3), & G^\ell_{z\overline{\zeta}} = \mathrm{O}_{z,\overline{z}}(2), & G^\ell_{\zeta\overline{\zeta}} = \mathrm{O}_{z,\overline{z}}(3), & G^\ell_{\overline{\zeta}v} = \mathrm{O}_{z,\overline{z}}(3), \\
G^\ell_v = \mathrm{O}_{z,\overline{z}}(3), & G^\ell_{zv} = \mathrm{O}_{z,\overline{z}}(2), & G^\ell_{\zeta v} = \mathrm{O}_{z,\overline{z}}(3), & G^\ell_{vv} = \mathrm{O}_{z,\overline{z}}(3).
\end{array}$$

To capture information about G^κ, we may truncate modulo $\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(\kappa+1)$

$$\begin{aligned}
m &\equiv m^2 + m^3 + \cdots + m^{\kappa-2} + m^{\kappa-1} + m^\kappa, \\
G &\equiv G^2 + G^3 + \cdots + G^{\kappa-2} + G^{\kappa-1} + G^\kappa,
\end{aligned}$$

where, for any formal

$$H = \sum_{a,b,c,d \geqslant 0} z^a \zeta^b \overline{z}^c \overline{\zeta}^d \, H_{a,b,c,d}(v),$$

and any $\mu \geqslant 0$, we set

$$\begin{aligned}
H^\mu &:= \sum_{a+b+c+d=\mu} z^a \zeta^b \overline{z}^c \overline{\zeta}^d \, H_{a,b,c,d}(v), \\
\pi^\mu(H) &:= \sum_{a+b+c+d\leqslant\mu} z^a \zeta^b \overline{z}^c \overline{\zeta}^d \, H_{a,b,c,d}(v).
\end{aligned}$$

We will insert $F = m + G$ in the Levi determinant and apply the projection $\pi^{\kappa-2}(\cdot)$ in order to capture $G^\kappa_{\zeta\overline{\zeta}}$.

Assertion 27.2. *Under the induction assumption,* $G^\kappa_{\zeta\overline{\zeta}} = \mathrm{O}_{z,\overline{z}}(3)$.

Proof. Some further preliminaries are necessary. At first, for any formal function $L = L(z, \zeta, \overline{z}, \overline{\zeta}, v)$ which is an $\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(\lambda)$ for some $\lambda \geqslant 0$, it holds, *with a shift*, that

$$\pi^{\kappa-2}\big(L \cdot H\big) = \pi^{\kappa-2}\Big(\pi^{\kappa-2}\big(L\big) \cdot \pi^{\kappa-2-\lambda}\big(H\big)\Big). \tag{27.1}$$

Next, with $_\cdot$ and $_{\cdot,\cdot}$ denoting partial derivatives with respect to any of the variables $z, \zeta, \overline{z}, \overline{\zeta}$, we have

$$\begin{aligned}
\pi^{\kappa-2}(m) &= m^2 + \cdots + m^{\kappa-2}, \\
\pi^{\kappa-2}\big(m_\cdot\big) &= m^2_\cdot + \cdots + m^{\kappa-2}_\cdot + m^{\kappa-1}_\cdot, \\
\pi^{\kappa-2}\big(m_{\cdot,\cdot}\big) &= m^2_{\cdot,\cdot} + \cdots + m^{\kappa-2}_{\cdot,\cdot} + m^{\kappa-1}_{\cdot,\cdot} + m^\kappa_{\cdot,\cdot},
\end{aligned}$$

and

$$\begin{aligned}
\pi^{\kappa-2}(G) &= G^2 + \cdots + G^{\kappa-2}, \\
\pi^{\kappa-2}\big(G_\cdot\big) &= G^2_\cdot + \cdots + G^{\kappa-2}_\cdot + G^{\kappa-1}_\cdot, \\
\pi^{\kappa-2}\big(G_{\cdot,\cdot}\big) &= G^2_{\cdot,\cdot} + \cdots + G^{\kappa-2}_{\cdot,\cdot} + G^{\kappa-1}_{\cdot,\cdot}, + G^\kappa_{\cdot,\cdot}.
\end{aligned}$$

Also, we will be using various values $\lambda = 0, 1, 2$ of the integer $\lambda \geqslant 0$ above

$$
\begin{aligned}
m_z &= \tfrac{\overline{z}+z\overline{\zeta}}{1-\zeta\overline{\zeta}} = \mathrm{O}_{z,\overline{z}}(1), & m_{\overline{z}} &= \tfrac{z+\overline{z}\zeta}{1-\zeta\overline{\zeta}} = \mathrm{O}_{z,\overline{z}}(1),\\
m_\zeta &= \tfrac{1}{2}\tfrac{(\overline{z}+z\overline{\zeta})^2}{(1-\zeta\overline{\zeta})^2} = \mathrm{O}_{z,\overline{z}}(2), & m_{\overline{\zeta}} &= \tfrac{1}{2}\tfrac{(z+\overline{z}\zeta)^2}{(1-\zeta\overline{\zeta})^2} = \mathrm{O}_{z,\overline{z}}(2),\\
m_{z\overline{z}} &= \tfrac{1}{1-\zeta\overline{\zeta}} = \mathrm{O}_{z,\overline{z}}(0), & m_{\zeta\overline{z}} &= \tfrac{\overline{z}+z\overline{\zeta}}{(1-\zeta\overline{\zeta})^2} = \mathrm{O}_{z,\overline{z}}(1),\\
m_{z\overline{\zeta}} &= \tfrac{z+\overline{z}\zeta}{(1-\zeta\overline{\zeta})^2} = \mathrm{O}_{z\overline{z}}(1), & m_{\zeta\overline{\zeta}} &= \tfrac{(z+\overline{z}\zeta)^2}{(1-\zeta\overline{\zeta})^3} = \mathrm{O}_{z,\overline{z}}(2).
\end{aligned}
$$

Indeed, we start from

$$
0 \equiv \pi^{\kappa-2}\left|\begin{array}{cccc}
0 & m_z + \sum\limits_{4\leqslant j\leqslant\kappa-1} G^j_z & m_\zeta + \sum\limits_{4\leqslant k\leqslant\kappa-1} G^k_\zeta & -\frac{1}{2} - \frac{i}{2}\sum\limits_{4\leqslant l\leqslant\kappa-2} G^l_v \\
m_{\overline{z}} + \sum\limits_{4\leqslant i\leqslant\kappa-1} G^i_{\overline{z}} & m_{z\overline{z}} + \sum\limits_{4\leqslant j\leqslant\kappa} G^j_{z\overline{z}} & m_{\zeta\overline{z}} + \sum\limits_{4\leqslant k\leqslant\kappa} G^k_{\zeta\overline{z}} & -\frac{i}{2}\sum\limits_{4\leqslant l\leqslant\kappa-1} G^l_{\overline{z}v} \\
m_{\overline{\zeta}} + \sum\limits_{4\leqslant i\leqslant\kappa-1} G^i_{\overline{\zeta}} & m_{z\overline{\zeta}} + \sum\limits_{4\leqslant j\leqslant\kappa} G^j_{z\overline{\zeta}} & m_{\zeta\overline{\zeta}} + \sum\limits_{4\leqslant k\leqslant\kappa} G^k_{\zeta\overline{\zeta}} & -\frac{i}{2}\sum\limits_{4\leqslant l\leqslant\kappa-1} G^l_{\overline{\zeta}v} \\
-\frac{1}{2} + \frac{i}{2}\sum\limits_{4\leqslant i\leqslant\kappa-2} G^i_v & \frac{i}{2}\sum\limits_{4\leqslant j\leqslant\kappa-1} G^j_{zv} & \frac{i}{2}\sum\limits_{4\leqslant k\leqslant\kappa-1} G^k_{\zeta v} & \frac{1}{4}\sum\limits_{4\leqslant l\leqslant\kappa-2} G^l_{vv}
\end{array}\right|.
$$

Let us expand this determinant along its first row, using (27.1) in order to take account of various useful *negative shifts* for the summations in the entries of the obtained 3×3 determinants

$$
\begin{aligned}
0 \equiv \pi^{\kappa-2}\Bigg(
&-\Big(m_z + \sum_{4\leqslant j\leqslant\kappa-1} G^j_z\Big)\left|\begin{array}{ccc}
m_{\overline{z}} + \sum\limits_{4\leqslant i\leqslant\kappa-2} G^i_{\overline{z}} & m_{\zeta\overline{z}} + \sum\limits_{4\leqslant k\leqslant\kappa-1} G^k_{\zeta\overline{z}} & -\frac{i}{2}\sum\limits_{4\leqslant l\leqslant\kappa-2} G^l_{\overline{z}v} \\
m_{\overline{\zeta}} + \sum\limits_{4\leqslant i\leqslant\kappa-2} G^i_{\overline{\zeta}} & m_{\zeta\overline{\zeta}} + \sum\limits_{4\leqslant k\leqslant\kappa-1} G^k_{\zeta\overline{\zeta}} & -\frac{i}{2}\sum\limits_{4\leqslant l\leqslant\kappa-2} G^l_{\overline{\zeta}v} \\
-\frac{1}{2} + \frac{i}{2}\sum\limits_{4\leqslant i\leqslant\kappa-3} G^i_v & \frac{i}{2}\sum\limits_{4\leqslant k\leqslant\kappa-2} G^k_{\zeta v} & \frac{1}{4}\sum\limits_{4\leqslant l\leqslant\kappa-3} G^l_{vv}
\end{array}\right| \\
&+\Big(m_\zeta + \sum_{4\leqslant k\leqslant\kappa-1} G^k_\zeta\Big)\left|\begin{array}{ccc}
m_{\overline{z}} + \sum\limits_{4\leqslant i\leqslant\kappa-3} G^i_{\overline{z}} & m_{z\overline{z}} + \sum\limits_{4\leqslant j\leqslant\kappa-2} G^j_{z\overline{z}} & -\frac{i}{2}\sum\limits_{4\leqslant l\leqslant\kappa-3} G^l_{\overline{z}v} \\
m_{\overline{\zeta}} + \sum\limits_{4\leqslant i\leqslant\kappa-3} G^i_{\overline{\zeta}} & m_{z\overline{\zeta}} + \sum\limits_{4\leqslant j\leqslant\kappa-2} G^j_{z\overline{\zeta}} & -\frac{i}{2}\sum\limits_{4\leqslant l\leqslant\kappa-3} G^l_{\overline{\zeta}v} \\
-\frac{1}{2} + \frac{i}{2}\sum\limits_{4\leqslant i\leqslant\kappa-4} G^i_v & \frac{i}{2}\sum\limits_{4\leqslant j\leqslant\kappa-3} G^j_{zv} & \frac{1}{4}\sum\limits_{4\leqslant l\leqslant\kappa-4} G^l_{vv}
\end{array}\right| \\
&-\Big(-\tfrac{1}{2} + \mathrm{O}_{z,\overline{z}}(3)\Big)\left|\begin{array}{ccc}
m_{\overline{z}} + \sum\limits_{4\leqslant i\leqslant\kappa-1} G^i_{\overline{z}} & m_{z\overline{z}} + \sum\limits_{4\leqslant j\leqslant\kappa} G^j_{z\overline{z}} & m_{\zeta\overline{z}} + \sum\limits_{4\leqslant k\leqslant\kappa} G^k_{\zeta\overline{z}} \\
m_{\overline{\zeta}} + \sum\limits_{4\leqslant i\leqslant\kappa-1} G^i_{\overline{\zeta}} & m_{z\overline{\zeta}} + \sum\limits_{4\leqslant j\leqslant\kappa} G^j_{z\overline{\zeta}} & m_{\zeta\overline{\zeta}} + \sum\limits_{4\leqslant k\leqslant\kappa} G^k_{\zeta\overline{\zeta}} \\
-\frac{1}{2} + \frac{i}{2}\sum\limits_{4\leqslant i\leqslant\kappa-2} G^i_v & \frac{i}{2}\sum\limits_{4\leqslant j\leqslant\kappa-1} G^j_{zv} & \frac{i}{2}\sum\limits_{4\leqslant k\leqslant\kappa-1} G^k_{\zeta v}
\end{array}\right|\Bigg).
\end{aligned}
$$

