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Some new results about left ideals of βS

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ABSTRACT. The smallest ideal $K(\beta S)$ of the Stone-Čech compactification of a discrete semigroup S is the union of pairwise isomorphic and homeomorphic minimal left ideals. We provide a simple characterization of semigroups for which the smallest ideal of βS is finite and some necessary conditions for the minimal left ideals to be finite. We investigate when the smallest ideal of the Stone-Čech compactification of a Cartesian product can be homeomorphic to a Cartesian product of the smallest ideal of Stone-Čech compactifications. We extend some known results about the fact that, if S is a countably infinite cancellative semigroup, every non-minimal semiprincipal left ideal in βS contains many semiprincipal left ideals defined by right cancelable elements of βS. We conclude with some observations about the topological properties of semiprincipal left ideals in βS.

CONTENTS

1. Introduction

We continue our study of the algebraic and topological structure of the Stone-Čech compactification βS of a discrete semigroup S, with emphasis on the left ideals of βS. The left ideals are of analytical interest for at least two main reasons. That is, they are *orbit closures* of the action of S on βS and, if they are minimal, they are *universal minimal dynamical systems*. We will provide a more detailed description of these things after we provide a brief introduction to the algebraic structure of βS.

We take the points of βS to be the ultrafilters on S, identifying a point $x \in S$ with the principal ultrafilter $e(x) = \{A \subseteq S : x \in A\}$. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$ and $A^* = \overline{A} \setminus A$, the set of nonprincipal ultrafilters with A as a member. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for

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the closed sets) in βS. With this topology, βS is a compact Hausdorff space with the property that if X is any compact Hausdorff space and $f : S \to X$, there is a continuous function $\widetilde{f} : \beta S \to X$ which extends f.

The operation $\cdot$ on S extends uniquely to an operation on βS so that $(\beta S, \cdot)$ is a right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous. Given points p and q in βS and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{s \in S : s^{-1}A \in q\} \in p$ where $s^{-1}A = \{t \in S : st \in A\}$. We can also characterize $p \cdot q$ as $\lim_{s \to p} \lim_{t \to q} st$, where s and t denote elements of S. From this point on, we will write pq for $p \cdot q$. If A is any subset of a semigroup, then $E(A) = \{x \in A : x \text{ is an idempotent}\}$.

As a compact Hausdorff right topological semigroup, βS has idempotents and contains a smallest two sided ideal $K(\beta S)$. An idempotent in $K(\beta S)$ is said to be a *minimal* idempotent. An idempotent in βS is minimal if and only if it is minimal with respect to the ordering of idempotents defined by $p \leq q$ if and only if $p = pq = qp$.

The smallest ideal $K(\beta S) = \bigcup\{L : L \text{ is a minimal left ideal of } \beta S\} = \bigcup\{R : R$ is a minimal right ideal of $\beta S\}$. If L is a minimal left ideal of βS and R is a minimal right ideal of βS, then $L \cap R$ is a group, and any two such groups are isomorphic. Then $L \cap R$ is referred to as the *structure group* of βS. If L and L' are minimal left ideals of βS and R is a minimal right ideal of βS, then $L \cap R$ and $L' \cap R$ are topologically isomorphic. (When we say that subsets of right topological semigroups are *topologically isomorphic* we mean that there is a function taking one to the other which is both an isomorphism and a homeomorphism.) Any two minimal left ideals of βS are isomorphic and homeomorphic. We do not know in general whether they are topologically isomorphic. Any two minimal right ideals of βS are isomorphic.

If L is a minimal left ideal of βS and $p \in L$, then $L = (\beta S)p = \rho_p[\beta S]$ so minimal left ideals of βS are compact. We will use without comment the fact that if $p \in E(L)$, then p is a right identity for L. (If $q \in (\beta S)p$, then $q = rp$ for some r, so $qp = rpp = rp = q$.)

See [8, Part I] for an elementary introduction to the algebra and topology of βS.

If $p \in \mathbb{N}^*$, the notion of p-limit provides a uniform way of taking limits of sequences. If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in a compact Hausdorff space X, then $p\text{-}\lim_{n \in \mathbb{N}} x_n = y \in X$ if and only if for every neighborhood U of y, $\{n \in \mathbb{N} : x_n \in U\} \in p$.

