Zeroth $\mathbb{A}^1$-homology of smooth proper varieties

Junnosuke Koizumi

ABSTRACT. We give an explicit formula for the zeroth $\mathbb{A}^1$-homology sheaf of a smooth proper variety. We also provide a simple proof of a theorem of Kahn-Sujatha which describes hom sets in the birational localization of the category of smooth varieties.

CONTENTS

1. Introduction

We fix a base field k. The sheaf of birational connected components $\pi_0^{br}(X)$ of a smooth proper variety X over k was introduced by Asok-Morel [AM11] to study $\mathbb{A}^1$-connected components of X.[1] It is a Nisnevich sheaf on the category of smooth schemes over k equipped with a morphism of sheaves $X \to \pi_0^{br}(X)$ which induces an isomorphism $\pi_0^{br}(X)(U) \cong X(k(U))/R$ for a smooth variety U over k, where $X(k(U))/R$ is the set of naive $\mathbb{A}^1$-homotopy classes of morphisms $\operatorname{Spec} k(U) \to X$ [AM11, Theorem 6.2.1].

On the other hand, the $\mathbb{A}^1$-homology sheaves $H_n^{\mathbb{A}^1}(X)$ of a smooth variety X over k were introduced by Morel [Mor05] as an $\mathbb{A}^1$-homotopy theoretic analogue of the homology groups of a topological space. When X is proper over k, the canonical morphism $X \to \pi_0^{br}(X)$ induces a morphism $H_0^{\mathbb{A}^1}(X) \to \mathbb{Z}(\pi_0^{br}(X))$ by the universal property of $H_0^{\mathbb{A}^1}(X)$ (see Lemma 4.3). Asok-Haesemeyer used this to prove that $H_0^{\mathbb{A}^1}(X)$ detects existence of a rational point on X [AH11, Corollary 2.9]. In this paper we refine this result:

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[1]It is written $\pi_0^{b\mathbb{A}^1}(X)$ in loc.cit.

Theorem 1.1 (see Theorem 4.5). *For any smooth proper variety X over a field k, the canonical morphism $X \to \mathbb{Z}(\pi_0^{br}(X))$ induces an isomorphism $H_0^{\mathbb{A}^1}(X) \cong \mathbb{Z}(\pi_0^{br}(X))$. In particular, $H_0^{\mathbb{A}^1}(X)(k)$ is a free abelian group of rank $\#X(k)/R$.*

Under resolution of sigularities, Theorem 1.1 was proved by Shimizu [Shi20, Theorem 4.1], and was also established in unpublished work of Hogadi. We do not assume resolution of sigularities here. The key of our proof is the following universal property of $\pi_0^{br}(X)$:

Theorem 1.2 (see Theorem 2.4). *For any smooth proper variety X over a field k, the canonical morphism $X \to \pi_0^{br}(X)$ is initial among morphisms from X to $\mathbb{A}^1$-invariant unramified sheaves over k. Moreover, if $X \to Y$ is a birational morphism between smooth proper varieties over k, then the induced morphism $\pi_0^{br}(X) \to \pi_0^{br}(Y)$ is an isomorphism.*

Section 1 contains a proof of this result. From Theorem 1.2 we also see that the unstable analogue $\pi_0^{\mathbb{A}^1}(X) \cong \pi_0^{br}(X)$ of Theorem 1.1 does not hold in general; there is an example of a birational morphism $X \to Y$ between smooth proper varieties such that $\pi_0^{\mathbb{A}^1}(X) \to \pi_0^{\mathbb{A}^1}(Y)$ is not an isomorphism (see [BHS15, Example 4.8]). We also need the following result:

Theorem 1.3 (see Theorem 3.7). *Let G be a strongly $\mathbb{A}^1$-invariant Nisnevich sheaf of groups on $\mathbf{Sm}_k$ (i.e. G and $H^1(-, G)$ are $\mathbb{A}^1$-invariant). Then its classifying space $\mathcal{B}G$ is $\mathbb{A}^1$-local.*

This theorem is stated in [Mor12] without proof, but requires a nontrivial computation of the homotopy groups of $\mathcal{B}G$. This is done in Section 2. Note that we only use the case where G is abelian and one can give a shorter proof in that case, but we record the general statement for future use.

Theorem 1.2 has another application. Let S_b denote the class of birational morphisms in the category $\mathbf{Sm}_k^{\mathrm{conn}}$ of smooth varieties over a field k. Using Theorem 1.2, we can provide a simple proof of the following result of Kahn-Sujatha [KS15, Theorem 6.6.3]:

Theorem 1.4 (see Theorem 4.7). *Let k be a field, $X, U \in \mathbf{Sm}_k^{\mathrm{conn}}$ and suppose that X is proper. Then there is a bijection*

$$\mathrm{Hom}_{S_b^{-1}\mathbf{Sm}_k^{\mathrm{conn}}}(U, X) \cong X(k(U))/R$$

which is compatible with the canonical maps from $X(U)$ to both sides.

Notations. For a scheme S, a *smooth scheme* over S is always assumed to be of finite type and separated over S. A *smooth variety* over S is a connected smooth scheme over S. Let $\mathbf{Sm}_S$ (resp. $\mathbf{Sm}_S^{\mathrm{conn}}$) denote the category of smooth schemes (resp. smooth varieties) over S.

