# Segre quartic surfaces and minitwistor spaces

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ABSTRACT. The Segre surfaces of the title are by definition those quartic surfaces in $\mathbb{CP}_4$ that arise as images of weak del Pezzo surfaces of degree four under the anti-canonical map. We first show that under some minimality condition, minitwistor spaces of genus one are exactly Segre quartic surfaces. By a kind of Penrose correspondence, Zariski open subsets of the projective dual varieties of these surfaces admit Einstein-Weyl structure. We investigate structures of these dual varieties in detail. In particular, we determine the degrees of these varieties (namely the classes of the Segre surfaces), as well as structure of several components of the divisors at infinity, which are the complements of the Einstein-Weyl spaces in the projective dual varieties.

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## 1. Introduction

In a classical terminology, a non-degenerate irreducible quartic surface in $\mathbb{CP}_4$ which is neither a cone over a quartic curve in $\mathbb{CP}_3$ nor a projection of a quartic surface in $\mathbb{CP}_5$ is called a Segre quartic surface. In a modern language, these are exactly the images of weak del Pezzo surfaces of degree four under the anti-canonical maps, and are realized in $\mathbb{CP}_4$ as complete intersections of pairs of quadrics. Segre quartic surfaces are classified into 16 kinds in terms of normalized quadratic equations of the complete intersections. Aside from the smooth examples, all these surfaces have isolated singularities, all of which are rational double points.

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The notion of minitwistor space was originally introduced by Hitchin [4] and Jones-Tod [9] as an analogue of Penrose's twistor space for a self-dual 4-manifold, and is a complex surface which has a smooth rational curve whose self-intersection number is two. Such a rational curve is called a minitwistor line, and they are parameterized by a 3-dimensional complex manifold. This complex manifold carries a natural geometric structure called an Einstein-Weyl structure, which is a pair consisting of a conformal structure and a compatible torsion free connection for which the symmetric-trace-free part of the Ricci tensor vanishes. A smooth quadric and a quadratic cone in $\mathbb{CP}_3$ are compact minitwistor spaces, and minitwistor lines are irreducible hyperplane sections of them. These two examples are essentially the only ones that arise from compact minitwistor spaces, as understood in Hitchin's original sense.

Generalizing Hitchin's framework, we then showed in [8] that, if we allow the rational curves to have *nodes* and at the same time increase their self-intersection numbers in such a way that the parameter space of the maximial family of nodal curves is 3-dimensional, then their parameter space still admits an Einstein-Weyl structure. We then generalized the meaning of the term minitwistor space by also applying it to surfaces that contain nodal curves of this kind. Confining our attention henceforth to *compact* complex surfaces, it then follows that these nodal curves, which we will persist in calling minitwistor lines, are mutually linearly equivalent. If we write $g$ for the number of nodes of minitwistor lines, then the linear system generated by minitwistor lines on a compact minitwistor space is $(3+g)$-dimensional, and the map induced by this linear system is always a birational morphism over the image. We call a compact minitwistor space *essential* if this birational morphism is an embedding. Hence, compact essential minitwistor spaces are naturally projective surfaces in $\mathbb{CP}_{3+g}$, and minitwistor lines on them are obtained as hyperplane sections of the surfaces. So the situation is quite similar to the classical case of $g = 0$, but when $g > 0$ a generic member of the linear system is not a minitwistor line because such a member is a smooth curve whose genus is exactly $g$. For this reason we call the number of nodes the *genus* of a minitwistor space.

In this article we shall investigate compact essential minitwistor spaces of genus one, and their associated Einstein-Weyl spaces. Our first main result means that such minitwistor spaces are exactly the Segre quartic surfaces (Theorem 2.7). This implies that, in contrast to the classical case of $g = 0$, there are a variety of compact minitwistor spaces with $g = 1$, but they can still be concretely classified. Minitwistor lines on a Segre quartic surface $S \subset \mathbb{CP}_4$ are obtained as hyperplane sections of the surface, where the hyperplanes are tangent to $S$ at exactly one point. A completion of the space of such hyperplanes is nothing but the projective dual variety of the Segre surface. Hence, the Einstein-Weyl spaces associated to the Segre quartic surfaces are realized as Zariski-open subsets of the dual varieties of the surfaces. From a reflexibility for the operation of taking the projective dual, a Penrose type correspondence

between compact essential minitwistor spaces of genus one and the associated Einstein-Weyl spaces can be understood as a projective duality.

By this reason, a large part of this article is devoted to investigate the dual varieties of Segre quartic surfaces. We show that all these dual varieties are rational threefolds (Proposition 2.8). Next in Section 2.3, we give a formula which expresses the degrees of the dual varieties of any Segre surfaces in terms of the types of singularities of the surfaces (Theorem 2.11). The degree of the dual variety will be smaller as the singularities of the surface become complicated.

The complements of the Einstein-Weyl spaces in the dual varieties of Segre surfaces are of special interest, because in the case of the smooth quadric $Q$ in $\mathbb{CP}_3$, the complement is exactly the dual quadric $Q^* \subset \mathbb{CP}_3^*$ of $Q$, and this can be regarded as a complexification of the ideal boundary of the hyperbolic space, which is the real form of the complex Einstein-Weyl space $\mathbb{CP}_3^* \backslash Q^*$. We call 2-dimensional components of the Einstein-Weyl space in the dual variety *divisors at infinity*.

In order to study these divisors at infinity, in Section 3.1, we recall Segre symbol by which all Segre quartic surfaces can be classified in a systematic way, and complete a classification of Segre surfaces (Proposition 3.1). In Section 3.2, we investigate double covering structure on many kinds of Segre surfaces, which can be detected from the Segre symbols. The base space of the double covering is either a smooth quadric or a quadratic cone in $\mathbb{CP}_3$ (Propositions 3.2 and 3.3). In Section 3.3 we investigate divisors at infinity on the dual varieties of Segre surfaces. We will find two kinds of such divisors. A first kind arises from straight lines lying on Segre surfaces, and the divisors are 2-planes (Proposition 3.4). The second kind comes from smooth conics lying on Segre surfaces, or equivalently, the double covering structure over a smooth quadric given in Section 3.2, and they are smooth quadric surfaces which are dual to the last smooth quadrics (Proposition 3.10). We determine the numbers of these divisors at infinity, for each type of Segre quartic surface (Tables 1, 2 and 3). Furthermore, by using deformation theoretic argument, we show that the dual varieties of Segre surfaces have ordinary double points along these two kinds of divisors at infinity (Proposition 3.12). In other words, the dual varieties of Segre quartic surfaces have self-intersection along the 2-planes and the smooth quadrics. We have to remark that these divisors do not exhaust all divisors at infinity in the dual varieties, because there can exist a divisor at infinity whose generic points correspond to rational curves having a cusp as their only singularity. These 'cuspidal locus' of the dual varieties of Segre surfaces is studied in detail in another article [7].

As mentioned at the beginning, Segre surfaces are classified into 16 types. In Section 4.1, we discuss several typical transitions between different types of Segre quartic surfaces. Finally, in Section 4.2, we give a remark about null surfaces in some of the present Einstein-Weyl spaces, and also pose a question about present minitwistor spaces, in connection with twistor spaces associated to self-dual 4-manifolds.

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## 2. Minitwistor spaces of genus one and Segre quartic surfaces

### 2.1. Minitwistor spaces and 3-dimensional Einstein-Weyl manifolds.
We begin with the definition of minitwistor spaces in the sense of [8]. This naturally includes minitwistor spaces in the original sense given in [4] as the simplest case. Let $g \geq 0$ be an integer. By a *g-nodal rational curve*, we mean a rational curve which has exactly $g$ nodes (i.e. ordinary double points) as its only singularities. In particular, when $g = 0$, it is just a smooth rational curve.

**Definition 2.1.** Let $g \geq 0$ be an integer and $S$ a normal, compact and irreducible complex surface. A $g$-nodal rational curve $C$ on $S$ is called a *minitwistor line* if it is contained in $S_{\mathrm{reg}}$, the smooth locus of $S$, and the self-intersection number satisfies $C^2 = 2 + 2g$. A normal compact complex surface having a minitwistor line with $g$ nodes as above is called a *minitwistor space of genus g*.

By [8, Proposition 2.6], any minitwistor space in this sense is a rational surface. Furthermore, by [8, Proposition 2.8], the linear system $|C|$ generated by minitwitor lines on a minitwistor space $S$ as above is $(3 + g)$-dimensional and base point free. Moreover, we have:

**Proposition 2.2.** *In the situation of the above definition, a generic member of the system $|C|$ is smooth and it is a curve of genus g.*

**Proof.** The smoothness of a generic member of $|C|$ follows from Bertini's theorem. Let $C \subset S$ be a minitwistor line, $\nu : \tilde{S} \to S$ the blowup of $S$ at the $g$ nodes of $C$, and $\tilde{C}$ the strict transform of $C$ into $\tilde{S}$. Evidently the curve $\tilde{C}$ is a smooth rational curve. If $E_1, \dots, E_g$ are the exceptional curves of $\nu$, we have

$$K_{\tilde{S}} \cdot \tilde{C} = \left( \nu^* K_S + \sum_{i=1}^{g} E_i \right) \cdot \left( \nu^* C - 2 \sum_{i=1}^{g} E_i \right) = K_S \cdot C + 2g \qquad (1)$$

and

$$\tilde{C}^2 = \left( \nu^* C - 2 \sum_{i=1}^{g} E_i \right)^2 = C^2 - 4g = 2 - 2g. \qquad (2)$$

Further, since $\tilde{C}$ is a smooth rational curve, we have by adjunction

$$K_{\tilde{S}} \cdot \tilde{C} + \tilde{C}^2 = -2.$$

Substituting (1) and (2) to this equality, we obtain $K_S \cdot C = -4$. Hence, if $C'$ is a smooth member of $|C|$, we have $K_S \cdot C' = -4$. Again by adjunction and using $(C')^2 = C^2 = 2 + 2g$, we obtain that the genus of the curve $C'$ is exactly $g$. $\square$

This is why we call $g$ the genus of a minitwistor space. Thus minitwistor lines are obtained as a degeneration of smooth members of the linear system $|C|$

into irreducible curves which have exactly $g$ nodes. We denote the morphism associated to the $(3 + g)$-dimensional system $|C|$ by

$$\phi : S \longrightarrow \mathbb{CP}_{3+g}. \tag{3}$$

By [8, Proposition 2.8], this morphism is always birational over the image. However, the morphism $\phi$ can contract curves on $S$. Indeed, any blow-up of a minitwistor space of genus $g$ is again a minitwistor space of genus $g$, but the map associated to the system generated by minitwistor lines contracts the exceptional curve of the blow-up. In order to ignore these redundant spaces, we call a minitwistor space $S$ *essential* if the birational morphism $\phi$ in (3) is an embedding, so that it does not contract any curves. This is a kind of minimality condition, but note that it is little stronger than the minimality introduced in [8] to the effect that in [8] minimality means that the curves contracted by $\phi$ were $(-1)$-curves only, while in the present paper we are allowing other curves (e.g. a $(-2)$-curve) to be contracted. The present notion seems more natural.

Since the system $|C|$ is base point free and the associated morphism $\phi$ is birational over the image as above, for the degree of the image surface $\phi(S) \subset \mathbb{CP}_{3+g}$, we always have

$$\deg \phi(S) = C^2 = 2 + 2g. \tag{4}$$

When $g = 0$, the conditions in Definition 2.1 mean that $C$ is a smooth rational curve satisfying $C^2 = 2$, and this agrees with the original definition of a minitwistor space given in [4]. In this case, the image surface $\phi(S) \subset \mathbb{CP}_4$ is either a smooth quadric or the cone over an irreducible conic. These are all examples of minitwistor spaces classically known, and essential minitwistor spaces of genus zero are exactly these two surfaces [8, Proposition 2.14].

Next we define a Severi variety of rational curves associated to a minitwistor space in the present sense.

**Definition 2.3.** Let $S \subset \mathbb{CP}_{3+g}$ be a essential minitwistor space of genus $g > 0$. Let $W_0$ be the subset of the dual projective space $\mathbb{CP}_{3+g}^*$ consisting of hyperplanes $H \subset \mathbb{CP}_{3+g}$ such that the hyperplane sections $S \cap H$ are minitwistor lines (with $g$ nodes). We write $W$ for the closure of $W_0$, taken in $\mathbb{CP}_{3+g}^*$. This is a subvariety in $\mathbb{CP}_{3+g}^*$, and we call it the *Severi variety of $g$-nodal rational curves* on $S$.

The space $W_0$ is a Zariski-open subset of the closure $W$. In other words, the Severi variety $W$ is a compactification of $W_0$. We note that in the above definition of the space $W_0$, since we are requiring that a minitwistor line is included in $S_{\text{reg}}$ as in Definition 2.1, we are requiring that the hyperplane sections $S \cap H$ do not pass through any singularity of $S$. If a hyperplane $H$ passes through a singularity of $S$, say $p$, then the section $S \cap H$ is always singular at $p$, and the section $S \cap H$ can be a nodal rational curve with a correct number of nodes. But in general such a section does not admit an equisingular displacement in the linear system $|C|$ which avoids the singularity of $S$, and in that case $H \notin S^*$. See Remark 2.6 for concrete examples of this kind.

**Remark 2.4.** Although it seems quite likely, we do not know whether the Zariski open subset $W_0$ in $W$ is precisely the smooth locus of the Severi variety $W$.