Now, apply the induction assumption, and simultaneously also, expand the last determinant along its first column

$$
\begin{aligned}
0 \;\equiv\; \pi^{\kappa-2}\Bigg(& -\mathrm{O}_{z,\bar z}(1)\begin{vmatrix} \mathrm{O}_{z,\bar z}(1) & \mathrm{O}_{z,\bar z}(1) & \mathrm{O}_{z,\bar z}(2) \\ \mathrm{O}_{z,\bar z}(2) & \mathrm{O}_{z,\bar z}(2) & \mathrm{O}_{z,\bar z}(3) \\ \mathrm{O}_{z,\bar z}(0) & \mathrm{O}_{z,\bar z}(3) & \mathrm{O}_{z,\bar z}(3) \end{vmatrix} \\
& +\mathrm{O}_{z,\bar z}(2)\begin{vmatrix} \mathrm{O}_{z,\bar z}(1) & \mathrm{O}_{z,\bar z}(0) & \mathrm{O}_{z,\bar z}(2) \\ \mathrm{O}_{z,\bar z}(2) & \mathrm{O}_{z,\bar z}(1) & \mathrm{O}_{z,\bar z}(3) \\ \mathrm{O}_{z,\bar z}(0) & \mathrm{O}_{z,\bar z}(2) & \mathrm{O}_{z,\bar z}(3) \end{vmatrix} \\
& +\Big(\tfrac{1}{2}+\mathrm{O}_{z,\bar z}(3)\Big)\Bigg\{\Big(m_{z\bar\zeta}+\sum_{4\leqslant i\leqslant\kappa-1} G^i_{z\bar\zeta}\Big)\begin{vmatrix} m_{z\bar\zeta}+\sum\limits_{4\leqslant j\leqslant\kappa-1} G^j_{z\bar\zeta} & m_{\zeta\bar\zeta}+\sum\limits_{4\leqslant k\leqslant\kappa-1} G^k_{\zeta\bar\zeta} \\ \frac{i}{2}\sum\limits_{4\leqslant j\leqslant\kappa-2} G^j_{zv} & \frac{i}{2}\sum\limits_{4\leqslant k\leqslant\kappa-2} G^k_{\zeta v} \end{vmatrix} \\
& -\Big(m_{\bar\zeta}+\sum_{4\leqslant i\leqslant\kappa-1} G^i_{\bar\zeta}\Big)\begin{vmatrix} m_{z\bar z}+\sum\limits_{4\leqslant j\leqslant\kappa-2} G^j_{z\bar z} & m_{\zeta\bar z}+\sum\limits_{4\leqslant k\leqslant\kappa-2} G^k_{\zeta\bar z} \\ \frac{i}{2}\sum\limits_{4\leqslant j\leqslant\kappa-3} G^j_{zv} & \frac{i}{2}\sum\limits_{4\leqslant k\leqslant\kappa-3} G^k_{\zeta v} \end{vmatrix} \\
& +\Big(-\tfrac{1}{2}+\mathrm{O}_{z,\bar z}(3)\Big)\begin{vmatrix} m_{z\bar z}+\sum\limits_{4\leqslant j\leqslant\kappa} G^j_{z\bar z} & m_{\zeta\bar z}+\sum\limits_{4\leqslant k\leqslant\kappa} G^k_{\zeta\bar z} \\ m_{z\bar\zeta}+\sum\limits_{4\leqslant j\leqslant\kappa} G_{z\bar\zeta} & m_{\zeta\bar\zeta}+\sum\limits_{4\leqslant k\leqslant\kappa} G^k_{\zeta\bar\zeta} \end{vmatrix}\Bigg\}\Bigg).
\end{aligned}
$$

Taking account of $0\equiv\left|\begin{smallmatrix} m_{z\bar z} & m_{\zeta\bar z} \\ m_{z\bar\zeta} & m_{\zeta\bar\zeta}\end{smallmatrix}\right|$ in the last 2×2 determinant, we may continue to expand

$$
\begin{aligned}
0 \;\equiv\; & \mathrm{O}_{z,\bar z}(3)+\mathrm{O}_{z,\bar z}(1)\begin{vmatrix} \mathrm{O}_{z,\bar z}(1) & \mathrm{O}_{z,\bar z}(2) \\ \mathrm{O}_{z,\bar z}(2) & \mathrm{O}_{z,\bar z}(3) \end{vmatrix}-\mathrm{O}_{z,\bar z}(2)\begin{vmatrix} \mathrm{O}_{z,\bar z}(0) & \mathrm{O}_{z,\bar z}(1) \\ \mathrm{O}_{z,\bar z}(2) & \mathrm{O}_{z,\bar z}(3) \end{vmatrix} \\
& +\Big(-\tfrac{1}{4}+\mathrm{O}_{z,\bar z}(3)\Big)\begin{Bmatrix} m_{z\bar z}\sum\limits_{4\leqslant k\leqslant\kappa} G^k_{\zeta\bar\zeta}+m_{\zeta\bar\zeta}\sum\limits_{4\leqslant j\leqslant\kappa-2} G^j_{z\bar z}+\Big(\sum\limits_{4\leqslant j\leqslant\kappa-2} G^j_{z\bar z}\Big)\Big(\sum\limits_{4\leqslant k\leqslant\kappa-2} G^k_{\zeta\bar\zeta}\Big) \\ -m_{\zeta\bar z}\sum\limits_{4\leqslant j\leqslant\kappa-1} G^j_{z\bar\zeta}-m_{z\bar\zeta}\sum\limits_{4\leqslant k\leqslant\kappa-1} G^k_{\zeta\bar z}-\Big(\sum\limits_{4\leqslant k\leqslant\kappa-2} G^k_{\zeta\bar z}\Big)\Big(\sum\limits_{4\leqslant j\leqslant\kappa-2} G^j_{z\bar\zeta}\Big) \end{Bmatrix},
\end{aligned}
$$

that is

$$
\begin{aligned}
\mathrm{O}_{z,\bar z}(3) \;\equiv\; & m_{z\bar z}\Big(\sum_{4\leqslant k\leqslant\kappa-1} G^k_{\zeta\bar\zeta}+G^\kappa_{\zeta\bar\zeta}\Big)+\mathrm{O}_{z,\bar z}(2)\,\mathrm{O}_{z,\bar z}(1)+\mathrm{O}_{z,\bar z}(1)\,\mathrm{O}_{z,\bar z}(3) \\
& -\mathrm{O}_{z,\bar z}(1)\,\mathrm{O}_{z,\bar z}(2)-\mathrm{O}_{z,\bar z}(1)\,\mathrm{O}_{z,\bar z}(2)-\mathrm{O}_{z,\bar z}(2)\,\mathrm{O}_{z,\bar z}(2),
\end{aligned}
$$

and reminding $m_{z\bar z}=\frac{1}{1-\zeta\bar\zeta}$, this gives the concluding identity

$$
\mathrm{O}_{z,\bar z}(3) \;=\; \tfrac{1}{1-\zeta\bar\zeta}\,G^\kappa_{\zeta\bar\zeta}. \qquad \square
$$

By integration, $G^\kappa=\lambda^\kappa(z,\zeta,\bar z,v)+\overline{\lambda}^\kappa(\bar z,\bar\zeta,z,v)+\mathrm{O}_{z,\bar z}(3)$. After absorption in $\mathrm{O}_{z,\bar z}(3)$, we can assume that λ^κ is of degree $\leqslant 2$ in $(z,\bar z)$, hence contains only monomials $z^a\zeta^b\bar z^c v^e$ with $a+c\leqslant 2$ and $a+b+c=\kappa$. So $b\geqslant\kappa-2$.

Further, $G^\kappa(z,\zeta,0,0,v)\equiv 0$ imposes $\lambda^\kappa(z,\zeta,0,v)\equiv 0$. So $1\leqslant c\leqslant 2$. Consequently, λ^κ can contain only three monomials

$$
\lambda^\kappa(z,\zeta,\bar z,v) \;=\; a(v)\,\bar z\zeta^{\kappa-1}+b(v)\,z\bar z\,\zeta^{\kappa-2}+c(v)\,\bar z^2\zeta^{\kappa-2}.
$$

Since $\kappa \geqslant 4$, we see that the conjugate $\overline{\lambda}^{\kappa}(\overline{z}, \overline{\zeta}, z, v)$ is multiple of $\overline{\zeta}^{\kappa-2 \geqslant 2}$, hence

$$G^{\kappa}\big(z, \zeta, \overline{z}, 0, v\big) \,=\, \lambda^{\kappa}(z, \zeta, \overline{z}, v) + \underline{\overline{\lambda}^{\kappa}(\overline{z}, 0, z, v)}_{\circ} + \mathrm{O}_{z,\overline{z}}(3).$$

Finally, because the prenormalized coordinates of Proposition 16.4 require $G^{\kappa}(z, \zeta, \overline{z}, 0, v) \,=\, \mathrm{O}_{z,\overline{z}}(3)$, we reach $\lambda^{\kappa}(z, \zeta, \overline{z}, v) \,=\, \mathrm{O}_{z,\overline{z}}(3)$, which forces $a \,=\, b = c = 0 = \lambda^{\kappa}$, so as asserted $G^{\kappa} = \mathrm{O}_{z,\overline{z}}(3)$. □

28. Consequence of prenormalization on equivalences

Thanks to Proposition 16.4, if we are given a holomorphic map

$$H\colon \quad (z, \zeta, w) \,\longmapsto\, (z', \zeta', w')$$

between two $\mathfrak{C}_{2,1}$ hypersurfaces $M^5 \subset \mathbb{C}^3$ and $M'^5 \subset \mathbb{C}'^3$, we can assume that both hypersurfaces are prenormalized. In particular, Proposition 27.1 tells us that the *whole* remainders after the GM-model part of their graphing functions is of order 3 in $(z, \overline{z})$

$$u \,=\, m + G \,=\, m + \mathrm{O}_{z,\overline{z}}(3) \qquad \text{and} \qquad u' \,=\, m' + G' \,=\, m + \mathrm{O}_{z',\overline{z}'}(3).$$

Observation 28.1. *Complex scalings* $(z, \zeta, w) \longmapsto \big(\lambda z, \, \frac{\lambda}{\overline{\lambda}}\zeta, \, \lambda\overline{\lambda}w\big)$ *with* $\lambda \in \mathbb{C}^*$ *preserve prenormalizations as in Proposition* 16.4. □

With $\lambda := \rho \in \mathbb{R}^*$, this is $\big(\rho^1 z, \, \rho^0\zeta, \, \rho^2 w\big)$. Hence this observation suggests naturally to assign the following weights to the three complex variables and their real and imaginary parts

$$[z] \,:=\, 1 \,=:\, [\overline{z}], \qquad\qquad [\zeta] \,:=\, 0 \,=:\, [\overline{\zeta}], \qquad\qquad [w] \,:=\, 2 \,=:\, [\overline{w}].$$

Accordingly, let us decompose the components (f, g, h) of H in weighted homogeneous parts

$$f \,=\, f_0 + f_1 + f_2 + f_3 \cdots, \qquad\qquad g \,=\, g_0 + g_1 + g_2 + \cdots,$$
$$h \,=\, h_0 + h_1 + h_2 + h_3 + h_4 + \cdots.$$

Proposition 28.2. *If both* M *and* M' *are prenormalized, possibly after composing with a complex dilation* $(z', \zeta', w') \longmapsto \big(\lambda z', \, \frac{\lambda}{\overline{\lambda}}\zeta', \lambda\overline{\lambda}w'\big)$*, one has* $f_0 = 0$*,* $f_1 = z$*,* $g_0 = \zeta$*,* $h_0 = 0$*,* $h_1 = 0$*,* $h_2 = w$*, and the weighted homogeneous components of* f*,* g*,* h *are*

$$f \,=\, z + f_2 + f_3 + \cdots, \qquad\qquad g \,=\, \zeta + g_1 + g_2 + \cdots,$$
$$h \,=\, w + h_3 + h_4 + \cdots.$$

Mind the fact that this does not mean that the map is $\mathtt{Id} + \mathrm{O}_{z,w,\zeta}(2)$, since in f_2, there can still be the linear term $f_{0,0,2}\, w$, and in $g_1 + g_2$, there can still be the linear terms $g_{1,0,0}\, z + g_{0,0,1}\, w$.