The word "sheaf" always means sheaf of sets in Nisnevich topology. Let $\mathbf{Sh}(\mathbf{Sm}_S)$ (resp. $\mathbf{Ab}(\mathbf{Sm}_S)$) denote the cateogory of sheaves (resp. sheaves of abelian groups) on $\mathbf{Sm}_S$. For $X \in \mathbf{Sm}_S$ and $M \in \mathbf{Ab}(\mathbf{Sm}_S)$, $H^n(X, M)$ denotes

the n-th Nisnevich cohomology group. For $X \in \mathbf{Sm}_k$ and $n \geq 0$, we write $X^{(n)}$ for the set of points of codimension n on X.

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2. Birational sheaves and unramified sheaves

First we recall the notion of birational sheaves defined in [AM11, Definition 6.1.1]. A presheaf S on $\mathbf{Sm}_k$ is called a *birational sheaf* if for any $X \in \mathbf{Sm}_k$ the restriction map $S(X) \to \prod_{\eta \in X^{(0)}} S(\eta)$ is a bijection; any birational sheaf is automatically a sheaf [AM11, Lemma 6.1.2].

Lemma 2.1. *Let S be a birational sheaf. Then S is $\mathbb{A}^1$-invariant; i.e. the canonical map $S(X) \to S(\mathbb{A}^1_X)$ is bijective for all $X \in \mathbf{Sm}_k$.*

Proof. This is [KS15, Appendix A] but we spell out the proof for the convinience of readers. Let $X \in \mathbf{Sm}_k$ and denote by $q \colon \mathbb{P}^1_X \to X$ the canonical projection. It suffices to show that $q^* \colon S(X) \to S(\mathbb{P}^1_X)$ is bijective. The injectivity can be seen by taking a section of q. Let us prove the surjectivity. Let W be the blow-up of $\mathbb{P}^1_X \times_X \mathbb{P}^1_X$ along $(\infty, \infty) \cong X$. Let $i \colon \mathbb{P}^1_X \to \mathbb{P}^1_X \times_X \mathbb{P}^1_X$ be the morphism $x \mapsto (x, \infty)$, and $\widetilde{i} \colon \mathbb{P}^1_X \to W$ the unique lift of i. Consider the commutative diagram

$$S(\mathbb{P}^1_X \times_X \mathbb{P}^1_X) \xrightarrow{i^*} S(\mathbb{P}^1_X).$$
$$\downarrow \qquad \nearrow \widetilde{i}^*$$
$$S(W)$$

Since i has a retraction, i^* is surjective and hence so is $\widetilde{i}^*$. Now let $\pi \colon W \to \mathbb{P}^2_X$ be the unique morphism extending the open immersion $\mathbb{A}^1_X \times_X \mathbb{A}^1_X \to \mathbb{P}^2_X; (x, y) \mapsto (x : y : 1)$. It fits in the following commutative diagram in $\mathbf{Sm}_k$:

$$\begin{array}{ccc} \mathbb{P}^1_X & \xrightarrow{\widetilde{i}} & W \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{(0:1:0)} & \mathbb{P}^2_X. \end{array}$$

Since π^* is bijective by assumption and $\widetilde{i}^*$ is surjective, we get the surjectivity of q^*. $\square$

Let $\mathcal{F}_k$ denote the category of finitely generated separable field extensions of k. Let X be a smooth proper variety over k and $F \in \mathcal{F}_k$. We say that two

F-points $b, b' \in X(F)$ are *naively $\mathbb{A}^1$-homotopic* or *R-equivalent* if there is a collection $\{\gamma_i : \mathbb{A}^1_F \to X\}_{i=1}^N$ of morphisms such that $\gamma_1(0) = b$, $\gamma_i(1) = \gamma_{i+1}(0)$ and $\gamma_N(1) = b'$. Let $X(F)/R$ denote the set of R-equivalence classes of F-points.

Lemma 2.2. *For any smooth proper variety X over k, there is a birational sheaf $\pi_0^{br}(X)$ together with a morphism $X \to \pi_0^{br}(X)$ such that for any smooth variety U there is a bijection $\pi_0^{br}(X)(U) \cong X(k(U))/R$ which is compatible with the canonical maps from $X(U)$ to both sides.*

Proof. See [AM11, Theorem 6.2.1]. $\square$

Next we recall from [Mor12] the notion of unramified sheaves. A sheaf S on $\mathbf{Sm}_k$ is called an *unramified sheaf* if the following conditions hold:

  (i) For any $X \in \mathbf{Sm}_k^{conn}$ the restriction map $S(X) \to S(k(X))$ is injective.
  (ii) For any $X \in \mathbf{Sm}_k^{conn}$ we have $S(X) = \bigcap_{x \in X^{(1)}} S(\mathcal{O}_{X,x})$ as subsets of $S(k(X))$.