Now from [8, Proposition 2.6 and Theorem 2.10], as a natural generalization of a well-known result in the case of genus zero given in [4], we have

**Proposition 2.5.** *The subset $W_0 \subset \mathbb{CP}^*_{3+g}$ is a 3-dimensional complex manifold, and it admits a torsion-free Einstein-Weyl structure.*

**2.2. Compact minitwistor spaces of genus one.** In the rest of this article, we are mainly interested in essential minitwistor spaces of genus one. In this case the Severi variety $W$ is a classical object that will be familiar to many readers. To explain this object precisely, let $S$ be an irreducible, non-denenerate 2-dimensional subvariety in $\mathbb{CP}_4$. We write $S^* \subset \mathbb{CP}^*_4$ for the dual variety of $S$. This is by definition [12, Def. 1.1] the closure in the dual projective space $\mathbb{CP}^*_4$ of the locus of hyperplanes which contain tangent planes of $S$ at some smooth points of $S$. (If $S$ is smooth, we do not need to take the closure.) Let $I(S)$ be the incidence variety for $S$ and $S^*$. Namely $I(S)$ is the closure of the set

$$\{(p, H) \in \mathbb{CP}_4 \times \mathbb{CP}^*_4 \mid p \in S_{\text{reg}}, \ T_p S \subset H\},$$

taken in $\mathbb{CP}_4 \times \mathbb{CP}^*_4$. This is a subvariety in $\mathbb{CP}_4 \times \mathbb{CP}^*_4$, and the dual variety $S^*$ is the image of $I(S)$ under the the projection to the second factor $\mathbb{CP}^*_4$. Hence, there is a double fibration

$$
\begin{array}{ccc}
 & I(S) & \\
 & \swarrow \qquad \searrow & \\
S & & S^*
\end{array}
\tag{5}
$$

Over the smooth locus $S_{\text{reg}}$ of $S$, the incidence variety $I(S)$ is a fiber bundle whose fibers are projective lines. In particular, $I(S)$ is 3-dimensional. Also, since $S$ is supposed to be irreducible, $I(S)$ is always an irreducible variety. Therefore, so is the dual variety $S^*$.

Now suppose that $S$ is a essential minitwistor space of genus one. So the morphism (3) provides a projective embedding $S \subset \mathbb{CP}_4$. If $H \subset \mathbb{CP}_4$ is a hyperplane which belongs to the subset $W_0$, the intersection $S \cap H$ is a minitwistor line from Definition 2.3. If $p$ is the node of this minitwistor line then we have $T_p S \subset H$ because otherwise $S \cap H$ would be smooth at the point $p$. Thus $H \in W_0$ implies $H \in S^*$. Namely $W_0 \subset S^*$. Therefore, since $S^*$ is closed, we obtain $W \subset S^*$ for the closure $W$ of $W_0$. Since $S^*$ is irreducible as above and at most 3-dimensional, this means that $W = S^*$. Thus *in the case of a minitwistor space of genus one, the Severi variety $W$ in Definition 2.3 is nothing but the dual variety $S^*$, and it is always irreducible.*

**Remark 2.6.** From this we can easily obtain an example of a 1-nodal rational curve on a minitwistor space of genus one, which does not belong to the Zariski-open subset $W_0$ (namely which is not a minitwistor line) as follows. Take any

essential minitwistor space $S \subset \mathbb{CP}_4$ of genus one which has at least one ordinary double point $p$. (See Section 3.2 for examples such minitwistor spaces.) Let $p^* \subset \mathbb{CP}_4^*$ be the hyperplane which is dual to $p$. So a hyperplane $H \subset \mathbb{CP}_4$ belongs to $p^*$ iff $p \in H$. By the irreducibility of $S^*$, we have $p^* \not\subset S^*$. Therefore, the hyperplane section $S^* \cap p^*$ is a divisor on $p^* = \mathbb{CP}_3$. If we take a generic $H \in p^*$ which does not belong to $S^*$, the section $S \cap H$ is a 1-nodal rational curve whose node is exactly $p$, and it cannot be deformed into a minitwistor line by any small displacement in $S$ since $H \notin S^*$. This implies that $S \cap H$ is not a minitwistor line.

As is already mentioned, the structure of essential minitwistor spaces of genus zero is strongly constrained. Next we would like to see that a constraint for the structure of essential minitwistor spaces of genus one is somewhat moderate, and they are exactly particular surfaces that are classically known. In order to explain what are these surfaces and put them in a broader context, we discuss classical results on del Pezzo surfaces of arbitrary degrees.

Suppose $n \geq 3$. All surfaces in $\mathbb{CP}_n$ appearing below are assumed to be irreducible and non-degenerate, but smoothness and even normality are not assumed. It is classically known [3, p. 174] that the degree of any surface $S$ in $\mathbb{CP}_n$ is bounded from below as

$$\deg S \geq n - 1.$$

Surfaces which attain the minimal degree $(n - 1)$ are classically classified [3, p. 525]. Surfaces with the second smallest degree, namely surfaces with degree $n$ in $\mathbb{CP}_n$ are also classified, and they belong to one of the following three kinds of surfaces [2, Section 8.1]:

(a) the images of surfaces of degree $n$ in $\mathbb{CP}_{n+1}$ under projections from points,

(b) the cones over irreducible curves of degree $n$ in $\mathbb{CP}_{n-1}$,

(c) surfaces not included in (a) nor (b).

A difference between these surfaces is that generic hyperplane sections of surfaces in (a) have arithmetic genus zero, while they are one for surfaces in (b) and (c). See [2, Section 8.1]. So an irreducible non-degenerate surface of degree $n$ in $\mathbb{CP}_n$ belongs to (c) iff a generic hyperplane section has arithmetic genus one and the surface is not the cone over a curve of degree $n$ in $\mathbb{CP}_{n-1}$.

According to [2, Definition 8.1.5], in a classical terminology, an irreducible non-degenerate surface in $\mathbb{CP}_n$ belonging to the case (c) is called a *del Pezzo surface* of degree $n$. The degree of a del Pezzo surface is at most 9 [2, Propositions 8.1.7 and 8.1.8]. Any del Pezzo surface is normal and has at worst rational double points. For smooth ones, these surfaces are exactly del Pezzo surfaces in modern definition. For singular ones, the minimal resolutions of del Pezzo surfaces in this classical sense have $(-2)$-curves, and in modern language they are often called weak del Pezzo surfaces.

Classically del Pezzo surfaces of degree four are called *Segre quartic surfaces* ([2, Section 8.6]). By [2, Theorem 8.6.2], any Segre quartic surface is a complete intersection of two quadrics in $\mathbb{CP}_4$. From this we readily obtain that the anti-canonical class of a Segre quartic surface is the class of hyperplane sections. Any Segre quartic surface has at most finite number of lines on it (see [2, Section 8.6.3]). Note that a complete intersection of two quadrics in $\mathbb{CP}_4$ is not necessarily a Segre surface because the cone over a quartic curve in $\mathbb{CP}_3$ is typically a complete intersection of two quadrics in $\mathbb{CP}_4$.

With these preliminaries, we have

**Theorem 2.7.** *Any essential minitwistor space of genus one is a Segre quartic surface. Conversely, any Segre quartic surface is a essential minitwistor space of genus one.*

**Proof.** Let $S$ be a essential minitwistor space of genus one. If $C$ is a minitwistor line on $S$, as is already remarked, the surface $S$ is embedded in $\mathbb{CP}_4$ by the complete linear system $|C|$, and in particular $S$ is non-degenerate in $\mathbb{CP}_4$. Since $C^2 = 2 + 2g = 4$, we have $\deg S = 4$. Thus, $S$ is a non-degenerate irreducible quartic surface in $\mathbb{CP}_4$. Hence, by letting $n = 4$ in the above classification of surfaces in $\mathbb{CP}_n$ of degree $n$, in order to show that $S$ is a Segre surface, it is enough to show that the surface $S$ does not belong to the above classes (a) nor (b). The class (a) is immediately rejected since the arithmetic genus of a generic hyperplane section of surfaces in (a) is zero, while it is one for minitwistor spaces of genus one by Proposition 2.2. Next, we show that the cones as in the class (b) cannot be a minitwistor space. Recall that we are supposing normality for a minitwistor space (see Definition 2.1). Therefore, because the cone over a singular curve is non-normal, the cone can be a minitwistor space of genus one only when the base curve is smooth. But even in that case the cone cannot be a minitwistor space of genus one because a hyperplane section of the cone is singular only when it passes through the vertex of the cone, and a section of the cone by such a hyperplane consists of generating lines of the cone and therefore it cannot be a minitwistor line. Hence the surface $S$ belongs to the class (c), and this means that $S$ has to be a Segre quartic surface.

To prove the converse, let $S \subset \mathbb{CP}_4$ be a Segre quartic surface. We first show that the dual variety $S^*$ of $S$ is 3-dimensional. Recall that a surface in $\mathbb{CP}^4$ is said to be ruled if any point of the surface is passed through by a line lying on the surface. By [12, Theorem 1.18], the strict inequality $\dim S^* < 3$ happens only for ruled surfaces But any Segre quartic surface is not ruled because it has only a finite number of lines on it. Therefore, $\dim S^* = 3$. As in [13, Section 2.1], if the dimension of the dual variety is maximal (i.e. it is a hypersurface in the dual projective space), then a hyperplane section of $S$ which corresponds to a generic element of the dual variety $S^*$ has precisely one node as its only singularity. From genericity such a hyperplane section can be assumed not to pass through any singularity of $S$, and if $C$ is such a hyperplane section, we have $C^2 = H.H.S = 4$. Moreover, the curve $C$ is rational since it has exactly

one node as its only singularity and its arithmetic genus is one. Therefore, $C$ is a minitwistor line with one node, and hence the surface $S$ is a essential minitwistor space of genus one. $\qquad\square$

Thus, essential minitwistor spaces of genus one are exactly Segre quartic surfaces. Hence from Proposition 2.5, the open subset $W_0$ of the dual variety (or equivalently of Severi variety) in Definition 2.3 of any Segre surface admits a torsion-free Einstein-Weyl structure. For a birational property of the dual varieties of Segre quartic surfaces, we have the following

**Proposition 2.8.** *The dual variety of any Segre quartic surface is rational.*

**Proof.** Let $S \subset \mathbb{CP}_4$ be a Segre quartic surface. Recall that, writing $I(S)$ for the incidence variety as before, we have the diagram

$$
\begin{array}{ccc}
& I(S) & \\
& \swarrow \qquad \searrow & \\
S & & S^*
\end{array}
\tag{6}
$$

Take and fix a point $q$ of $\mathbb{CP}_4$ in which $S$ is embedded. Then for a generic point $p \in S_{\mathrm{reg}}$, we have $q \notin T_p S$. Hence the linear subspace $H(p)$ spanned by a 2-plane $T_p S$ and the point $q$ is a hyperplane, and it contains $T_p S$. Hence the pair $(p, H(p)) \in \mathbb{CP}_4 \times \mathbb{CP}_4^*$ belongs to $I(S)$. So the assignment $p \mapsto (p, H(p))$ defines a rational section of the projection $I(S) \to S$. Generic fibers of this projection are projective lines, and therefore, the presence of the rational section implies [1, Lemma 3.4] that $I(S)$ is birational to the product $S \times \mathbb{CP}_1$. Moreover, as mentioned right after Definition 2.1, any minitwistor space is a rational surface. Hence the variety $I(S)$ is rational. Since the variety $S^*$ is the image of $I(S)$ under the projection to $\mathbb{CP}_4^*$, this means that $S^*$ is unirational. For the rationality, it is enough to show that the projection $I(S) \to S^*$ is of degree-one. If the degree is greater than one, a hyperplane $H \in S^*$ which is generic in $S^*$ contains at least two distinct tangent spaces of $S$. This means that $H \cap S$ has at least two singularities. But as mentioned in the last part of the proof of Theorem 2.7, the intersection $H \cap S$ has exactly one node as its only singularity for generic $H \in S^*$. Hence the projection $I(S) \to S^*$ is of degree-one. $\qquad\square$

Note that the proof works for any rational variety $X \subset \mathbb{CP}_N$ which is not ruled.

**2.3. The class formula for Segre quartic surfaces.** In this subsection we determine the degrees of the dual varieties of Segre quartic surfaces, namely the *classes* of the surfaces, by blowing up the surfaces at the intersection with a generic 2-plane in $\mathbb{CP}_4$ and then investigating singular fibers of the resulting elliptic fibrations induced on the blowups. This idea was used in [10] to calculate the classes for Segre surfaces which have at most two nodes as their only singularity, but the argument there for proving non-existence of reducible

fibers of the elliptic fibration seems to require a clarification as we will provide in Lemma 2.9 below in full generality.