Proof. The fundamental identity expressing that we have a map $M \longrightarrow M'$ reads

$$\begin{aligned} h_0 + h_1 + \cdots + \overline{h}_0 + \overline{h}_1 + \cdots \;=\; 2F'\Big(& f_0 + f_1 + \cdots,\, g_0 + g_1 + \cdots, \\ & \overline{f}_0 + \overline{f}_1 + \cdots,\, \overline{g}_0 + \overline{g}_1 + \cdots, \\ & \tfrac{1}{2i}\big(h_0 + h_1 + \cdots - \overline{h}_0 - \overline{h}_1 - \cdots\big)\Big). \end{aligned} \tag{28.1}$$

Observe that $f_0 = f_0(\zeta), g_0 = g_0(\zeta), h_0 = h_0(\zeta)$ depend only on ζ. This identity projected to weight 0 becomes

$$h_0(\zeta) + \overline{h}_0(\overline{\zeta}) \;\equiv\; 2F'\Big(f_0(\zeta),\, g_0(\zeta),\, \overline{f}_0(\overline{\zeta}),\, \overline{g}_0(\overline{\zeta}),\, \tfrac{1}{2i}h_0(\zeta) - \tfrac{1}{2i}\overline{h}_0(\overline{\zeta})\Big).$$

Put $\overline{\zeta} := 0$, use the assumption that there are no pluriharmonic terms (coordinates are prenormalized), namely that $0 \equiv F'(z', \zeta', 0, 0, v')$, and get $h_0(\zeta) \equiv 0$.

Once again, look at (28.1), and get from $F' = m' + G' = m' + \mathrm{O}_{z',\overline{z}'}(3)$

$$0 \;\equiv\; \frac{2\,f_0(\zeta)\overline{f}_0(\overline{\zeta}) + f_0(\zeta)^2\,\overline{g}_0(\overline{\zeta}) + \overline{f}_0(\overline{\zeta})^2\,g_0(\zeta)}{1 - g_0(\zeta)\overline{g}_0(\overline{\zeta})} + \mathrm{O}_{f_0(\zeta),\overline{f}_0(\overline{\zeta})}(3).$$

We claim that $f_0(\zeta) \equiv 0$. Otherwise, $f_0 = e\,\zeta^\tau + \mathrm{O}_\zeta(\tau+1)$ with $e \neq 0$ and $\tau \in \mathbb{N}_{\geqslant 1}$. Hence

$$0 \;\equiv\; 2e\overline{e}\,\zeta^\tau\overline{\zeta}^\tau\,\big(1 + \mathrm{O}_{\zeta,\overline{\zeta}}(1)\big) + \zeta^{2\tau}(\cdots) + \overline{\zeta}^{2\tau}(\cdots) + \mathrm{O}_{\zeta,\overline{\zeta}}\big(3\,\tau\big),$$

and this forces $e\overline{e} = 0$. So $f_0(\zeta) \equiv 0$, and (28.1) at weight 0, namely the identity above, reduces to $0 = 0$.

Next, examine weight 1. Certainly, $f_1 = z f_1(\zeta)$ and $h_1 = z h_1(\zeta)$, while g will not participate here. Since m' is weighted 2-homogeneous, as it contains $z\overline{z}, \overline{z}^2, z^2$ times functions of $(\zeta, \overline{\zeta})$, we have $F' = \mathrm{O}_{z',\overline{z}'}(2)$, so the identity

$$z\,h_1(\zeta) + \overline{z}\,\overline{h}_1(\overline{\zeta}) \;\equiv\; \mathrm{O}_{z f_1(\zeta),\overline{z}\overline{f}_1(\overline{\zeta})}(2) \;=\; \mathrm{O}_{z,\overline{z}}(2),$$

forces $h_1(\zeta) \equiv 0$.

Next, expand in powers of z, w

$$\begin{aligned} f \;&=\; z\,f_1(\zeta) + z^2(\cdots) + w(\cdots), \qquad g \;=\; g_0(\zeta) + z\,g_1(\zeta) + z^2(\cdots) + w(\cdots), \\ h \;&=\; h_2 + h_3 + \cdots, \qquad h_2 \;=\; z^2\,\varphi(\zeta) + w\,\psi(\zeta). \end{aligned}$$

The holomorphic Jacobian at the origin is assumed to be invertible

$$0 \;\neq\; \begin{vmatrix} f_z(0) & f_\zeta(0) & f_w(0) \\ g_z(0) & g_\zeta(0) & g_w(0) \\ h_z(0) & h_\zeta(0) & h_w(0) \end{vmatrix} \;=\; \begin{vmatrix} f_1(0) & 0 & f_w(0) \\ g_1(0) & g_0'(0) & g_w(0) \\ 0 & 0 & h_w(0) \end{vmatrix},$$

whence $h_w(0) \neq 0$ and $g_0'(0) \neq 0$ and also $f_1(0) \neq 0$. Then the fundamental identity (28.1) becomes in weight 2

$$h_2 + \overline{h}_2 \equiv 2\,m'\big(zf_1(\zeta),\, g_0(\zeta),\, \overline{z}\overline{f}_1(\overline{\zeta}),\, \overline{g}_0(\overline{\zeta})\big),$$

that is, after replacing $w = m + iv$ in h_2

$$\begin{aligned} z^2\,\varphi(\zeta) + \overline{z}^2\,\overline{\varphi}(\overline{\zeta}) + m(z,\zeta,\overline{z},\overline{\zeta})\big[\psi(\zeta) + \overline{\psi}(\overline{\zeta})\big] + i\,v\big[\psi(\zeta) - \overline{\psi}(\overline{\zeta})\big] &\equiv \\ \equiv \frac{2\,z\,f_1(\zeta)\,\overline{z}\,\overline{f}_1(\overline{\zeta}) + \overline{z}^2\,\overline{f}_1(\overline{\zeta})\,g_0(\zeta) + z^2\,f_1(\zeta)^2\,\overline{g}_0(\overline{\zeta})}{1 - g_0(\zeta)\,\overline{g}_0(\overline{\zeta})}&, \end{aligned}$$

this holding identically in $\mathbb{C}\{z,\zeta,\overline{z},\overline{\zeta},v\}$. This forces $\psi(\zeta) \equiv \rho$ to be constant, with $\rho \in \mathbb{R}^*$, and then $\varphi(\zeta) \equiv 0$ necessarily.

It remains an identity

$$m(z,\zeta,\overline{z},\overline{\zeta})\,2\rho \equiv 2\,m'\big(zf_1(\zeta),\, g_0(\zeta),\, \overline{z}\overline{f}_1(\overline{\zeta}),\, \overline{g}_0(\overline{\zeta})\big),$$

which expresses that the map $(z,\zeta,w) \longmapsto \big(zf_1(\zeta),\, g_0(\zeta),\, \rho\,w\big)$ is an *automorphism* — in fact a *rigid* automorphism, *cf.* [8] — of the Gaussier-Merker model. But we know from Section 5, see the fractional expression of w' there, that this requires $\alpha = 0$ and $r = 0$, while only $\lambda \in \mathbb{C}^*$ is free. Consequently, the map is of the form $(f_1, g_0, h_2) = \big(\lambda z,\, \frac{\lambda}{\overline{\lambda}}\zeta,\, \lambda\overline{\lambda}w\big)$. Post-composing by the inverse map yields the conclusion. □

29. Uniqueness of normal form

Starting with a $\mathcal{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$ which is 2-nondegenerate and of constant Levi rank 1, at any point $p \in M$, it is elementary to find holomorphic coordinates (z,ζ,w) vanishing at p in which M has equation

$$u \,=\, F \,=\, z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(5). \tag{29.1}$$

Such an equation can hence freely be taken as the starting point towards a complete normalization of $F(z,\zeta,\overline{z},\overline{\zeta},v)$.

In the preceding sections, we have in fact established the *existence* of a *normal form* for M. We can now present a final *uniqueness* statement which will terminate our article.

Theorem 29.1. *Given $M^5 \subset \mathbb{C}^3$ in the class $\mathfrak{C}_{2,1}$ with $0 \in M$ of the form*

$$u \,=\, z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(5),$$

there exists a biholomorphism $(z,\zeta,w) \longmapsto (z',\zeta',w')$ *fixing* 0 *which maps* $(M,0)$ *into* $(M',0)$ *of normalized equation*

$$\begin{aligned}
u' \;=\; & z'\overline{z}' + \tfrac{1}{2}\overline{z}'^{2}\zeta' + \tfrac{1}{2}z'^{2}\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}' + \tfrac{1}{2}\overline{z}'^{2}\zeta'\zeta'\overline{\zeta}' + \tfrac{1}{2}z'^{2}\overline{\zeta}'\zeta'\overline{\zeta}' + z'\overline{z}'\zeta'\overline{\zeta}'\zeta'\overline{\zeta}' \\
& + 0 + z'^{3}\overline{z}'^{3}\,\mathrm{O}_{z',\overline{z}'}(1) \\
& + 2\,\mathrm{Re}\Big\{0 + 0 + z'^{3}\overline{\zeta}'^{2}\,F'_{3,0,0,2}(v') + \zeta'\overline{\zeta}'\,\big(3\,z'^{2}\overline{z}'\overline{\zeta}'\,F'_{3,0,0,2}(v')\big)\Big\} \\
& + 2\,\mathrm{Re}\Big\{z'^{5}\overline{\zeta}'\,F'_{5,0,0,1}(v') + \quad 0 \quad + z'^{4}\overline{\zeta}'^{2}\,F'_{4,0,0,2}(v') \\
& \qquad\qquad + z'^{3}\overline{z}'^{2}\overline{\zeta}'\,F'_{3,0,2,1}(v') + z'^{3}\overline{z}'\overline{\zeta}'^{2}\,F'_{3,0,1,2}(v') \\
& \qquad\qquad\qquad + z'^{3}\overline{\zeta}'^{3}\,F'_{3,0,0,3}(v')\Big\} \\
& + \overline{z}'^{3}\zeta'\,\mathrm{O}_{z',\zeta',\overline{z}'}(3) + z'^{3}\overline{\zeta}'\,\mathrm{O}_{z',\overline{z}',\overline{\zeta}'}(3) + \zeta'\overline{\zeta}'\,\mathrm{O}_{z',\zeta',\overline{z}'}(3)\,\mathrm{O}_{z',\zeta',\overline{z}',\overline{\zeta}'}(2).
\end{aligned}$$

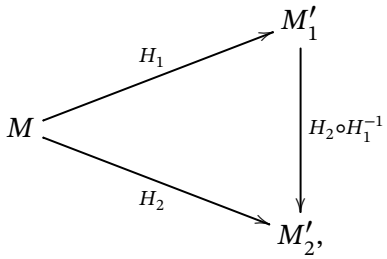
Furthermore, the map exists and is unique *if it is assumed to be of the form*

$$\begin{aligned}
z' \;&:=\; z + f_{\geqslant 2}(z,\zeta,w) \quad \zeta' \;:=\; \zeta + g_{\geqslant 1}(z,\zeta,w), \qquad w' \;:=\; w + h_{\geqslant 3}(z,\zeta,w), \\
0 \;&=\; f_{w}(0), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \;=\; \mathrm{Im}\, h_{ww}(0).
\end{aligned}$$

Here of course, $f_{\geqslant 2}$ is of weight $\geqslant 2$, while $g_{\geqslant 1}$ is of weight $\geqslant 1$, and $h_{\geqslant 3}$ is of weight $\geqslant 3$ for the currently useful weighting $[z] := 1$, $[\zeta] := 0$, $[w] := 2$.