If $p \in \beta S$, then $\{\lambda_s(p) : s \in S\}$ is the *orbit of p under S*, and the *orbit closure*, $c\ell\{\lambda_s(p) : s \in S\}$ is of substantial analytical interest. This orbit closure is $c\ell(Sp) = c\ell\rho_p[S] = (\beta S)p$, the *semiprincipal left ideal* generated by p. (The *principal left ideal* generated by p is $\{p\} \cup (\beta S)p$.)

If L is a minimal left ideal of βS and for $s \in S$, λ'_s is the restriction of λ_s to L, then $(L, \langle \lambda'_s \rangle_{s\in S})$ is a *universal minimal dynamical system* for S as described in [8, Section 19.1].

Given a set X, we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X. A set S is a *right zero semigroup* provided $xy = y$ for all x and y in S. A set S is a *left zero semigroup* provided $xy = x$ for all x and y in S.

We let $\omega = \mathbb{N} \cup \{0\}$. Then ω is the first infinite cardinal. Recall that cardinals are ordinals and each ordinal is the set of its predecessors.

In Section 2, we deal with the question of under what conditions the minimal left ideals of βS are finite. That characterization is simple in the event that $K(\beta S) \cap S \neq \emptyset$, namely that S has a finite left ideal. Much more interesting is the situation in which $K(\beta S) \subseteq S^*$. We obtain some necessary conditions for the existence of finite left ideals and conclude the section by showing that these conditions are far from sufficient, since they may hold with the left ideals having cardinality 2^{2^κ} for any infinite cardinal κ.

In Section 3, we obtain several results about the Stone-Čech compactification of Cartesian products of semigroups, Cartesian products of Stone-Čech compactifications of semigroups, and the relations between them and their smallest ideals.

In Section 4, we obtain some results about countably infinite cancellative semigroups including the fact that if S is such a semigroup, then the left ideals of βS that are contained in the smallest ideal are characterized as being those ideals that are the union of groups.

We conclude the introduction with some basic facts about F-spaces that we will need.

Definition 1.1. Let X be a completely regular Hausdorff space. Then $C(X)$ is the set of continuous real valued functions with domain X.

Recall that subsets A and B of a completely regular Hausdorff space X are *completely separated* if and only if there exists $f \in C(X)$ such that for all $x \in A$, $f(x) = 0$ and for all $x \in B$, $f(x) = 1$. (If A and B are nonempty, this assertion is equivalent to the statment that $f[A] = \{0\}$ and $f[B] = \{1\}$. We write it the way that we do because we want $\emptyset$ to be an F-space. If it is not, then Theorem 1.3(2) fails.)

Definition 1.2. (1) A topological space X is an F-space if and only if X is a completely regular Hausdorff space and for every $f \in C(X)$, $\{x \in X : f(x) < 0\}$ and $\{x \in X : f(x) > 0\}$ are completely separated.
(2) A topological space X is a P-space if and only if X is a completely regular Hausdorff space and every G_δ subset of X is open.

We will need the following well known result.

Theorem 1.3. (1) *If S is a discrete space, then βS is an F-space.*
(2) *A compact subset of an F-space is an F-space.*
(3) *If X and Y are nonempty completely regular Hausdorff spaces and $X \times Y$ is an F-space, then X and Y are F-spaces.*

(4) *If X and Y are nonempty completely regular Hausdorff spaces and $X \times Y$ is an F-space, then either X or Y is a P-space.*
(5) *Any compact P-space is finite.*

Proof. (1) This follows from [5, Theorem 14.25] and the fact that a discrete space is an F-space.

(2) Let Y be a compact subspace of the F-space X and let $f \in C(Y)$. By [5, Item 3.11(c)], pick $g \in C(X)$ such that $g_{|Y} = f$. Pick $h \in C(X)$ such that h completely separates $\{x \in X : g(x) < 0\}$ and $\{x \in X : g(x) > 0\}$. Then $h_{|Y}$ completely separates $\{x \in Y : f(x) < 0\}$ and $\{x \in Y : f(x) > 0\}$.

(3) Assume that $X \times Y$ is an F-space. It suffices to show that X is an F-space, so let $f \in C(X)$ be given. Define $g : X \times Y \to \mathbb{R}$ by $g(x, y) = f(x)$. Then $g \in C(X \times Y)$ so pick $h \in C(X \times Y)$ such that h completely separates $\{(x, y) \in X \times Y : f(x, y) < 0\}$ and $\{(x, y) \in X \times Y : f(x, y) > 0\}$. Pick $y \in Y$ and define $k \in C(X)$ by $k(x) = h(x, y)$. Then k completely separates $\{x \in X : f(x) < 0\}$ and $\{x \in X : f(x) > 0\}$.