First, we explain how one can obtain an elliptic fibration. Let $S \subset \mathbb{CP}_4$ be any Segre quartic surface, and $P \subset \mathbb{CP}_4$ a 2-plane which is sufficiently general so that $P$ intersects $S$ transversally at any point of $S \cap P$. This in particular means that $S \cap P$ consists of four points as $\deg S = 4$, and $S$ is smooth at these four points. If $p$ is any one of these points, from the transversality we have $T_p S \cap T_p P = 0$ for the intersection of tangent spaces. Therefore, for any hyperplane $H$ containing $P$, we have $T_p S \not\subset H$, and hence the hyperplane section

$$S_H := S \cap H$$

is smooth at any of the four points $S \cap P$. These hyperplane sections are quartic curves in $H = \mathbb{CP}_3$, and belong to the anti-canonical class of $S$. Further for any two distinct hyperplanes containing $P$, the anti-canonical curves intersect transversally at the four points $S \cap P$. Let $\mathbb{CP}_4 \to \mathbb{CP}_1$ be the generic projection from the 2-plane $P$. Fibers of this projection are hyperplanes that contain $P$, and by intersecting with $S$, we obtain a pencil of anti-canonical curves on $S$. This pencil has the four points $S \cap P$ as the base locus. Let $S' \to S$ be the blowing-up at the four points $S \cap P$. Equivalently the surface $S'$ is the strict transform of $S$ under the blowing-up of $\mathbb{CP}_4$ along the 2-plane $P$. By the transversality the base points of the above pencil on $S$ are eliminated through the blowup, and we obtain a morphism. We write it as

$$f' : S' \to \mathbb{CP}_1. \tag{7}$$

All fibers of $f'$ are isomorphic to the corresponding members of the original pencil on $S$. By Bertini's theorem fibers of $f'$ are smooth except for finite ones. Obviously all fibers of $f'$ are anti-canonical curves on $S'$. In particular it is an elliptic curve as long as it is smooth. Thus the morphism $f' : S' \to \mathbb{CP}_1$ is an elliptic fibration. Since all blown-up points on $S$ are smooth points of $S$, the surface $S'$ has the same singularities as $S$ has.

Let $\tilde{S}' \to S'$ be the minimal resolution of all singularities of $S'$. If $S$ is smooth we promise $\tilde{S}' = S'$. Since all singularities of $S'$ are rational double points, all components of the exceptional divisors of the resolution are $(-2)$-curves. We write

$$\tilde{f}' : \tilde{S}' \to \mathbb{CP}_1 \tag{8}$$

for the composition $\tilde{S}' \to S' \xrightarrow{f'} \mathbb{CP}_1$. This is also an elliptic fibration, but this time $\tilde{S}'$ is smooth. Since all singularities of $S'$ are rational double points, fibers of $\tilde{f}'$ are still anti-canonical curves on $\tilde{S}'$, and therefore we have $K^2 = 0$ for the surface $\tilde{S}'$. By Hartogs theorem this readily means that any fiber of $\tilde{f}'$ does not contain a $(-1)$-curve. Namely the elliptic fibration $\tilde{f}'$ in (8) is relatively minimal.

Obviously, these constructions depend only on the choice of the 2-plane $P \subset \mathbb{CP}_4$, where the 2-plane has to satisfy the transversality for intersection with $S$.

But for the purpose of calculating the degrees of the dual varieties of the Segre surfaces, we need to choose the plane $P$ more carefully to make singular fibers of the elliptic fibration (7) or equivalently (8) in most generic forms. For this purpose, we show

**Lemma 2.9.** *There exists a plane $P \subset \mathbb{CP}_4$ such that the (singular) elliptic fibration $f' : S' \to \mathbb{CP}_1$ induced by $P$ as in (7) satisfies the following properties.*

   (i) *Two singularities of the surface $S'$ do not belong to the same fiber of $f'$.*

   (ii) *Any singular fiber of $f'$ on which a singularity of $S'$ belongs has no singularity other than that singularity of $S'$.*

   (iii) *If a singular fiber of $f'$ has no singularity of $S'$ on it, the fiber is of type $I_1$.*

**Proof.** This is a problem of the existence of a line in the dual space $\mathbb{CP}_4^*$ which defines a pencil on $S$ whose associated morphism $f' : S' \to \mathbb{CP}_1$ satisfies the three genericity conditions in the lemma.

First let $\mathscr{S}$ be the set of hyperplanes in $\mathbb{CP}_4$ such that the sections of $S$ by the hyperplanes are singular. By Bertini's theorem $\mathscr{S}$ is a strict subvariety of the dual space $\mathbb{CP}_4^*$. Obviously the dual variety $S^*$ is an irreducible component of the subvariety $\mathscr{S}$. Also if $p_1, \dots, p_k$ are all singularities of $S$, the dual hyperplanes $p_1^*, \dots, p_k^* \subset \mathbb{CP}_4^*$, which are the sets of hyperplanes in $\mathbb{CP}_4$ which pass through the singular points $p_1, \dots, p_k$ respectively, are components of $\mathscr{S}$. To show that these are all components of $\mathscr{S}$, let $H \subset \mathbb{CP}_4$ be any hyperplane such that $S_H := S \cap H$ is singular, and suppose that $H$ does not belong to $S^* \cup (p_1^* \cup \cdots \cup p_k^*)$. If there is a singularity, say $p$, of $S_H$ which is a smooth point of $S$, we have $T_p S \subset H$ and from the definition of the dual variety $S^*$, this means $H \in S^*$ which contradicts our choice of $H$. Hence, all singularities of $S_H$ have to be singular points of $S$. This implies $H \in p_i^*$ for some $i$, and again this cannot happen from our choice of $H$. Hence we have

$$\mathscr{S} = S^* \cup \left( p_1^* \cup \cdots \cup p_k^* \right). \tag{9}$$

For a portion of the locus in the dual space $\mathbb{CP}_4^*$ which should be avoided from a line to pass, for any different indices $i, j \leq k$, we put $P_{ij} := p_i^* \cap p_j^*$. This is a 2-plane in $\mathbb{CP}_4^*$ and is the space of hyperplanes which pass through $p_i$ and $p_j$. When the surface $S$ has at most one singularity, we do not need these in the following argument. In particular, for any $H \in P_{ij}$, the section $S_H$ has singularities least at $p_i$ and $p_j$. For each index $i \leq k$, let $D_i$ be the subset of the dual hyperplane $p_i^*$ such that if $H \in D_i$, the section $S_H$ has a singularity other than $p_i$. We show that for each $i = 1, \dots, k$,

$$\overline{D}_i = (S^* \cap p_i^*) \cup \left( \bigcup_{j \neq i} P_{ij} \right) \tag{10}$$

holds, where $\overline{D}_i$ is the closure of $D_i$ in $\mathbb{CP}_4^*$. The inclusion '⊃' is obvious. For the reverse inclusion, take any $H \in D_i$ which does not belong to $P_{ij}$ for any $j \neq i$. Then the section $S_H$ has a singularity which is necessarily a smooth point of $S$. This means $H \in S^*$ and hence, $H \in S^* \cap p_i^*$. Therefore, $D_i$ is included in RHS

of (10). Since $\overline{D}_i$ is obviously a subvariety of $\mathbb{CP}_4^*$ and in particular closed, this implies the inclusion '$\subset$' in (10). Thus, the equality (10) holds.

Next, as in Definition 2.1, let $W_0$ be the subset of $S^*$ formed by hyperplanes $H \subset \mathbb{CP}_4$ such that $S \cap H$ is a minitwistor line (with one node in the present situation). From the definition of a minitwistor line, the node is not a singularity of $S$. We denote $A := S^* \backslash W_0$. Since $W_0$ is Zariski-open in $W = S^*$ and nonempty, $A$ is a strict subvariety of $S^*$. In particular any irreducible component of $A$ is at most 2-dimensional.

Finally, we consider the set of 2-planes in $\mathbb{CP}_4$ which do not intersect transversally with $S$. This is a strict subvariety of the Grassmanian of 2-planes in $\mathbb{CP}_4$. Let $T$ be the complement of this subvariety in the Grassmanian, and $T^*$ the subset of Grassmanian of lines in $\mathbb{CP}_4^*$ whose elements are lines which are dual to 2-planes belonging to $T$. This is a Zariski-open subset of the last Grassmanian. This finishes preliminary considerations.

Since any irreducible component of the subvarieties $A$ and $\overline{D}_1, \ldots, \overline{D}_k$ are at most 2-dimensional, by a dimensional reason, there exists a line $l \subset \mathbb{CP}_4^*$ which does not intersect any of these subvarieties. Moreover, the line $l$ can be taken from the subset $T^*$ since $T^*$ is Zariski-open in the Grassmanian of lines in $\mathbb{CP}_4^*$. Let $P \subset \mathbb{CP}_4$ be the 2-plane which is dual to $l$, and $f' : S' \to \mathbb{CP}_1 \simeq l$ the elliptic fibration determined by the 2-plane $P$ as in (7). Since $l \in T^*$, the intersection $S \cap P$ is transversal, and in particular we have $p_i \notin P$ for any $i = 1, \ldots, k$. For $i = 1, \ldots, k$, we put $H_i := l \cap p_i^*$. In other words, $H_i$ is a hyperplane in $\mathbb{CP}_4$ spanned by $P$ and the point $p_i$. Since $l \cap \overline{D}_i = \emptyset$, from (10) we have $l \cap P_{ij} = \emptyset$ for any $j \neq i$. This means $p_j \notin H_i$ for any $j \neq i$. Hence, for any $i = 1, \ldots, k$, on the section $S \cap H_i$ there is no singular point of $S$ other than the point $p_i \in S$. This means the property (i) in the lemma. Moreover, the assumption $l \cap \overline{D}_i = \emptyset$ means that the section $S \cap H_i$ does not have a singularity other than the point $p_i$. This means that the fibration $f'$ satisfies the property (ii) in the lemma.

Next, let $H \in l$ be a hyperplane on which no singularity of $S$ belongs. Because a hyperplane section $S_H$ is singular only when $H$ belongs to the subvariety $\mathscr{S}$, by (9) the fiber $(f')^{-1}(H) \simeq S_H$ is singular only when $H \in S^*$ or $H = H_i$ for some $i = 1, \ldots, k$. From the choice of $H$, the latter cannot occur. Moreover, since we have chosen a line $l$ which satisfies $l \cap A = \emptyset$, we have $H \notin A = S^* \backslash W_0$ and therefore if $H \in S^*$ then $H \in W_0$. Therefore, the section $S_H$ has a unique node as its only singularity and it is a smooth point of $S$. In particular the fiber $(f')^{-1}(H)$ is of type $\mathrm{I}_1$ and no singularity of $S'$ belongs to the same fiber. This means that the fibration $f'$ satisfies the property (iii) in the lemma. $\square$

By taking the minimal resolutions of all singularities for the elliptic surface $S'$ which satisfies the three properties in the lemma, we immediately obtain the following

**Corollary 2.10.** *Let $S \subset \mathbb{CP}_4$ be any Segre quartic surface and $p_1, \ldots, p_k$ the singularities of $S$. Then there exists a 2-plane $P \subset \mathbb{CP}_4$ which intersects $S$ transversally at four points and for which the induced elliptic fibration $\tilde{f}' : \tilde{S}' \to \mathbb{CP}_1$ in (8) satsisfies the following properties.*

- (i) *If $X_i$ is the dual graph of the exceptional curves of the singularity $p_i$, then the dual graph of the singular fiber which includes the exceptional curves of that singularity is the extended Dynkin diagram $\tilde{X}_i$.*
- (ii) *All other singular fibers of $\tilde{f}'$ are of type $I_1$.*

Now we are able to prove the class formula for Segre surfaces.

**Theorem 2.11.** *Let $S \subset \mathbb{CP}_4$ and $X_1, \ldots, X_k$ be as in the previous corollary, and $e_1, \ldots, e_k$ the topological Euler characteristics of the singular fibers of the elliptic fibration (8), so that the dual graphs of the fibers are of type $\tilde{X}_1, \ldots, \tilde{X}_k$ respectively. Then we have the formula*

$$\deg S^* = 12 - (e_1 + \cdots + e_k). \tag{11}$$

*In particular we have $\deg S^* = 12$ if the Segre surface $S$ is smooth.*

**Proof.** We take a 2-plane $P \subset \mathbb{CP}_4$ as in the previous corollary and let $\tilde{f}' : \tilde{S}' \to \mathbb{CP}_1$ be the associated elliptic fibration. Then the set of critical values of $\tilde{f}'$ consists of hyperplanes $H_i = l \cap p_i^*, 1 \leq i \leq k$, as well as the points corresponding to the singular fibers of type $I_1$. The singular fibers over the former kind of critical values are reducible since each of them includes the exceptional curves of the minimal resolution of the singularity as components. Hence the two kinds of critical values do not have a common point. Moreover, we have $H_i \notin S^*$ since if not, we have $H_i \in S^*$ but from our choice we have $l \cap A = \emptyset$ and therefore the hyperplane $H_i$ would belong to $W_0$, which contradicts $p_i^* \cap W_0 = \emptyset$. On the other hand, if $H \subset \mathbb{CP}_4$ is a hyperplane for which $S \cap H$ corresponds to a singular fiber of type $I_1$, the node of this singular fiber is a smooth point of $S$. This means that $H \in W_0$. These imply $l \cap S^* = l \cap W_0$, and that points belonging to the intersection $l \cap W_0$ are in one-to-one correspondence with the singular fibers of type $I_1$ of $\tilde{f}'$.

