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Properly infinite corona algebras

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ABSTRACT. We investigate conditions under which the corona algebra of a nonunital, simple C*-algebra is properly infinite. Among other things, we prove that if $\mathcal{B}$ is a nonunital, separable, simple, $\mathcal{Z}$-stable C*-algebra with Property I and having an approximate unit consisting of projections, then $\mathcal{C}(\mathcal{B})$ is properly infinite if and only if $T(\mathcal{B})$ is compact. We also provide other characterizations.

CONTENTS

1. Introduction

Let $\mathcal{B}$ be a separable, stable C*-algebra. It is an elementary fact from operator theory that

$$1_{\mathcal{M}(\mathcal{B})} \sim 1_{\mathcal{M}(\mathcal{B})} \oplus 1_{\mathcal{M}(\mathcal{B})},$$

where $\mathcal{M}(\mathcal{B})$ is the multiplier algebra of $\mathcal{B}$ and $\sim$ here is Murray–von Neumann equivalence of projections in $\mathbb{M}_2 \otimes \mathcal{M}(\mathcal{B})$. This is the basic observation underlying the *Brown–Douglas–Fillmore (BDF) sum* which led to the extension semigroup $Ext(\mathcal{A}, \mathcal{B})$, which is a group when $\mathcal{A}$ is separable and nuclear. When $\mathcal{B} = \mathcal{K}$ and $\mathcal{A} = C(X)$ for X a compact subset of the plane, BDF used the functorial properties of this object in their outstanding classification of all essentially normal operators via Fredholm indices ([4]).

Perhaps, as witnessed above, one of the reasons for the success of the BDF theory is that their multiplier algebra $\mathcal{M}(\mathcal{K}) = B(l_2)$ and corona algebra $\mathbb{B}(l_2)/\mathcal{K}$ have particularly nice structure. For example, the BDF-Voiculescu absorption theorem, which roughly says that all essential extensions are absorbing, would not be true if the Calkin algebra $\mathbb{B}(l_2)/\mathcal{K}$ were not simple ([31]).

Thus, structural properties of multiplier and corona algebras are indispensible for the advancement of extension theory and the associated operator theory

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beyond the small number of successful classical cases. This idea was well understood by previous workers, and has had its most successful realizations in the definitive work of Lin (see, e.g., [13], [14], [15], [16], [17], [20]). One problem of the current moment is the case where the canonical ideal need not be stable. One of the insights of previous workers is that, while this nonstable case is very interesting in itself, it is also indispensible for progress in the classical case of stable canonical ideals (see, e.g., [14], [17], [20]). For instance, in the classical stable case, under a nuclearity hypothesis, Kasparov's KK^1 only classifies the absorbing extensions – a very thin class, and thus misses many relevant essential extensions. To delve further, even in the classical stable case, requires finer examination of the structure of the corona algebras and more delicate nonstable absorption theory. Following in the footsteps of previous workers, this has been the program that we have been pursuing (e.g., [11], [25], [27]).

A unital C*-algebra $\mathcal{C}$ is said to be properly infinite if $1_{\mathcal{C}} \succeq 1_{\mathcal{C}} \oplus 1_{\mathcal{C}}$, where here $\preceq$ is Murray–von Neumann subequivalence of projections in $\mathbb{M}_2 \otimes \mathcal{C}$. It was observed in [6] that when a corona algebra $\mathcal{C}(\mathcal{B})$ was properly infinite, there is a generalized BDF sum on the class of extensions which may serve the needs of extension theory even for nonstable $\mathcal{B}$. This anticipated later works (e.g., [14], [17]) where definitive nonstable generalizations of the BDF index theorem were proven. In this paper, we give some characterizations of proper infiniteness for corona algebras, generalizing the results of [6] who worked in the AF case.

We note that aside from connections to extension theory, proper infiniteness of a C*-algebra (especially a corona algebra) is in itself an interesting and fundamental structural property, which is connected to many other interesting properties. For example, it is an open question whether every properly infinite unital C*-algebra is K_1-injective ([3]). Among other things, K_1-injectivity of the Paschke dual algebras (which are properly infinite) imply interesting uniqueness theorems and generalizations of the BDF essential codimension result (e.g., see [23]).

We end this introduction by introducing some notations that are to be used in this paper. This paper uses only elementary techniques and should be accessible to a reader with basic knowledge of C*-algebra theory – modulo knowing about multiplier algebras, strict topology, Choquet simplexes, lower semicontinuous affine functions on compact convex sets, and basic notions and regularity properties (like AF-algebras, irrational rotations algebras, real rank zero, strict comparison, stable rank one) from the current theory of simple C*-algebras. We recall some notation here, and recall others in later parts of the paper.

For a nonunital C*-algebra $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ and $\mathcal{C}(\mathcal{B}) =_{df} \mathcal{M}(\mathcal{B})/\mathcal{B}$ denote the multiplier and corona algebras (resp.) of $\mathcal{B}$. Recall that the multiplier algebra $\mathcal{M}(\mathcal{B})$, of $\mathcal{B}$, is roughly speaking, the largest unital C*-algebra containing $\mathcal{B}$ as an essential ideal. Good references for multiplier algebras, corona algebras, strict topology and associated subjects are [18] and [32].

For a compact convex set K, let $Aff(K)$ denote the vector space of all affine continuous functions from K to $\mathbb{R}$. Note that, with the uniform norm, $Aff(K)$ is a Banach space. $LAff(K)$ denotes the vector space of all lower semicontinuous, affine functions from K to $(-\infty, \infty]$. $Aff(K)_+$ (resp. $LAff(K)_+$) denotes all $f \in Aff(K)$ (resp. $LAff(K)$) such that $f \geq 0$. $Aff(K)_{++}$ (resp. $LAff(K)_{++}$) denotes all $f \in Aff(K)_+$ (resp. $LAff(K)_+$) such that $f(x) > 0$ for all $x \in K$. References for the above material, especially for how it is used in this paper, are [1], [9], [10], [11] and the references therein.

For a C*-algebra $\mathcal{D}$ (unital or nonunital), we let $T(\mathcal{D})$ denote the tracial state space of $\mathcal{D}$, given the weak* topology. We will be interested in $T(\mathcal{B})$, $T(\mathcal{M}(\mathcal{B}))$ and $T(\mathcal{C}(\mathcal{B}))$ (some or all of which could be empty) for some nonunital $\mathcal{B}$. Note that when $\mathcal{D}$ is unital, then $T(\mathcal{D})$ is a compact convex set – in fact, if $\mathcal{D}$ is additionally separable, then $T(\mathcal{D})$ is a metrizable Choquet simplex. Suppose that $\mathcal{D}$ is additionally separable. For an element $e \in Ped(\mathcal{D})_+ - \{0\}$, we let $T_e(\mathcal{D})$ denote all densely defined, norm-lower semicontinuous traces $\mathcal{D}_+ \to [0, \infty]$ which are normalized at e. Recall that $Ped(\mathcal{D})$ denotes the Pedersen ideal of $\mathcal{D}$; and when $\mathcal{D}$ is separable, then $T_e(\mathcal{D})$, with the topology of pointwise convergence on $Ped(\mathcal{D})$, is a metrizable Choquet simplex. Recall also that any densely defined, norm lower semicontinuous trace τ on $\mathcal{D}$ has a unique extension to a strictly lower semicontinuous trace on $\mathcal{M}(\mathcal{D})_+$. Unless otherwise specified, we will also denote this extension trace by "τ".

For any element $A \in \mathcal{M}(\mathcal{D})_+ - \{0\}$, A induces an element $\widehat{A} \in LAff(T_e(\mathcal{D}))_{++}$ via

$$\widehat{A}(\tau) =_{df} \tau(A)$$

for all $\tau \in T_e(\mathcal{D})$. In a similar manner, A induces elements in $Aff(T(\mathcal{M}(\mathcal{D})))_+$ and $Aff(T(\mathcal{D}))_+$, which we will also denote by $\widehat{A}$.

References for the above material are again [9], [10], [11] and the references therein. We will assume that all our simple, separable C*-algebras have the property that every quasitrace is a trace.

We caution the reader that in this paper, we use one terminology different from what is in the papers [9], [10], [11], *and other works: In* [9], [10], *and* [11], *$T(\mathcal{D})$ means $T_e(\mathcal{D})$ for some $e \in Ped(\mathcal{D})_+ - \{0\}$, but that is **NOT** the case in this paper.*

We note that in this paper, when we write "$T_e(\mathcal{D})$", we just mean the aforementioned object with some element $e \in Ped(\mathcal{D})_+ - \{0\}$. For our results, it will not matter which positive nonzero element e of the Pedersen ideal is used. Good basic references for the theory of simple C*-algebras are [5] and [18].

Finally, many of the ideas of this paper are generalizations of those from the paper [6], though we need the comparison theory for multiplier algebras as from [11], [22] and the references therein.

