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Direct products of null semigroups and rectangular bands in $\beta\mathbb{N}$

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Abstract. We show that, for every $m \in \mathbb{N}$, the direct product of the m-element null semigroup and the $2^{\mathfrak{c}} \times 2^{\mathfrak{c}}$ rectangular band has copies in $\beta\mathbb{N}$. In particular, the direct product of the 2-element null semigroup and the 2×2 rectangular band has copies in $\beta\mathbb{N}$. We also point out a Ramsey theoretic consequence of the latter fact.

The addition of the discrete semigroup $\mathbb{N}$ of natural numbers extends to the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ so that for each $a \in \mathbb{N}$, the left translation $\beta\mathbb{N} \ni x \mapsto a+x \in \beta\mathbb{N}$ is continuous, and for each $q \in \beta\mathbb{N}$, the right translation $\beta\mathbb{N} \ni x \mapsto x+q \in \beta\mathbb{N}$ is continuous.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying the principal ultrafilters with the points of $\mathbb{N}$. For every $A \subseteq \mathbb{N}$, $\overline{A} = \{p \in \beta\mathbb{N} : A \in p\}$ and $A^* = \overline{A} \setminus A$. The subsets $\overline{A}$, where $A \subseteq \mathbb{N}$, form a base for the topology of $\beta\mathbb{N}$, and $\overline{A}$ is the closure of A. For $p, q \in \beta\mathbb{N}$, the ultrafilter $p+q$ has a base consisting of subsets of the form $\bigcup_{x\in A}(x+B_x)$, where $A \in p$ and for each $x \in A$, $B_x \in q$.

Being a compact Hausdorff right topological semigroup, $\beta\mathbb{N}$ has a smallest two sided ideal $K(\beta\mathbb{N})$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. Every right (left) ideal of $\beta\mathbb{N}$ contains a minimal right (left) ideal, the intersection of a minimal right ideal and a minimal left ideal is a group, and the idempotents in a minimal right (left) ideal form a right (left) zero semigroup, that is, $x+y=y$ ($x+y=x$) for all x, y.

The semigroup $\beta\mathbb{N}$ is interesting both for its own sake and for its applications to Ramsey theory and to topological dynamics. The first application to Ramsey theory was the proof of Hindman's theorem: whenever $\mathbb{N}$ is finitely colored, there is an infinite subset all of whose sums are monochrome. An elementary introduction to $\beta\mathbb{N}$ can be found in [1].

In [3] D. Strauss showed that if φ is a continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$, then $\varphi(\beta\mathbb{N})$ is finite and $\varphi(\mathbb{N}^*)$ is a group. In 1996 the author proved that $\beta\mathbb{N}$ contains no nontrivial finite groups (see [1, Theorem 7.17]). In contrast, it does contain bands (= semigroups of idempotents). For example, apart from

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mentioned already left (right) zero semigroups, it contains chains of idempotents ($x \le y$ if and only if $x + y = y + x = x$). A large enough class of finite bands that have copies in $\beta\mathbb{N}$ was constructed in [4]. It includes, in particular, all finite rectangular bands (= direct products of a left zero semigroup and a right zero semigroup, so $(x_1, y_1) + (x_2, y_2) = (x_1, y_2)$). In [2] it was shown that $\beta\mathbb{N}$ contains even $2^{\mathfrak{c}} \times 2^{\mathfrak{c}}$ rectangular bands. The question of whether there are finite semigroups in $\beta\mathbb{N}$ distinct from bands is equivalent to asking whether there exist nontrivial continuous homomorphisms from $\beta\mathbb{N}$ to $\mathbb{N}^*$ and it was an open problem since 1992. It was solved in [6] by constructing a 2-element null semigroup ($x+x = y+y = x+y = y+x = y$) in $\beta\mathbb{N}$. In [7] it was shown that all finite null semigroups have copies in $\beta\mathbb{N}$ and a connection of finite semigroups in $\beta\mathbb{N}$ with Ramsey theory was established.

In this paper we modify construction in [7] and show that

Theorem 1. *For every $m \in \mathbb{N}$, the direct product of the m-element null semigroup and the $2^{\mathfrak{c}} \times 2^{\mathfrak{c}}$ rectangular band has copies in $\beta\mathbb{N}$.*

In particular, by Theorem 1, the direct product of the 2-element null semigroup and the 2×2 rectangular band has copies in $\beta\mathbb{N}$. This fact and [7, Theorem 4.4] give us the following Ramsey theoretic consequence.