Since the elliptic surface $\tilde{S}'$ is rational and relatively minimal as seen before, the topological Euler characteristic of $\tilde{S}'$ is 12. Hence, from the additivity of the topological Euler characteristic to the elliptic fibration $\tilde{f}' : \tilde{S}' \to \mathbb{CP}_1$, we obtain that the number of singular fibers which are of type $I_1$ is exactly $12 - (e_1 + \cdots + e_k)$. From the conclusion in the first paragraph, this directly implies the desired equality (11). $\qquad\square$

## 3. The divisors at infinity in the dual varieties of Segre surfaces

**3.1. The Segre symbol.** As is already mentioned, any Segre quartic surface is a complete intersection of two quadrics in $\mathbb{CP}_4$. In general, complete intersections of two quadrics in $\mathbb{CP}_n$ for arbitrary $n$ can be systematically investigated by using so called the Segre symbol. In this section, following [5, Chapter XIII,

Section 10] and [2, Section 8.6], We first recall Segre symbol and next present a classification of Segre surfaces in terms of the Segre symbol.

Let $X_0, \dots, X_n$ be homogeneous coordinates on $\mathbb{CP}_n$. Any quadric in $\mathbb{CP}_n$ is defined as the zero locus of a quadratic form in $X_0, \dots, X_n$, and quadratic forms are in one-to-one correspondence with symmetric matrices of size $(n+1) \times (n+1)$ in a standard way. Let $Y$ be a complete intersection of two quadrics in $\mathbb{CP}_n$. We write $\mathcal{Q}$ for the pencil of quadrics generated by equations of $Y$. Let $U$ and $V$ be symmetric matrices which correspond to distinct two elements of $\mathcal{Q}$. Assume that some member (and hence a generic member) of $\mathcal{Q}$ is smooth. Then we may suppose $|V| \neq 0$. If $\lambda$ is an indeterminate, the determinant $|U - \lambda V|$ is a polynomial in $\lambda$ whose degree is precisely $n + 1$. Let

$$(\lambda - \alpha_1)^{e_1}, \ (\lambda - \alpha_2)^{e_2}, \ \dots, \ (\lambda - \alpha_s)^{e_s}$$

be all elementary divisors of the matrix $U - \lambda V$, so that $\alpha_1, \dots, \alpha_s$ are roots of the equation $|U - \lambda V| = 0$. Unlike the presentation in [2] we do not assume $\alpha_i \neq \alpha_j$ for $i \neq j$, but we put indices for the roots $\alpha_i$ in such a way that the same roots are adjacent in the sense that $\alpha_i = \alpha_j$ for some $i < j$ implies $\alpha_i = \alpha_{i+1} = \alpha_{i+2} = \cdots = \alpha_j$. From $|V| \neq 0$, we have

$$0 < s \leq n + 1 \quad \text{and} \quad \sum_{i=1}^{s} e_i = n + 1.$$

For a positive integer $e$, we define two $e \times e$ matrices by

$$P_e(\alpha) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \alpha \\ 0 & 0 & \dots & 0 & \alpha & 1 \\ 0 & 0 & \dots & \alpha & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \alpha & \cdots & 0 & 0 & 0 \\ \alpha & 1 & \dots & 0 & 0 & 0 \end{pmatrix}, \quad Q_e = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha$ is any complex number. Note that the elementary divisor of $P_e(\alpha) - \lambda Q_e$ is $(\lambda - \alpha)^e$. The matrices $U$ and $V$ can be simultaneously normalized in the sense of the theory of quadratic forms respectively into the matrices

$$\begin{pmatrix} P_{e_1}(\alpha_1) & 0 & \dots & 0 \\ 0 & P_{e_2}(\alpha_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{e_s}(\alpha_s) \end{pmatrix}, \quad \begin{pmatrix} Q_{e_1} & 0 & \dots & 0 \\ 0 & Q_{e_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_{e_s} \end{pmatrix}.$$

Under this situation, the *Segre symbol* for the complete intersection $Y$ is given by

$$[e_1 e_2 \dots e_s],$$

with the exception that if some of the roots $\alpha_i$ are equal, namely if $\alpha_i = \alpha_{i+1} = \alpha_{i+2} = \cdots = \alpha_j$ for some $i < j$ and if all other roots are different from this common number, then the sequence of the entries $e_i e_{i+1} e_{i+2} \dots e_j$ is enclosed by round brackets.

For example, if $n = 4$ and the Segre symbol is [2111], the elementary divisors of the matrix $U - \lambda V$ are $(\lambda - \alpha_1)^2, \lambda - \alpha_2, \lambda - \alpha_3$ and $\lambda - \alpha_4$ for some distinct numbers $\alpha_1, \dots, \alpha_4$, and we have $e_1 = 2, e_2 = e_3 = e_4 = 1$. Hence the symmetric matrices $U$ and $V$ are simultaneously normalized respectively into the matrices

$$\begin{pmatrix} 0 & \alpha_1 & 0 & 0 & 0 \\ \alpha_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & 0 & \alpha_4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the complete intersection $Y$ is defined by the equations

$$2\alpha_1 X_0 X_1 + X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2 + \alpha_4 X_4^2 = 2X_0 X_1 + X_2^2 + X_3^2 + X_4^2 = 0.$$

As the second example, if $n = 4$ and the Segre symbol is [32], all elementary divisors of the matrix $U - \lambda V$ are $(\lambda - \alpha_1)^3$ and $(\lambda - \alpha_2)^2$ for some distinct numbers $\alpha_1$ and $\alpha_2$, and we have $e_1 = 3, e_2 = 2$. The symmetric matrices $U$ and $V$ are simultaneously normalized respectively into the matrices

$$\begin{pmatrix} 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 1 & 0 & 0 \\ \alpha_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & 0 & \alpha_2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the complete intersection $Y$ is defined by

$$2\alpha_1 X_0 X_2 + \alpha_1 X_1^2 + 2X_1 X_2 + 2\alpha_2 X_3 X_4 + X_4^2 = 2X_0 X_2 + X_1^2 + 2X_3 X_4 = 0.$$

Instead, if the Segre symbol of $Y$ is [(32)], we just need to let $\alpha_1 = \alpha_2$ in this argument. But the case [(32)] does not give a Segre surface since it turns out from the normalized equations that it is a cone over a quartic curve.

A complete intersection of two quadrics in $\mathbb{CP}_4$ is smooth iff its Segre symbol is [11111]. A list of Segre symbols for complete intersections of two quadrics in $\mathbb{CP}_4$ which define Segre quartic surfaces and whose pencil $\mathscr{Q}$ of quadrics contains a smooth member is given in Dolgachev's book [2, p. 398], and they consist of 16 kinds as in the following list in terms of the number of distinct roots of the equation $|U - \lambda V| = 0$, or equivalently the number of singular members of the pencil $\mathscr{Q}$.

- [11111]
- [2111]   [(11)111]
- [(11)(11)1]   [(11)21]   [311]   [221]   [(12)11]
- [41]   [(31)1]   [3(11)]   [32]   [(12)2]   [(12)(11)]
- [5]   [(41)]

The above assumption on smoothness for members of the pencil $\mathscr{Q}$ is in effect not necessary (i.e. the above list covers all Segre quartic surfaces) by the following

**Proposition 3.1.** *If the pencil $\mathscr{Q}$ of quadrics in $\mathbb{CP}_4$ does not have a smooth member, the complete intersection defined by $\mathscr{Q}$ is not a Segre quartic surface.*

**Proof.** We use the notations and results in [5, Chap. XIII, Sect. 10 & 11]. Under the hypothesis on $\mathscr{Q}$, we have $|V| = 0$. We again let $(\lambda - \alpha_1)^{e_1}, \dots, (\lambda - \alpha_s)^{e_s}$ be the elementary divisors of the matrix $U - \lambda V$, where in the present situation $s = 0$ if $|U - \lambda V| = 0$. There exist integers $k > 0$ and $r_0 \geq 0$ as well as a collection of integers $r_1, \dots, r_k > 0$ which are determined from $U$ and $V$, such that the relation

$$\sum_{i=1}^{s} e_i = n + 1 - 2 \sum_{i=1}^{k} r_i - k - r_0 \tag{12}$$

holds ([5, p.290]). The integer $r_0$ is the number of variables among $X_0, \dots, X_n$ which do not appear in any equations of quadrics in $\mathscr{Q}$, and since Segre surfaces are not a cone we have $r_0 = 0$. Even so, since we have $n = 4$ (as we are considering complete intersections in $\mathbb{CP}_4$), if $k > 1$ would hold, RHS of (12) would be negative which is a contradiction. So we have $k = 1$, and the relation (12) becomes

$$\sum_{i=1}^{s} e_i = 4 - 2r_1. \tag{13}$$

Since LHS of this equation is non-negative, we have $r_1 = 1, 2$.

If $r_1 = 1$, we have $\sum_{i=1}^{s} e_i = 2$ and the possibilities for all elementary divisors of $U - \lambda V$ are

$$\begin{array}{lll} s = 1 & (\therefore e_1 = 2) & \Rightarrow & (\lambda - \alpha_1)^2, \\ s = 2 & (\therefore e_1 = e_2 = 1) & \Rightarrow & (\lambda - \alpha_1), \ (\lambda - \alpha_2). \end{array}$$

Segre symbols of these cases are respectively $[2; 1]$ if $s = 1$, and $[11; 1]$ or $[(11); 1]$ if $s = 2$ according as $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2$ respectively. (The numbers put after the semicolon are $r_1, \dots, r_k$ in general.) The normalized equations for these complete intersections are (see [5, p.294])

$$[2; 1] \Rightarrow 2X_0X_1 + 2\alpha_1 X_3X_4 + X_4^2 = 2X_1X_2 + 2X_3X_4 = 0,$$

$$[11; 1] \Rightarrow 2X_0X_1 + \alpha_1 X_3^2 + \alpha_2 X_4^2 = 2X_1X_2 + X_3^2 + X_4^2 = 0,$$

$$[(11); 1] \Rightarrow 2X_0X_1 + \alpha_1 X_3^2 + \alpha_1 X_4^2 = 2X_1X_2 + X_3^2 + X_4^2 = 0,$$

where $\alpha_1 \neq \alpha_2$ for the middle one. The first one is reducible since it contains the 2-plane $X_1 = X_4 = 0$. The last one is also reducible since it contains the 2-plane $X_1 = X_3 + iX_4 = 0$ (and also the plane $X_1 = X_3 - iX_4 = 0$). So these are not Segre surfaces. For the middle one it can be readily seen that the surface has singularities along the line $X_1 = X_3 = X_4 = 0$. Since any singularity of a Segre surface is isolated, this means that the surface is not a Segre quartic surface.

Finally, if $r_1 = 2$, the Segre symbol of the complete intersection is $[; 2]$, and the normalized equations are

$$2X_0X_1 + 2X_2X_3 = 2X_1X_2 + 2X_3X_4 = 0.$$

This is again reducible since it contains the plane $X_1 = X_3 = 0$, and hence does not give a Segre surface. Therefore, $|V| \neq 0$. $\qquad\square$

**3.2. Double covering structures on Segre quartic surfaces.** As explained in the previous subsection, the Segre symbol is useful for a systematic study of complete intersections of two quadrics. Especially, it provides us a pair of normalized forms for quadratic polynomials which define the surface. In this subsection by using the normalized equations, we see that most Segre quartic surfaces have a structure of a double covering over an irreducible quadric in $\mathbb{CP}_3$. This structure can be used to determine the types of singularities for most Segre surfaces. We note that in [2, p.401] the types of all singularities of any Segre surfaces are presented without a proof. The double covering structure will also be used in the next subsection to investigate the divisors at infinity for the Einstein-Weyl spaces associated to Segre surfaces.

We begin with an easy observation which is useful for our purpose.

**Proposition 3.2.** *Suppose that the symbol of a Segre quartic surface $S$ has at least one '1' not living in a pair of round brackets. Then by a generating projection from a point not lying on the surface, $S$ has a structure of a finite double covering over a smooth quadric in $\mathbb{CP}_3$, whose branch curve $B$ is a complete intersection of the smooth quadric and another quadric. Further, the Segre symbol of the curve $B$ is obtained from that of $S$ by just removing the '1'.*

For example, if the Segre symbol of a Segre surface $S$ is [1112], since this contains (indeed three) '1' and each of them do not live in a pair of round brackets, $S$ is a double cover over a smooth quadric surface whose branch is a complete intersection with another quadric, and the Segre symbol of the branch curve is [112]. On the contrary, if the Segre symbol of a Segre surface $S$ is [(11)(12)] for instance, then since all '1' in the symbol are living in a pair of round brackets, we cannot apply the proposition to such an $S$.