The above give the basic references required for understanding the contents of this paper. To understand, for example, the connections with KK theory, extension theory and operator theory, which requires a bit more work, we recommend beginning with the basics in [4], [12], [14], [17], [18], [19], [31], and moving on to the more advanced theory from later references.

2. Properly infinite corona algebras

The first result is straightforward. For the convenience of the reader, we provide a proof.

Lemma 2.1. *Let K be a compact convex set in a locally convex topological vector space V such that $\alpha\tau \in K \Rightarrow \alpha = 1$ for all $\tau \in K$ and all $\alpha \geq 0$. Let $f \in Aff(K)$ and g be the unique nonnegatively homogeneous extension of f to $\mathbb{R}^+K$, where $\mathbb{R}^+ =_{df} [0, \infty)$. Then g is continuous on $\mathbb{R}^+K$.*

Proof. Since V is locally convex, let $\{\|.\|_\alpha\}_{\alpha\in I}$ be a family of seminorms on V which induces the topology on V.

Suppose that $\lambda_j\tau_j \to \lambda\tau$ in $\mathbb{R}^+K$. To show that $\lim g(\lambda_j\tau_j) = g(\lambda\tau)$, it suffices to prove that every subnet has a subnet for which the equality holds. Hence, passing to a subnet if necessary, we may assume that $\lambda_j \to \lambda' \in \mathbb{R}^+ \cup \{\infty\}$ and $\tau_j \to \tau' \in K$. (Recall that we are assuming that K is compact.)

Claim: $\lambda' \neq \infty$.

Suppose, to the contrary, that $\lambda' = \infty$. Choose $\alpha \in I$ so that $\|\tau'\|_\alpha > 0$. Then

$$\|\lambda\tau\|_\alpha = \lim \|\lambda_j\tau_j\|_\alpha = \lim |\lambda_j|\|\tau_j\|_\alpha = (\lim |\lambda_j|)(\lim \|\tau_j\|_\alpha) = \infty\|\tau'\|_\alpha = \infty$$

which is a contradiction. This completes the proof of the Claim.

From the Claim, $0 \leq \lambda' < \infty$ and $\lambda'\tau' = \lambda\tau$. Hence,

$$g(\lambda_j\tau_j) = \lambda_j g(\tau_j) = \lambda_j f(\tau_j) \to \lambda' f(\tau') = \lambda' g(\tau') = g(\lambda'\tau') = g(\lambda\tau).$$

□

Remark 2.2. *The function g in Lemma* 2.1 *is also additive. With notation from Lemma* 2.1, *here is a short argument: Let $\tau, \tau' \in K$, $\alpha > 0$ and $\beta \geq 0$. Then*

$$\begin{aligned} g(\alpha\tau + \beta\tau') &= g\left((\alpha+\beta)\frac{\alpha\tau + \beta\tau'}{\alpha+\beta}\right) \\ &= (\alpha+\beta) g\left(\frac{\alpha\tau + \beta\tau'}{\alpha+\beta}\right) \\ &= (\alpha+\beta) f\left(\frac{\alpha\tau + \beta\tau'}{\alpha+\beta}\right) \\ &= (\alpha+\beta)\left(\frac{\alpha f(\tau)}{\alpha+\beta} + \frac{\beta f(\tau')}{\alpha+\beta}\right) \\ &= \alpha f(\tau) + \beta f(\tau') \\ &= g(\alpha\tau) + g(\beta\tau'). \end{aligned}$$

Corollary 2.3. *Let K be a metrizable, compact convex set in a locally convex topological vector space such that $\alpha\tau \in K \Rightarrow \alpha = 1$ for all $\tau \in K$ and all $\alpha \geq 0$. If $h \in LAff(K)_{++}$, then h has an extension to a nonnegatively homogeneous, additive, lower semicontinuous function $\widetilde{h}$ on $\mathbb{R}^+K$. (Here, we use the convention $0 * \infty = 0$.)*

Proof. There exists a sequence $\{h_n\}$ in $Aff(K)_{++}$ with $0 \leq h_n \leq h_{n+1}$ for all n such that $h_n \to h$ pointwise on K.

By Lemma 2.1, for all n, let $\widetilde{h}_n$ be the unique nonnegatively homogeneous, continuous extension of h_n to $\mathbb{R}^+K$. Then $0 \leq \widetilde{h}_n \leq \widetilde{h}_{n+1}$ for all n and

$$\widetilde{h}(x) =_{df} \lim_{n\to\infty} \widetilde{h}_n(x), \qquad x \in \mathbb{R}^+K,$$

is the desired extension. □

We require some notions that are taken from [11].

Definition 2.4. *Let K be a compact convex set and let $f, g \in LAff(K)_{++}$. f is said to be* complemented under g *if there exists an $h \in LAff(K)_{++}$ such that $f + h = g$.*

Definition 2.5. *Let $\mathcal{B}$ be a nonunital, separable, simple C*-algebra.*

(1) *$\mathcal{B}$ is said to be* projection surjective *if for every $f \in LAff(T_e(\mathcal{B}))_{++}$ which is complemented under $\widehat{1_{\mathcal{M}(\mathcal{B})}}$, there exists a projection $P \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ such that $\widehat{P} = f$.*

(2) *$\mathcal{B}$ is said to be* projection injective *if for all projections $P, Q \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$,*

$$\widehat{P} = \widehat{Q} \text{ on } T_e(\mathcal{B})$$

implies that P is Murray–von Neumann equivalent to Q in $\mathcal{M}(\mathcal{B})$.

(3) *$\mathcal{B}$ is said to be* stably projection surjective (resp. injective) *if $\mathcal{B} \otimes \mathcal{K}$ is projection surjective (resp. injective).*

From [11] and [22], we have that any nonunital, simple, separable, finite, $\mathcal{Z}$-stable C*-algebra, with stable rank one, is both projection injective and projection surjective.

Remark 2.6. *For the benefit of the reader, we elaborate on a degenerate case for Definition 2.5, which should be immediate. Nonetheless, for expository purposes, it is better to make it more explicit.*

In the above definition, suppose that $\mathcal{B}$ is stably projection injective and we are in the degenerate case where $T(\mathcal{B}) = \emptyset$. Then for every full projection $P \in \mathcal{M}(\mathcal{B})$ and for every $\tau \in T_e(\mathcal{B})$ (which can be empty or nonempty), $\tau(P) = \infty$. (For otherwise, since P is full in $\mathcal{M}(\mathcal{B})$, we would have a nonzero bounded trace on $P\mathcal{B}P$, which can be extended to a nonzero bounded trace on $\mathcal{B}$, and thus, $T(\mathcal{B}) \neq \emptyset$ which gives a contradiction.) However, identifying $\mathcal{M}(\mathcal{B})$ with $\mathcal{M}(\mathcal{B}) \otimes e_{1,1} \subseteq \mathcal{M}(\mathcal{B} \otimes \mathcal{K})$ and identifying e with $e \otimes e_{1,1}$, we have that $T_e(\mathcal{B}) = T_{e\otimes e_{1,1}}(\mathcal{B} \otimes \mathcal{K})$. So $\tau(P \otimes e_{1,1}) = \infty = \tau(1_{\mathcal{M}(\mathcal{B}\otimes\mathcal{K})})$, for all full projections $P \in \mathcal{M}(\mathcal{B})$ and for all $\tau \in T_{e\otimes e_{1,1}}(\mathcal{B} \otimes \mathcal{K})$. Thus, stable projection injectivity trivially implies that for all full projections $P \in \mathcal{M}(\mathcal{B})$, $P \otimes e_{1,1}$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(\mathcal{B}\otimes\mathcal{K})}$ in $\mathcal{M}(\mathcal{B} \otimes \mathcal{K})$. Thus, for any full projection $P \in \mathcal{M}(\mathcal{B})$,

$$P\mathcal{B}P \cong (P \otimes e_{1,1})(\mathcal{B} \otimes \mathcal{K})(P \otimes e_{1,1}) \cong \mathcal{B} \otimes \mathcal{K},$$

which is stable. So $\mathcal{M}(\mathcal{B})$ is properly infinite and $T(\mathcal{M}(\mathcal{B})) = \emptyset$.

Let $\mathcal{D}$ be a nonunital, separable, simple C*-algebra. For any densely defined, (norm) lower semicontinuous trace τ on $\mathcal{D}$, we let $\|\tau\| = \tau(1_{\mathcal{M}(\mathcal{D})})$. Of course, as stated in the introduction, we are identifying τ with its strictly lower semicontinuous extension to $\mathcal{M}(\mathcal{D})_+$.

The next result is a variation on [6] Lemma 2.9.