Proof. By choosing a chain at $0 \in M$ whose first jet is flat, directed along the v-axis, one can verify (exercise) that all the constructions done in the preceding sections do indeed give a biholomorphism of this specific form. So our job is to establish uniqueness.

Suppose therefore that two such normalizations $H_\iota :\ (z,\zeta,w) \longmapsto (z+f_\iota,\ \zeta + g_\iota, w + h_\iota)$, $\iota = 1,2$, are given



with $0 = f_{\iota,w}(0)$ and $0 = \mathrm{Re}\, h_{\iota,ww}(0)$ for $\iota = 1,2$. We leave to the reader to verify that, then, $H := H_2 \circ H_1^{-1}$ is also of the form $(z,\zeta,w) \longmapsto \big(z + f_{\geqslant 2},\ \zeta + g_{\geqslant 1},\ w + h_{\geqslant 3}\big)$ also with $0 = f_w(0)$ and $0 = \mathrm{Im}\, h_{ww}(0)$. For this, one has to take account of (29.1).

The theorem asserts that $H_1 = H_2$. Equivalently, $H_2 \circ H_1^{-1} = \mathrm{Id}$. This will be offered by the next independent key uniqueness statement. $\square$

Theorem 29.2. *For a given $M^5 \subset \mathbb{C}^3$ in the class $\mathfrak{C}_{2,1}$, if two normal forms N and N, at some point $p \in M$ are constructed, with N having normalized equation*

$$\begin{aligned} u \,=\, & z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta} + z\overline{z}\zeta\overline{\zeta}\zeta\overline{\zeta} \\ & + 0 + z^3\overline{z}^3\,\mathrm{O}_{z,\overline{z}}(1) \\ & + 2\,\mathrm{Re}\Big\{0 + 0 + z^3\overline{\zeta}^2\,F_{3,0,0,2}(v) + \zeta\overline{\zeta}\,\big(3\,z^2\overline{z}\overline{\zeta}\,F_{3,0,0,2}(v)\big)\Big\} \\ & + 2\,\mathrm{Re}\Big\{z^5\overline{\zeta}\,F_{5,0,0,1}(v) + \quad 0 \quad + z^4\overline{\zeta}^2\,F_{4,0,0,2}(v) \\ & \qquad\qquad + z^3\overline{z}^2\overline{\zeta}\,F_{3,0,2,1}(v) + z^3\overline{z}\overline{\zeta}^2\,F_{3,0,1,2}(v) \\ & \qquad\qquad\qquad + z^3\overline{\zeta}^3\,F_{3,0,0,3}(v)\Big\} \\ & + \overline{z}^3\zeta\,\mathrm{O}_{z,\zeta,\overline{z}}(3) + z^3\overline{\zeta}\,\mathrm{O}_{z,\overline{z},\overline{\zeta}}(3) + \zeta\overline{\zeta}\,\mathrm{O}_{z,\zeta,\overline{z}}(3)\,\mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(2), \end{aligned}$$

and with N, having similarly normalized equation, and if the map $(z,\zeta,w) \longmapsto (z',\zeta',w')$ between them is of the form

$$\begin{aligned} z' \,&:=\, z + f_{\geqslant 2}(z,\zeta,w) \quad \zeta' \,:=\, \zeta + g_{\geqslant 1}(z,\zeta,w), \qquad w' \,:=\, w + h_{\geqslant 3}(z,\zeta,w), \\ 0 \,&=\, f_w(0), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \,=\, \mathrm{Im}\, h_{ww}(0), \end{aligned}$$

then the map $(z',\zeta',w') = (z,\zeta,w)$ is the identity, and the two normal forms $N = N$, coincide.

Proof. Equivalently, the graphing function $F = \sum_{a,b,c,d} z^a\zeta^b\overline{z}^c\overline{\zeta}^d\,F_{a,b,c,d}(v)$ of N satisfies the general *prenormalization conditions*

$$\begin{aligned} 0 \,&\equiv F_{a,b,0,0}(v) \equiv F_{0,0,c,d}(v), \\ 0 \,&\equiv F_{a,b,1,0}(v) \equiv F_{1,0,c,d}(v), \\ 0 \,&\equiv F_{a,b,2,0}(v) \equiv F_{2,0,c,d}(v), \end{aligned}$$

with the obvious two exceptions $F_{1,0,1,0}(v) \equiv 1$ and $F_{0,1,2,0}(v) \equiv \frac{1}{2} \equiv F_{2,0,0,1}(v)$, together with the *sporadic normalization conditions*, listed by increasing order 4, 5, 6

$$\begin{aligned} 0 \,&\equiv F_{3,0,0,1}(v) \equiv F_{0,1,3,0}(v), \\ 0 \,&\equiv F_{4,0,0,1}(v) \equiv F_{0,1,4,0}(v), \qquad\qquad 0 \equiv F_{3,0,1,1}(v) \equiv F_{1,1,3,0}(v), \\ 0 \,&\equiv F_{4,0,1,1}(v) \equiv F_{1,1,4,0}(v), \qquad\qquad 0 \equiv F_{3,0,3,0}(v), \end{aligned}$$

and the same holds about F'.

Accordingly, let us introduce

$$\begin{aligned} S \,:=\, & \Big\{(a,b,0,0),\,(0,0,c,d),\,(a,b,1,0),\,(1,0,c,d),\,(a,b,2,0),\,(2,0,c,d)\Big\} \\ & \bigcup\Big\{(3,0,0,1),\,(0,1,3,0),\;(4,0,0,1),\,(0,1,4,0),\,(3,0,1,1),\,(1,1,3,0), \\ & \qquad\qquad (4,0,1,1),\,(1,1,4,0),\,(3,0,3,0)\Big\}. \end{aligned}$$

Notice that S takes *no* dependent derivatives $\zeta\overline{\zeta}(\cdots)$, namely one always has $b + d \leqslant 1$ for any $(a,b,c,d) \in S$.

For a general real converging power series vanishing at the origin $(z,\zeta,\overline{z},\overline{\zeta},v) = (0,0,0,0,0)$ we have

$$H \;=\; \sum_{a,b,c,d,e} H_{a,b,c,d,e}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d v^e \qquad (\overline{H_{c,d,a,b,e}} \;=\; H_{a,b,c,d,e}),$$

i.e. with $H_{0,0,0,0,0} = 0$, introduce the projection

$$\Pi_S(H) \;:=\; \sum_{(a,b,c,d)\in S} \sum_{e=0}^{\infty} H_{a,b,c,d,e}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d v^e,$$

so that

$$\Pi_S(F) \;=\; z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2} z^2 \overline{\zeta} \qquad \text{and} \qquad \Pi_S(F') \;=\; z'\overline{z}' + \tfrac{1}{2}\overline{z}'^2\zeta' + \tfrac{1}{2} z'^2 \overline{\zeta}'.$$

By assumption (or because of Proposition 28.2), the map is of the form

$$z' \;=\; z + f_2 + f_3 + \cdots, \qquad \zeta' \;=\; \zeta + g_1 + g_2 + \cdots, \qquad w' \;=\; w + h_3 + h_4 + \cdots,$$

that is, more precisely

$$\begin{aligned} f \;&=\; \sum_{\nu\geqslant 3} f_{\nu-1} \;=\; \sum_{\nu\geqslant 3} \Big(\sum_{a+b+2e=\nu-1} f_{a,b,e}\, z^a \zeta^b w^e \Big), \\ g \;&=\; \sum_{\nu\geqslant 3} g_{\nu-2} \;=\; \sum_{\nu\geqslant 3} \Big(\sum_{a+b+2e=\nu-2} g_{a,b,e}\, z^a \zeta^b w^e \Big), \\ h \;&=\; \sum_{\nu\geqslant 3} h_{\nu} \;=\; \sum_{\nu\geqslant 3} \Big(\sum_{a+b+2e=\nu} h_{a,b,e}\, z^a \zeta^b w^e \Big). \end{aligned}$$

Let us introduce the projections

$$\pi_{\nu-1}(f) \;:=\; f_{\nu-1}, \qquad \pi_{\nu-2}(g) \;:=\; g_{\nu-2}, \qquad \pi_{\nu}(h) \;:=\; h_{\nu},$$

and also

$$\pi_\nu(H) \;:=\; \sum_{a+b+c+d+2e=\nu} H_{a,b,c,d,e}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d v^e,$$

so that

$$\Pi_S\big(\pi_\nu(F)\big) \;=\; 0 \;=\; \Pi_S\big(\pi_\nu(F')\big) \qquad (\forall\, \nu \;\geqslant\; 3).$$

Also, let us introduce

$$\pi^\nu \;:=\; \pi_2 + \cdots + \pi_\nu.$$

Now, remind that $m \;=\; \frac{z\overline{z} + \frac{1}{2}\overline{z}^2\zeta + z^2\overline{\zeta}}{1-\zeta\overline{\zeta}}$ is homogeneous of weight 2. Thanks to Proposition 27.1, we may write

$$u \;=\; F \;=\; m + \sum_{\nu\geqslant 3} G_\nu.$$

Then for a holomorphic function $e_\mu = e_\mu(z,\zeta,w)$ which is weighed μ-homogeneous, it holds (exercise)

$$\pi^\mu\Big(e_\mu\Big(z,\,\zeta,\,\,i\,v+m(z,\zeta,\overline{z},\overline{\zeta})+\sum_{\nu\geqslant 3}G_\nu(z,\zeta,\overline{z},\overline{\zeta},v)\Big)\Big)\;=\;e_\mu(z,\zeta,\,i\,v+m). \tag{29.2}$$

Now, the fundamental identity expressing that $(z+f,\zeta+g,w+h)$ is a map $N\longrightarrow N$, writes

$$\begin{aligned}0\;\equiv\;&-\operatorname{Re}\big(w+h_3+h_4+\cdots\big)\\&+F'\Big(z+f_2+f_3+\cdots,\;\zeta+g_1+g_2+\cdots,\\&\qquad\overline{z}+\overline{f}_2+\overline{f}_3+\cdots,\;\overline{\zeta}+\overline{g}_1+\overline{g}_2+\cdots,\;\operatorname{Im}\big(w+h_3+h_4+\cdots\big)\Big).\end{aligned} \tag{29.3}$$

In order to prove that $(f,g,h)=(0,0,0)$, we may proceed progressively, by induction on $\nu\geqslant 3$

$(\bullet_3)$ $(f_2,g_1,h_3)=(0,0,0)$;

$(\bullet_{\nu-1})$ $\big(f_{\mu-1},g_{\mu-2},h_\mu\big)=(0,0,0)$ for $\mu=3,\dots,\nu-1$ and some $\nu\geqslant 4$ implies that $\big(f_{\nu-1},g_{\nu-2},h_\nu\big)=(0,0,0)$.