Let $\mathbf{Sh}^{ur}(\mathbf{Sm}_k)$ denote the full subcategory of $\mathbf{Sh}(\mathbf{Sm}_k)$ consisting of unramified sheaves.
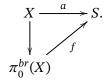
Let $\mathcal{V}_k$ denote the class of all pairs (F, v) where $F \in \mathcal{F}_k$ and v is a discrete valuation on F such that there are some $X \in \mathbf{Sm}_k$ and $x \in X^{(1)}$ such that $\mathcal{O}_v \cong \mathcal{O}_{X,x}$. An *unramified $\mathcal{F}_k$-datum* is a functor $S : \mathcal{F}_k \to \mathbf{Set}$ together with a subset $S(\mathcal{O}_v) \subset S(F)$ and a specialization map $s_v : S(\mathcal{O}_v) \to S(k(v))$ for each $(F, v) \in \mathcal{V}_k$ satisfying some compatibility conditions (see [Mor12, Definition 2.9] for details). A morphism of $\mathcal{F}_k$-data $f : S \to S'$ is a natural transformation satisfying $f_F(S(\mathcal{O}_v)) \subset S'(\mathcal{O}_v)$ for every $(F, v) \in \mathcal{V}_k$ and compatible with specialization maps. Let $\mathbf{Data}^{ur}$ denote the category of unramified $\mathcal{F}_k$-data.

Lemma 2.3. *The restriction functor $\mathbf{Sh}^{ur}(\mathbf{Sm}_k) \to \mathbf{Data}^{ur}$ is an equivalence of categories.*

Proof. See [Mor12, Theorem 2.11]. $\square$

Theorem 2.4. *For any smooth proper variety X over k, the morphism $X \to \pi_0^{br}(X)$ is initial among morphisms from X to $\mathbb{A}^1$-invariant unramified sheaves. Moreover, if $X \to Y$ is a birational morphism between smooth proper varieties over k, then the induced morphism $\pi_0^{br}(X) \to \pi_0^{br}(Y)$ is an isomorphism.*

Proof. Firstly, $\pi_0^{br}(X)$ itself is $\mathbb{A}^1$-invariant by Lemma 2.1 and unramified by definition. For any $\mathbb{A}^1$-invariant unramified sheaf S and $a \in S(X)$, we prove that there exists a unique morphism of sheaves $f : \pi_0^{br}(X) \to S$ which makes the following diagram commute:

$$
\begin{array}{ccc}
X & \xrightarrow{\ a\ } & S. \\
\downarrow & \nearrow{\scriptstyle f} & \\
\pi_0^{br}(X) & &
\end{array}
$$

Since S is unramified, commutativity of the above diagram is equivalent to that of

$$
\begin{array}{ccc}
X(F) & \xrightarrow{\;a_F\;} & S(F) \\
\downarrow & \nearrow{\scriptstyle f_F} & \\
X(F)/R =\!\!=\!\!= \pi_0^{br}(X)(F) & &
\end{array}
$$

for each $F \in \mathscr{F}_k$. This proves the uniqueness of f_F and hence the uniqueness of f.

We prove the existence of f. First we construct f_F for each $F \in \mathscr{F}_k$ so that the above diagram becomes commutative. Consider the following commutative diagram for $\varepsilon = 0, 1$:

$$
\begin{array}{ccc}
X(\mathbb{A}_F^1) & \xrightarrow{\;a_{\mathbb{A}_F^1}\;} & S(\mathbb{A}_F^1) \\
{\scriptstyle i_\varepsilon^*}\downarrow & & \downarrow{\scriptstyle i_\varepsilon^*} \\
X(F) & \xrightarrow{\;a_F\;} & S(F).
\end{array}
$$

Here i_ε denote the F-valued point $F \to \mathbb{A}_F^1$ with coordinate ε. Since S is $\mathbb{A}^1$-invariant we have $i_0^* = i_1^* : S(\mathbb{A}_F^1) \to S(F)$. This shows that a_F descends to R-equivalence classes and gives a map $f_F : \pi_0^{br}(X)(F) \to S(F)$ for each $F \in \mathscr{F}_k$.

Next we have to show that $\{f_F\}_{F \in \mathscr{F}_k}$ extends to a morphism of sheaves. Since $\pi_0^{br}(X)$ and S are both unramified, it suffices to check that $\{f_F\}_{F \in \mathscr{F}_k}$ gives a morphism in $\mathbf{Data}^{ur}$ by Lemma 2.3. Let $(F, v) \in \mathcal{V}_k$. Consider the following diagram:

$$
\begin{array}{ccccc}
X(\mathcal{O}_v) & \longrightarrow & \pi_0^{br}(X)(\mathcal{O}_v) & \xrightarrow{\;a_{\mathcal{O}_v}\;} & S(\mathcal{O}_v) \\
\downarrow & & {\scriptstyle \cong}\downarrow & & \downarrow \\
X(F) & \longrightarrow\!\!\!\rightarrow & \pi_0^{br}(X)(F) & \xrightarrow{\;f_F\;} & S(F).
\end{array}
$$

The left square and the total rectangle are commutative. By the valuative criterion of properness, the left vertical map is bijective. Hence we have

$$
f_F(\pi_0^{br}(X)(\mathcal{O}_v)) \subset S(\mathcal{O}_v),
$$

and the map $X(\mathcal{O}_v) \to \pi_0^{br}(X)(\mathcal{O}_v)$ is surjective. Consider the following diagram:

$$
\begin{array}{ccccc}
X(\mathcal{O}_v) & \longrightarrow\!\!\!\rightarrow & \pi_0^{br}(X)(\mathcal{O}_v) & \xrightarrow{\;f_F\;} & S(\mathcal{O}_v) \\
\downarrow & & \downarrow & & \downarrow \\
X(k(v)) & \longrightarrow & \pi_0^{br}(X)(k(v)) & \xrightarrow{\;f_{k(v)}\;} & S(k(v)).
\end{array}
$$

Again the left square and the total rectangle are commutative. Hence the right square is also commutative and we get a morphism in $\mathbf{Data}^{ur}$. This proves the first statement.