Lemma 2.7. *Let $\mathcal{B}$ be a nonunital, separable, simple C*-algebra with projection injectivity and projection surjectivity such that $T(\mathcal{B})$ is compact. Suppose that $f \in Aff(T(\mathcal{B}))_{++}$ is such that $f(\tau) < 1$ for all $\tau \in T(\mathcal{B})$. Then there exists a projection $P \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ such that $f(\tau) = \tau(P)$ for all $\tau \in T(\mathcal{B})$.*

Proof. Since $T(\mathcal{B})$ is compact, let $\epsilon > 0$ be such that $\epsilon < f < 1 - \epsilon$. Fix $e \in Ped(\mathcal{B})_+ - \{0\}$ and define $\overline{f} : T_e(\mathcal{B}) \to (0, \infty]$ by

$$\overline{f}(\tau) =_{df} \begin{cases} \infty & \text{if } \|\tau\| = \infty \\ f\left(\frac{\tau}{\|\tau\|}\right)\|\tau\| & \text{if } \|\tau\| < \infty. \end{cases}$$

Firstly, we will show that $\overline{f}$ is an affine lower semicontinuous function on $T_e(\mathcal{B})$. To simplify notation, we also denote by f the unique nonnegatively homogeneous extension of f to the convex cone $\mathbb{R}^+T(\mathcal{B})$, which takes the value 0 at 0. Here, recall that $\mathbb{R}^+ = [0, \infty)$. By Lemma 2.1 and Remark 2.2, f is continuous and additive on $\mathbb{R}^+T(\mathcal{B})$.

Claim 1: $\overline{f}$ is affine and lower semicontinuous on $T_e(\mathcal{B})$.

Proof of Claim 1: Firstly, note that for all $\tau \in T_e(\mathcal{B})$, if $\overline{f}(\tau) < \infty$, then $\overline{f}(\tau) = f(\tau)$. Hence, since f is nonnegatively homogeneous, additive and continuous on $\mathbb{R}^+T(\mathcal{B})$, $\overline{f}$ is affine and lower semicontinuous whereever it is finite.

Secondly, for all $\tau \in T_e(\mathcal{B})$, $\overline{f}(\tau) = \infty$ if and only if $\|\tau\| = \infty$. Hence, $\overline{f}$ is affine on $T_e(\mathcal{B})$. Here, we use the convention $0 * \infty = 0$.

Finally, since $f \geq \epsilon$ on $T(\mathcal{B})$, we must have that $f(\tau) \geq \epsilon\|\tau\|$ for all $\tau \in \mathbb{R}^+T(\mathcal{B})$. Therefore, $\overline{f}(\tau) \geq \epsilon\|\tau\|$ for all $\tau \in T_e(\mathcal{B})$. Since the map $T_e(\mathcal{B}) \to (0, \infty] : \tau \mapsto \|\tau\|$ is lower semicontinuous, $\overline{f}$ is lower semicontinuous whereever it takes the value $+\infty$. This ends the proof of Claim 1.

Let $g \in Aff(T(\mathcal{B}))$ be given by $g =_{df} 1 - f$. Then $\epsilon < g < 1 - \epsilon$. Let $\overline{g} : T_e(\mathcal{B}) \to (0, \infty]$ be given by

$$\overline{g}(\tau) =_{df} \begin{cases} +\infty & \text{if } \|\tau\| = \infty \\ g\left(\frac{\tau}{\|\tau\|}\right)\|\tau\| & \text{if } \|\tau\| < \infty. \end{cases}$$

By the same argument as that for $\overline{f}$, we can show that $\overline{g}$ is an affine lower semicontinuous function on $T_e(\mathcal{B})$. (See Claim 1.)

Claim 2: Viewed as affine functions on $T_e(\mathcal{B})$, we have that

$$\overline{f} + \overline{g} = \widehat{1_{\mathcal{M}(\mathcal{B})}}.$$

Proof of Claim 2: Since $g = 1 - f$, we have that $f(\tau') + g(\tau') = 1$ for all $\tau' \in T(\mathcal{B})$. Thus

$$f\left(\frac{\tau}{\|\tau\|}\right) + g\left(\frac{\tau}{\|\tau\|}\right) = 1$$

for all $\tau \in T_e(\mathcal{B})$ for which $\|\tau\| < \infty$. Therefore,

$$f\left(\frac{\tau}{\|\tau\|}\right)\|\tau\| + g\left(\frac{\tau}{\|\tau\|}\right)\|\tau\| = \|\tau\| = \widehat{1_{\mathcal{M}(\mathcal{B})}}(\tau)$$

for all $\tau \in T_e(\mathcal{B})$ for which $\|\tau\| < \infty$. So

$$\overline{f}(\tau) + \overline{g}(\tau) = \widehat{1_{\mathcal{M}(\mathcal{B})}}(\tau)$$

for all $\tau \in T_e(\mathcal{B})$ for which $\|\tau\| < \infty$.

Also,

$$\overline{f}(\tau) + \overline{g}(\tau) = \infty = \widehat{1_{\mathcal{M}(\mathcal{B})}}(\tau)$$

for all $\tau \in T_e(\mathcal{B})$ for which $\|\tau\| = \infty$. Therefore,

$$\overline{f} + \overline{g} = \widehat{1_{\mathcal{M}(\mathcal{B})}}.$$

This ends the proof of Claim 2.

By Claim 2, $\overline{f}$ is complemented under $\widehat{1_{\mathcal{M}(\mathcal{B})}}$. Therefore, since $\mathcal{B}$ has projection surjectivity, there exists a projection $P \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ such that $\overline{f} = \widehat{P}$, or $\tau(P) = \overline{f}(\tau)$ for all $\tau \in T_e(\mathcal{B})$. As a consequence, $\tau(P) = f(\tau)$ for all $\tau \in T(\mathcal{B})$. □

We note that the statement of Lemma 2.9 in [6] is itself incorrect, and we will provide a counterexample in Proposition 3.11. We note, though, that most of the arguments and the main result of their paper, [6, Theorem 3.1], are still correct and their main argument can be patched up by following the argument of the present paper (which uses many of the ideas of [6]).

Remark 2.8. *The projection P, in Lemma 2.7, satisfies that $\tau(P) = f(\tau)$ for all $\tau \in T(\mathcal{B})$ and $\tau(P) = \infty$ for all $\tau \in T_e(\mathcal{B})$ for which $\|\tau\| = \infty$.*

Recall that if $\mathcal{B}$ is a nonunital, separable C*-algebra, and if τ is any densely defined, (norm) lower semicontinuous trace on $\mathcal{B}$, then, unless otherwise stated, we also denote by "τ" the unique strictly lower semicontinuous extension on $\mathcal{M}(\mathcal{B})_+$ (which is a trace). In particular, for all $B \in \mathcal{M}(\mathcal{B})_+$, for every approximate unit $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{B}$, $\tau(B) = \tau(e_n B e_n)$.

Recall also that if τ is bounded on $\mathcal{B}$ then its strictly lower semicontinuous extension is also bounded on $\mathcal{M}(\mathcal{B})$, and $\|\tau\|_{\mathcal{B}^*} = \|\tau\|_{\mathcal{M}(\mathcal{B})^*}$.

Traces on $\mathcal{M}(\mathcal{B})$, as in the above, are essentially the "computable" traces, since they arise, in a natural, concrete way, via strict topology approximations, from traces on $\mathcal{B}$. Part of what makes the subject matter of the present paper difficult is that we will also be dealing with "noncomputable" traces in $T(\mathcal{M}(\mathcal{B}))$. We presently have no concrete way of getting a handle on these latter "noncomputable traces", and as a result will have to postulate principles like Property I. (See Definition 2.12.)

Before moving on, let us prove some more results about the "computable" traces.

Lemma 2.9. *Let $\mathcal{B}$ be a nonunital, separable, simple C*-algebra, and suppose that τ is a bounded trace on $\mathcal{B}$. Then the strictly lower semicontinuous extension of τ to $\mathcal{M}(\mathcal{B})$ is the smallest extension of τ to a bounded trace on $\mathcal{M}(\mathcal{B})$. Equivalently, if σ is a bounded trace on $\mathcal{M}(\mathcal{B})$ such that $\sigma|_{\mathcal{B}} = \tau$, then for all $B \in \mathcal{M}(\mathcal{B})_+$, $\tau(B) \leq \sigma(B)$.*

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be an approximate unit for $\mathcal{B}$, let $B \in \mathcal{M}(\mathcal{B})_+$ be arbitrary, and let $\epsilon > 0$ be given. Choose $N \geq 1$ so that

$$\tau(B) < \tau(B^{1/2} e_N B^{1/2}) + \epsilon.$$

Then

$$\begin{aligned} \tau(B) &< \tau(B^{1/2} e_N B^{1/2}) + \epsilon \\ &= \sigma(B^{1/2} e_N B^{1/2}) + \epsilon \\ &\leq \sigma(B) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have $\tau(B) \leq \sigma(B)$. □

Next, we have a natural embedding

$$T(\mathcal{B}) \hookrightarrow T(\mathcal{M}(\mathcal{B})) : \tau \mapsto \tau.$$

Of course, we are bringing $\tau \in T(\mathcal{B})$ to its strictly lower semicontinuous extension trace $\tau \in T(\mathcal{M}(\mathcal{B}))$.