Therefore, let us examine first the fundamental identity in weight $\nu=3$, remembering that this identity already holds true in weights 0, 1, 2 — according to (the proof of) Proposition 28.2, or according to our hypothesis —

$$\begin{aligned}0\;\equiv\;&\pi^3\Big(-\operatorname{Re}\big(w+h_3\big)+m'\big(z+f_2,\zeta+g_1,\overline{z}+\overline{f}_2,\overline{\zeta}+\overline{g}_1\big)\\&\qquad\qquad+F'_3\big(z+f_2,\zeta+g_1,\overline{z}+\overline{f}_2,\overline{\zeta}+\overline{g}_1\big)\Big)\\\equiv\;&\pi^3\Big(-m-F_3-\operatorname{Re}h_3+m'\big(z+f_2,\zeta+g_1,\overline{z}+\overline{f}_2,\overline{\zeta}+\overline{g}_1\big)\Big)\\&\qquad\qquad+F'_3(z,\zeta,\overline{z},\overline{\zeta}),\end{aligned}$$

since m' is weighted homogeneous of degree 2, since we use here (29.2). Equivalently

$$\begin{aligned}F_3(z,\zeta,\overline{z},\overline{\zeta})-F'_3(z,\zeta,\overline{z},\overline{\zeta})\;\equiv\;&\pi^3\Big(m'\big(z+f_2,\zeta+g_1,\overline{z}+\overline{f}_2,\overline{\zeta}+\overline{g}_1\big)\\&\qquad-m\big(z,\zeta,\overline{z},\overline{\zeta}\big)\Big)-\operatorname{Re}h_3\big(z,\zeta,m+i\upsilon\big).\end{aligned}$$

Generally, for any $\nu\geqslant 3$, starting from the induction assumption expressed by $(\bullet_{\nu-1})$ above, the same reasoning (exercise) conducts to the identity

$$\begin{aligned}F_\nu(z,\zeta,\overline{z},\overline{\zeta})-F'_\nu(z,\zeta,\overline{z},\overline{\zeta})\;\equiv\;&\pi^\nu\Big(m'\big(z+f_{\nu-1},\zeta+g_{\nu-2},\overline{z}+\overline{f}_{\nu-1},\overline{\zeta}+\overline{g}_{\nu-2}\big)\\&\qquad-m\big(z,\zeta,\overline{z},\overline{\zeta}\big)\Big)-\operatorname{Re}h_\nu\big(z,\zeta,m+i\upsilon\big).\end{aligned}$$

Observe that

$$m_z \,=\, \tfrac{\overline{z}+z\overline{\zeta}}{1-\zeta\overline{\zeta}} \qquad\qquad \text{and} \qquad\qquad m_\zeta \,=\, \tfrac{1}{2}\,\tfrac{(\overline{z}+z\overline{\zeta})^2}{(1-\zeta\overline{\zeta})^2}.$$

Lemma 29.3. *One has*

$$\begin{aligned}\pi^\nu\Big(m'\big(z+f_{\nu-1},\zeta+g_{\nu-2},\overline{z}+\overline{f}_{\nu-1},\overline{\zeta}+\overline{g}_{\nu-2}\big)-m\big(z,\zeta,\overline{z},\overline{\zeta}\big)\Big)\\ =\; 2\,\mathrm{Re}\Big\{\tfrac{\overline{z}+z\overline{\zeta}}{1-\zeta\overline{\zeta}}\,f_{\nu-1}\big(z,\zeta,\,m+i\upsilon\big)+\tfrac{1}{2}\,\tfrac{(\overline{z}+z\overline{\zeta})^2}{(1-\zeta\overline{\zeta})^2}\,g_{\nu-2}\big(z,\zeta,\,m+i\upsilon\big)\Big\}.\end{aligned}$$

Proof. The reader is referred to [8, Prp. 6.2] which provides all arguments. □

Next, let us apply $\Pi_S(\cdot)$ to the above identity, multiplied by 2, namely to

$$2F_\nu - 2F'_\nu \,\equiv\, \pi^\nu\big(2m'-2m\big)-2\,\mathrm{Re}\,h_\nu,$$

so that all monomials in the left-hand side disappear due to our assumption that both N and $N_,$ are in normal form

$$\begin{aligned}0 \,&\equiv\, \Pi_S\Big(\pi^\nu\big(2m'-2m\big)-2\,\mathrm{Re}\,h_\nu\Big)\\ &\equiv\, \Pi_S\Big(2\,\mathrm{Re}\Big\{2\,\tfrac{\overline{z}+z\overline{\zeta}}{1-\zeta\overline{\zeta}}\,f_{\nu-1}\big(z,\zeta,\,m+i\upsilon\big)+\tfrac{(\overline{z}+z\overline{\zeta})^2}{(1-\zeta\overline{\zeta})^2}\,g_{\nu-2}\big(z,\zeta,\,m+i\upsilon\big)\\ &\qquad\qquad -h_\nu\big(z,\zeta,\,m+i\upsilon\big)\Big\}\Big).\end{aligned}$$

Then for all monomials $z^a\zeta^b\overline{z}^c\overline{\zeta}^d\,\upsilon^e$ with $(a,b,c,d)\in S$ and $a+b+c+d+2e=\nu$, we obtain a system of linear equations

$$(\mathsf{E}_\nu):\qquad 0 \,=\, L_{a,b,c,d,e}\Big(\big\{f_{a',b',e'}\big\}_{a'+b'+2e'=\nu-1},\,\big\{g_{a',b',e'}\big\}_{a'+b'+2e'=\nu-2},\,\big\{h_{a',b',e'}\big\}_{a'+b'+2e'=\nu}\Big).$$

On the other hand, by considering the complete $f=f_2+f_3+\cdots$, the complete $g=g_1+g_2+\cdots$, and the complete $h=h_3+h_4+\cdots$ — not to be confused with the previous $(z',\zeta',w')\,=\,(z,\zeta,w)+(f,g,h)$ —, we can introduce the analog 'complete' linear system

$$0 \,\equiv\, \Pi_S\Big(2\,\mathrm{Re}\Big\{2\,\tfrac{\overline{z}+z\overline{\zeta}}{1-\zeta\overline{\zeta}}\,f\big(z,\zeta,\,m+i\upsilon\big)+\tfrac{(\overline{z}+z\overline{\zeta})^2}{(1-\zeta\overline{\zeta})^2}\,g\big(z,\zeta,\,m+i\upsilon\big)-h\big(z,\zeta,\,m+i\upsilon\big)\Big\}\Big),$$

which, similarly, after extracting the coefficients of all monomials $z^a\zeta^b\overline{z}^c\overline{\zeta}^d\,\upsilon^e$ with $(a,b,c,d)\in S$ and any $e\in\mathbb{N}$, can be abbreviated as

$$(\mathsf{E}):\qquad 0 \,=\, L_{a,b,c,d,e}\big(f_{\cdot,\cdot,\cdot},\,g_{\cdot,\cdot,\cdot},\,h_{\cdot,\cdot,\cdot}\big)\qquad\qquad \big((a,b,c,d)\,\in\,S,\;e\,\in\,\mathbb{N}\big).$$

The key and elementary observation is that, because $m+i\upsilon$ is (weighted) 2-homogeneous, the full system (E) *splits into the linear subsystems* (E_ν) *having separate unknowns* $\big(f_{\nu-1},\,g_{\nu-2},\,h_\nu\big)$

$$(\mathsf{E}) \,=\, (\mathsf{E}_3)\cup(\mathsf{E}_4)\cup\cdots\cup(\mathsf{E}_\nu)\cup\cdots.$$

Therefore

$$\Big((\mathsf{E}) \implies (f,g,h)=(0,0,0)\Big) \qquad \Longleftrightarrow$$
$$\Longleftrightarrow \qquad \Big((\mathsf{E}_\nu) \implies (f_{\nu-1},g_{\nu-2},h_\nu)=(0,0) \quad \text{for all } \nu\geqslant 3\Big).$$

The interest of this equivalence is that one will be able to gather all powers v^e for $e\in\mathbb{N}$ in order to deal with functions of the real variable $v\in\mathbb{R}$, and hence, extract only coefficients of powers $z^a\zeta^b\overline{z}^c\overline{\zeta}^d$, as we will see in a while.

Thus, we are left with establishing the following main technical statement, which will close the proof of Theorem 29.2. □

Theorem 29.4. *In weighted expansions, assume that $f=f_2+f_3+\cdots$, that $g=g_1+g_2+\cdots$, and that $h=h_3+h_4+\cdots$ vanish at the origin and satisfy in addition*

$$0=f_w(0) \qquad \text{and} \qquad 0=\operatorname{Im} h_{ww}(0).$$

If, for all $(a,b,c,d)\in S$ and all $e\in\mathbb{N}$

$$0=\big[z^a\zeta^b\overline{z}^c\overline{\zeta}^d v^e\big]\Big(2\operatorname{Re}\Big\{2\tfrac{\overline{z}+z\overline{\zeta}}{1-\zeta\overline{\zeta}}f\big(z,\zeta,m+iv\big)+\tfrac{(\overline{z}+z\overline{\zeta})^2}{(1-\zeta\overline{\zeta})^2}g\big(z,\zeta,m+iv\big) - h\big(z,\zeta,m+iv\big)\Big\}\Big),$$

then $(f,g,h)=(0,0,0)$.

Proof. For some reason of technical simplification, to be explained in a little interlude below, we now decide to 'shift' to the representation $v=F(z,\zeta,\overline{z},\overline{\zeta},u)$ instead of $u=F(z,\zeta,\overline{z},\overline{\zeta},v)$, where $u=\operatorname{Re} w$ and $v=\operatorname{Im} w$ as always.

The hypotheses become (exercise), instead

$$0=f_w(0) \qquad \text{and} \qquad 0=\operatorname{Re} h_{ww}(0),$$

and also, for all $(a,b,c,d)\in S$ and all $e\in\mathbb{N}$

$$0=\big[z^a\zeta^b\overline{z}^c\overline{\zeta}^d u^e\big]\Big(2\operatorname{Re}\Big\{2\tfrac{\overline{z}+z\overline{\zeta}}{1-\zeta\overline{\zeta}}f\big(z,\zeta,u+im\big)+\tfrac{(\overline{z}+z\overline{\zeta})^2}{(1-\zeta\overline{\zeta})^2}f\big(z,\zeta,u+im\big) + ih\big(z,\zeta,u+im\big)\Big\}\Big).$$

Because S does not contain any dependent-jet monomial $\zeta\overline{\zeta}(\cdots)$ by its very definition given above, we may compute everything modulo $\zeta\overline{\zeta}(\cdots)$, and this will simplify our task. Thus, by expanding

$$\begin{aligned} m &= \frac{z\overline{z}+\frac{1}{2}\overline{z}^2\zeta+\frac{1}{2}z^2\overline{\zeta}}{1-\zeta\overline{\zeta}} \\ &= z\overline{z}+\tfrac{1}{2}\overline{z}^2\zeta+\tfrac{1}{2}z^2\overline{\zeta}+z\overline{z}\zeta\overline{\zeta}+\tfrac{1}{2}\overline{z}^2\zeta\zeta\overline{\zeta}+\tfrac{1}{2}z^2\overline{\zeta}\zeta\overline{\zeta}+z\overline{z}\zeta\overline{\zeta}\zeta\overline{\zeta}+\cdots, \end{aligned}$$

we visibly have

$$\begin{aligned}
m &\equiv z\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta + \tfrac{1}{2}z^2\overline{\zeta},\\
m_z &\equiv \overline{z} + z\overline{\zeta},\\
m_\zeta &\equiv \tfrac{1}{2}\overline{z}^2 + z\overline{z}\overline{\zeta} + \tfrac{1}{2}z^2\overline{\zeta}^2,\\
m_{\overline{z}} &\equiv z + \overline{z}\zeta,\\
m_{\overline{\zeta}} &\equiv \tfrac{1}{2}z^2 + z\zeta\overline{z} + \tfrac{1}{2}\overline{z}^2\zeta^2.
\end{aligned}$$