Now we prove the second statement. By the universal property we have just proved, it suffices to show that $S(Y) \to S(X)$ is bijective for any $\mathbb{A}^1$-invariant unramified sheaf S. Let F be the function field of X. Considering $S(X)$ and $S(Y)$ as subsets of $S(F)$, we have an inclusion $S(Y) \subset S(X)$. To prove $S(X) \subset S(Y)$, it suffices to prove $S(X) \subset S(\mathcal{O}_{Y,y})$ for any $y \in Y^{(1)}$. By the valuative criterion of properness, there is a morphism $\operatorname{Spec} \mathcal{O}_{Y,y} \to X$ extending $\operatorname{Spec} F \to X$. This implies the required inclusion. $\qquad\square$

3. $\mathbb{A}^1$-locality of classifying spaces

In this section we work over a noetherian base scheme S of finite Krull dimension. We prove an extension of [MV99, Section 4 Proposition 1.16] and use it to prove Theorem 3.7. Much of the discussion here goes back to work of Giraud and is folklore in the literature on stacks and homotopy theory. For related results in a different formulation, see [Jar15, Chapter 9]. We also note that most of the results here are not special to the Nisnevich topology on smooth schemes.

We write $\mathbf{sSh}(\mathbf{Sm}_S)$ for the category of simplicial sheaves on $\mathbf{Sm}_S$ equipped with the simplicial model structure [MV99, Section 2 Definition 1.2], and $\mathcal{H}_s(S)$ for its homotopy category. For a sheaf of groups G on $\mathbf{Sm}_S$ and $\mathcal{X} \in \mathbf{sSh}(\mathbf{Sm}_S)$, we write $P(\mathcal{X}, G)$ for set of isomorphism classes of Nisnevich G-torsors on $\mathcal{X}$.

Lemma 3.1. *Let G be a sheaf of groups on $\mathbf{Sm}_S$. Then there is a fibrant simplicial sheaf $\mathcal{B}G \in \mathbf{sSh}(\mathbf{Sm}_S)$ together with a G-torsor $\mathcal{E}G$ on it such that*

$$\operatorname{Hom}_{\mathbf{sSh}(\mathbf{Sm}_S)}(\mathcal{X}, \mathcal{B}G) \to P(\mathcal{X}, G); \ f \mapsto f^* \mathcal{E}G$$

induces a bijection $\operatorname{Hom}_{\mathcal{H}_s(S)}(\mathcal{X}, \mathcal{B}G) \cong P(\mathcal{X}, G)$.

Proof. See [MV99, Section 4 Proposition 1.15]. $\qquad\square$

Let us fix such $\mathcal{B}G$ and $\mathcal{E}G$. For any $U \in \mathbf{Sm}_S$ the simplicial set $\mathcal{B}G(U)$ is a Kan complex since $\mathcal{B}G$ is fibrant. We want to describe this Kan complex explicitly.

Lemma 3.2. *We have $\pi_n(\mathcal{B}G(U), p) = 0$ $(n \geq 2)$ for any vertex $p \in \mathcal{B}G(U)_0$.*

Proof. It suffices for us to prove that the restriction map from $[\Delta^{n+1}, \mathcal{B}G(U)]$ to $[\partial\Delta^{n+1}, \mathcal{B}G(U)]$ is surjective for $n \geq 2$. By Lemma 3.1 this is equivalent to saying that any G-torsor on $\partial\Delta^{n+1} \times U$ can be extended to $\Delta^{n+1} \times U$. Let $\mathcal{Y} \to \partial\Delta^{n+1} \times U$ be a G-torsor, and consider the evaluation morphism $\operatorname{ev} : \partial\Delta^{n+1} \times \mathcal{H}\mathrm{om}_{\partial\Delta^{n+1}}(\partial\Delta^{n+1}, \mathcal{Y}) \to \mathcal{Y}$. For any $V \in \mathbf{Sm}_S$ and a point $x \in V$, $\operatorname{ev}_x : \partial\Delta^{n+1} \times \operatorname{Hom}_{\partial\Delta^{n+1}}(\partial\Delta^{n+1}, \mathcal{Y}_x) \to \mathcal{Y}_x$ is an isomorphism since any G_x-torsor on $\partial\Delta^{n+1} \times U_x$ is trivial. Hence $\mathcal{Y} \cong \partial\Delta^{n+1} \times \mathcal{H}\mathrm{om}_{\partial\Delta^{n+1}}(\partial\Delta^{n+1}, \mathcal{Y})$ and the claim is trivial. $\qquad\square$