Lemma 2.10. *The affine map*

$$T(\mathcal{B}) \hookrightarrow T(\mathcal{M}(\mathcal{B})) : \tau \mapsto \tau$$

is weak-weak* continuous.*

Proof. This is Lemma 2.8 in [6]. □

With the embedding $T(\mathcal{B}) \hookrightarrow T(\mathcal{M}(\mathcal{B}))$, we may also view $T(\mathcal{B})$ as a subspace of $T(\mathcal{M}(\mathcal{B}))$. In what follows, we sometimes do this implicitly.

For nonexperts, we state the following well-known and standard result:

Lemma 2.11. *Let K be a compact convex set, and suppose that $E \subseteq Aff(K)$ is a linear subspace that contains 1 and separates the points of K. Then E is uniformly dense in $Aff(K)$.*

Proof. This is a standard result. But we note, for the convenience of the reader, that it follows immediately from [33] Corollary I.1.11 (on page 7) and the Hahn–Banach theorem for Banach spaces. (The result [33] Corollary I.1.11 says that every bounded linear functional on $Aff(K)$ has the form $\alpha ev_\tau - \beta ev_\mu$, where $\alpha, \beta \in [0, \infty)$, $\tau, \mu \in K$, and $ev_\tau, ev_\mu \in Aff(K)^*$ are the evaluations at τ, μ respectively. This is not hard to prove. But since [33] is an unpublished manuscript, we note that [33] Corollary I.1.11 can be replaced with [1] Theorem I.2.6, the Hahn–Banach theorem, and the Riesz representation theorem for $C(K)^*$ which can be found in [30] 13.5.25.) □

We next define a notion which will be used prominently in this paper:

Definition 2.12. *Let $\mathcal{B}$ be a nonunital, separable, simple C*-algebra. We say that $\mathcal{B}$ has* Property I *if the natural map*

$$K_0(\mathcal{M}(\mathcal{B})) \to Aff(T(\mathcal{M}(\mathcal{B})))$$

has image which separates the points of $T(\mathcal{M}(\mathcal{B}))$.

Of course, in the above, if $T(\mathcal{M}(\mathcal{B})) = \emptyset$ then, trivially, $\mathcal{B}$ has Property I. See Remark 2.6 for remarks on this degenerate case.

Property I is a crucial assumption in the main result of this paper. As stated previously, one key reason why we will need to assume Property I in our work is the presence of mysterious traces inside $T(\mathcal{M}(\mathcal{B}))$. These mysterious traces are not as accessible as the "computable" traces which are strict extensions of traces on $\mathcal{B}$. We will show in subsection 3.3 that many simple C*-algebras have Property I.

The next result and its proof is essentially a variation on [6, Theorem 2.10] (which was for AF algebras), except that we work with more general algebras, and we bring in comparison theory for multiplier algebras to make the argument work in the more general context. We provide the proof for the convenience of the reader.

Theorem 2.13. *Let $\mathcal{B}$ be a nonunital, separable, simple C*-algebra with stable projection injectivity and surjectivity such that $T(\mathcal{B})$ is compact. Suppose, in addition, that $\mathcal{B}$ has Property I. Then $T(\mathcal{B}) = T(\mathcal{M}(\mathcal{B}))$.*

Proof. By Remark 2.6, $T(\mathcal{B}) = \emptyset$ if and only if $T(\mathcal{M}(\mathcal{B})) = \emptyset$. So we may assume that $T(\mathcal{B}) \neq \emptyset \neq T(\mathcal{M}(\mathcal{B}))$. This implies that $T_e(\mathcal{B}) \neq \emptyset$.

Since $K_0(\mathcal{M}(\mathcal{B}))$ separates the points of $T(\mathcal{M}(\mathcal{B}))$, it follows, by Lemma 2.11, that $\mathbb{Q}$ times the image of $K_0(\mathcal{M}(\mathcal{B}))$ is uniformly dense in $Aff T(\mathcal{M}(\mathcal{B}))$.

Let $x \in K_0(\mathcal{M}(\mathcal{B}))$ be arbitrary. Suppose that $x = [P] - [Q]$, where P, Q are projections in $\mathcal{M}(\mathcal{B}) \otimes \mathcal{K} - \mathcal{B} \otimes \mathcal{K}$. By Lemma 2.7, we know that if x is strictly greater than 0 on $T(\mathcal{B})$, then there exists a projection $R \in \mathcal{M}(\mathcal{B}) \otimes \mathcal{K} - \mathcal{B} \otimes \mathcal{K}$ such that $\tau(x) = \tau(R)$ for all $\tau \in T(\mathcal{B})$. Hence,

$$\widehat{P} + \widehat{1_{\mathcal{M}(\mathcal{B})}} = \widehat{Q} + \widehat{R} + \widehat{1_{\mathcal{M}(\mathcal{B})}}$$

on $T_e(\mathcal{B})$, for some $e \in Ped(\mathcal{B})_+ - \{0\}$.

Since $\mathcal{B} \otimes \mathcal{K}$ has projection injectivity,

$$P \oplus 1_{\mathcal{M}(\mathcal{B})} \sim Q \oplus R \oplus 1_{\mathcal{M}(\mathcal{B})},$$

where $\sim$ is Murray–von Neumann equivalence of projections in $\mathcal{M}(\mathcal{B}) \otimes \mathcal{K}$. It follows that $x = [R]$, i.e., x is nonnegative in $Aff(T(\mathcal{M}(\mathcal{B})))$.

If an element x in $\mathbb{Q}$ times the image of $K_0(\mathcal{M}(\mathcal{B}))$ in $Aff(T(\mathcal{M}(\mathcal{B})))$ is strictly positive on $T(\mathcal{B})$, then x is nonnegative in $Aff(T(\mathcal{M}(\mathcal{B})))$. Since $\mathbb{Q}$ times the image of $K_0(\mathcal{M}(\mathcal{B}))$ is uniformly dense in $Aff(T(\mathcal{M}(\mathcal{B})))$, we have that if an arbitrary element of $Aff(T(\mathcal{M}(\mathcal{B})))$ is nonnegative on $T(\mathcal{B})$ then it is nonnegative on $T(\mathcal{M}(\mathcal{B}))$. Here is the argument: Let $f \in Aff(T(\mathcal{M}(\mathcal{B})))$. If $f \geq 0$ on $T(\mathcal{B})$ then for every $\epsilon > 0$, $f + \epsilon > 0$ on $T(\mathcal{B})$ and hence, $f + \epsilon \geq 0$ on $T(\mathcal{M}(\mathcal{B}))$. Hence, $f \geq 0$ on $T(\mathcal{M}(\mathcal{B}))$.)

By the Hahn–Banach separation theorem (applied to $Aff(T(\mathcal{M}(\mathcal{B})))^*$ with the weak* topology), it follows that $T(\mathcal{B})$ is weak* dense in $T(\mathcal{M}(\mathcal{B}))$. Since $T(\mathcal{B})$ is compact, we have that $T(\mathcal{B}) = T(\mathcal{M}(\mathcal{B}))$. □

We recall some more terminology from the theory of simple C*-algebras. Let $\mathcal{D}$ be a separable C*-algebra. For all $n \geq 1$, we have a *-embedding

$$\mathbb{M}_n \otimes \mathcal{D} \hookrightarrow \mathbb{M}_{n+1} \otimes \mathcal{D}$$

given by $b \mapsto diag(b, 0)$. We let $\mathbb{M}_\infty(\mathcal{D})$ denote the *-algebra

$$\mathbb{M}_\infty(\mathcal{D}) =_{df} \bigcup_{n=1}^{\infty} \mathbb{M}_n \otimes \mathcal{D}.$$

We have a subequivalence relation on positive elements, which generalizes Murray-von Neumann subequivalence of projections, that is given as follows: For all $a, b \in \mathbb{M}_\infty(\mathcal{D})_+$, $a \preceq b$ means that there exists an $N \geq 1$ with $a, b \in \mathbb{M}_N \otimes \mathcal{D}$ and a sequence $\{x_k\}$ in $\mathbb{M}_N \otimes \mathcal{D}$ such that $x_k b x_k^* \to a$. We define $a \sim b$ to mean $a \preceq b$ and $b \preceq a$.