Define $r : \mathbb{N} \to \{1, 2, 3, 4\}$ by $n \equiv r(n) \pmod 4$.

Corollary 2. *There exists a partition $\{A_1, \dots, A_8\}$ of $\mathbb{N}$ with the following property: for any finite partitions $\mathcal{B}_i$ of A_i, there exist $B_i \in \mathcal{B}_i$ and a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \in B_{r(n)} \cap 2^n\mathbb{N}$ for each $n \in \mathbb{N}$ and for each finite $F \subseteq \mathbb{N}$ with $|F| \ge 2$, if $j = r(\min F)$ and $k = r(\max F)$, then $\sum_{n\in F} x_n \in B_i$, where*

$$i = \begin{cases} 5 & \text{if } j \in \{1,4\} \text{ and } k \in \{1,2\} \\ 6 & \text{if } j \in \{2,3\} \text{ and } k \in \{1,2\} \\ 7 & \text{if } j \in \{2,3\} \text{ and } k \in \{3,4\} \\ 8 & \text{if } j \in \{1,4\} \text{ and } k \in \{3,4\}. \end{cases}$$

Proof. Let S be a subsemigroup of $\mathbb{N}^*$ splitting into the direct product of a null semigroup $\{a_1, a_0\}$ and a rectangular band $\{b_{10}, b_{00}, b_{01}, b_{11}\}$. Enumerate S as

$$S = \{q_1, \dots, q_8\} = \{a_1b_{10}, a_1b_{00}, a_1b_{01}, a_1b_{11}, a_0b_{10}, a_0b_{00}, a_0b_{01}, a_0b_{11}\}.$$

Then for any $j, k \in \{1, 2, 3, 4\}$, one has $q_j + q_k = q_i$, where i is as in statement of Corollary 2. Pick a partition $\{A_1, \dots, A_8\}$ of $\mathbb{N}$ such that $A_i \in q_i$. Let $(z_n)_{n=1}^{\infty}$ be a sequence guaranteed by [7, Theorem 4.4] and take the subsequence

$$z_1, z_2, z_3, z_4, z_9, z_{10}, z_{11}, z_{12}, z_{17}, z_{18}, z_{19}, z_{20}, \dots$$

as $(x_n)_{n=1}^{\infty}$. □

Notice that it is not true that if each of two finite semigroups has copies in $\beta\mathbb{N}$, so does their direct product. Indeed, the direct product of the 2-element chain of idempotents with itself contains a 3-element semilattice which has no copy in $\beta\mathbb{N}$ [5, Lemma 3].

In the rest of the paper we prove Theorem 1. In fact, we prove a bit stronger result.

Any $x \in \mathbb{N}$ can be uniquely written as $x = \sum_{n \in F} 2^n$ for some finite nonempty $F \subseteq \omega$. Let $\operatorname{supp} x = F$, $\phi(x) = \max F$, and $\theta(x) = \min F$. We shall need the continuous extension $\beta\mathbb{N} \to \beta\omega$ of the function ϕ and we denote it by the same letter ϕ. If $x, y \in \mathbb{N}$ and $\phi(x) < \theta(y)$, then $\phi(x+y) = \phi(y)$. If $\phi(x) \leq \phi(y)$, then $\phi(x+y) \in \{\phi(y), \phi(y)+1\}$, and if $\phi(x)+1 < \phi(y)$, then $\phi(y-x) \in \{\phi(y), \phi(y)-1\}$. It then follows that for any $v \in \mathbb{N}^*$ and $w \in \beta\mathbb{Z}$, $\phi(w+v) \in \{\phi(v)-1, \phi(v), \phi(v)+1\}$, and if $v \in \mathbb{H}$, where $\mathbb{H} = \bigcap_{n<\omega} \overline{2^n\mathbb{N}}$, and $w \in \beta\mathbb{N}$, then $\phi(w+v) = \phi(v)$ (see [1, Lemma 6.8 and Lemma 13.4]). The last statement implies that for every $u \in \omega^*$, $\phi^{-1}(u) \cap \mathbb{H}$ is a left ideal of $\mathbb{H}$ (since $\phi(2^n) = n$, $\phi(\mathbb{H}) = \omega^*$).