**Proof of Proposition 3.2.** The effect of changes of the order of the entry numbers in Segre symbol is just the exchanges for the variables in homogeneous coordinates on a projective space. Hence, we may suppose that the '1' corresponds to the first variable $X_0$. Since the '1' is not living in a pair of round brackets, the corresponding elementary divisor is of the form $\lambda - \alpha_1$ for some $\alpha_1 \in \mathbb{C}$. Let $F$ and $G$ be a pair of normalized quadratic polynomials of $S$ in these coordinates. From the above elementary divisor, monomials in $F$ and $G$ which contain the variable $X_0$ are (a constant multiple of) $X_0^2$ only, and we have

$$F = \alpha_1 X_0^2 + f, \quad G = X_0^2 + g, \tag{14}$$

where the residual quadratic polynomials $f$ and $g$ do not contain $X_0$. Let $\pi : \mathbb{CP}_4 \to \mathbb{CP}_3$ be the generic projection from the point $(1, 0, 0, 0, 0)$. Namely $\pi$ is the map which drops the coordinate $X_0$. Eliminating $X_0^2$ from the equations $F = G = 0$, we obtain
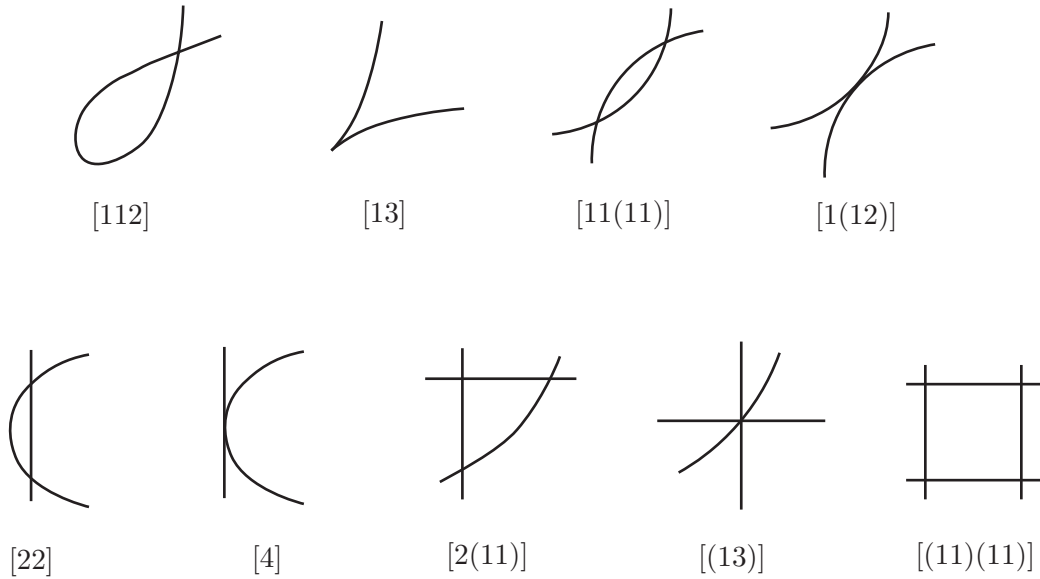
$$f - \alpha_1 g = 0. \tag{15}$$

This is a quadratic equation in $X_1, X_2, X_3, X_4$, and the corresponding symmetric $4 \times 4$ matrix is readily seen to be of full rank (i.e. rank 4) by using that $\alpha_1$ is different from other roots appearing in the polynomial $f$. Hence if $Q \subset \mathbb{CP}_3$ is the quadric which is defined by (15), it is smooth. From (14), the generating point of the projection $\pi$ is not on $S$, and the restriction $\pi|_S$ provides $S$ a structure of finite double cover over $Q$, with the branch divisor $B$ being defined by $f - \alpha_1 g = g = 0$. Hence the branch divisor is a complete intersection of two quadrics in $\mathbb{CP}_3$. The last equations are equivalent to $f = g = 0$, so we have $B = \{f = g = 0\}$. Moreover, obviously the two polynomials $f$ and $g$ are already in normalized forms, and the Segre symbol of the complete intersection $B$ is exactly the one given in the proposition. $\qquad\square$

When the symbol of a Segre surface is as in the proposition, principal structure of the surface may be read off from the structure of the branch curve $B$ in the smooth quadric $Q$. In turn structure of the curve $B$ is known from the Segre symbol of $B$, because the symbol provides normalized forms of the quadratic equations of $B$, and it is not difficult to obtain concrete structure from the equations, by projecting $B$ to $\mathbb{CP}_2$ from a point. Below we present a list of structure of $B$ which actually arise from Segre surfaces as in Proposition 3.2. These can also be found in [5, pp. 305–8] where arbitrary complete intersections of two quadrics in $\mathbb{CP}_3$ are treated. See also Figure 1 for these curves.

$[1111] \Rightarrow$ smooth elliptic curve, $[112] \Rightarrow$ 1-nodal rational curve, $[13] \Rightarrow$ 1-cuspidal rational curve, $[11(11)] \Rightarrow$ two conics intersecting transversally at two points, $[1(12)] \Rightarrow$ two conics touching at one point, $[22] \Rightarrow$ one line and one rational normal curve intersecting transversally at two points, $[4] \Rightarrow$ one line and one rational normal curve touching at one point, $[2(11)] \Rightarrow$ two lines and one conic, forming a 'triangle', $[(13)] \Rightarrow$ two lines and one conic, sharing one point, $[(11)(11)] \Rightarrow$ a 'square' of four lines.

From these, the types of all singularities of Segre surfaces whose symbols satisfy the property in Proposition 3.2 are obtained. We present them in the second column of Table 1. Then by using Theorem 2.11, the classes of these Segre surfaces are obtained, and they are listed in the third column in Table 1. The forth column of the table presents the numbers of '1' in the symbols which are not living in a pair of round brackets. These are in one-to-one correspondence with generating projections from a point which induces the double covering map to the smooth quadric surface as in Proposition 3.2. We will show in Section 3.3 that the dual quadric $Q^*$ of the smooth quadric $Q$ is always contained in the dual variety $S^*$, and this is why we are writing $Q^*$ in the table. The numbers of lines on $S$ are listed in the fifth column, and they are taken from [2, p. 401]. Among them the number of lines on $S$ which do not pass through any singularity of $S$ are listed in the sixth column. In Section 3.3 we will also show that the 2-plane in $\mathbb{CP}_4^*$ which is formed by hyperplanes in $\mathbb{CP}_4$ that contain such a line is always contained in $S^*$.

FIGURE 1. The branch curves $B$

| Segre symbol | Sing $S$ | deg $S^*$ | #$\{Q^* \subset S^*\}$ | #$\{\text{lines} \subset S\}$ | #$\{\mathbb{CP}_2 \subset S^*\}$ | $\text{Aut}_e S$ |
|:---:|:---:|:---:|:---:|:---:|:---:|:---:|
| $[11111]$ | none | 12 | 5 | 16 | 16 | $\{\text{id}\}$ |
| $[1112]$ | $A_1$ | 10 | 3 | 12 | 8 | $\{\text{id}\}$ |
| $[111(11)]$ | $2A_1$ | 8 | 3 | 8 | 0 | $\mathbb{C}^*$ |
| $[12(11)]$ | $3A_1$ | 6 | 1 | 6 | 0 | $\mathbb{C}^*$ |
| $[1(11)(11)]$ | $4A_1$ | 4 | 1 | 4 | 0 | $\mathbb{C}^* \times \mathbb{C}^*$ |
| $[113]$ | $A_2$ | 8 | 2 | 8 | 4 | $\{\text{id}\}$ |
| $[122]$ | $2A_1$ | 9 | 1 | 9 | 4 | $\{\text{id}\}$ |
| $[11(12)]$ | $A_3$ | 8 | 2 | 4 | 0 | $\{\text{id}\}$ |
| $[14]$ | $A_3$ | 8 | 1 | 5 | 2 | $\mathbb{C}^*$ |
| $[1(13)]$ | $D_4$ | 6 | 1 | 2 | 0 | $\mathbb{C}^*$ or $\{\text{id}\}$ |

TABLE 1. Segre quartic surfaces which are realizable as a dou-
ble cover over a smooth quadric surface

Next, we discuss the case where the Segre symbol contains '1' but where it
lives in a pair of round brackets. Then the situation is little different from that
in the previous proposition, but it happens to be still simple.

**Proposition 3.3.** *Suppose that the Segre symbol of a Segre quartic surface S in-
cludes '1' which is enclosed by a pair of round brackets. Then by a generating pro-
jection from a point not lying on the surface, S has a structure of a finite double
covering over the cone over an irreduble conic, and the branch curve is a complete*

*intersection of the cone with another quadric. Further, the Segre symbol of the branch curve is obtained from that of S by removing the '1' and next removing the pair of round brackets in which the '1' lives.*

For example, the symbol [(11)12] (resp. [(14)]) enjoys the assumption of the proposition, and the symbol of the branch curve on the cone is [112] (resp. [4]). Combining with Proposition 3.2, this means that if the symbol has two '1', and if one of them is included in a pair of round brackets while the other one is not (like [(11)12]), then $S$ is a double cover of a smooth quadric and also is a double cover of the cone over an irreducible conic.

**Proof of Proposition 3.3.** From the list for the Segre symbols of all Segre surfaces given right before Proposition 3.1, the Segre symbols in the situation of the present proposition are of the forms $[(11)\boldsymbol{e}], [(12)\boldsymbol{e}']$ for some $\boldsymbol{e}$ and $\boldsymbol{e}'$, or exactly one of $[(13)1]$ and $[(14)]$. We verify the proposition on case-by-case basis depending on these distinctions.

For the first case (in which the symbol is of the form $[(11)\boldsymbol{e}]$), $\boldsymbol{e}$ is one of

$$111, \ 12, \ 3, \ (11)1, \ (12). \tag{16}$$

The pair of normalized symmetric matrices which define the surface $S$ are of the forms

$$U = \left( \begin{array}{cc|c} \alpha_1 & 0 & \mathbf{0} \\ 0 & \alpha_1 & \\ \hline \mathbf{0} & & U' \end{array} \right), \quad V = \left( \begin{array}{cc|c} 1 & 0 & \mathbf{0} \\ 0 & 1 & \\ \hline \mathbf{0} & & V' \end{array} \right), \tag{17}$$

where $U'$ and $V'$ are $2 \times 2$ matrices which are determined from $\boldsymbol{e}$. In terms of these symmetric matrices, an elimination of the variable $X_0$ (or equivalently $X_1$) from the pair of the quadratic equations corresponds to considering the matrix $U - \alpha_1 V$, and we obtain a quadratic polynomial in $X_2, X_3, X_4$. The $3 \times 3$ symmetric matrix corresponding to this polynomial is exactly $U' - \alpha_1 V'$, and it is not difficult to see from the list (16) using $\alpha_i \neq \alpha_j$ for $i \neq j$, that this matrix is of full rank. So the quadratic polynomial defines an irreducible conic in $\mathbb{CP}_2$. The point $(1, 0, 0, 0, 0)$ is again not on $S$, and if $\pi : \mathbb{CP}_4 \to \mathbb{CP}_3$ is the projection from this point, the image $\pi(S) \subset \mathbb{CP}_3$ is contained in the cone over the last conic. The restriction $\pi|_S : S \to \pi(S)$ is of degree-two over the image, so $\pi(S)$ is 2-dimensional. Hence we obtain that the image $\pi(S)$ is the cone over the conic. Moreover, we find that the branch divisor of $\pi|_S$ is a complete intersection of two quadrics in $\mathbb{CP}_3$ whose symmetric matrices are

$$\begin{pmatrix} 0 & 0 \\ 0 & U' - \alpha_1 V' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & V' \end{pmatrix}. \tag{18}$$

Again, by using $\alpha_1 \neq \alpha_i$ if $i > 1$, it is readily seen (again by case-by case checking relying on (16)) that this pair is already in normal forms, and that the corresponding Segre symbol is exactly $[1\boldsymbol{e}]$, the first '1' being corresponding to the $1 \times 1$-components of the upper-left in the matrices (18). Thus, the Segre symbol of the branch curve of $\pi|_S$ is certainly as in the proposition.

Next, if the Segre symbol of $S$ is of the form $[(12)\boldsymbol{e}']$ for some $\boldsymbol{e}'$, all the possibilities for $\boldsymbol{e}'$ are $11, (11)$ and $2$. Then we have

$$
U = \left(\begin{array}{ccc|c}
\alpha_1 & 0 & 0 & \\
0 & 0 & \alpha_1 & 0 \\
0 & \alpha_1 & 1 & \\
\hline
& 0 & & U'
\end{array}\right), \quad
V = \left(\begin{array}{ccc|c}
1 & 0 & 0 & \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & \\
\hline
& 0 & & V'
\end{array}\right), \qquad (19)
$$

where $U'$ and $V'$ are two symmetric matrices that are determined from $\boldsymbol{e}'$. By the same procedure as in the last case of $[(11)\boldsymbol{e}]$, we obtain that the projection $\pi : \mathbb{CP}_4 \to \mathbb{CP}_3$ from the point $(1, 0, 0, 0, 0) \notin S$ induces a double covering map to the cone over an irreducible conic, and the branch curve is a complete intersection of two quadrics in $\mathbb{CP}_3$ defined by two symmetric matrices

$$
\left(\begin{array}{cc|c}
0 & 0 & \\
0 & 1 & 0 \\
\hline
& 0 & U' - \alpha_1 V'
\end{array}\right) \quad \text{and} \quad
\left(\begin{array}{cc|c}
0 & 1 & 0 \\
1 & 0 & \\
\hline
& 0 & V'
\end{array}\right). \qquad (20)
$$

Again, by using $\alpha_1 \neq \alpha_i$ for any $i > 1$, we can see that this pair is already in normal forms, and the corresponding Segre symbol is $[2\boldsymbol{e}']$, where the first '2' being corresponding to the $2 \times 2$ matrices in the upper-left in the matrices (20). Thus, the Segre symbol of the branch curve of $\pi|_S$ is again as in the proposition.