Thus the canonical map $\mathcal{B}G(U) \to \mathrm{N}(\Pi_1(\mathcal{B}G(U)))$ is a homotopy equivalence, where Π_1 denotes the fundamental groupoid and N denotes its nerve. Let $G\text{-}\mathbf{Tors}_U$ denote the groupoid of G-Torsors on U. We construct a functor $\Phi : \Pi_1(\mathcal{B}G(U)) \to G\text{-}\mathbf{Tors}_U$ as follows. For an object $p \in \mathcal{B}G(U)_0$, we define $\Phi(p) = p^*\mathcal{E}G$. Let $e \in \mathcal{B}G(U)_1$ an edge from $p_0 \in \mathcal{B}G(U)_0$ to $p_1 \in \mathcal{B}G(U)_0$. Then $e^*\mathcal{E}G$ is a G-torsor on $\Delta^1 \times U$ which restricts to $p_\varepsilon^*\mathcal{E}G$ over $\Delta^{\{i\}} \times U$ for $i = 0, 1$. Using the next lemma, we get a unique isomorphism of G-torsors $\rho_e : \Delta^1 \times p_0^*\mathcal{E}G \to e^*\mathcal{E}G$ which restricts to the identity over $\Delta^{\{0\}} \times U$. We define $\Phi(e)$ to be the restriction of ρ_e over $\Delta^{\{1\}} \times U$. This indeed gives a functor $\Phi : \Pi_1(\mathcal{B}G(U)) \to G\text{-}\mathbf{Tors}_U$, again by the next lemma.

Lemma 3.3. *Let $\mathcal{Y} \to \Delta^n \times U$ be a G-torsor and $\mathcal{Y}_0$ its restriction to $\Delta^{\{0\}} \times U$. Then there exists a unique isomorphism of G-torsors $\rho : \Delta^n \times \mathcal{Y}_0 \to \mathcal{Y}$ which restricts to the identity over $\Delta^{\{0\}} \times U$.*

Proof. By the same argument as in the proof of Lemma 3.2, we may assume $\mathcal{Y} \cong \Delta^n \times \mathcal{Y}_0$. It suffices to show that if $\rho : \Delta^n \times \mathcal{Y}_0 \to \Delta^n \times \mathcal{Y}_0$ is an isomorphism which restricts to the identity on $\Delta^{\{0\}} \times U$ then $\rho = \mathrm{id}$. This can be checked by taking stalks. $\square$

The following lemma is easy.

Lemma 3.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be groupoids and $\varphi : \mathcal{C} \to \mathcal{D}$ a functor. Then φ is an equivalence of groupoids if and only if the following hold:*

(i) *φ is essentially surjective and conservative.*
(ii) *For any $c \in \mathcal{C}$, $\mathrm{Aut}_{\mathcal{C}}(c) \to \mathrm{Aut}_{\mathcal{D}}(\varphi(c))$ is an isomorphism.*

Lemma 3.5. *$\Phi : \Pi_1(\mathcal{B}G(U)) \to G\text{-}\mathbf{Tors}_U$ is an equivalence of groupoids.*

Proof. We check the conditions in Lemma 3.4. (i) is an immediate consequence of Lemma 3.1. Take any vertex $p \in \mathcal{B}G(U)_0$. If $e \in \mathcal{B}G(U)_1$ is an edge from p to p such that $\Phi(e) = \mathrm{id}_{p^*\mathcal{E}G}$, then the pullback of $\mathcal{E}G$ by $e : S^1 \times U \to \mathcal{B}G$ is a trivial G-torsor. Thus there is some homotopy from $e : S^1 \to \mathcal{B}G(U)$ to a constant map. This shows that $[e] = [\mathrm{id}_p]$ in $\Pi_1(\mathcal{B}G(U))$, and hence $\mathrm{Aut}_{\Pi_1(\mathcal{B}G(U))}(p) \to \mathrm{Aut}_{G\text{-}\mathbf{Tors}_U}(p^*\mathcal{E}G)$ is injective. Let $\rho : p^*\mathcal{E}G \to p^*\mathcal{E}G$ be an automorphism of G-torsors. We define $\mathcal{Y}$ to be the coequalizer of

$$p^*\mathcal{E}G \underset{(1,\rho)}{\overset{(0,\mathrm{id})}{\rightrightarrows}} \Delta^1 \times p^*\mathcal{E}G$$

which is a G-torsor on $S^1 \times U$. Let $f : S^1 \to \mathcal{B}G(U)$ be a loop classifying $\mathcal{Y}$ and p' its endpoint. Since $p'^*\mathcal{E}G$ is isomorphic to $p^*\mathcal{E}G$, there is an edge $g \in \mathcal{B}G(U)_1$ from p to p'. Then $\Phi([g]^{-1} \circ [f] \circ [g]) = \rho$ by construction and hence $\mathrm{Aut}_{\Pi_1(\mathcal{B}G(U))}(p) \to \mathrm{Aut}_{G\text{-}\mathbf{Tors}_U}(p^*\mathcal{E}G)$ is surjective. $\square$

We obtain an extension of [MV99, Section 4 Proposition 1.16]:

Corollary 3.6. *Let G be a sheaf of groups on $\mathbf{Sm}_S$ and $U \in \mathbf{Sm}_S$. Then there is a canonical homotopy equivalence $\mathcal{B}G(U) \to \mathrm{N}(G\text{-}\mathbf{Tors}_U)$ sending $p \in \mathcal{B}G(U)_0$ to $p^*\mathcal{E}G$. In particular, the homotopy groups of $\mathcal{B}G(U)$ are given by*

$$\pi_n(\mathcal{B}G(U), p) \cong \begin{cases} H^1(U, G) & (n = 0) \\ \mathrm{Aut}_{G\text{-}\mathbf{Tors}_U}(p^*\mathcal{E}G) & (n = 1) \\ 0 & (n \geq 2). \end{cases}$$

We say that a sheaf of groups G on $\mathbf{Sm}_S$ is *strongly $\mathbb{A}^1$-invariant* if G and $H^1(-, G)$ are $\mathbb{A}^1$-invariant. An object $\mathcal{X} \in \mathbf{sSh}(\mathbf{Sm}_S)$ is said to be *$\mathbb{A}^1$-local* if the map

$$\mathrm{Hom}_{\mathcal{H}_s(S)}(\mathcal{Y}, \mathcal{X}) \to \mathrm{Hom}_{\mathcal{H}_s(S)}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$$

induced by the canonical projection is bijective for every $\mathcal{Y} \in \mathbf{sSh}(\mathbf{Sm}_S)$.

Theorem 3.7. *Let G be a strongly $\mathbb{A}^1$-invariant sheaf of groups on $\mathbf{Sm}_S$. Then $\mathcal{B}G$ is $\mathbb{A}^1$-local.*

Proof. By [MV99, Section 2 Proposition 3.19] it suffices to show that

$$\pi_n(\mathcal{B}G(U), p) \to \pi_n(\mathcal{B}G(\mathbb{A}^1_U), r^*p)$$

is bijective for any $U \in \mathbf{Sm}_S$ and $p \in \mathcal{B}G(U)_0$, where $r : \mathbb{A}^1_U \to U$ is the canonical projection. The cases $n = 0$ and $n \geq 2$ follows from Corollary 3.6 and the $\mathbb{A}^1$-invariance of $H^1(-, G)$. It remains to show that $\mathrm{Aut}_{G\text{-}\mathbf{Tors}_U}(p^*\mathcal{E}G) \to \mathrm{Aut}_{G\text{-}\mathbf{Tors}_{\mathbb{A}^1_U}}(r^*p^*\mathcal{E}G)$ is an isomorphism. Take a Nisnevich covering $q : V \to U$ such that $q^*p^*\mathcal{E}G$ is a trivial G-torsor. Let $\varphi : \mathrm{pr}_1^*q^*p^*\mathcal{E}G \xrightarrow{\cong} \mathrm{pr}_2^*q^*p^*\mathcal{E}G$ be the descent data for $\mathcal{E}G$; it is an isomorphism of G-torsors on $V \times_U V$. A choice of an isomorphism $\sigma : q^*p^*\mathcal{E}G \cong V \times G$ enables us to identify $\mathrm{Aut}_{G\text{-}\mathbf{Tors}_V}(q^*p^*\mathcal{E}G)$ with $G(V)$ and φ with an element $h \in G(V \times_U V)$. By Nisnevich descent for G-torsors, we obtain an isomorphism

$$\mathrm{Aut}_{G\text{-}\mathbf{Tors}_U}(p^*\mathcal{E}G) \cong \{g \in G(V) \mid h(\mathrm{pr}_1^*g) = (\mathrm{pr}_2^*g)h\}.$$

On the other hand, σ also induces an identification $q'^*r^*p^*\mathcal{E}G \cong \mathbb{A}^1_V \times G$ where $q' : \mathbb{A}^1_V \to \mathbb{A}^1_U$ is the morphism induced by q. We thus get a similar isomorphism

$$\mathrm{Aut}_{G\text{-}\mathbf{Tors}_{\mathbb{A}^1_U}}(r^*p^*\mathcal{E}G) \cong \{g \in G(\mathbb{A}^1_V) \mid h'(\mathrm{pr}_1^*g) = (\mathrm{pr}_2^*g)h'\}.$$

where h' is the image of h in $G(\mathbb{A}^1_V \times_{\mathbb{A}^1_U} \mathbb{A}^1_V)$. The morphism under consideration is identified with the morphism

$$\{g \in G(V) \mid h(\mathrm{pr}_1^*g) = (\mathrm{pr}_2^*g)h\} \to \{g \in G(\mathbb{A}^1_V) \mid h'(\mathrm{pr}_1^*g) = (\mathrm{pr}_2^*g)h'\}$$

induced by the canonical projection. Since G is $\mathbb{A}^1$-invariant, this is an isomorphism. $\qquad\square$

We now return to the theory over a field k.