(4) This is [5, Exercise 14Q(1)]. (This is an easy exercise following the hint and using the fact that a space X is a P-space if and only if for every $f \in C(X)$, $\{x \in X : f(x) = 0\}$ is open.)

(5) This is a consequence of [5, Exercise 4K(1)]. □

2. Finite minimal left ideals

In this section, we present several results about semigroups S that have the property that βS has finite minimal left ideals. There are some obvious examples of semigroups with this property, such as any semigroup with a right zero or $(\mathbb{N}, \max)$. As we shall see, this property is preserved by all finite products and by some infinite products. In [10], we had a section titled *Finitely many minimal right ideals*. As we see now, these notions are equivalent.

Lemma 2.1. *Let S be an infinite discrete semigroup and let L be a minimal left ideal of βS. The following statements are equivalent.*
(a) *L is finite.*
(b) *$E(L)$ is finite.*
(c) *There are finitely many minimal right ideals in βS.*

Proof. It is trivial that (a) implies (b). If R is a minimal right ideal of βS, then $R \cap L$ is a group and distinct minimal right ideals are disjoint, so (b) implies (c). If L were infinite, then by [8, Theorem 6.39], βS would contain at least $2^{\mathfrak{c}}$ minimal right ideals, so (c) implies (a). □

We shall need the following lemma which is analogous to [8, Theorem 2.23]. Since it is a purely algebraic statement, the corresponding statement with "left" replaced by "right" is also valid.

Lemma 2.2. *Let $\langle S_i \rangle_{i \in I}$ be a family of semigroups and let $S = \times_{i \in I} S_i$.*
(1) *If for each $i \in I$, L_i is a minimal left ideal of S_i, then $\times_{i \in I} L_i$ is a minimal left ideal of S.*

(2) *Assume L is a minimal left ideal of S and pick $\vec{x} \in L$. For each $i \in I$, let $L_i = S_i x_i$. Then $L = \times_{i\in I} L_i$ and for each $i \in I$, L_i is a minimal left ideal of S_i.*

Proof. (1) Let $L = \times_{i\in I} L_i$. Trivially L is a left ideal of S. To see that L is minimal, by [8, Lemma 1.52(a)] it suffices to show that for all $\vec{x} \in L$, $L = L\vec{x}$, so let $\vec{x} \in L$. Then $L\vec{x} \subseteq L$. To see that $L \subseteq L\vec{x}$, let $\vec{y} \in L$. For each $i \in I$, pick $z_i \in L_i$ such that $y_i = z_i x_i$. Then $\vec{y} = \vec{z}\vec{x} \in L\vec{x}$.

(2) To see that $L \subseteq \times_{i\in I} L_i$, let $\vec{y} \in L$. Then by [8, Lemma 1.52(a)], pick $\vec{z} \in L$ such that $\vec{y} = \vec{z}\vec{x}$. Then for each $i \in I$, $y_i = z_i x_i \in L_i$. To see that $\times_{i\in I} L_i \subseteq L$, let $\vec{y} \in \times_{i\in I} L_i$ and for $i \in I$, pick $\vec{z}_i \in S_i$ such that $y_i = z_i x_i$. Then $\vec{y} = \vec{z}\vec{x} \in L$.

Finally, let $i \in I$. To see that L_i is a minimal left ideal of S_i suppose instead there is a left ideal J of S_i properly contained in L_i. For $j \in I$, let $M_j = \begin{cases} J & \text{if } j = i \\ L_j & \text{if } j \neq i. \end{cases}$ Then $\times_{j\in I} M_j$ is a left ideal of S properly contained in L, a contradiction. □

We obtain a simple characterization of when $K(\beta S)$ is finite. Recall that we are identifying the points of S with the principal ultrafilters on S. Recall also that if $p \in \beta S$, then $||p|| = \min\{|A| : A \in p\}$.

Theorem 2.3. *Let S be an infinite discrete semigroup. If $K(\beta S)$ is finite, then $K(\beta S) \subseteq S$. Consequently, $K(\beta S)$ is finite if and only if there is a finite ideal of S.*

Proof. If I is a finite ideal of S, then by [8, Corollary 4.18], $c\ell(I)$ is an ideal of βS so $K(\beta S) \subseteq c\ell(I) = I$.

Now assume that $K(\beta S)$ is finite. It suffices to show that $K(\beta S) \cap S \neq \emptyset$, since then $K(\beta S) \cap S$ is a finite ideal of S.