The cases where the Segre symbol of $S$ is $[(13)1]$ or $[(14)]$ can be shown in the same manner, and we omit them. □

When Proposition 3.3 can be applied to a Segre surface $S$, since the double covering map is finite, $S$ always has singularity over the vertex of the cone. Let $v$ be the vertex of the cone and $B$ the branch divisor on the cone. If $v \notin B$, then the surface $S$ has two $A_1$-singularities over $v$. If $v \in B$, then the surface $S$ has exactly one singularity over $v$. Since the curve $B$ can have a singularity at a smooth point of the cone, $S$ can have other singularity in general, but again the types of them are known from that of the branch curve $B$. When $v \in B$, the type of the singularity of $S$ over $v$ is known by transforming $B$ through blowing up at $v$, and noticing that the exceptional curve of the blowup is always included in the branch locus of the new double covering. This way the types of all singularities of $S$ are again known, and we display them in Table 2. Again, the classes of the surfaces are obtained from Theorem 2.11, and we list them in the third column. Positioning of the vertex $v$ and the branch curve $B$ is shown in the fifth column. We note that it is possible to show that any line on $S$ passes through a singularity of $S$ for any Segre surface $S$ which appears in Table 2, and therefore we obtain no plane in the dual variety $S^*$ in the situation of Proposition 3.3. Also, we note that $[(11)(12)]$ and $[(12)(11)]$ represent Segre surfaces of the same kind, and we are taking different projections to the cone.

Evidently at least one of Propositions 3.2 and 3.3 is applied if the Segre symbol contains at least one '1'. Among 16 symbols for Segre quartic surfaces, there are

| Segre symbol | Sing $S$ | deg $S^*$ | #{lines $\subset S$} | the vertex $v$ | Aut$_e S$ |
|---|---|---|---|---|---|
| $[(11)111]^*$ | $2A_1$ | 8 | 8 | $v \notin B$ | $\mathbb{C}^*$ |
| $[(12)11]^*$ | $A_3$ | 8 | 4 | $v$ is a node of $B$ | $\{id\}$ |
| $[(11)12]^*$ | $3A_1$ | 6 | 6 | $v \notin B$ | $\mathbb{C}^*$ |
| $[(11)(11)1]^*$ | $4A_1$ | 4 | 4 | $v \notin B$ | $\mathbb{C}^* \times \mathbb{C}^*$ |
| $[(11)3]$ | $2A_1 + A_2$ | 5 | 4 | $v \notin B$ | $\mathbb{C}^* \times \mathbb{C}^*$ |
| $[(13)1]^*$ | $D_4$ | 6 | 2 | $v$ is a cusp of $B$ | $\mathbb{C}^*$ or $\{id\}$ |
| $[(12)2]$ | $A_1 + A_3$ | 6 | 3 | $v$ is a node of $B$ | $\mathbb{C}^*$ |
| $[(11)(12)]$ | $2A_1 + A_3$ | 4 | 2 | $v$ is a node of $B$ | $\mathbb{C}^* \times \mathbb{C}^*$ |
| $[(12)(11)]$ | $2A_1 + A_3$ | 4 | 2 | $v \notin B$ | $\mathbb{C}^* \times \mathbb{C}^*$ |
| $[(14)]$ | $D_5$ | 5 | 1 | Sing $B = \{v\}$ | $\mathbb{C}^*$ or $\{id\}$ |

TABLE 2. Segre quartic surfaces which are realizable as a double cover of the cone over an irreducible conic. Symbols which appeared in Table 1 are associated by '$*$'.

exactly two symbols which do not contain any '1', and they are

$$[23] \quad \text{and} \quad [5]. \tag{21}$$

According to [2, Table 8.6], the former surface has exactly two singularities and they are of types $A_1$ and $A_2$, and the latter surface has exactly one singularity, which is of type $A_4$. For these Segre surfaces, we cannot obtain a double covering structure over a quadric by a generic projection from a point which does not belong to $S$. Similarly to the previous two tables, we display principal structures of these surfaces in Table 3.

| Segre symbol | Sing $S$ | deg $S^*$ | #{lines $\subset S$} | #{$\mathbb{CP}_2 \subset S^*$} | Aut$_e S$ |
|---|---|---|---|---|---|
| $[23]$ | $A_1 + A_2$ | 7 | 6 | 2 | $\mathbb{C}^*$ |
| $[5]$ | $A_4$ | 7 | 3 | 1 | $\mathbb{C}^*$ |

TABLE 3. Segre quartic surfaces which cannot be realized as a double cover over a quadric surface

## 3.3. Self-intersection loci of the dual varieties.

Recall that if $S \subset \mathbb{CP}_4$ is a Segre quartic surface, the dual variety $S^* \subset \mathbb{CP}_4^*$ contains a Zariski-open subset $W_0$ parameterizing 1-nodal rational curves that are contained in $S_{\text{reg}}$, the smooth locus of $S$, and $W_0$ is a complex 3-manifold which has an Einstein-Weyl structure (see Definition 2.3 and Proposition 2.5). The subset $W_0$ cannot coincide with the full set $S^*$ since 1-nodal rational curves can always be deformed in $S$ to a 2-nodal (and hence reducible) curve or a 1-cuspidal rational curve. So the complement $S^* \backslash W_0$ is always a non-empty subvariety in $S^*$. This subvariety may be regarded as the natural boundary set or a kind of conformal

infinity of the complex Einstein-Weyl space $W_0$. As in the introduction, we call 2-dimensional components of $S^* \backslash W_0$ *divisors at infinity*. In this subsection we find several components of the divisors at infinity in concrete forms, and show that the dual varieties have self-intersection along these divisors.

In order to find and describe divisors at infinity, we first make use of lines lying on a Segre surface. Let $S \subset \mathbb{CP}_4$ be a Segre quartic surface and $l$ a line on $S$. The number of lines on Segre surfaces are presented in Tables 1–3. We write $l^*$ for the set of hyperplanes in $\mathbb{CP}_4$ which contain the line $l$. So $l^*$ is a 2-plane in the dual space $\mathbb{CP}_4^*$. If $H \in l^*$, the line $l$ is contained in the hyperplane section $S \cap H$, and since $S$ is of degree four, the section is reducible. This means that the section $S \cap H$ is not a minitwistor line. So it does not belong to the open subset $W_0$ of $S^*$. But we have

**Proposition 3.4.** *Suppose that the line $l \subset S$ does not pass through any singularity of $S$. Then the 2-plane $l^* \subset \mathbb{CP}_4^*$ is contained in the dual variety $S^*$. In particular, it is a divisor at infinity.*

For the proof of the proposition, we use the following three lemmas, for which we omit the proofs as they can be shown in a standard way.

**Lemma 3.5.** *Let $S \subset \mathbb{CP}_4$ be a Segre quartic surface. Then any 2-plane in $\mathbb{CP}_4$ does not contain a curve on $S$ whose degree in $\mathbb{CP}_4$ is greater than two.*

**Lemma 3.6.** *If a Segre surface $S \subset \mathbb{CP}_4$ contains an irreducible conic $C$ which does not pass through any singularity of $S$, we have $C^2 = 0$ on $S$.*

**Lemma 3.7.** *Let $S \subset \mathbb{CP}_4$ be a Segre surface. For each singularity of $S$, conics on $S$ which pass through that singularity constitute at most 1-dimensional family.*

By using these three lemmas, we next show

**Lemma 3.8.** *A generic secant $\overline{pq}$, $p, q \in S$, of any Segre quartic surface $S \subset \mathbb{CP}_4$ has the properties that it does not intersect any line on $S$, and that any 2-plane which contains $\overline{pq}$ does not include any curve on $S$.*

**Proof.** By Lemmas 3.6 and 3.7, any component of the variety of conics in $\mathbb{CP}_4$ which are contained in $S$ is at most 1-dimensional. A generic element $C$ of this variety defines a 2-plane $P \subset \mathbb{CP}_4$ by the condition $C \subset P$. Hence the variety of conics on $S$ is identified with the variety of 2-planes in $\mathbb{CP}_4$ which contains a conic on $S$. We then define $V$ to be the variety consisting of lines in $\mathbb{CP}_4$ which are contained in some 2-plane belonging to the last variety of 2-planes. In other words, $V$ is the set of lines lying on the same 2-plane in $\mathbb{CP}_4$ as some conic on $S$. Then obviously each component of $V$ is at most 3-dimensional. Moreover, if $W$ denotes the variety formed by secants of the surface $S$, then we have $\dim W = 4$, and moreover $V \subset W$.
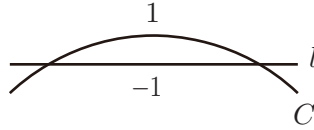
Next let $K$ be the set of secants of $S$ which intersect some line in $S$. Evidently $K$ is a 3-dimensional subvariety of $W$. By a dimensional reason, we may choose a secant $\overline{pq} \in W$ which does not belong to the subvariety $V \cup K$. Then $\overline{pq} \cap l = \emptyset$ for any line $l \subset S$. Let $P \subset \mathbb{CP}_4$ be any 2-plane containing $\overline{pq}$. By Lemma 3.5,

$P$ does not contain any curve on $S$ whose degree is greater than two. Moreover from $\overline{pq} \notin V$, the 2-plane $P$ does not contain any conic on $S$. $\qquad\square$

**Proof of Proposition 3.4.** Let $l$ be a line on a Segre surface $S \subset \mathbb{CP}_4$. Let $H$ be a generic hyperplane which contains $l$. We first show that there exists a secant of $S$ which is included in $H$ and which satisfies the property in Lemma 3.8. Since $S$ is quartic we have $S \cap H = l \cup C$, where $C$ is a cubic curve. By Lemma 3.5, $C$ is non-degenerate in the sense that it is not contained in a 2-plane. Any secant of $S$ included in $H$ is a secant of the hyperplane section $S \cap H = l \cup C$. Any secant of the cubic curve $C$ is of course such a secant, and since $C$ is a curve they constitute 2-dimensional family of secants of $S$. Because $C$ is non-degenerate as above, generic two distinct secants in the family do not intersect. If a generic secant of $S$ in $H$ does not satisfy the property in Lemma 3.8, then to each secant of $C$ we have a conic in $S$ which lies on the same 2-plane as the secant. These mean that there exists a 2-dimensional family of conics in $S$, which contradicts Lemmas 3.6 and 3.7. Therefore, there is a secant $\overline{pq}$ of $S$ which is contained in $H$ and which satisfies the property in Lemma 3.8. Obviously, we can choose such a secant which does not pass through the line $l$. Then the hyperplane $H$ is spanned by $l$ and $\overline{pq}$.

We fix any one of such secants $\overline{pq} \subset H$, and let $\varpi : \mathbb{CP}_4 \to \mathbb{CP}_2$ be the projection from the line $\overline{pq}$. Fibers of $\varpi$ are 2-planes which contain $\overline{pq}$. We use the same letter $\varpi$ for the restriction of the projection to the surface $S$. Then $\varpi : S \to \mathbb{CP}_2$ has the two points $p$ and $q$ as the set of indeterminacy, and if $S'$ is the blowing-up of $S$ at these two points, then the composition $S' \to S \xrightarrow{\varpi} \mathbb{CP}_2$ is a morphism. We write $\varpi'$ for this morphism. This is the anti-canonical map from $S'$. Since $S$ is of degree four, the morphism $\varpi'$ is of degree two. Moreover, since there exists no 2-plane in $\mathbb{CP}_4$ which contains a curve on $S$, the degree-two morphism $\varpi' : S' \to \mathbb{CP}_2$ does not contract any curve on $S'$. Namely $\varpi'$ is a finite morphism. Since $K_S^2 = \deg S = 4$, we have $K_{S'}^2 = 2$. From these we readily obtain that the branch divisor of $\varpi' : S' \to \mathbb{CP}_2$ is a quartic curve.

Now assume that the line $l$ on $S$ does not to pass through any singularity of $S$. Then $l$ is a $(-1)$-curve on $S$. Let $l'$ be the strict transform of $l$ into $S'$. Then since the secant $\overline{pq}$ does not intersect $l$ from our choice, $l'$ is still a $(-1)$-curve. Hence we have $K_{S'}^{-1}.l' = 1$. So the image $\varpi'(l') = \varpi(l) \subset \mathbb{CP}_2$ is a line, and $l'$ is isomorphic to this line by $\varpi'$. Further the pullback $(\varpi')^{-1}(\varpi'(l'))$ is an anti-canonical curve on $S'$. On the other hand, since the hyperplane $H$ contains the center of the projection $\varpi$, the image $\varpi(H)$ is also a line, and as $l \subset H$, we have $\varpi'(l') = \varpi(H)$. If the line $\varpi'(l')$ would be contained in the branch quartic of $\varpi'$, the branch curve is singular at some point on the line $\varpi'(l')$. Since $\varpi'$ does not contract any curve on $S'$, this means that there would exist a singular point of $S'$ on $l'$. Hence $S$ would have a singularity at some point of the line $l$, which contradicts the assumption on $l$. Therefore, the line $\varpi'(l')$ is not contained in the branch quartic of $\varpi'$. Hence, the curve $(\varpi')^{-1}(\varpi'(l'))$ on $S'$ has a component other than the line $l$ which is mapped to the bitangent

$\varpi'(l')$ isomorphically. Since the branch curve is of degree four, this can happen exactly when the line $\varpi'(l') = \varpi(l)$ is a bitangent of the branch quartic. Now we move the line along the branch quartic in a way that it is a regular tangent of the branch quartic. The preimages of such a tangent under $\varpi$ and $\varpi'$ are still anti-canonical curves on $S$ and $S'$ respectively, and generically have respectively an ordinary double point over the tangent point as their only singularity. This means that the preimage to $S$ of the tangent is a minitwistor line on $S$. Therefore, the curve $\varpi^{-1}(\varpi(l))$ on $S$ is a limit of minitwistor lines on $S$. Since $S^*$ is closed and $\varpi^{-1}(\varpi(l)) = H \cap S$, this means $H \in S^*$. Hence $l^* \subset S^*$, as desired.                                                                        □