Corollary 3.8. *Any strongly $\mathbb{A}^1$-invariant sheaf of groups on $\mathbf{Sm}_k$ is unramified.*

Proof. If G is a strongly $\mathbb{A}^1$-invariant sheaf of groups on $\mathbf{Sm}_k$, then $\mathcal{B}G$ is $\mathbb{A}^1$-local by Lemma 3.7. So by the proof of [Mor12, Theorem 6.1], $G = \pi_1^{\mathbb{A}^1}(\mathcal{B}G, *)$ is unramified. $\qquad\square$

4. Main result

We recall the definition of $\mathbb{A}^1$-homology sheaves. A sheaf M of abelian groups on $\mathbf{Sm}_k$ is *strictly $\mathbb{A}^1$-invariant* if its cohomology presheaves $H^n(-, M)$ ($n \geq 0$) are all $\mathbb{A}^1$-invariant. Let $\mathbf{Ab}^{\mathbb{A}^1}(\mathbf{Sm}_k)$ denote the full subcategory of $\mathbf{Ab}(\mathbf{Sm}_k)$ consisting of strictly $\mathbb{A}^1$-invariant sheaves. Let $\mathbf{D}(\mathbf{Sm}_k)$ denote the unbounded derived category of $\mathbf{Ab}(\mathbf{Sm}_k)$. An object $C \in \mathbf{D}(\mathbf{Sm}_k)$ is called $\mathbb{A}^1$-local if for any $D \in \mathbf{D}(\mathbf{Sm}_k)$ the map

$$\mathrm{Hom}_{\mathbf{D}(\mathbf{Sm}_k)}(D, C) \to \mathrm{Hom}_{\mathbf{D}(\mathbf{Sm}_k)}(D \otimes \mathbb{Z}(\mathbb{A}^1), C)$$

is bijective. For example, a sheaf of abelian groups (viewed as an object of $\mathbf{D}(\mathbf{Sm}_k)$) is $\mathbb{A}^1$-local if and only if it is strictly $\mathbb{A}^1$-invariant. Morel's $\mathbb{A}^1$-derived category $\mathbf{D}_{\mathbb{A}^1}(k)$ is the full triangulated subcategory of $\mathbf{D}(\mathbf{Sm}_k)$ consisting of $\mathbb{A}^1$-local complexes. The inclusion functor $\mathbf{D}_{\mathbb{A}^1}(k) \to \mathbf{D}(\mathbf{Sm}_k)$ admits a left adjoint $L_{\mathbb{A}^1}$ (see [Mor12, Corollary 6.19]). We say that $C \in \mathbf{D}(\mathbf{Sm}_k)$ is (-1)-*connected* if $H_n C = 0$ for all $n < 0$.

Theorem 4.1. *If $C \in \mathbf{D}(\mathbf{Sm}_k)$ is (-1)-connected, then so is $L_{\mathbb{A}^1}C$.*

Proof. See [Mor12, Theorem 6.22]. $\qquad\square$

Corollary 4.2. *An object $C \in \mathbf{D}(\mathbf{Sm}_k)$ is $\mathbb{A}^1$-local if and only if its homology sheaves are all strictly $\mathbb{A}^1$-invariant.*

Proof. This is well-known, but we recall the proof for the convinience of readers. The 'if' part is clear. To prove the 'only if' part, it suffices to show that if $C \in \mathbf{D}(\mathbf{Sm}_k)$ is $\mathbb{A}^1$-local then its truncation $\tau_{\geq n}C$ is also $\mathbb{A}^1$-local for each $n \in \mathbb{Z}$. By Theorem 4.1, $L_{\mathbb{A}^1}\tau_{\geq n}C$ is $(n-1)$-connected. Thus the canonical morphism $L_{\mathbb{A}^1}\tau_{\geq n}C \to L_{\mathbb{A}^1}C \cong C$ factors through $\tau_{\geq n}C$. Since the composition $\tau_{\geq n}C \to L_{\mathbb{A}^1}\tau_{\geq n}C \to \tau_{\geq n}C$ is the identity, it follows that $\tau_{\geq n}C$ is $\mathbb{A}^1$-local. $\qquad\square$

This implies that the canonical t-structure on $\mathbf{D}(\mathbf{Sm}_k)$ restricts to a t-structure on $\mathbf{D}_{\mathbb{A}^1}(k)$ whose heart is $\mathbf{Ab}^{\mathbb{A}^1}(\mathbf{Sm}_k)$. The *n-th $\mathbb{A}^1$-homology sheaf* $H_n^{\mathbb{A}^1}(X)$ of $X \in \mathbf{Sm}_k$ is defined to be the *n*-th homology sheaf of $L_{\mathbb{A}^1}\mathbb{Z}(X)$, which is strictly $\mathbb{A}^1$-invariant and vanishes for $n < 0$. There is a canonical morphism of sheaves $X \to H_0^{\mathbb{A}^1}(X)$.

Lemma 4.3. *Let $X \in \mathbf{Sm}_k$. Then the canonical morphism $X \to H_0^{\mathbb{A}^1}(X)$ is initial among morphisms from X to strictly $\mathbb{A}^1$-invariant sheaves of abelian groups.*

Proof. This is originally due to Morel, and a proof can be found in [Aso12, Lemma 3.3]. We recall the proof for the convinience of readers. For any $M \in$