We note that, when a and b are projections, $a \sim b$, as above defined, is *not* the same as Murray–von Neumann equivalence of projections. In fact, in any simple purely infinite C*-algebra (e.g., O_∞), any two nonzero positive elements a, b will satisfy $a \sim b$, as above defined – this includes the case where a, b are nonzero projections that are not Murray–von Neumann equivalent. *In the rest of the paper, we will let $\sim$ be as above defined (even for projections), and when we have Murray–von Neumann equivalent projections, we will explicitly say so.*

For all $a \in \mathbb{M}_\infty(\mathcal{D})_+$, we let $[a]$ be the equivalence class of a under $\sim$ in $\mathbb{M}_\infty(\mathcal{D})$ and let

$$W(\mathcal{D}) =_{df} \{[a] : a \in \mathbb{M}_\infty(\mathcal{D})_+\}.$$

$W(\mathcal{D})$ is an ordered semigroup under the order induced by $\preceq$ and with addition given by

$$[a] + [b] =_{df} [diag(a, b)].$$

Note that $W(\mathcal{D})$ is a generalization of the Murray–von Neumann semigroup (which consists of equivalence classes of projections).

Suppose that $\mathcal{D}$ is, additionally, simple. Then for all $[a] \in W(\mathcal{D}) - \{0\}$, $[a]$ induces an element

$$\widehat{[a]} \in LAff(T_e(\mathcal{D}))_{++}$$

given by

$$\widehat{[a]}(\tau) =_{df} d_\tau(a)$$

where

$$d_\tau(a) =_{df} \lim_{n\to\infty} \tau(a^{1/n}), \qquad \tau \in T_e(\mathcal{D}).$$

By the same procedure, $[a]$ also induces elements in $LAff(T(\mathcal{D}))_+$ as well as in $LAff(T(\mathcal{M}(\mathcal{D})))_+$ which we also denote by $\widehat{[a]}$.

Let

$$\mathbb{M}_\infty(\mathcal{D})\widehat{}_+ =_{df} \{\widehat{a} : a \in (\mathbb{M}_n \otimes \mathcal{D})_+ \text{ and } n \geq 1\}.$$

Suppose, in addition, that $\mathcal{D}$ is unital. We let $S(W(\mathcal{D}))$ denote the collection of all order preserving, semigroup maps $\rho : W(\mathcal{D}) \to [0, \infty)$ such that $\rho([1]) = 1$.

The next result, Theorem 2.15, is a main result of this paper. Before working on this theorem, we elaborate, for the benefit of the reader, a degenerate case. As witnessed by Remark 2.14, this case leads to trivially true statements. However, for expository purposes, it is better to make it explicit.

Remark 2.14. *Let $\mathcal{B}$ be a separable, simple, nonunital, $\mathcal{Z}$-stable C*-algebra with an approximate unit consisting of projections. Since $\mathcal{B}$ is $\mathcal{Z}$-stable, it is stably projection injective ([22]). By Remark 2.6, $T(\mathcal{B}) = \emptyset$ if and only if $T(\mathcal{M}(\mathcal{B})) = \emptyset$.* ***Suppose that we have*** *$T(\mathcal{B}) = \emptyset$. It follows, from the discussion of Remark 2.6, that $\mathcal{B}$ is stable and hence, $\mathcal{M}(\mathcal{B})$ contains a unital copy of O_2.*

From the above, it follows trivially that $C(\mathcal{B})$ is properly infinite, $T(\mathcal{B}) = \emptyset$ is weak-compact (by definition), the image of $T(\mathcal{B}) = \emptyset$ in $T(\mathcal{M}(\mathcal{B})) = \emptyset$ is weak* compact (by definition), $T(\mathcal{M}(\mathcal{B})) = \emptyset = T(\mathcal{B})$, $T(\mathcal{C}(B)) = \emptyset$, for every $r \in Proj(\mathcal{B})_+ - \{0\}$, $\mathbb{M}_\infty(r\mathcal{B}r)\widehat{}_+$ is uniformly dense in $Aff(T(\mathcal{M}(\mathcal{B})))_{++} \cup \{0\} = \emptyset$, $D_W(\mathcal{C}(\mathcal{B})) = \{[a] \in W(\mathcal{C}(\mathcal{B})) : a \in \mathcal{C}(\mathcal{B})\}$ is a semigroup, and $S(W(\mathcal{C}(\mathcal{B}))) = \emptyset$.* ***In other words, all the statements in Theorem 2.15 are trivially satisfied.***

Theorem 2.15. *Let $\mathcal{B}$ be a separable, simple, nonunital, $\mathcal{Z}$-stable C*-algebra with an approximate unit consisting of projections. Suppose, further, that $\mathcal{B}$ has Property I. Then the following statements are equivalent:*

1. *$\mathcal{C}(\mathcal{B})$ is properly infinite.*
2. *$T(\mathcal{B})$ is weak*-compact.*
3. *The image of $T(\mathcal{B})$ in $T(\mathcal{M}(\mathcal{B}))$ is weak*-compact.*
4. *$T(\mathcal{M}(\mathcal{B})) = T(\mathcal{B})$*
5. *$T(\mathcal{C}(\mathcal{B})) = \emptyset$.*
6. *For every $r \in Proj(\mathcal{B})_+ - \{0\}$, $\mathbb{M}_\infty(r\mathcal{B}r)\widehat{}_+$ is uniformly dense in*
$$Aff(T(\mathcal{M}(\mathcal{B})))_{++} \cup \{0\}.$$
7. *$D_W(\mathcal{C}(\mathcal{B})) =_{df} \{[a] \in W(\mathcal{C}(\mathcal{B})) : a \in \mathcal{C}(\mathcal{B})_+\}$ is a semigroup.*
8. *$S(W(\mathcal{C}(\mathcal{B}))) = \emptyset$.*

Proof. By Remark 2.14, we may assume $T(\mathcal{B}) \neq \emptyset \neq T(\mathcal{M}(\mathcal{B}))$. So $T_e(\mathcal{B}) \neq \emptyset$.

(5) ⇒ (4): Let $\tau \in T(\mathcal{M}(\mathcal{B}))$ be arbitrary. Let τ' be the unique strictly lower semicontinuous extension of $\tau|_{\mathcal{B}}$ to a bounded trace on $\mathcal{M}(\mathcal{B})$. By Lemma 2.9,

$\tau - \tau'$ is a (positive) bounded trace on $T(\mathcal{M}(\mathcal{B}))$ which is zero on $\mathcal{B}$. Hence, $\tau - \tau'$ induces a (positive) bounded trace on $\mathcal{C}(\mathcal{B})$. Since $T(\mathcal{C}(\mathcal{B})) = \emptyset$, we have $\tau - \tau' = 0$, i.e., $\tau = \tau'$. So $\tau \in T(\mathcal{B})$.

(4) ⇒ (3): This implication is immediate since $T(\mathcal{M}(\mathcal{B}))$ is weak* compact.

(3) ⇒ (2): This implication follows from the fact that the restriction map

$$T(\mathcal{M}(\mathcal{B})) \to (\text{bounded traces on } \mathcal{B}) : \tau \mapsto \tau|_{\mathcal{B}}$$

is weak*-weak* continuous. Note that the above map takes the image of $T(\mathcal{B})$ in $T(\mathcal{M}(\mathcal{B}))$ back to $T(\mathcal{B})$.

(4) ⇒ (6): Let $f \in Aff(T(\mathcal{M}(\mathcal{B})))_{++} = Aff(T(\mathcal{B}))_{++}$ be arbitrary. Since $\mathcal{B}$ has projection surjectivity, by Lemma 2.7, there exists a $P \in Proj(\mathcal{M}(\mathcal{B})) - \mathcal{B}$ such that $\tau(P) = f(\tau)$ for all $\tau \in T(\mathcal{B})$.

Let $b \in (P\mathcal{B}P)_+$ with $\|b\| = 1$ be a strictly positive element. Then $d_\tau(b) = f(\tau)$ for all $\tau \in T(\mathcal{B})$. Therefore,

$$\widehat{b^{1/n}} \nearrow f$$

pointwise on $T(\mathcal{B})$. Since f is continuous and $T(\mathcal{B})$ is compact,

$$\widehat{b^{1/n}} \nearrow f$$

uniformly on $T(\mathcal{B})$.

Let $\epsilon > 0$ be given and choose $N \geq 1$ so that

$$\widehat{b^{1/N}} \approx_{\frac{\epsilon}{2}} f$$

on $T(\mathcal{B})$. By a similar argument,

$$\widehat{(b-\gamma)_+^{1/N}} \nearrow \widehat{b^{1/N}}$$

uniformly on $T(\mathcal{B})$ as $\gamma \to 0+$. Choose $\delta > 0$ so that

$$\widehat{(b-\delta)_+^{1/N}} \approx_\epsilon f$$

on $T(\mathcal{B})$. Since f and ϵ were arbitrary and since $\mathcal{B}$ is simple, the $\widehat{\mathbb{M}_\infty(r\mathcal{B}r)}_+$ is uniformly dense in $Aff(T(\mathcal{B}))_{++} \cup \{0\}$. Since $T(\mathcal{B}) = T(\mathcal{M}(\mathcal{B}))$, we are done.