Pick an increasing sequence $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_m = \omega$ of infinite subsets of ω such that $U_{i+1} \setminus U_i$ is infinite for each $i \in \{0, \ldots, m-1\}$. Define a function h from $\mathbb{N}$ onto the decreasing $(m+1)$-element chain $0 > 1 > \ldots > m$ of idempotents (with the operation $i \wedge j = \max\{i, j\}$) by

$$h(x) = \min\{i \in \{0, 1, \ldots, m\} : \operatorname{supp} x \subseteq U_i\}$$

(here max and min refer to the usual order, and $\wedge$ is the operation induced by the order $0 > 1 > \ldots > m$) and let the same letter h denote its continuous extension $\beta\mathbb{N} \to \{0, 1, \ldots, m\}$. If $x, y \in \mathbb{N}$ and $\phi(x) < \theta(y)$, then $h(x+y) = h(x) \wedge h(y)$. Consequently, for any $v \in \mathbb{H}$ and $w \in \beta\mathbb{N}$, $h(w+v) = h(w) \wedge h(v)$, in particular, the restriction of h to $\mathbb{H}$ is a homomorphism. For each $i \in \{0, 1, \ldots, m\}$, let $T_i = h^{-1}(\{0, \ldots, i\}) \cap \mathbb{H}$.

Lemma 3. *For each $i \in \{0, 1, \ldots, m\}$, $h(K(T_i)) = \{i\}$, and $K(T_m) = K(\beta\mathbb{N}) \cap T_m$.*

Proof. This is [7, Lemma 3.1]. □

We thus have that $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_m = \mathbb{H}$ is an increasing sequence of closed subsemigroups of $\mathbb{H}$ such that $T_i \cap \overline{K(T_{i+1})} = \emptyset$ for each $i \in \{0, \ldots, m-1\}$ and $K(T_m) = K(\beta\mathbb{N}) \cap T_m$, and for every $u \in U_0^*$, $\phi^{-1}(u) \cap T_0$ is a left ideal of T_0.

Pick an injective sequence $(u_n)_{n<\omega}$ in U_0^*. Choose a minimal right ideal R_0 of T_0, and for every $n < \omega$, a minimal left ideal $L(n)$ of T_0 contained in $\phi^{-1}(u_n) \cap T_0$, and let $p(n)$ be the identity of the group $R_0 \cap L(n)$. Then $\{p(n) : n < \omega\}$ is a right zero semigroup. Let $p_0 = p(0)$.

Enumerate $\{2^n : n \in U_1 \setminus U_0\}^*$ without repetitions as $\{r_\alpha : \alpha < 2^{\mathfrak{c}}\}$.

Lemma 4. $(p_0 + r_\alpha + T_m) \cap (p_0 + r_\beta + T_m) = \emptyset$ *if* $\alpha \neq \beta$.

Proof. This is [7, Lemma 3.2]. □

For every $\alpha < 2^{\mathfrak{c}}$, choose a minimal right ideal $R_{1,\alpha}$ of T_1 contained in $p_0 + r_\alpha + T_1$, and choose a minimal left ideal L_1 of T_1 contained in $T_1 + p_0$, and let $p_{1,\alpha}$ denote the identity of the group $R_{1,\alpha} \cap L_1$ and $p_1 = p_{1,0}$. Then by Lemma 4, $p_{1,\alpha} \neq p_{1,\beta}$ if $\alpha \neq \beta$, $p_{1,\alpha} + p_0 = p_0 + p_{1,\alpha} = p_{1,\alpha}$, and $\{p_{1,\alpha} : \alpha < 2^{\mathfrak{c}}\}$ is a left zero semigroup.

Inductively, for each $i \in \{2, \dots, m\}$, choose a minimal right ideal R_i of T_i contained in $p_{i-1} + T_i$ and a minimal left ideal L_i of T_i contained in $T_i + p_{i-1}$, let p_i denote the identity of the group $R_i \cap L_i$, and for every $\alpha < 2^{\mathfrak{c}}$, let $p_{i,\alpha} = p_{1,\alpha} + p_i$. Then $p_i + p_{i-1} = p_{i-1} + p_i = p_i$, so $p_0 > p_1 > \dots > p_i$ is a chain, and $p_{i,0} = p_i$. By Lemma 4, $p_{i,\alpha} \neq p_{i,\beta}$ if $\alpha \neq \beta$, and since $p_{i,\alpha} \in K(T_i)$, it follows that all elements $p_{i,\alpha}$, where $i \in \{1, \dots, m\}$ and $\alpha < 2^{\mathfrak{c}}$, are distinct.