$\mathbf{Ab}^{\mathbb{A}^1}(\mathbf{Sm}_k)$ we have a sequence of natural isomorphisms

$$\mathrm{Hom}_{\mathbf{Sh}(\mathbf{Sm}_k)}(X, M) \cong \mathrm{Hom}_{\mathbf{D}(\mathbf{Sm}_k)}(\mathbb{Z}(X), M)$$
$$\cong \mathrm{Hom}_{\mathbf{D}_{\mathbb{A}^1}(k)}(L_{\mathbb{A}^1}\mathbb{Z}(X), M)$$
$$\cong \mathrm{Hom}_{\mathbf{D}_{\mathbb{A}^1}(k)}(\tau_{\leq 0}L_{\mathbb{A}^1}\mathbb{Z}(X), M).$$

Since $L_{\mathbb{A}^1}\mathbb{Z}(X)$ is (-1)-connected, we have $\tau_{\leq 0}L_{\mathbb{A}^1}\mathbb{Z}(X) \cong H_0^{\mathbb{A}^1}(X)$. Thus the last group is isomorphic to $\mathrm{Hom}_{\mathbf{Ab}^{\mathbb{A}^1}(\mathbf{Sm}_k)}(H_0^{\mathbb{A}^1}(X), M)$. $\qquad \square$

Lemma 4.4. *Any birational sheaf of abelian groups is strictly $\mathbb{A}^1$-invariant.*

Proof. Let M be a birational sheaf of abelian groups. Then M is $\mathbb{A}^1$-invariant by Lemma 2.1. Moreover we have $H^n(-, M) = 0$ $(n > 0)$ since the Čech cohomology groups vanish for $n > 0$ by birationality (see [AH11, Lemma 2.4] for details). Therefore M is strictly $\mathbb{A}^1$-invariant. $\qquad \square$

Theorem 4.5. *For any smooth proper variety X over a field k, the canonical morphism $X \to \mathbb{Z}(\pi_0^{br}(X))$ induces an isomorphism $H_0^{\mathbb{A}^1}(X) \cong \mathbb{Z}(\pi_0^{br}(X))$. In particular, $H_0^{\mathbb{A}^1}(X)(k)$ is a free abelian group of rank $\#X(k)/R$.*

Proof. By Lemma 4.4 the sheaf $\mathbb{Z}(\pi_0^{br}(X))$ is strictly $\mathbb{A}^1$-invariant. By Theorem 2.4 and Corollary 3.8, the canonical morphism $X \to \mathbb{Z}(\pi_0^{br}(X))$ is initial among morphisms from X to strictly $\mathbb{A}^1$-invariant sheaves of abelian groups. Hence by Lemma 4.3 we get the required isomorphism. $\qquad \square$

Next we give another application of Theorem 2.4. Let S_b denote the class of birational morphisms in $\mathbf{Sm}_k^{\mathrm{conn}}$. Then the category of birational sheaves is equivalent to the category $\mathbf{PSh}(S_b^{-1}\mathbf{Sm}_k^{\mathrm{conn}})$ of presheaves on $S_b^{-1}\mathbf{Sm}_k^{\mathrm{conn}}$. For any smooth variety X over k, define $h_X \in \mathbf{PSh}(S_b^{-1}\mathbf{Sm}_k^{\mathrm{conn}})$ to be the presheaf represented by X. Let $\widetilde{h}_X$ denote the corresponding birational sheaf. There is a canonical morphism $X \to \widetilde{h}_X$.

Lemma 4.6. *Let X be a smooth variety over k. Then the canonical morphism $X \to \widetilde{h}_X$ is initial among morphisms from X to birational sheaves.*

Proof. Let $X \to S$ be a morphism to a birational sheaf on $\mathbf{Sm}_k$. Write S' for the presheaf on $S_b^{-1}\mathbf{Sm}_k^{\mathrm{conn}}$ corresponding to S. Then the claim follows from the sequence of natural bijections

$$\mathrm{Hom}_{\mathbf{Sh}(\mathbf{Sm}_k)}(X, S) \cong S(X) \cong S'(X)$$
$$\cong \mathrm{Hom}_{\mathbf{PSh}(S_b^{-1}\mathbf{Sm}_k^{\mathrm{conn}})}(h_X, S')$$
$$\cong \mathrm{Hom}_{\mathbf{Sh}(\mathbf{Sm}_k)}(\widetilde{h}_X, S)$$

where the first and the third bijection is due to the Yoneda lemma. $\qquad \square$

We obtain a simple proof of the following result of Kahn-Sujatha [KS15, Theorem 6.6.3]:

Theorem 4.7. *Let* $X, U \in \mathbf{Sm}_k^{\mathrm{conn}}$ *and suppose that* X *is proper. Then there is a bijection*

$$\mathrm{Hom}_{S_b^{-1}\mathbf{Sm}_k^{\mathrm{conn}}}(U, X) \cong X(k(U))/R$$

which is compatible with the canonical maps from $X(U)$ *to both sides.*

Proof. By Theorem 2.4 and Lemma 2.1, the canonical morphism $X \to \pi_0^{br}(X)$ is initial among morphisms from X to birational sheaves. Hence by Lemma 4.6 there is a unique isomorphism $\widetilde{h}_X \cong \pi_0^{br}(X)$ which is compatible with the canonincal morphisms from X to both sides. Evaluating at $U \in \mathbf{Sm}_k^{\mathrm{conn}}$ we obtain the required bijection. $\square$

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(Junnosuke Koizumi) GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU, TOKYO 153-8914, JAPAN
`jkoizumi@ms.u-tokyo.ac.jp`