(6) ⇒ (5): Suppose, to the contrary, that $T(\mathcal{C}(\mathcal{B})) \neq \emptyset$. So let $\tau \in T(\mathcal{M}(\mathcal{B}))$ be such that $\tau|_{\mathcal{B}} = 0$. Since $\widehat{\mathbb{M}_\infty(r\mathcal{B}r)}_+$ is uniformly dense in

$$Aff(T(\mathcal{M}(\mathcal{B})))_{++} \cup \{0\}.$$

Let $n \geq 1$ and $b \in \mathbb{M}_n(\mathcal{B})_+$ such that

$$|\sigma(b) - \sigma(1_{\mathcal{M}(\mathcal{B})})| < 1/3$$

for all $\sigma \in T(\mathcal{M}(\mathcal{B}))$, that is, $|\sigma(b) - 1| < 1/3$ for all $\sigma \in T(\mathcal{M}(\mathcal{B}))$. Hence, $\sigma(b) > 2/3$ for all $\sigma \in T(\mathcal{M}(\mathcal{B}))$. But this contradicts that $\tau(b) = 0 \leq 2/3$. Thus we must have $T(\mathcal{C}(\mathcal{B})) = \emptyset$.

(2) ⇒ (1): Let $\{e_n\}$ be an approximate unit for $\mathcal{B}$, consisting of an increasing sequence of projections. Note that

$$\widehat{e_n} \nearrow \widehat{1_{\mathcal{M}(\mathcal{B})}}$$

pointwise on $T(\mathcal{B})$.

Since $\widehat{e_n}$ and $\widehat{1_{\mathcal{M}(\mathcal{B})}}$ are continuous on $T(\mathcal{B})$, and since $T(\mathcal{B})$ is compact, we can find $N \geq 1$ so that

$$0 < 2\tau(1_{\mathcal{M}(\mathcal{B})} - e_N) < 1$$

for all $\tau \in T(\mathcal{B})$. By Lemma 2.7, there exists a projection $P \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ such that

$$\tau(P) = 2\tau(1_{\mathcal{M}(\mathcal{B})} - e_N)$$

for all $\tau \in T(\mathcal{B})$. Hence,

$$1 = \tau(1 - P) + 2\tau(1_{\mathcal{M}(\mathcal{B})} - e_N)$$

for all $\tau \in T(\mathcal{B})$ and

$$\tau(1_{\mathcal{M}(\mathcal{B})}) = \tau(1 - P) + 2\tau(1 - e_N)$$

for all $\tau \in T_e(\mathcal{B})$.

Since $\mathcal{B}$ is $\mathcal{Z}$-stable, $\mathcal{B} \otimes \mathcal{K}$ has projection injectivity. So we have a Murray-von Neumann equivalence of projections

$$1_{\mathcal{M}(\mathcal{B})} \sim (1_{\mathcal{M}(\mathcal{B})} - P) \oplus (1_{\mathcal{M}(\mathcal{B})} - e_N) \oplus (1_{\mathcal{M}(\mathcal{B})} - e_N).$$

Hence,

$$1_{\mathcal{C}(\mathcal{B})} \succeq 1_{\mathcal{C}(\mathcal{B})} \oplus 1_{\mathcal{C}(\mathcal{B})}$$

in $\mathbb{M}_2 \otimes \mathcal{C}(\mathcal{B})$ and $\mathcal{C}(\mathcal{B})$ is properly infinite, as required.

(1) ⇒ (7): Since $\mathcal{C}(\mathcal{B})$ is properly infinite, let $S, T \in \mathcal{C}(\mathcal{B})$ be isometries with pairwise orthogonal ranges such that

$$SS^* + TT^* \leq 1_{\mathcal{C}(\mathcal{B})}.$$

Let $[a], [b] \in D_W(\mathcal{C}(B))$ be arbitrary, where $a, b \in \mathcal{C}(B)_+$. Then

$$a \oplus b \sim SaS^* + TbT^*$$

in $\mathbb{M}_2 \otimes \mathcal{C}(B)$. Since $SaS^* + TbT^* \in \mathcal{C}(\mathcal{B})_+$, we have that

$$[a] + [b] = [SaS^* + TbT^*] \in D_W(\mathcal{C}(\mathcal{B})).$$

Since $[a], [b] \in D_W(\mathcal{C}(\mathcal{B}))$ were arbitrary, we have that $D_W(\mathcal{C}(\mathcal{B}))$ is a semigroup.

(7) ⇒ (1): Since $D_W(\mathcal{C}(\mathcal{B}))$ is a semigroup, let $a \in \mathcal{C}(\mathcal{B})_+$ with $||a|| < 1$ be such that $[1] + [1] = [a]$. Then $1 \oplus 1 \sim a \leq 1$, so $\mathcal{C}(\mathcal{B})$ is properly infinite.

(1) ⇒ (8): Since $\mathcal{C}(\mathcal{B})$ is properly infinite, we have $[1]+[1] \leq [1]$ in $W(\mathcal{C}(\mathcal{B}))$. If we apply any $\rho \in S(W(\mathcal{C}(\mathcal{B})))$, we get that $1 + 1 \leq 1$, i.e., $1 \leq 0$, which is false.

(8) ⇒ (5): This is immediate, since any $\tau \in T(\mathcal{C}(\mathcal{B}))$ induces an element of $S(W(\mathcal{C}(\mathcal{B})))$ which is given by

$$W(\mathcal{C}(\mathcal{B})) \to [0, \infty) : [C] \mapsto d_\tau(C).$$

□

Remark 2.16. *In Theorem* 2.15*, the hypothesis that $\mathcal{B}$ has an approximate unit consisting of projections is only used in the implication (2) ⇒ (1). The hypothesis of $\mathcal{Z}$-stability is only used in the implications (4) ⇒ (6) and (2) ⇒ (1). In fact, in (2) ⇒ (1), the hypothesis of $\mathcal{Z}$-stability can be replaced by the weaker hypotheses of stable projection injectivity and surjectivity for $\mathcal{B}$.*

Remark 2.17. *In analogy with condition (v) of Theorem 3.1 in* [6]*, it is tempting to believe that the following condition is equivalent to the statements in Theorem* 2.15*:*

(9) For some $b \in \mathcal{B}_+$, $\tau(b) \geq 1$ for all $\tau \in T(\mathcal{B})$.

However, we do not at present know how to prove the equivalence. Also, given the gap in the argument of [6] *(see the discussion after the proof of Lemma* 2.7 *and Proposition* 3.11*), the statement may not be equivalent.*

3. Examples

3.1. Examples of corona algebras which are not properly infinite. We here construct a large class of nonproperly infinite corona algebras. In fact, we will show that whenever $\mathcal{B}$ is a simple, finite, stable C*-algebra with standard regularity properties and infinite $\partial_{ext}T_e(\mathcal{B})$, then $\mathcal{B}$ always has a hereditary C*-subalgebra with a corona algebra which is not properly infinite.

The construction involves using many ideas from [11], and we first review some notations introduced in that and previous papers.

Let $\mathcal{B}$ be a nonunital, separable, simple C*-algebra. Then $\mathcal{M}(\mathcal{B})$ contains a unique smallest ideal $\mathcal{I}_{min}$ properly containing $\mathcal{B}$ (see, for example, [13]). Let $e \in Ped(\mathcal{B})_+ - \{0\}$. When $\mathcal{B}$ is, additionally, nonelementary and has strict comparison of positive elements by traces, then $\mathcal{I}_{min}$ is the ideal of $\mathcal{M}(\mathcal{B})$ generated by

$$\{A \in \mathcal{M}(\mathcal{B})_+ : \widehat{A} \text{ is continuous on } T_e(\mathcal{B})\}.$$

See [10] Theorem 5.6.

Next, $\mathcal{I}_{fin}$ is the ideal of $\mathcal{M}(\mathcal{B})$ generated by

$$\{A \in \mathcal{M}(\mathcal{B})_+ : \widehat{A}(\tau) < \infty, \forall \tau \in \partial_{ext}T_e(\mathcal{B})\}.$$

Note that $\mathcal{I}_{min} \subseteq \mathcal{I}_{fin}$.