We then obtain that $p_{i,\alpha} + p_0 = p_0 + p_{i,\alpha} = p_{i,\alpha}$ and

$$\begin{aligned} p_{i,\alpha} + p_{j,\beta} &= p_{1,\alpha} + p_i + p_{1,\beta} + p_j = p_{1,\alpha} + (p_i + p_1) + p_{1,\beta} + p_j \\ &= p_{1,\alpha} + p_i + (p_1 + p_{1,\beta}) + p_j = p_{1,\alpha} + p_i + p_1 + p_j = p_{1,\alpha} + p_{i \wedge j} \\ &= p_{i \wedge j,\alpha}. \end{aligned}$$

For every $i \in \{1, \dots, m\}$ and $\alpha < 2^{\mathfrak{c}}$, let $D_{i,\alpha} = \{p_{i,\alpha} + p(n) : n < \omega\}$ and pick $q_{i,\alpha} \in \overline{D_{i,\alpha}} \setminus D_{i,\alpha}$. Notice that $\phi(p_{i,\alpha} + p(n)) = \phi(p(n)) = u_n$. (It is easy to see, although it is not directly important to us, that $D_{i,\alpha}$ is a right zero semigroup.)

Lemma 5. *$q_{i,\alpha} + p_0 = p_{i,\alpha}$, and so $q_{i,\alpha} + p_{j,\beta} = p_{i \wedge j,\alpha}$.*

Proof. Since the right translation by p_0 is continuous and

$$(p_{i,\alpha} + p(n)) + p_0 = p_{i,\alpha} + (p(n) + p_0) = p_{i,\alpha} + p_0 = p_{i,\alpha},$$

one has $q_{i,\alpha} + p_0 = p_{i,\alpha}$. Then

$$q_{i,\alpha} + p_{j,\beta} = q_{i,\alpha} + (p_0 + p_{j,\beta}) = (q_{i,\alpha} + p_0) + p_{j,\beta} = p_{i,\alpha} + p_{j,\beta} = p_{i \wedge j,\alpha}.$$

□

Define $Q \subseteq \mathbb{N}^*$ by

$$Q = \{p_{i,\alpha} + q_{j,\beta} : i, j \in \{1, \dots, m\} \text{ and } \alpha, \beta < 2^{\mathfrak{c}}\}.$$

Using Lemma 5, we obtain that

$$\begin{aligned} (p_{i,\alpha} + q_{j,\beta}) + (p_{k,\gamma} + q_{l,\delta}) &= p_{i,\alpha} + (q_{j,\beta} + p_{k,\gamma}) + q_{l,\delta} = p_{i,\alpha} + p_{j \wedge k,\beta} + q_{l,\delta} \\ &= p_{i \wedge j \wedge k,\alpha} + q_{l,\delta}. \end{aligned}$$

Now we shall show that all elements $p_{i,\alpha} + q_{j,\beta}$ of the semigroup Q are distinct.

An ultrafilter $p \in \mathbb{Z}^*$ is

(i) *prime* if $p \notin \mathbb{Z}^* + \mathbb{Z}^*$, and
(ii) *right cancelable* if the right translation of $\beta\mathbb{Z}$ by p is injective.

An ultrafilter $p \in \mathbb{Z}^*$ is right cancelable if and only if $p \notin \mathbb{Z}^* + p$ (see [7, Lemma 3.5]). Thus, every prime ultrafilter is right cancelable.

Lemma 6. *Let D be a countable subset of $\mathbb{H}$ and suppose that ϕ is injective on D. Then every $q \in \overline{D} \setminus D$ is prime.*

Proof. Assume the contrary. Then $q \in \mathbb{Z}^* + v$ for some $v \in \mathbb{Z}^*$. Since $-\mathbb{N}^*$ is a left ideal of $\beta\mathbb{Z}$, one has $v \in \mathbb{N}^*$. Let $Z = \{n \in \mathbb{Z} : n + v \notin \mathbb{H}\}$ and let $D' = \{p \in D : \phi(p) \notin \{\phi(v) - 1, \phi(v), \phi(v) + 1\}\}$. Notice that $|\mathbb{Z} \setminus Z| \leq 1$ and $|D \setminus D'| \leq 3$. We then have that $q \in \overline{D'} \cap \overline{Z + v}$, so by [1, Theorem 3.40], either

$n + v \in \overline{D'}$ for some $n \in Z$ or $p \in \overline{Z + v} = \overline{Z} + v$ for some $p \in D'$. In the first case, $n + v \in \mathbb{H}$. In the second, $p = w + v$ for some $w \in \overline{Z}$, so

$$\phi(p) = \phi(w + v) \in \{\phi(v) - 1, \phi(v), \phi(v) + 1\}.$$

In either case we have a contradiction. □

Statement (3) of the next lemma tells us that all elements $p_{i,\alpha} + q_{j,\beta}$ of the semigroup Q are distinct.