From the proof, it is not difficult to obtain the following

**Proposition 3.9.** *If $l$ is a line on a Segre surface $S \subset \mathbb{CP}_4$ which does not pass through any singularity of $S$ as in Proposition* 3.4, *for a generic hyperplane $H$ which contains $l$, the section $S \cap H$ is as in Figure* 3.3, *where the numbers are self-intersection numbers in $S$.*

**Proof.** We keep the notations in the previous proof. The image $\varpi(l) = \varpi'(l')$ was a bitangent of the branch curve of the double covering $\varpi' : S' \to \mathbb{CP}_2$, and the preimage $(\varpi')^{-1}(\varpi'(l'))$ is of the form $l' + C'$ for some smooth rational curve $C' \subset S'$ which is mapped isomorphically to the bitangent by $\varpi'$. Moreover the intersection $l' \cap C'$ consists of two points, which are over the tangent points of the bitangent, and both intersections are transversal. Let $C$ be the image of $C'$ by the blow-down $S' \to S$. Since $S|_H = l + C$, $C$ is a cubic curve and by rationality it is a rational normal curve in $H$. Also $C$ is contained in the smooth locus of $S$, and the intersection $C \cap l$ consists of two points and both intersections are transversal. For finishing a proof of the proposition, since $l$ is a $(-1)$-curve, it remains to see that $C^2 = 1$. Since $C + l$ is a hyperplane section of $S$, we have $(C + l)^2 = \deg S = 4$. Moreover we have $C.l = 2$ as above. This means $C^2 = 1$, as desired.                                                                        □

Next we find another type of divisors at infinity. For this purpose we use the double covering structure over a smooth quadric obtained in Proposition 3.2. So let $S$ be any one of Segre surfaces listed in Table 1. As in the proof of Proposition 3.2 we denote $\pi : \mathbb{CP}_4 \to \mathbb{CP}_3$ for the generating projection from a point that induces the double covering map from $S$ to a smooth quadric $Q \subset \mathbb{CP}_3$. We denote the generating point of $\pi$ and its dual hyperplane respectively by

$$w \quad \text{and} \quad w^*. \tag{22}$$

In the coordinates of the proof of Proposition 3.2, $w = (1, 0, 0, 0, 0)$. The projection $\pi$ induces the dual inclusion

$$\pi^* : \mathbb{CP}_3^* \hookrightarrow \mathbb{CP}_4^*, \tag{23}$$

and we have $\pi^*(\mathbb{CP}_3^*) = w^*$. In the following we often think subvarieties in $\mathbb{CP}_3^*$ as those in $\mathbb{CP}_4^*$ through this inclusion.

If $B \subset \mathbb{CP}_3$ is an irreducible curve and is not a straight line, the dual variety $B^* \subset \mathbb{CP}_3^*$ is defined in the same way to the case for the surface $S$ explained in Section 2, and it is irreducible and 2-dimensional. If $B \subset \mathbb{CP}_3$ is reducible and does not have a straight line as a component, we define the dual variety $B^*$ to be the union of the dual varieties of all its irreducible components. If $B$ is reducible and have a straight line as a component, we define $B^*$ to be the union of the dual varieties of all its irreducible components which are not a straight line. So $B^*$ is empty if all components of $B$ are lines.

**Proposition 3.10.** *Assume that a Segre surface $S$ is a finite double cover over a smooth quadric surface $Q$ by a projection from a point of $\mathbb{CP}_4$ as in Proposition 3.2. Let $w \in \mathbb{CP}_4$ and $w^* \subset \mathbb{CP}_4^*$ be as in (22). Then if $B \subset Q$ denotes the branch divisor of the double cover, the hyperplane section by $w^*$ of the dual variety $S^*$ satisfies*
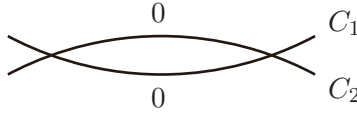
$$Q^* \cup B^* \subset S^* \cap w^*, \tag{24}$$

*where $Q^*$ and $B^*$ are the dual varieties of $Q$ and $B$ respectively.*

**Proof.** The inclusion $Q^* \cup B^* \subset w^*$ is obvious because we are using the inclusion (23), so it is enough to show $Q^* \cup B^* \subset S^*$.

First, we show $B^* \subset S^*$. From the above convention for the dual varieties, it is enough to show that for any irreducible component $B_1$ of $B$ which is not a straight line, the inclusion $B_1^* \subset S^*$ holds. In the following for simplicity we write $B$ for $B_1$. Then since $B$ is not a line, as a curve on $Q \simeq \mathbb{CP}_1 \times \mathbb{CP}_1$, it is not a curve of bidegree $(1, 0)$ nor $(0, 1)$. Therefore, a smooth generic point $q$ of $B$ satisfies $T_q B \not\subset Q$. For such a point $q \in B$, the intersection of $Q$ with a 2-plane $h \subset \mathbb{CP}_3$ containing $T_q B$ is a smooth $(1, 1)$-curve on $Q$ unless $h = T_q Q$. Moreover except for a finite number of such 2-planes, the intersection $h \cap B$ consists of three points, and one of them is the tangent point $q$, while the other two intersections are transversal. If we put $H := \pi^{-1}(h)$, then the hyperplane section $S \cap H$ has a node at the point $(\pi|_S)^{-1}(q)$ because the curve $Q \cap h$ is tangent to $B$ at $q$, and it has no other singularity because $h$ intersects transversally at the other two intersection points with $B$. This means $H \in W_0$, where as before $W_0$ is the Zariski-open subset of $S^*$ which parameterizes minitwistor lines (see Definition 2.3 and Proposition 2.5). Hence, for a generic $h \in B^*$, we have $\pi^{-1}(h) \in S^*$. Since $B^*$ is closed, this implies $B^* \subset S^*$.

It remains to show $Q^* \subset S^*$. So let $h \subset \mathbb{CP}_3$ be a tangent plane to $Q$, and $q$ the tangent point. The section $Q \cap h$ consists of two lines intersecting transversally at the point $q$. Let $l_1$ and $l_2$ be these lines. We choose the tangent plane $h$ in such a generic way that $q \notin B$ and that the two lines $l_1$ and $l_2$ intersect the

branch divisor $B$ transversally. Since $B$ is a cut of $Q$ by a quadric by Proposition 3.2, the intersections $B \cap l_1$ and $B \cap l_2$ consist of two points respectively. From these the preimages

$$C_1 := (\pi|_S)^{-1}(l_1) \quad \text{and} \quad C_2 := (\pi|_S)^{-1}(l_2) \tag{25}$$

are smooth rational curves on $S$, and they intersect transversally at the two points $(\pi|_S)^{-1}(q)$. We write $p_1$ and $p_2$ for these two points, and put $H = \pi^{-1}(h)$. The latter is a hyperplane spanned by the generating point $w$ of the projection $\pi$ and the tangent plane $h$, and contains the tangent spaces $T_{p_1}S$ and $T_{p_2}S$ since $T_qQ = h$ and $q \notin B$. Obviously we have $H|_S = C_1 + C_2$, $C_1 \cap C_2 = \{p_1, p_2\}$. Moving the tangent point $q \in Q$, we obtain a 2-dimensional family of reducible curves belonging to the system $\left|\mathcal{O}_{\mathbb{CP}_4}(1)|_S\right| = \left|K_S^{-1}\right|$, whose members have two nodes as their only singularity. In particular, $H \notin W_0$.

In order to show $Q^* \subset S^*$ it is enough to show that the curve $C_1 + C_2$ is a limit of curves which belong to the Zariski-open subset $W_0$ of $S^* = W$. Since $l_1^2 = l_2^2 = 0$ on $S$, from (25), we have $C_1^2 = C_2^2 = 0$ on $S$. Hence by adjunction we have $K_S . C_1 = K_S . C_2 = -2 < 0$. From [11, Prop. (2.11)], this means that any one of the two nodes $p_1$ and $p_2$ of the curve $C_1 + C_2$ can be smoothed out under a small displacement by moving the curve on $S$ while the other node is not smoothed out. Clearly the curve obtained as such a partial smoothing of the curve $C_1 + C_2$ belongs to the subset $W_0$ of $S^*$. This means that $C_1 + C_2$ is a limit of curves which belong to $W_0$, and so we obtain $Q^* \subset S^*$, as desired.  $\square$

Note that from the proof, generic points of $B^*$ belong to $W_0$. Therefore, $W_0$ is not entirely contained in the affine space $\mathbb{CP}_4^* \backslash w^* (\simeq \mathbb{C}^4)$. The following proposition is also obvious from the proof.

**Proposition 3.11.** *In the situation of the previous proposition, for a generic hyperplane $H$ that belongs to the dual quadric $Q^*$, the section $S \cap H$ is as in Figure 3.3.*

Thus, we have found two kinds of divisors at infinity, and both consist of two smooth rational curves intersecting transversally at two points. These two kinds of curves are distinguished by the self-intersection numbers of the two components.

We next show that the dual variety $S^*$ intersects itself along the dual plane $l^*$ and the dual quadric $Q^*$. More precisely, we show

**Proposition 3.12.** *Let $l \subset S$ and $Q \subset \mathbb{CP}_3$ be as in Propositions 3.4 and 3.10 respectively. Then the dual variety $S^*$ has ordinary double points along the dual plane $l^*$ and the dual quadric $Q^*$.*

**Proof.** We use results and notations in [11]. (In [11] the letter $D$ is used instead of $C$.) The proofs for $l^*$ and $Q^*$ are almost the same, so for a generic hyperplane $H$ belonging to $l^*$ or $Q^*$ we write $H|_S = C = C_1 + C_2$ and $C_1 \cap C_2 = \{p_1, p_2\}$. By choosing $H \in l^* \cup Q^*$ in a sufficiently generic way, we may suppose that $C \cap \operatorname{Sing} S = \emptyset$. Put $N_C := \mathscr{O}_C(C)$. This is an invertible sheaf on the curve $C$, and there is a natural isomorphism

$$T_H |\mathscr{O}_{\mathbb{CP}_4}(1)| \simeq H^0(N_C) \simeq \mathbb{C}^4.$$

Moreover from the exact sequence

$$0 \longrightarrow \mathscr{O}_S \longrightarrow \mathscr{O}_S(C) \longrightarrow N_C \longrightarrow 0 \tag{26}$$

and rationality of $S$, we have $H^1(N_C) \simeq H^1(\mathscr{O}_S(C))$. Let $\tilde{S} \to S$ be the minimal resolution of all singularities of $S$ and $\tilde{C}$ the strict transform of $C$ into $\tilde{S}$. Then $\tilde{C}$ is an anti-canonical curve on $\tilde{S}$, and it is nef and big. Hence by Kodaira-Ramanujan vanishing, we have $H^1(\mathscr{O}_{\tilde{S}}(\tilde{C})) = 0$. So as $C \cap \operatorname{Sing} S = \emptyset$, we have $H^1(\mathscr{O}_S(C)) = 0$, and hence from (26) we obtain $H^1(N_C) = 0$.

Next write $T_C^1 := \mathscr{E}xt_{\mathscr{O}_C}^1(\Omega_C, \mathscr{O}_C)$, where $\Omega_C$ is the sheaf of Kähler differentials on $C$. Then from the standard exact sequence $0 \to \mathscr{O}_S(-C) \xrightarrow{d} \Omega_S|_C \to \Omega_C \to 0$, there is a natural homomorphism $N_C \to T_C^1$. Since $C$ is normal crossing at $p_1$ and $p_2$, we have $T_C^1 \simeq \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}$ and since $S$ is smooth at these points, the homomorphism $N_C \to T_C^1$ is surjective. Let $N_C'$ be the kernel sheaf of this homomorphism, so that we have an exact sequence

$$0 \longrightarrow N_C' \longrightarrow N_C \longrightarrow T_C^1 \longrightarrow 0. \tag{27}$$

The Zariski-tangent space at the point $[C]$ of the locus in the linear system $|K_S^{-1}| \simeq |\mathscr{O}_{\mathbb{CP}_4}(1)|$ which corresponds to equisingular displacements of $C$ is given by $H^0(N_C')$, and the obstruction for smoothness of this locus is in $H^1(N_C')$.

Now in case $H \in l^*$, namely in case $l \subset H$, as in the proof of Proposition 3.4, the image $\varpi(l)$ is a bitangent of the branch quartic of the double cover $\varpi' : S' \to \mathbb{CP}_2$ obtained by choosing a secant $\overline{pq}$ as in Lemma 3.8, and the two singularities $p_1$ and $p_2$ of the curve $C$ are mapped to the tangent points of the bitangent. Evidently there are two ways of moving the bitangent to a regular tangent by respecting the tangency at $p_1$ or $p_2$. Accordingly, the reducible curve $C = C_1 + C_2$ admits a displacement in $S$ for which exactly any one of the singularities $p_1$ and $p_2$ is smoothed out. This means that in case $H \in l^*$ the natural map $H^0(N_C) \to H^0(T_C^1) \simeq \mathbb{C}^2$ is surjective. In case $H \in Q^*$, as in the last part of the proof of Proposition 3.10, the curve $C$ admits a displacement in $S$ which induces a smoothing for exactly any one of the singularities $p_1$ and $p_2$. Hence the natural map $H^0(N_C) \to H^0(T_C^1) \simeq \mathbb{C}^2$ is again surjective. Hence, regardless of $H \in l^*$ or $H \in Q^*$, from (27) and $H^1(N_C) = 0$, we obtain $H^1(N_C') = 0$. We also obtain $h^0(N_C') = h^0(N_C) - h^0(T_C^1) = 4 - 2 = 2$.