Proposition 3.1. *Let $\mathcal{B}$ be a nonelementary, separable, simple, stable C*-algebra with strict comparison of positive elements by traces, and projection surjectivity and injectivity such that $T_e(\mathcal{B})$ has infinitely many extreme points for some $e \in Ped(\mathcal{B})_+ - \{0\}$. Then there exists a nonunital hereditary C*-subalgebra $\mathcal{D} \subseteq \mathcal{B}$ such that $\mathcal{C}(\mathcal{D})$ is not properly infinite.*

Proof. By Corollary 7.12 of [11], $\mathcal{I}_{min} \neq \mathcal{I}_{fin}$. Let $A \in \mathcal{M}(\mathcal{B})_+$ be such that $A \in \mathcal{I}_{fin} - \mathcal{I}_{min}$. Since $\mathcal{B}$ is projection surjective, let $P \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ be a projection such that $\widehat{P} = \widehat{A}$. As $A \notin \mathcal{I}_{min}$, $\widehat{P} = \widehat{A}$ is not continuous. By Proposition

5.2(iii) of [10], $P \in \mathcal{I}_{fin} - \mathcal{I}_{min}$. Therefore, by Proposition 7.2 of [11], $\pi(P)$ is not properly infinite, where $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{C}(\mathcal{B})$ is the natural quotient map. Hence, if we define $\mathcal{D} = P\mathcal{B}P$, then $\mathcal{D}$ is a nonunital hereditary C*-subalgebra of $\mathcal{B}$ such that $\mathcal{C}(\mathcal{D})$ is not properly infinite. □

Remark 3.2. *Proposition* 3.1 *provides lots of examples. For example, if $\mathcal{A}$ is a unital, separable, simple C*-algebra with infinite $\partial_{ext}T(\mathcal{A})$ such that either i. $\mathcal{A}$ is an AF algebra, or ii. $\mathcal{A}$ is the crossed product from a Cantor minimal system, then $\mathcal{A} \otimes \mathcal{K}$ has a nonunital hereditary C*-subalgebra $\mathcal{D}$ for which $\mathcal{C}(\mathcal{D})$ is not properly infinite.*

In fact, if $\mathcal{A}$ is any simple, unital, separable, nuclear, $\mathcal{Z}$-stable C-algebra with infinite $\partial_{ext}T(\mathcal{A})$, then $\mathcal{A} \otimes \mathcal{K}$ has a nonunital hereditary C*-subalgebra $\mathcal{D}$ for which $\mathcal{C}(\mathcal{D})$ is not properly infinite.*

3.2. Properly infinite corona algebras which are not purely infinite. Recall that a nonzero C*-algebra $\mathcal{C}$ is said to be *purely infinite* if $c \oplus c \preceq c$ for all $c \in \mathcal{C}_+$, where the relation $\preceq$ is a subequivalence relation between positive elements in $\mathcal{C} \otimes \mathcal{K}$ which generalizes Murray–von Neumann subequivalence for projections. Here is the precise definition (again): For all $a, b \in (\mathcal{C} \otimes \mathcal{K})_+$, $a \preceq b$ if there exists a sequence $\{x_n\}$ in $\mathcal{C} \otimes \mathcal{K}$ such that $x_n b x_n^* \to a$.

Following a long history of much previous research (e.g., [13]; see also the references in [11]), purely infinite corona algebras were characterized in [11]. This was partly motivated by the longstanding recognition by previous workers that, in the stable classical extension theory, the only cases for which we have a complete classification are the ones for which the corona algebras are simple purely infinite (e.g., see [14], [21], [8] and the references therein). Thus, following previous authors, the study in [11] was meant to map out a class of algebras for which we expect to develop a "nicest extension theory". Nonetheless, properly infinite corona algebras, which are not necessary purely infinite, have suitable features which should allow for the development of a reasonable theory, and this is one motivation for the present paper.

Among other things, the classical stable case itself gives many examples of properly infinite, but not purely infinite, corona algebras. This follows immediately from the examples in the previous section.

Proposition 3.3. *Let $\mathcal{B}$ be a separable, simple, stable C*-algebra with strict comparison of positive elements by traces, and projection surjectivity and injectivity such that $T_e(\mathcal{B})$ has infinitely many extreme points for some $e \in Ped(\mathcal{B})_+ - \{0\}$. Then $\mathcal{C}(\mathcal{B})$ is properly infinite but not purely infinite.*

Proof. Since $\mathcal{B}$ is stable, $\mathcal{C}(\mathcal{B})$ is certainly properly infinite. By the proof of Proposition 3.1, $\mathcal{C}(\mathcal{B})$ cannot be purely infinite, since every nonzero hereditary subalgebra of a purely infinite C*-algebra is purely infinite. (With notation as in the proof of Proposition 3.1, $\mathcal{D} = P\mathcal{B}P$. So $\mathcal{C}(\mathcal{D}) \cong \pi(P)\mathcal{C}(\mathcal{B})\pi(P)$.) □

While Proposition 3.3 is nice, of greater interest are examples where the canonical ideal is not stable. Here is a such a result:

Proposition 3.4. *Let X be a compact metric space and let K be the Bauer simplex with extreme boundary $\partial_{ext}K = X$. Suppose that $C \subseteq X$ is a closed subset such that $X - C$ has infinitely many points. And suppose that $\mathcal{B}$ is a separable, simple, stable C*-algebra with real rank zero, and projection surjectivity and injectivity such that $T_e(\mathcal{B}) \cong K$. Then there exists a nonunital, nonstable hereditary C*-subalgebra $\mathcal{D} \subseteq \mathcal{B}$ such that $\mathcal{C}(\mathcal{D})$ is properly infinite but not purely infinite.*

Proof. Let $f \in LAff(K)_{++}$ be given by

$$f(\tau) =_{df} \begin{cases} 1 & \tau \in C \\ \infty & \tau \in X - C. \end{cases}$$

Since $\mathcal{B}$ is projection surjective, let $P \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ be a projection such that $\widehat{P} = f$. Let $\mathcal{D} =_{df} P\mathcal{B}P$.

Note that

$$T(\mathcal{D}) \cong \overline{Conv(C)},$$

where the closure is taken in K. By Theorem 2.15 and Remark 2.16, $\mathcal{C}(\mathcal{D})$ is properly infinite. Since $T(\mathcal{D}) \neq \emptyset$, $\mathcal{D}$ is not stable. Finally, by [11] Theorem 7.11, $\mathcal{C}(\mathcal{D})$ cannot be purely infinite. □

3.3. Simple C*-algebras with property I. Recall that one of the present difficulties is the presence of mysterious traces in $T(\mathcal{M}(\mathcal{B}))$ which are not as accessible. These mysterious traces are difficult to get a handle on, since they are not like the "computable traces" which are strictly lower semicontinuous extensions of traces on $\mathcal{B}$. (See Lemma 2.9 and the paragraphs before that.) As a consequence, in our main result, we need to have Property I as an assumption. We will presently prove that many simple C*-algebras have Property I, and thus, this hypothesis is not too stringent.

Recall that a nonunital, separable, simple C*-algebra $\mathcal{B}$ has *Property I* if the image of $K_0(\mathcal{M}(\mathcal{B}))$, in $Aff(T(\mathcal{M}(\mathcal{B})))$, separates the points of $T(\mathcal{M}(\mathcal{B}))$.

Proposition 3.5. *If $\mathcal{B}$ is a separable, simple, and stable C*-algebra, then $\mathcal{B}$ has Property I.*

Proof. This follows immediately from the fact that since $\mathcal{B}$ is stable, $\mathcal{M}(\mathcal{B})$ is properly infinite and so, $T(\mathcal{M}(\mathcal{B})) = \emptyset$. □

Proposition 3.6. *If $\mathcal{B}$ is a nonunital, separable, simple C*-algebra with real rank zero, then $\mathcal{B}$ has Property I.*

Proof. This follows immediately from Theorem 1.1 in [34], which implies that $\mathcal{M}(\mathcal{B})$ is the closed linear span of its projections. □

Remark 3.7. *We note that the class in Proposition 3.6 is a large class which includes all nonunital hereditary C*-subalgebras of $\mathcal{A} \otimes \mathcal{K}$, where $\mathcal{A}$ can be a simple nonelementary AF algebra, an irrational rotation algebra, a simple noncommutative torus, a crossed product coming from a Cantor minimal system, and many other C*-algebras.*

The next result and its proof is, in itself, of interest.

Lemma 3.8. *Let $\mathcal{B}$ be a nonunital, separable, simple, finite, $\mathcal{Z}$-stable C*-algebra with stable rank one such that $T(\mathcal{C}(\mathcal{B})) = \emptyset$. Then every element of $\mathcal{M}(\mathcal{B})$ is a finite linear combination of projections.*

Proof. Let $A \in \mathcal{M}(\mathcal{B})_+$. We will show that A is a finite linear combination of projections. We may assume that $\|A\| < 1$.

Since $\mathcal{B}$ has projection-surjectivity, let $P \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ be a projection such that $\tau(P) = \tau(A)$ for all $\tau \in T_e(\mathcal{B})$, where $e \in Ped(\mathcal{B})_+ - \{0\}$. Consider $B = A - P$. It suffices to prove that B is a finite linear combination of projections.