We will also need (little exercise), still modulo $\zeta\overline{\zeta}(\cdots)$

$$\begin{aligned}
m^2 &\equiv z^2\overline{z}^2 + z\zeta\overline{z}^3 + z^3\overline{z}\overline{\zeta} + \tfrac{1}{4}z^4\overline{\zeta}^2 + \tfrac{1}{4}\zeta^2\overline{z}^4,\\
m^3 &\equiv z^3\overline{z}^3 + \tfrac{3}{2}z^2\zeta\overline{z}^4 + \tfrac{3}{2}z^4\overline{z}^2\overline{\zeta} + \tfrac{3}{4}z^5\overline{z}\overline{\zeta}^2 + \tfrac{3}{4}z\overline{z}^5\overline{\zeta}^2 + \tfrac{1}{8}z^6\overline{\zeta}^3 + \tfrac{1}{8}\zeta^3\overline{z}^6,\\
m\,m_\zeta &\equiv \tfrac{1}{2}z\overline{z}^3 + \tfrac{1}{4}\zeta\overline{z}^4 + \tfrac{5}{4}z^2\overline{z}\overline{\zeta} + z^3\overline{z}\overline{\zeta}^2 + \tfrac{1}{4}z^4\overline{\zeta}^3,\\
m\,m_z &\equiv z\overline{z}^2 + \tfrac{1}{2}\zeta\overline{z}^3 + \tfrac{3}{2}z^2\overline{z}\overline{\zeta} + \tfrac{1}{2}z^3\overline{\zeta}^2,\\
m^2\,m_z &\equiv z^2\overline{z}^3 + z\zeta\overline{z}^4 + 2\,z^3\overline{z}^2\overline{\zeta} + \tfrac{1}{4}\zeta^2\overline{z}^5 + \tfrac{5}{4}z^4\overline{z}\overline{\zeta}^2 + \tfrac{1}{4}z^5\overline{\zeta}^3.
\end{aligned}$$

Assuming therefore that the graphing equation $v = F = m + G$ is solved with respect to v, not to u, with arguments $(z, \zeta, w) = (z, \zeta, u + i\,m)$, the Moser (linear) operator is defined as

$$L(f, g, h) \;:=\; 2\,m_z\,f + 2\,m_{\overline{z}}\,\overline{f} + 2\,m_\zeta\,g + 2\,m_{\overline{\zeta}}\,\overline{g} + i\,h - i\,\overline{h}.$$

Given a holomorphic function $e = e(w)$, we may Taylor expand at u

$$\begin{aligned}
e\big(u + i\,m\big) \;&=\; e(u) + e_w(u)\,[i\,m] + e_{ww}(u)\,\big[-\tfrac{m^2}{2}\big] + e_{www}(u)\,\big[-i\,\tfrac{m^3}{6}\big] + \cdots\\
&=:\; e + e'\,[i\,m] + e''\,\big[-\tfrac{m^2}{2}\big] + e'''\,\big[-i\,\tfrac{m^3}{6}\big] + \cdots,
\end{aligned}$$

and we can abbreviate derivatives using primes, even without writing the argument u. Let us now make the promised little interlude.

The other choice of graphing $u = F = m + G$ leads to $e(w) = e(iv + m)$ which expands as

$$e\big(iv + m\big) \;=\; e(iv) + e_w(iv)\,[m] + e_{ww}(iv)\,\big[\tfrac{m^2}{2}\big] + e_{www}(iv)\,\big[\tfrac{m^3}{6}\big] + \cdots.$$

It is then convenient to consider the *composed* function of one real variable

$$v \;\longmapsto\; e(iv) \;=:\; E(v),$$

which satisfies

$$\begin{aligned}
\tfrac{d}{dv}E(v) &= i\,e_w(iv) && \Longleftrightarrow && -i\,E'(v) = e_w(iv),\\
\tfrac{d^2}{dv^2}E(v) &= -e_{ww}(iv) && \Longleftrightarrow && -E''(v) = e_{ww}(iv),\\
\tfrac{d^3}{dv^3}E(v) &= -i\,e_{www}(iv) && \Longleftrightarrow && i\,E'''(v) = e_{www}(iv).
\end{aligned}$$

Thus

$$\begin{aligned}
e\big(iv+m\big) &= E(v) - i\,E'(v)\,[m] - E''(v)\big[\tfrac{m^2}{2}\big] + i\,E'''(v)\big[\tfrac{m^3}{6}\big] + \cdots,\\
\overline{e}\big(-iv+m\big) &= \overline{E}(v) + i\,\overline{E}'(v)\,[m] - \overline{E}''(v)\big[\tfrac{m^2}{2}\big] - i\,\overline{E}'''(v)\big[\tfrac{m^3}{6}\big] + \cdots,
\end{aligned}$$

and similarly for the conjugate. If by convention, we then make the abuse of notation to denote e instead of E, that is $e(v)$ instead of $E(v) = e(iv)$, we can abbreviate, without writing the arguments iv or $-iv$

$$\begin{aligned}
e\big(iv+m\big) &= e + e'\big[-i\,m\big] + e''\big[-\tfrac{m^2}{2}\big] + e'''\big[i\,\tfrac{m^3}{6}\big] + \cdots,\\
\overline{e}\big(-iv+m\big) &= \overline{e} + \overline{e}'\big[i\,m\big] + \overline{e}''\big[-\tfrac{m^2}{2}\big] + \overline{e}'''\big[-i\,\tfrac{m^3}{6}\big] + \cdots.
\end{aligned}$$

This can be applied to functions $e = f_{j,k}$ or $e = g_{j,k}$ or $e = h_{j,k}$ in the useful expansions

$$f = \sum_j \sum_k z^j \zeta^k\, f_{j,k}(w), \quad g = \sum_j \sum_k z^j \zeta^k\, g_{j,k}(w), \quad h = \sum_j \sum_k z^j \zeta^k\, h_{j,k}(w).$$

But in these last paragraphs of our paper, we decided to choose $v = F$ in order to simplify a bit the presentation, so that $e = e(u) = E(u)$ and there will be no abuse of notation.

We can write the Moser operator as

$$L(f,g,h) = \mathsf{T}_1 + \overline{\mathsf{T}}_1 + \mathsf{T}_2 + \overline{\mathsf{T}}_2 + \mathsf{T}_3 + \overline{\mathsf{T}}_3.$$

Computing modulo $\zeta\overline{\zeta}(\cdots)$, start with

$$\begin{aligned}
\mathsf{T}_3 &\equiv \sum_j \sum_k z^j \zeta^k \, i\, h_{j,k}(u + i\,m) \\
&\equiv \sum_j \sum_k z^j \zeta^k \Big\{ i\, h_{j,k} + h'_{j,k}\,[-m] + h''_{j,k}\,[-\tfrac{i}{2}\,m^2] + h'''_{j,k}\,[\tfrac{1}{6}\,m^3] + \cdots \Big\} \\
&\equiv \sum_j \sum_k z^j \zeta^k \Big\{ h_{j,k}\,[i] + h'_{j,k}\big[- z\overline{z} - \tfrac{1}{2}\overline{z}^2\zeta - \tfrac{1}{2}z^2\overline{\zeta}\big] \\
&\qquad\qquad + h''_{j,k}\big[- \tfrac{i}{2}z^2\overline{z}^2 - \tfrac{i}{2}z\zeta\overline{z}^3 - \tfrac{i}{2}z^3\overline{z}\overline{\zeta} - \tfrac{i}{8}z^4\overline{\zeta}^2 - \tfrac{i}{8}\zeta^2\overline{z}^4\big] \\
&\qquad\qquad + h'''_{j,k}\big[\tfrac{1}{6}z^3\overline{z}^3 + \tfrac{1}{4}z^2\zeta\overline{z}^4 + \tfrac{1}{4}z^4\overline{z}^2\overline{\zeta}^2 + \tfrac{1}{8}z^5\overline{z}\overline{\zeta}^2 + \tfrac{1}{8}z\overline{z}^5\overline{\zeta}^2 + \tfrac{1}{48}z^6\overline{\zeta}^3 + \tfrac{1}{48}\zeta^3\overline{z}^6\big] \\
&\qquad\qquad\qquad\qquad + \cdots \Big\} \\
&\equiv \sum_j \sum_k \Big\{ h_{j,k}[iz^j\zeta^k] + h'_{j,k}\big[- z^{j+1}\zeta^k\overline{z} - \tfrac{1}{2}z^j\zeta^{k+1}\overline{z}^2 - \tfrac{1}{2}z^{j+2}\zeta^k\overline{\zeta}\big] \\
&\qquad + h''_{j,k}\big[- \tfrac{i}{2}z^{j+2}\zeta^k\overline{z}^2 - \tfrac{i}{2}z^{j+1}\zeta^{k+1}\overline{z}^3 - \tfrac{i}{2}z^{j+3}\zeta^k\overline{z}\overline{\zeta} - \tfrac{i}{8}z^{j+4}\zeta^k\overline{\zeta}^2 - \tfrac{i}{8}z^j\zeta^{k+2}\overline{z}^4\big] \\
&\qquad + h'''_{j,k}\big[\tfrac{1}{6}z^{j+3}\zeta^k\overline{z}^3 + \tfrac{1}{4}z^{j+2}\zeta^{k+1}\overline{z}^4 + \tfrac{1}{4}z^{j+4}\zeta^k\overline{z}^2\overline{\zeta}^2 + \tfrac{1}{8}z^{j+5}\zeta^k\overline{z}\overline{\zeta}^2 + \tfrac{1}{8}z^{j+1}\zeta^k\overline{z}^5\overline{\zeta}^2 \\
&\qquad\qquad + \tfrac{1}{48}z^{j+6}\zeta^k\overline{\zeta}^3 + \tfrac{1}{48}z^j\zeta^{k+3}\overline{z}^6\big] + \cdots \Big\}.
\end{aligned}$$

The useful expression of $\overline{\mathsf{T}}_3$ is obtained by plain complex conjugation.