Lemma 7. (1) *All subsets $\overline{D_{i,\alpha}}$, where $i \in \{1, \dots, m\}$ and $\alpha < 2^{\mathfrak{c}}$, are pairwise disjoint.*

(2) *All elements $q_{i,\alpha}$, where $i \in \{1, \dots, m\}$ and $\alpha < 2^{\mathfrak{c}}$, are distinct.*

(3) *All elements $p_{i,\alpha} + q_{j,\beta}$, where $i, j \in \{1, \dots, m\}$ and $\alpha, \beta < 2^{\mathfrak{c}}$, are distinct.*

Proof. (1) Assume on the contrary that $\overline{D_{i,\alpha}} \cap \overline{D_{j,\beta}} \neq \emptyset$ for some $(i, \alpha) \neq (j, \beta)$. Then either $D_{i,\alpha} \cap \overline{D_{j,\beta}} \neq \emptyset$ or $\overline{D_{i,\alpha}} \cap D_{j,\beta} \neq \emptyset$. It suffices to consider the first case. Since $D_{i,\alpha} \cap D_{j,\beta} = \emptyset$, it follows that $p_{i,\alpha} + p(n) = q$ for some $n < \omega$ and $q \in \overline{D_{j,\beta}} \setminus D_{j,\beta}$. But by Lemma 6, q is prime, a contradiction.

(2) is immediate from (1).

(3) Suppose that $p_{i,\alpha} + q_{j,\beta} = p_{k,\gamma} + q_{l,\delta}$. Then by [1, Corollary 6.21], either $q_{j,\beta} \in \beta\mathbb{N} + q_{l,\delta}$ or $q_{l,\delta} \in \beta\mathbb{N} + q_{j,\beta}$. In either case $q_{j,\beta} = q_{l,\delta}$, since both of them are prime and in $\mathbb{H}$, so by (2), $(j, \beta) = (l, \delta)$. We thus have that $p_{i,\alpha} + q_{j,\beta} = p_{k,\gamma} + q_{j,\beta}$. But then $p_{i,\alpha} = p_{k,\gamma}$, since $q_{j,\beta}$ is right cancelable, and so $(i, \alpha) = (k, \gamma)$. □

We have constructed Q as a subsemigroup of $\mathbb{N}^*$. We now describe it without mentioning ultrafilters.

Given a semilattice I and a cardinal κ, let $S = S(I, \kappa)$ denote the semigroup whose underlying set is $I \times \kappa \times I \times \kappa$ and the operation is defined by

$$(i, \alpha, j, \beta) + (k, \gamma, l, \delta) = (i \wedge j \wedge k, \alpha, l, \delta).$$

The semigroup S decomposes into the semilattice I of the subsemigroups

$$S_t = \{(i, \alpha, j, \beta) \in S : i \wedge j = t\},$$

where $t \in I$ (that is, $S_i + S_j \subseteq S_{i \wedge j}$). For every $(i, \alpha, j, \beta) \in S_t$, if $i = t$, then

$$(t, \alpha, j, \beta) + (t, \alpha, j, \beta) = (t, \alpha, j, \beta),$$

so (t, α, j, β) is an idempotent, and if $i \neq t$, then

$$\begin{aligned}(i, \alpha, j, \beta) + (i, \alpha, j, \beta) &= (t, \alpha, j, \beta)\\ &= (i, \alpha, j, \beta) + (t, \alpha, j, \beta) = (t, \alpha, j, \beta) + (i, \alpha, j, \beta),\end{aligned}$$

so $\{(i, \alpha, j, \beta), (t, \alpha, j, \beta)\}$ is a null semigroup.