Since $T(\mathcal{C}(\mathcal{B})) = \emptyset$, it follows, by [28], that there exist $X_1, X_2, ..., X_n \in \mathcal{M}(\mathcal{B})$ such that

$$\pi(B) = \sum_{j=1}^{n} \pi([X_j, X_j^*]).$$

Let $b \in \mathcal{B}_{SA}$ be such that

$$B = \sum_{j=1}^{n} [X_j, X_j^*] + b.$$

Since $\tau(B) = 0$ for all $\tau \in T(\mathcal{B})$, it follows that $\tau(b) = 0$ for all $\tau \in T(\mathcal{B})$. Hence, by [26], there exist $m \geq 1$ and $y_j, z_j \in \mathcal{B}$, $1 \leq j \leq m$, such that

$$b = \sum_{j=1}^{m} [y_j, z_j].$$

Thus B is the sum of $n + m$ commutators in $\mathcal{M}(\mathcal{B})$.

Let $f \in LAff(T_e(\mathcal{B}))_{++}$ be given by $f = \widehat{1_{\mathcal{M}(\mathcal{B})}}/2$. Since $\mathcal{B}$ has projection surjectivity, let $Q \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ be a projection such that $\widehat{Q} = f$ on $T_e(\mathcal{B})$. Since $\mathcal{B}$ has stable projection injectivity, $1_{\mathcal{M}(\mathcal{B})} \sim Q \oplus Q$. Hence, there exist orthogonal projections $Q_1, Q_2 \in \mathcal{M}(\mathcal{B})$ such that $1_{\mathcal{M}(\mathcal{B})} = Q_1 + Q_2$ and $Q_1 \sim Q_2$. It follows that

$$\mathcal{M}(\mathcal{B}) \cong \mathbb{M}_2 \otimes Q_1 \mathcal{M}(\mathcal{B}) Q_1.$$

By Theorem 5.6 of [24], every commutator in $\mathcal{M}(\mathcal{B})$ is a finite linear combination of projections. So B is a finite linear combination of projections. □

Proposition 3.9. *Let $\mathcal{B}$ be a nonunital, separable, simple, finite, $\mathcal{Z}$-stable C*-algebra with stable rank one such that $T(\mathcal{C}(\mathcal{B})) = \emptyset$. Then $\mathcal{B}$ has Property I.*

Proof. This follows immediately from Lemma 3.8. □

Remark 3.10. *From Proposition* 3.9*, if $\mathcal{A}$ is any simple, separable, stable rank one, $\mathcal{Z}$-stable, finite C*-algebra, then any hereditary C*-subalgebra of $\mathcal{A} \otimes \mathcal{K}$, with continuous scale or quasicontinous scale (see* [11]*), will have Property I. For example, any nonunital hereditary C*-subalgebra of a unital, simple, separable, $\mathcal{Z}$-stable C*-algebra will have Property I. If $\mathcal{B}$ is any nonunital, simple, separable, $\mathcal{Z}$-stable C*-algebra with $T_e(\mathcal{B})$ having finitely many extreme points, then any nonunital hereditary C*-subalgebra of $\mathcal{B} \otimes \mathcal{K}$ has Property I.*

3.4. Counterexample to a statement of Elliott–Handelman. We now find a counterexample to [6, Lemma 2.9]. Before continuing, we note, again, that most of the proofs and the main result of [6] are still correct, and the main argument of [6] can be patched up by following the argument of the present paper (which uses many ideas from [6]).

Proposition 3.11. *There exist a nonunital, nonelementary, separable, simple AF algebra $\mathcal{B}$, an $f \in Aff(T(\mathcal{M}(\mathcal{B})))$, and $\epsilon > 0$ such that $\epsilon \leq f(\tau) \leq 1 - \epsilon$ for all $\tau \in T(\mathcal{B})$, and there exists no projection $P \in \mathcal{M}(\mathcal{B})$ for which $f(\tau) = \tau(P)$ for all $\tau \in T(\mathcal{B})$.*

Proof. Let K be the Bauer simplex with extreme boundary

$$\partial_{ext} K = \{0\} \cup \{\frac{1}{n} : n \geq 1\}.$$

By [2], there exists a unital, simple, nonelementary AF algebra $\mathcal{A}$ such that $T(\mathcal{A}) = K$. Identifying $\mathcal{A}$ with $\mathcal{A} \otimes e_{1,1}$ in $\mathcal{A} \otimes \mathcal{K}$, we have that

$$T_{1_{\mathcal{A}} \otimes e_{1,1}}(\mathcal{A} \otimes \mathcal{K}) = T(\mathcal{A}) = K.$$

We will denote the extreme trace corresponding to the point $1/n$ (or 0) by $\tau_{1/n}$ (resp. τ_0). Since $\mathcal{A}$ is AF, $\mathcal{A} \otimes \mathcal{K}$ has projection surjectivity. Let

$$Q, Q_1, Q_2 \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}) - (\mathcal{A} \otimes \mathcal{K})$$

be projections such that the following statements are true:

(1) $\widehat{Q}\left(\tau_{1/n}\right) = 1$ for all $n \geq 1$, and $\widehat{Q}(\tau_0) = 1/2$.
(2) $\widehat{Q_1} = \widehat{Q}/5$.
(3) $\widehat{Q_2}(\tau) = 2/5$ for all $\tau \in T_{1_{\mathcal{A}} \otimes e_{1,1}}(\mathcal{A} \otimes \mathcal{K})$.

Since

$$\widehat{Q_1} + \frac{4}{5}\widehat{Q} = \widehat{Q},$$

$\widehat{Q_1}$ is complemented under $\widehat{Q}$. Since $\mathcal{A} \otimes \mathcal{K}$ has projection injectivity and surjectivity, we have $Q_1 \preceq Q$. Thus we may assume that $Q_1 \leq Q$.

Similarly, we have that $\widehat{Q_2} + g = \widehat{Q}$, where

$$g : T_{1_{\mathcal{A}} \otimes e_{1,1}}(\mathcal{A} \otimes \mathcal{K}) \to (0, 1)$$

is the affine lower semicontinuous function given by $g = \widehat{Q} - (2/5)$. Hence, again, $\widehat{Q_2}$ is complemented under $\widehat{Q}$. Since $\mathcal{A} \otimes \mathcal{K}$ has projection injectivity and surjectivity, we have $Q_2 \preceq Q$. Thus we may again assume that $Q_2 \leq Q$.

Let $\mathcal{B} = Q(\mathcal{A} \otimes \mathcal{K})Q$. Then $Q_1, Q_2 \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ and

$$\widehat{Q_2} - \widehat{Q_1} \in Aff(T(\mathcal{M}(\mathcal{B})).$$

Note that

$$\frac{1}{5} \leq \tau(Q_2) - \tau(Q_1) \leq \frac{3}{10}$$

and

$$\frac{1}{2} \leq \tau(Q) \leq 1$$

for all $\tau \in T_{1_{\mathcal{A}} \otimes e_{1,1}}(\mathcal{A} \otimes \mathcal{K})$. The map

$$T_{1_{\mathcal{A}} \otimes e_{1,1}}(\mathcal{A} \otimes \mathcal{K}) \to T(\mathcal{B}) : \tau \mapsto \frac{1}{\tau(Q)} \tau|_{\mathcal{B}}$$

is a bijective map. Hence,

$$\frac{1}{5} \leq \tau(Q_2 - Q_1) \leq \frac{3}{5}$$

for all $\tau \in T(\mathcal{B})$.

Let $f \in Aff(T(\mathcal{M}(\mathcal{B})))$ be given by $f(\tau) = \tau(Q_2 - Q_1)$ for all $\tau \in T(\mathcal{M}(\mathcal{B}))$. Then we certainly have that there exists an $\epsilon > 0$ such that $\epsilon \leq f(\tau) \leq 1 - \epsilon$ for all $\tau \in T(\mathcal{B})$. Now, assume, to the contrary, that there exists a projection $P \in \mathcal{M}(\mathcal{B})$ such that $f(\tau) = \tau(P)$ for all $\tau \in T(\mathcal{B})$. Then

$$\widehat{P} + \widehat{Q_1} = \widehat{Q_2} \quad \text{on} \quad T_{1_{\mathcal{A}} \otimes e_{1,1}}(\mathcal{A} \otimes \mathcal{K}).$$

This is impossible, because $\widehat{Q_2}$ is continuous on $T_{1_{\mathcal{A}} \otimes e_{1,1}}(\mathcal{A} \otimes \mathcal{K})$, $\widehat{P}$ and $\widehat{Q_1}$ are both lower semicontinuous, but $\widehat{Q_1}$ is not continuous on $T_{1_{\mathcal{A}} \otimes e_{1,1}}(\mathcal{A} \otimes \mathcal{K})$. □

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