Finally, we define the sheaves $N_C''(p_1)$ and $N_C''(p_2)$ on $C$ by the properties

$$N_C''(p_1)|_{C\setminus\{p_1\}} \simeq N_C|_{C\setminus\{p_1\}} \quad \text{and} \quad N_C''(p_1)|_{C\setminus\{p_2\}} \simeq N_C'|_{C\setminus\{p_2\}},$$
$$N_C''(p_2)|_{C\setminus\{p_2\}} \simeq N_C|_{C\setminus\{p_2\}} \quad \text{and} \quad N_C''(p_2)|_{C\setminus\{p_1\}} \simeq N_C'|_{C\setminus\{p_1\}}.$$

Since $N_C$ and $N_C'$ are canonically isomorphic over $C\setminus\{p_1, p_2\}$, these two sheaves are well-defined. Note that the these sheaves are slightly larger than $N_C'$, and if $\{i, j\} = \{1, 2\}$, we have exact sequences

$$0 \longrightarrow N_C' \longrightarrow N_C''(p_i) \longrightarrow \mathbb{C}_{p_j} \longrightarrow 0. \tag{28}$$

As in [11, Remark (1.7)], for $i = 1, 2$, the space $H^0(N_C''(p_i))$ is the Zariski-tangent space at the point $[C]$ of the locus in $|\mathscr{O}_S(C)|$ which corresponds to equisingular displacements of $C$, where equisingularity is imposed only for the node $p_i$. Moreover the space $H^1(N_C''(p_i))$ is the obstruction space for such displacements. We then have natural inclusions

$$H^0(N_C') \subset H^0(N_C''(p_i)) \subset H^0(N_C), \quad i = 1, 2, \tag{29}$$

and the codimensions of the two inclusions are both one. Since the nodes $p_1$ and $p_2$ can be independently smoothed out as above, we have $H^0(N_C''(p_1)) \neq H^0(N_C''(p_2))$ as subspaces in $H^0(N_C)$. Hence we obtain the transversality

$$H^0(N_C''(p_1)) \cap H^0(N_C''(p_2)) = H^0(N_C'). \tag{30}$$

Since $H^1(N_C') = 0$ as above, from (28), we obtain $H^1(N_C''(p_i)) = 0$ for $i = 1, 2$. Thus, the above two equisingular displacements of $C$ in $S$ are unobstructed, and each partially equisingular deformations constitute a smooth threefold in the linear system $|\mathscr{O}_S(C)|$. By the transversality (30), these two components intersect transversally along the locus of equisingular displacements of $C$, where this time equisingularity is imposed on both $p_1$ and $p_2$. Evidently the last locus is identified with a neighborhood of the point $H$ in the dual 2-plane $l^*$ or the dual quadric $Q^*$. Therefore, the dual variety $S^*$ has ordinary double points along $l^*$ or $Q^*$. $\qquad\square$

When the quadric $\pi(S) \subset \mathbb{CP}_3$ is not smooth but the cone over an irreducible conic as in Proposition 3.3, we write $\Lambda$ for an irreducible conic, and $\mathrm{Cone}(\Lambda) \subset \mathbb{CP}_3$ for the cone over $\Lambda$. Also, similarly to the notation in Proposition 3.10, we write $w$ for the generic point of the projection which induces the double covering map $S \to \mathrm{Cone}(\Lambda)$, and $w^* \subset \mathbb{CP}_4^*$ for the dual hyperplane to $w$. Then the following proposition can be shown in the same way to the assertion $B^* \subset S^* \cap w^*$ in Proposition 3.10.

**Proposition 3.13.** *Assume that a Segre surface $S$ is a finite double cover over* $\mathrm{Cone}(\Lambda)$ *by a projection from the point $w \in \mathbb{CP}_4$ as in Proposition 3.3. Then if* $B \subset \mathrm{Cone}(\Lambda)$ *denotes the branch divisor of the double cover, then the hyperplane section by $w^*$ of the dual variety of $S$ satisfies*

$$B^* \subset S^* \cap w^*. \tag{31}$$

## 4. Examples of transitions and concluding remarks

**4.1. Typical transitions between Segre quartic surfaces.** Next, we discuss adjacent relations (or degeneration relations) between some Segre quartic surfaces. First, the transitions $[1111] \to [112] \to [13]$ for complete intersections of two quadrics in $\mathbb{CP}_3$ provide a standard degeneration of a smooth elliptic curve into cuspidal rational curve via a nodal rational curve, and accordingly, through the double cover, the transitions $[11111] \to [1112] \to [113]$ for Segre surfaces provide modest degenerations of the surfaces. From Table 1, in this degeneration, the classes of the Segre surfaces decrease as $12 \to 10 \to 9$, while those of the branch curves can be seen to decrease as $8 \to 6 \to 5$. Thus, all decreases of the classes of Segre surfaces come from those of the branch curves.

For another typical degenerations of complete intersections of two quadrics in $\mathbb{CP}_3$, we take

$$[112] \to [11(11)] \to [2(11)] \to [(11)(11)]. \tag{32}$$

These are 1, 2, 3 and 4-nodal curves respectively, and the numbers of irreducible components of the curves are 1,2,3 and 4 respectively. The first one is a 1-nodal rational curve, and the second one consists of two conics (i.e. $(1, 1)$-curves in $Q \simeq \mathbb{CP}_1 \times \mathbb{CP}_1$) intersecting transversally at two points. The third one consists of two lines and one conic, and they form a 'triangle' of smooth rational curves. The final one consists of four lines, forming a 'square' of rational curves. Using that the dual variety of a smooth conic in $\mathbb{CP}_3$ is the cone over the conic and in particular quadratic, we obtain that the degrees of the dual varieties of these quartic curves are respectively $6$, $2 + 2 = 4$, $2$ and $0$. Adding a single '1' to each symbol in the series (32), we obtain a series of Segre surfaces whose symbols are

$$[1112] \to [111(11)] \to [12(11)] \to [1(11)(11)]. \tag{33}$$

From the above description of the quartic curves, these Segre surfaces have one, two, three and four $A_1$-singularities respectively, and have no other singularities. Hence by Theorem 2.11, the classes of these surfaces are $10, 8, 6, 4$ respectively. Thus, again the decreases of the classes of the surfaces exactly come from those of the branch quartic curves.

For one more interesting example of a degeneration, we take the transition $[22] \to [4]$ for complete intersections in $\mathbb{CP}_3$. Both curves consist of one rational normal curve and one line in $\mathbb{CP}_3$, but for the symbol $[22]$ the two components intersect transversally at two points, while for the symbol $[4]$ the two components are tangent at one point. If we consider the transition $[122] \to [14]$ for Segre surfaces, the singularities are two $A_1$ for the former and a single $A_3$ for the latter, Hence, from Theorem 2.11, for both kinds of surfaces, we have $\deg S^* = 12 - 4 = 8$ (see Table 1). Thus, no decrease occurs for the classes in this degeneration.

As a final example of a degeneration, we consider the transition $[(11)3] \to [(13)1]$ for Segre surfaces. Both of these kinds of Segre surfaces have a structure

of double covering over a quadratic cone as in Proposition 3.13. The branch curves have [13] as the Segre symbol for both, and this is a cuspidal rational curves. The vertex $v$ of the cone is on the branch curve only for the latter kind of surfaces. So certainly, the latter kind of surfaces is obtained from the former kind of surfaces as a degeneration. The former kind of surface has two $A_1$-singularities and one $A_2$-singularity, and hence the class is $12 - 2 - 2 - 3 = 5$ by Theorem 2.11. On the other hand, the latter kind of surfaces have exactly one singularity which is of type $D_4$ (This is over the vertex of the cone.) Hence, we have $\deg S^* = 12 - 6 = 6$ by Theorem 2.11. Thus, the class increases under this degeneration, The added one is a consequence of the fact that, the surface $[(13)1]$ includes the the plane $v^*$ formed by hyperplanes passing through the vertex $v$ is included in $S$, while the surface $[(11)3]$ does not. In other words, in the case of $[(13)1]$, a hyperplane section of $S$ which passes through the $D_4$-singularity admits a displacement which avoids the singularity and which gives a 1-nodal curve which is really a minitwistor line. This is in contrast with the situation mentioned in Remark 2.6.

**4.2. Concluding remarks.** We end this article by giving two remarks. The first one is about a relation between null surfaces in Einstein-Weyl space associated to the Segre surfaces and their divisors at infinity. For this, we recall that each Segre quartic surface $S \subset \mathbb{CP}_4$ has a structure of a minitwistor space in the sense of Definition 2.1 and therefore the 3-dimensional complex manifold $W_0$ which parameterizes minitwistor lines in $S$ has an Einstein-Weyl structure. Many Segre surfaces admit double covering structure over a smooth quadric $Q$ as in Proposition 3.2. As in Propositions 3.10 and 3.12. the dual quadric $Q^*$, considered as included in the dual space $\mathbb{CP}_4^*$ via the generic projection, is contained in the completion $W = S^*$ of $W_0$ as a self-intersection locus. We see that *the completions of null surfaces in these Einstein-Weyl spaces $W_0$ are always tangent to the dual quadrics $Q^* \subset S^*$.*

Before doing so, we first consider the case where the minitwistor space is a smooth quadric $Q \subset \mathbb{CP}_3$. In this case, a hyperplane $H \subset \mathbb{CP}_3$ cuts out a minitwistor line in the original sense iff it is not tangent to $Q$. Tangent planes to $Q$ are in one-to-one correspondence with its tangent point of $Q$, and thus a hyperplane section $H \subset \mathbb{CP}_3$ does not cut out a minitwistor line iff $H \in Q^*$, where $Q^* \subset \mathbb{CP}_3^*$ is the dual quadric. This means $W_0^* = \mathbb{CP}_3^* \backslash Q^*$. So we call $Q^*$ a quadric at infinity of $W_0 = \mathbb{CP}_3^* \backslash Q^*$. Null surfaces in the Einstein-Weyl space $W_0 = \mathbb{CP}_3^* \backslash Q^*$ are nothing but the intersections with the dual planes $p^* \subset \mathbb{CP}_3^*$, where $p$ is chosen from the minitwistor space $Q$. For such a dual plane $p^*$, the intersection $p^* \cap Q^*$ consists of the two lines consisting of planes in $\mathbb{CP}_3$ which is tangent to $Q$ at some point on the two lines that pass through the point $p$. This implies that the completion of any null surface in the Einstein-Weyl space $\mathbb{CP}_3^* \backslash Q^*$ is tangent to the infinite quadric $Q^*$.

Returning to the case of genus one, let $S$ be a Segre quartic surface which has a structure of double covering over a smooth quadric $Q \subset \mathbb{CP}_3$ as in Proposition 3.2. As before let $\pi : \mathbb{CP}_4 \to \mathbb{CP}_3$ be the generic projection from the point

$w \notin S$ which induces the double covering map. By the dual embedding $\pi^*$ : $\mathbb{CP}_3^* \hookrightarrow \mathbb{CP}_4^*$ we identify the dual quadric $Q^* \subset \mathbb{CP}_3^*$ with its image $\pi^*(Q^*)$. We call this a quadric at infinity. Again null surfaces in $W_0$ are of the form $p^* \cap W_0$, where $p^* \subset \mathbb{CP}_4^*$ is the dual hyperplane to the point $p$ chosen from $S$. Since the completion $W$ of $W_0$ contains the dual quadric $\pi^*(Q^*)$, this means that the intersections of the completions of the null surfaces with the quadric at infinity are of the form

$$p^* \cap \pi^*(Q^*) = \left\{ \pi^{-1}(h) \mid h \in Q^*,\, \pi(p) \in h \right\}, \quad p \in S. \tag{34}$$

The RHS can be identified with the union of two lines on $Q \subset \mathbb{CP}_3$ which pass through the point $\pi(p)$. Of course, these two lines intersect transversally at the point $\pi(p)$. This means that the hyperplane $p^* \subset \mathbb{CP}_4^*$ and the completion of the null surface $p^* \cap W_0$ is tangent to the quadric $\pi^*(Q^*)$ at the point $\pi^{-1}(T_{\pi(p)}Q) \in \pi^*(Q^*)$.

The second remark is about a relationship between Segre quartic surfaces and twistor spaces associated to self-dual metrics on 4-manifolds. By Jones-Tod [9], a 3-dimensional Einstein-Weyl manifold is obtained from a self-dual 4-manifold as a quotient space under an action of 1-dimensional Lie group preserving the self-dual structure. Correspondingly, a minitwistor space is obtained from the twistor space of a self-dual 4-manifold as a quotient space with respect to the holomorphic action of 1-dimensional complex Lie group. Among the 16 kinds of the Segre surfaces, only the ones whose symbol is [111(11)] are known to be obtained this way. There, the source 4-manifolds are the connected sums of arbitrary number of complex projective planes, and their twistor spaces were constructed in [6]. It might be an interesting question as to whether other Segre surfaces can be obtained as quotient spaces of compact twistor spaces.

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