Next, going only to derivatives of $g_{j,k}$ up to order 1, which will be enough, we obtain, without intermediate explanations

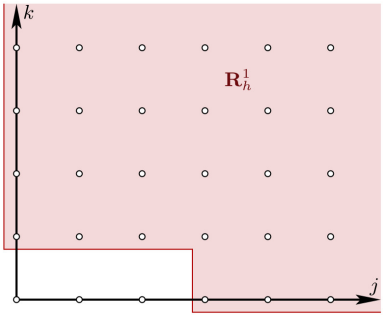
$$\begin{aligned}
\mathsf{T}_2 \equiv \sum_j \sum_k \Big\{ & g_{j,k}\big[z^j\zeta^k\overline{z}^2 + 2z^{j+1}\zeta^k\overline{z}\overline{\zeta} + z^{j+2}\zeta^k\overline{\zeta}^2\big] \\
& + g'_{j,k}\big[iz^{j+1}\zeta^k\overline{z}^3 + \tfrac{i}{2}z^j\zeta^{k+1}\overline{z}^4 + \tfrac{5i}{2}z^{j+2}\zeta^k\overline{z}^2\overline{\zeta} + 2iz^{j+3}\zeta^k\overline{z}\overline{\zeta}^2 + \tfrac{i}{2}z^{j+4}\zeta^k\overline{\zeta}^3\big] + \cdots \Big\}.
\end{aligned}$$

Lastly, going to derivatives of order 2 of the $f_{j,k}$

$$\begin{aligned}
\mathsf{T}_1 \equiv \sum_j \sum_k \Big\{ & f_{j,k}\big[2z^j\zeta^k\overline{z} + 2z^{j+1}\zeta^k\overline{\zeta}\big] \\
& + f'_{j,k}\big[2iz^{j+1}\zeta^k\overline{z}^2 + iz^j\zeta^{k+1}\overline{z}^3 + 3iz^{j+2}\zeta^k\overline{z}\overline{\zeta} + iz^{j+3}\zeta^k\overline{z}^2\big] \\
& + f''_{j,k}\big[- \tfrac{1}{2}z^{j+2}\zeta^k\overline{z}^3 - \tfrac{1}{2}z^{j+1}\zeta^{k+1}\overline{z}^4 - z^{j+3}\zeta^k\overline{z}^2\overline{\zeta} - \tfrac{1}{8}z^j\zeta^{k+2}\overline{z}^5 \\
& \qquad - \tfrac{5}{8}z^{j+4}\zeta^k\overline{z}\overline{\zeta}^2 - \tfrac{1}{8}z^{j+5}\zeta^k\overline{\zeta}^3\big] + \cdots \Big\}.
\end{aligned}$$

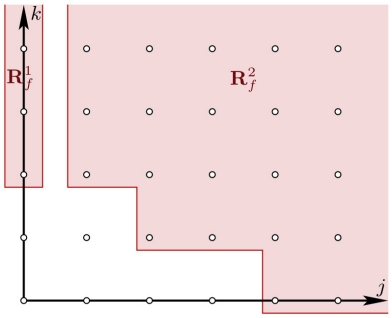
Now, patiently, in $0 = \mathsf{T}_1 + \mathsf{T}_2 + \mathsf{T}_3 + \overline{\mathsf{T}}_1 + \overline{\mathsf{T}}_2 + \overline{\mathsf{T}}_3$, we chase coefficients $z^a\zeta^b\overline{z}^c\overline{\zeta}^d$ for all $(a,b,c,d) \in S$, and each time, we obtain linear combinations of (differentiated) functions of u. Using a computer helps to avoid mistakes.

We hence obtain several groups of linear differential equations in the functions $f_{j,k}(u)$, $g_{j,k}(u)$, $h_{j,k}(u)$. We begin with three major groups coming from (part of) the prenormalization assumption and which imply a certain agreeable *'nilpotency phenomenon'*, well known to also hold for Levi nondegenerate hypersurfaces ([12, 24, 32]). Figures help to grasp the inequalities we are stating below, which show certain *regions* $\mathbf{R}^*_h$, $\mathbf{R}^*_f$, $\mathbf{R}^*_g$.



$(\mathbf{R}_h^1)$ $\boxed{0 = i\,h_{j,k}(u)}$ for $(j,k,0,0) \in S$ with $j \geqslant 3$ or with $k \geqslant 1$. This yields, without writing the argument u of the $h_{j,k}(u)$, that h is a relative polynomial in (z,ζ)

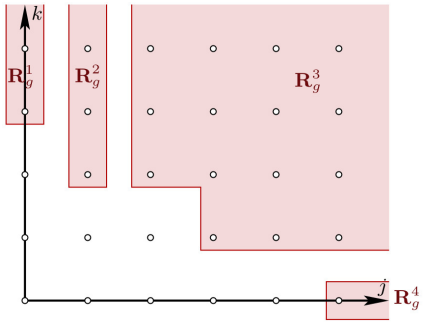
$$h = h_{0,0} + h_{1,0}\,z + h_{2,0}\,z^2.$$



$(\mathbf{R}_f^1)$ $\boxed{0 = 2\,f_{0,k}(u)}$ for $(j,k,1,0) \in S$ with $j = 0$ and $k \geqslant 2$.

$(\mathbf{R}_f^2)$ $\boxed{0 = 2\,f_{j,k}(u) - h'_{j-1,k}(u)}$ for $(j,k,1,0) \in S$ with $j \geqslant 1$ and: with $k \geqslant 2$ when $j = 1$; with $k \geqslant 1$ when $j = 2,3$; with $k \geqslant 0$ when $j \geqslant 4$. This yields relative polynomialness of

$$\begin{aligned} f = {} & f_{0,1}\,\zeta + f_{1,1}\,z\zeta \\ & + f_{0,0} + f_{1,0}\,z + f_{2,0}\,z^2 + f_{3,0}\,z^3. \end{aligned}$$



$(\mathbf{R}_g^1)$ $\boxed{0 = g_{0,k} - \frac{1}{2} h'_{0,k-1}}$ for $(j,k,2,0) \in S$ with $j = 0$ and $k \geqslant 3$.

$(\mathbf{R}_g^2)$ $\boxed{0 = g_{1,k} - \frac{1}{2} h'_{1,k-1} + 2i\, f'_{0,k}}$ for $(j,k,2,0) \in S$ with $j = 1$ and $k \geqslant 2$.

$(\mathbf{R}_g^3)$ $\boxed{0 = g_{j,k} - \frac{1}{2} h'_{j,k-1} + 2i f'_{j-1,k} - \frac{i}{2} h''_{j-2,k}}$ for $(j,k,2,0) \in S$ with $j \geqslant 2$ and $k \geqslant 1$ excepting $(j,k) = (2,1)$.

$(\mathbf{R}_g^4)$ $\boxed{0 = g_{j,0} + 2i f'_{j-1,0} - \frac{i}{2} h''_{j-2,0}}$, for $(j,k,0,0) \in S$ with $j \geqslant 5$ and $k = 0$.

All this also yields relative polynomialness of

$$\begin{aligned} g \;=\; & g_{0,2}\, \zeta^2 \\ & + g_{0,1}\, \zeta + g_{1,1}\, z\zeta + g_{2,1}\, z^2\zeta \\ & + g_{0,0} + g_{1,0}\, z + g_{2,0}\, z^2 + g_{3,0}\, z^3 + g_{4,0}\, z^4. \end{aligned}$$

To prove that $(f,g,h) = (0,0,0)$, it suffices to prove that the $3+6+9$ remaining functions of u, namely $h_{0,0}$, $h_{1,0}$, $h_{2,0}$ and $f_{0,1}$, $f_{1,1}$, $f_{0,0}$, $f_{1,0}$, $f_{2,0}$, $f_{3,0}$, and $g_{0,2}$, $g_{0,1}$, $g_{1,1}$, $g_{2,1}$, $g_{0,0}$, $g_{1,0}$, $g_{2,0}$, $g_{3,0}$, $g_{4,0}$ are identically zero.

For this, we have to examine the remaining groups of linear ordinary differential equations with $(a,b,c,d) \in S$.

Firstly (first group), the equations for $(j,k,0,0) \in S$ outside the region $\mathbf{R}_h^1$ are

$$\begin{aligned} 0 \;&=\; i\, h_{0,0} - i\, \overline{h}_{0,0}, && (0,0,0,0) \\ 0 \;&=\; 2\, \overline{f}_{0,0} + i\, h_{1,0}, && (1,0,0,0) \\ 0 \;&=\; \overline{g}_{0,0} + i\, h_{2,0}. && (2,0,0,0) \end{aligned}$$

The conjugate equations are not written, should be understood, and will in fact be considered later.

Secondly (second group), the equations for $(j,k,1,0) \in S$ outside $\mathbf{R}_f^1 \cup \mathbf{R}_f^2$ are

$$\begin{aligned} 0 \;&=\; 2\, f_{0,0} - i\, \overline{h}_{1,0} \qquad \text{[Already seen]}, && (0,0,1,0) \\ 0 \;&=\; 2\, f_{1,0} - h'_{0,0} - \overline{h}'_{0,0} + 2\, \overline{f}_{1,0}, && (1,0,1,0) \\ 0 \;&=\; 2\, f_{2,0} + \overline{g}_{1,0} - h'_{1,0} - 2i\, \overline{f}'_{0,0}, && (2,0,1,0) \\ 0 \;&=\; 2\, f_{3,0} - h'_{2,0} - i\, \overline{g}'_{0,0}, && (3,0,1,0) \\ 0 \;&=\; 2\, f_{0,1} + 2\, \overline{f}_{0,0}, && (0,1,1,0) \\ 0 \;&=\; 2\, f_{1,1} - \underline{h'_{0,1}}_{\circ} + 2\, \overline{g}_{0,0}. && (1,1,1,0) \end{aligned}$$

Notice that the last equation let appear $h_{0,1}(u)$, which we already know is identically zero. Again, the conjugate equations are understood.

Thirdly (third group), the equations for $(j,k,2,0)$ outside $\mathbf{R}_g^1 \cup \mathbf{R}_g^2 \cup \mathbf{R}_g^3 \cup \mathbf{R}_g^4$ are

$$0 = g_{0,0} - i\,\overline{h}_{2,0} \qquad \text{[Already seen]}, \qquad (0,0,2,0)$$

$$0 = g_{1,0} + 2\,\overline{f}_{2,0} - \overline{h}'_{1,0} + 2i\,f'_{0,0} \qquad \text{[Already seen]}, \qquad (1,0,2,0)$$

$$0 = g_{2,0} + \overline{g}_{2,0} - 2i\,\overline{f}'_{1,0} + 2i\,f'_{1,0} - \tfrac{i}{2}\,h''_{0,0} + \tfrac{i}{2}\,\overline{h}''_{0,0}, \qquad (2,0,2,0)$$

$$0 = g_{3,0} - i\,\overline{g}'_{1,0} + 2i\,f'_{2,0} - \tfrac{i}{2}\,h''_{1,0} - \overline{f}''_{0,0}, \qquad (3,0,2,0)$$

$$0 = g_{4,0} + 2i\,f'_{3,0} - \tfrac{i}{2}\,h''_{2,0} - \tfrac{1}{2}\,\overline{g}''_{0,0}, \qquad (4,0,2,0)$$

$$0 = g_{0,1} + 2\,\overline{f}_{1,0} - \tfrac{1}{2}\,\overline{h}'_{0,0} - \tfrac{1}{2}\,h'_{0,0}, \qquad (0,1,2,0)$$

$$0 = g_{1,1} + 2\,\overline{g}_{1,0} - \tfrac{1}{2}\,h'_{1,0} - 3i\,\overline{f}'_{0,0} + 2i\,f'_{0,1}, \qquad (1,1,2,0)$$

$$0 = g_{2,1} - \tfrac{1}{2}\,h'_{2,0} - \tfrac{5i}{2}\,\overline{g}'_{0,0} + 2i\,f'_{1,1} - \tfrac{i}{2}\,\underline{h''_{0,1}}_{\circ}, \qquad (2,1,2,0)$$

$$0 = g_{0,2} + \overline{g}_{0,0} - \tfrac{1}{2}\,\underline{h'_{0,1}}_{\circ}. \qquad (0,2,2,0)$$

Notice that the last two equations let appear $h_{0,1}(u)$, which we already know is identically zero.