If I is a decreasing chain $1 > \dots > m$, we write $S(m, \kappa)$ instead of $S(I, \kappa)$. For each $t \in \{1, \dots, m\}$, the component S_t of $S = S(m, \kappa)$ is the union of $\kappa \times (\{1, \dots, t\} \times \kappa)$ rectangular band

$$B_t = \{(t, \alpha, j, \beta) : j \in \{1, \dots, t\} \text{ and } \alpha, \beta < \kappa\},$$

which is the smallest ideal of S_t, and the subsemigroup

$$S_{t,t} = \{(i, \alpha, t, \beta) : i \in \{1, \ldots, t\} \text{ and } \alpha, \beta < \kappa\}.$$

The intersection of B_t and $S_{t,t}$ is $\kappa \times \kappa$ rectangular band

$$B_{t,t} = \{(t, \alpha, t, \beta) : \alpha, \beta < \kappa\},$$

which is the smallest ideal of $S_{t,t}$, and $S_{t,t}$ is a disjoint union of t-element null subsemigroups $\{(i, \alpha, t, \beta) : i \in \{1, \ldots, t\}\}$, where $\alpha, \beta < \kappa$, so $S_{t,t}$ is isomorphic to the direct product of t-element null semigroup and $B_{t,t}$.

Define $\varepsilon : S(m, 2^{\mathfrak{c}}) \to Q$ by

$$\varepsilon(i, \alpha, j, \beta) = p_{i,\alpha} + q_{j,\beta}.$$

Then ε is an isomorphism. Furthermore,

$$\varepsilon(m, \alpha, j, \beta) = p_{m,\alpha} + q_{j,\beta} \in K(\beta\mathbb{N})$$

because $p_{m,\alpha} \in K(\beta\mathbb{N})$, and

$$\varepsilon(i, \alpha, m, \beta) = p_{i,\alpha} + q_{m,\beta} \in \overline{K(\beta\mathbb{N})}$$

because $q_{m,\beta} \in \overline{D_{m,\beta}} \subseteq \overline{K(\beta\mathbb{N})}$ and $\overline{K(\beta\mathbb{N})}$ is an ideal of $\beta\mathbb{N}$ [1, Theorem 4.44].

Thus, we have established the following result.

Theorem 8. *Let $m \in \mathbb{N}$ and $S = S(m, 2^{\mathfrak{c}})$. Then there is an isomorphic embedding $\varepsilon : S \to \mathbb{N}^*$. Furthermore, ε can be chosen so that $\varepsilon(S_m) \subseteq \overline{K(\beta\mathbb{N})}$ and $\varepsilon(K(S_m)) \subseteq K(\beta\mathbb{N})$.*

Since $S_{m,m}$ is isomorphic to the direct product of the m-element null semigroup and the $2^{\mathfrak{c}} \times 2^{\mathfrak{c}}$ rectangular band, Theorem 1 is a partial case of Theorem 8.

References

[1] HINDMAN, NEIL; STRAUSS, DONA. Algebra in the Stone-Čech Compactification. Theory and Applications. Second revised and extended edition. De Gruyter Textbook series. *Walter de Gruyter & Co., Berlin*, 2012. xviii+591 pp. ISBN: 978-3-11-025623-9. MR2893605, Zbl 1241.22001, doi: 10.1515/9783110258356. 610, 612, 613, 614, 615

[2] HINDMAN, NEIL; STRAUSS, DONA; ZELENYUK, YEVHEN. Large rectangular semigroups in Stone-Čech compactifications. *Trans. Amer. Math. Soc.* **355** (2003), no. 7, 2795–2812. MR1975400, Zbl 1028.22004, doi: 10.1090/S0002-9947-03-03276-8. 611

[3] STRAUSS, DONA. $\mathbb{N}^*$ does not contain an algebraic and topological copy of $\beta\mathbb{N}$. *J. London Math. Soc. (2)* **46** (1992), no. 3, 463–470. MR1190430, Zbl 0723.54035, doi: 10.1112/jlms/s2-46.3.463. 610

[4] ZELENYUK, YEVHEN. On subsemigroups of $\beta\mathbb{N}$ and absolute coretracts. *Semigroup Forum* **63** (2001), no. 3, 457–465. MR1851825, Zbl 1009.22005, doi: 10.1007/S002330010073. 611

[5] ZELENYUK, YEVHEN; ZELENYUK, YULIYA. Three-element bands in $\beta\mathbb{N}$. *New York J. Math.* **21** (2015), 1263–1267. MR3441641, Zbl 1330.22004. 611

[6] ZELENYUK, YEVHEN. Elements of order 2 in $\beta\mathbb{N}$. *Fund. Math.* **252** (2021), no. 3, 355–359. MR4178872, Zbl 1475.22005, doi: 10.4064/fm902-5-2020. 611

[7] ZELENYUK, YEVHEN Finite semigroups in $\beta\mathbb{N}$ and Ramsey theory. *Bull. London Math. Soc.* **53** (2021), no. 3, 710–722. MR4275083, Zbl 07381903, doi: 10.1112/blms.12453. 611, 612, 613

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