Fourthly (fourth group) and lastly, we list the sporadic equations

$$0 \equiv 2\,f_{2,0} - \tfrac{1}{2}\,h'_{1,0} - i\,\overline{f}'_{0,0}, \qquad (3,0,0,1)$$

$$0 \equiv \tfrac{1}{6}\,h'''_{0,0} + \tfrac{1}{6}\,\overline{h}'''_{0,0} - f''_{1,0} - \overline{f}''_{1,0} + i\,g'_{2,0} - i\,\overline{g}'_{2,0}, \qquad (3,0,3,0)$$

$$0 \equiv 2\,f_{3,0} - \tfrac{1}{2}\,h'_{2,0} - \tfrac{i}{2}\,\overline{g}'_{0,0}, \qquad (4,0,0,1)$$

$$0 \equiv 2\,g_{2,0} - i\,\overline{g}'_{0,1} - i\,\overline{f}'_{1,0} + 3i\,f'_{1,0} + \tfrac{i}{2}\,\overline{h}''_{0,0} - \tfrac{i}{2}\,h''_{0,0}, \qquad (3,0,1,1)$$

$$0 \equiv 2\,g_{3,0} - \tfrac{i}{2}\,\overline{g}'_{1,0} + 3i\,f'_{2,0} - \tfrac{i}{2}\,h''_{1,0} - \overline{f}''_{0,0}. \qquad (4,0,1,1)$$

Now, the assumptions of Theorem 29.4 can be reformulated by comparing the two representations

$$f_{\geqslant 2} = \sum_j \sum_k z^j \zeta^k\, f_{j,k}(u), \qquad g_{\geqslant 1} = \sum_j \sum_k z^j \zeta^k\, g_{j,k}(u),$$

$$h_{\geqslant 3} = \sum_j \sum_k z^j \zeta^k\, h_{j,k}(u),$$

and one realizes that

$$\begin{aligned}
0 = f(0,0,0) = g(0,0,0) = h(0,0,0) \quad &\Longleftrightarrow \quad 0 = f_{0,0}(0) = g_{0,0}(0) = h_{0,0}(0),\\
f = f_2 + f_3 + \cdots \quad &\Longrightarrow \quad f_{1,0}(0) = 0,\\
h = h_3 + h_4 + \cdots \quad &\Longrightarrow \quad h'_{0,0}(0) = 0,\\
f_w(0) = 0 \quad &\Longleftrightarrow \quad f'_{0,0}(0) = 0,\\
\operatorname{Re} h_{ww}(0) = 0 \quad &\Longleftrightarrow \quad \operatorname{Re} h''_{0,0}(0) = 0.
\end{aligned}$$

The proof of Theorem 29.4 will hence be finished with the next statement. □

Proposition 29.5. *If* $3+6+9$ *analytic functions* $h_{0,0}$, $h_{1,0}$, $h_{2,0}$ *and* $f_{0,1}$, $f_{1,1}$, $f_{0,0}$, $f_{1,0}$, $f_{2,0}$, $f_{3,0}$, *and* $g_{0,2}$, $g_{0,1}$, $g_{1,1}$, $g_{2,1}$, $g_{0,0}$, $g_{1,0}$, $g_{2,0}$, $g_{3,0}$, $g_{4,0}$ *of the real variable* $u \in \mathbb{R}$ *with*

$$\begin{aligned} 0 &= f_{0,0}(0) = f_{1,0}(0), \qquad 0 = g_{0,0}(0), \qquad 0 = h_{0,0}(0), \\ 0 &= f'_{0,0}(0), \qquad\qquad\qquad\qquad 0 = h'_{0,0}(0) = \operatorname{Re} h''_{0,0}(0), \end{aligned}$$

satisfy the above system of four groups of linear ordinary differential equations, then they all vanish identically.

Proof. From the first two groups of equations *and conjugate equations*, we may solve

$$\begin{aligned} \overline{h}_{0,0} &:= h_{0,0}, \\ h_{1,0} &:= 2i\,\overline{f}_{0,0}, & \overline{h}_{1,0} &:= -2i\,f_{0,0}, \\ h_{2,0} &:= i\,\overline{g}_{0,0}, & \overline{h}_{2,0} &:= -i\,g_{0,0}, \\ \overline{f}_{1,0} &:= -f_{1,0} + h'_{0,0}, \\ f_{2,0} &:= -\tfrac{1}{2}\,\overline{g}_{1,0} + 2i\,\overline{f}'_{0,0}, & \overline{f}_{2,0} &:= -\tfrac{1}{2}\,g_{1,0} - 2i\,f'_{0,0}, \\ f_{3,0} &:= i\,\overline{g}'_{0,0}, & \overline{f}_{3,0} &:= -i\,g'_{0,0}, \\ f_{0,1} &:= -\overline{f}_{0,0}, & \overline{f}_{0,1} &:= -f_{0,0}, \\ f_{1,1} &:= -\overline{g}_{0,0}, & \overline{f}_{1,1} &:= -g_{0,0}. \end{aligned}$$

Once this is done, these first two groups of equations become just $0 = 0$, while the third group becomes[1]

$$\begin{aligned} 0 &\overset{2020}{=} g_{2,0} + \overline{g}_{2,0} - 2i\,h''_{0,0} + 4i\,f'_{1,0}, \\ 0 &\overset{3020}{=} g_{3,0} - 2i\,\overline{g}'_{1,0} - 4\,\overline{f}''_{0,0}, & 0 &\overset{2030}{=} \overline{g}_{3,0} + 2i\,g'_{1,0} - 4\,f''_{0,0}, \\ 0 &\overset{4020}{=} g_{4,0} - 2\,\overline{g}''_{0,0}, & 0 &\overset{2040}{=} \overline{g}_{4,0} - 2\,g''_{0,0}, \\ 0 &\overset{0120}{=} g_{0,1} - 2\,f_{1,0} + h'_{0,0}, & 0 &\overset{2001}{=} \overline{g}_{0,1} + 2\,f_{1,0} - h'_{0,0}, \\ 0 &\overset{1120}{=} g_{1,1} + 2\,\overline{g}_{1,0} - 6i\,\overline{f}'_{0,0}, & 0 &\overset{2011}{=} \overline{g}_{1,1} + 2\,g_{1,0} + 6i\,f'_{0,0}, \\ 0 &\overset{2120}{=} g_{2,1} - 5i\,\overline{g}'_{0,0}, & 0 &\overset{2021}{=} \overline{g}_{2,1} + 5i\,g'_{0,0}, \\ 0 &\overset{0220}{=} g_{0,2} + \overline{g}_{0,0}, & 0 &\overset{2002}{=} \overline{g}_{0,2} + g_{0,0}, \end{aligned}$$

[1] — mind the fact that because we have sometimes solved $\overline{e}$ in terms of e for certain functions $e = e(u)$, the obtained equations are *not* all pairwise conjugates on certain lines, and this is normal —

and the fourth, last, sporadic group becomes

$$
\begin{aligned}
0 &\overset{3001}{=} 2i\,\overline{f}'_{0,0} - \overline{g}_{1,0}, & 0 &\overset{0130}{=} -2i\,f'_{0,0} - g_{1,0},\\
0 &\overset{3030}{=} -\tfrac{2}{3}\,h'''_{0,0} + i\,g'_{2,0} - i\,\overline{g}'_{2,0},\\
0 &\overset{4001}{=} i\,\overline{g}'_{0,0}, & 0 &\overset{0140}{=} -i\,g'_{0,0},\\
0 &\overset{3011}{=} 2\,g_{2,0} - i\,\overline{g}'_{0,1} - i\,h''_{0,0} + 4i\,f'_{1,0}, & 0 &\overset{1130}{=} 2\,\overline{g}_{2,0} + i\,g'_{0,1} - 3i\,h''_{0,0} + 4i\,f'_{1,0},\\
0 &\overset{4011}{=} 2\,g_{3,0} - 6\,\overline{f}''_{0,0} - 2i\,\overline{g}'_{1,0}, & 0 &\overset{1140}{=} 2\,\overline{g}_{3,0} - 6\,f''_{0,0} + 2i\,g'_{1,0}.
\end{aligned}
$$

Hence, from the third group, we can solve

$$
\begin{aligned}
\overline{g}_{2,0} &:= -g_{2,0} - 4i\,f'_{1,0} + 2i\,h''_{0,0},\\
g_{3,0} &:= 2i\,\overline{g}'_{1,0} + 4\,\overline{f}''_{0,0}, & \overline{g}_{3,0} &:= -2i\,g'_{1,0} + 4\,f''_{0,0},\\
g_{4,0} &:= 2\,\overline{g}''_{0,0}, & \overline{g}_{4,0} &:= 2\,g''_{0,0},\\
g_{0,1} &:= 2\,f_{1,0} - h'_{0,0}, & \overline{g}_{0,1} &:= -2\,f_{1,0} + h'_{0,0},\\
g_{1,1} &:= 6i\,\overline{f}'_{0,0} - 2\,\overline{g}_{1,0}, & \overline{g}_{1,1} &:= -6i\,f'_{0,0} - 2\,g_{1,0},\\
g_{2,1} &:= 5i\,\overline{g}'_{0,0}, & \overline{g}_{2,1} &:= -5i\,g'_{0,0},\\
g_{0,2} &:= -\overline{g}_{0,0}, & \overline{g}_{0,2} &:= -g_{0,0},
\end{aligned}
$$

and after that, all equations of the third group reduce to $0 = 0$. Then the equations of the fourth group become

$$
\begin{aligned}
0 &\overset{3001}{=} 2i\,\overline{f}'_{0,0} - \overline{g}_{1,0}, & 0 &\overset{0130}{=} -2i\,f'_{0,0} - g_{1,0},\\
0 &\overset{3030}{=} \tfrac{4}{3}\,h'''_{0,0} + 2i\,g'_{2,0} - 4\,f''_{1,0},\\
0 &\overset{4001}{=} i\,\overline{g}'_{0,0}, & 0 &\overset{0140}{=} -i\,g'_{0,0},\\
0 &\overset{3011}{=} 2\,g_{2,0} - 2i\,h''_{0,0} + 6i\,f'_{1,0}, & 0 &\overset{1130}{=} -2\,g_{2,0} - 2i\,f'_{1,0},\\
0 &\overset{4011}{=} 2\,\overline{f}''_{0,0} + 2i\,\overline{g}'_{1,0}, & 0 &\overset{1140}{=} 2\,f''_{0,0} - 2i\,g'_{1,0}.
\end{aligned}
$$

From this, we can solve, thanks to the assumption $g_{0,0}(0) = 0$

$$
\begin{aligned}
g_{0,0} &= 0, & \overline{g}_{0,0} &= 0,\\
g_{1,0} &= -2i\,f'_{0,0}, & \overline{g}_{1,0} &= 2i\,\overline{f}'_{0,0},\\
g_{2,0} &:= -2i\,f'_{1,0}.
\end{aligned}
$$

The remaining equations become

$$\begin{aligned}
0 &\overset{3030}{=} \tfrac{4}{3}\, h'''_{0,0} - 2\, f''_{1,0},\\
0 &\overset{3011}{=} -2i\, h''_{0,0} + 4i\, f'_{1,0},\\
0 &\overset{4011}{=} -2\, \overline{f}''_{0,0},\\
0 &\overset{1140}{=} -2\, f''_{0,0}.
\end{aligned}$$

Differentiating once the second equation, using $0 \neq \left| \begin{smallmatrix} \frac{4}{3} & -2 \\ -2i & 4i \end{smallmatrix} \right|$, we get

$$h'''_{0,0} \,=\, 0, \qquad\qquad f''_{1,0} \,=\, 0.$$

But we have assumed $0 = h_{0,0}(0) = h'_{0,0}(0) = \operatorname{Re} h''_{0,0}(0)$, and we know from the beginning that $\overline{h}_{0,0} = h_{0,0}$ is real. So $h_{0,0} = 0$.

Back to $\overset{3011}{=}$ above, we get $f'_{1,0} = 0$. Also, we have assumed that $f_{1,0}(0) = 0$. So $f_{1,0} = 0$.

Lastly, $f''_{0,0} = 0$ together with $f'_{0,0}(0) = 0$ gives $f_{0,0} = 0$. This concludes everything. □

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This paper is available via http://nyjm.albany.edu/j/2022/28-11.html.