So, suppose that $K(\beta S) \subseteq S^*$. Pick an idempotent $p \in K(\beta S)$ and let L be the minimal left ideal of βS such that $p \in L$. Let $\kappa = ||p||$. Then $\kappa \geq \omega$. Pick $D \in p$ such that $|D| = \kappa$.

Since L is finite, p is isolated in L. Pick $A \in p$ such that $\overline{A} \cap L = \{p\}$. Then $\overline{A} \in pp = \rho_p(p)$, so pick $C \in p$ such that $\overline{C}p \subseteq \overline{A}$. If $x \in C$, then $xp \in \overline{A} \cap L$ so $xp = p$. Thus for each $x \in C$, $\lambda_x(p) = p$ so by [8, Theorem 3.35], $B_x = \{a \in S : xa = a\} \in p$.

We claim that $\bigcap_{x\in C\cap D} \overline{B_x} \subseteq K(\beta S)$. This will contradict [8, Theorem 3.62] which says that $|\bigcap_{x\in C\cap D} \overline{B_x}| = 2^{2^\kappa}$. So, let $q \in \bigcap_{x\in C\cap D} \overline{B_x}$. Then for each $x \in C \cap D$, λ_x is the identity on B_x so $xq = q$. Then ρ_q is constantly equal to q on $C \cap D$, so $pq = q$ and thus $q \in K(\beta S)$ as claimed. □

We get an equally simple characterization of the existence of finite minimal left ideals of βS in the event that $K(\beta S) \cap S \neq \emptyset$.

Theorem 2.4. *Let S be an infinite discrete semigroup and assume that $K(\beta S) \cap S \neq \emptyset$. Then βS has finite minimal left ideals if and only if S has a finite left ideal.*

Proof. Sufficiency. Let L be a finite left ideal of S. By [8, Corollary 4.18], $c\ell(L)$ is a left ideal of βS which contains a minimal left ideal, and $c\ell(L) = L$.

Necessity. Pick $p \in K(\beta S) \cap S$ and let $L = \beta S p$. Then L is a minimal left ideal of βS. Given $s \in S$, $sp \in L \cap S$ so $L \cap S$ is a left ideal of S. □

Let $S = (\mathbb{N}, \vee)$ where $x \vee y = \max\{x, y\}$. Then by [8, Exercise 4.1.11], $K(\beta S) = \mathbb{N}^*$ and $K(\beta S)$ is a right zero semigroup. In particular, the fact that the minimal left ideals of βS are finite (in this case singletons) does not imply that $K(\beta S) \cap S \neq \emptyset$.

As a consequence of Theorem 2.4, we are interested in characterizing semigroups S such that $K(\beta S) \subseteq S^*$ and βS has finite minimal left ideals. The following simple result is one such characterization. However, it is not very satisfactory, since it depends on determining the existence of two nonprincipal ultrafilters.

Theorem 2.5. *Let S be an infinite discrete semigroup such that $K(\beta S) \subseteq S^*$. Then βS has finite minimal left ideals if and only if there exist a finite partition $\mathcal{R}$ of S and $q \in S^*$ such that for every $A \in \mathcal{R}$ there is some $r \in S^*$ such that $Aq = \{r\}$.*

Proof. Sufficiency. Pick $\mathcal{R}$ and q as specified. Let

$$L = \left\{r \in S^* : (\exists A \in \mathcal{R})(Aq = \{r\})\right\}.$$

Then $Sq \subseteq L$ so $\beta Sq = c\ell(Sq) \subseteq L$ and L is finite.

Necessity. Let L be a finite minimal left ideal of βS and pick $q \in L$. Then $Sq \subseteq L$ so $\beta Sq \subseteq L \subseteq S^*$. For $r \in L$, let $A_r = \{s \in S : sq = r\}$ and let $\mathcal{R} = \{A_r : r \in L\}$. □

We do get satisfactory characterizations of the existence of minimal left ideals as singletons.

Theorem 2.6. *Let S be an infinite discrete semigroup such that $K(\beta S) \subseteq S^*$. For $s \in S$, let $C_s = \{t \in S : st = t\}$. Then the minimal left ideals of βS are singletons if and only if $\{C_s : s \in S\}$ has the infinite finite intersection property.*

Proof. Necessity. Pick a minimal left ideal L of βS and pick $q \in S^*$ such that $L = \{q\}$. Then for each $s \in S$, $sq = q$ so by [8, Theorem 3.35], $C_s \in q$. Since q is closed under finite intersections and contains no finite sets the conclusion holds.

Sufficiency. Assume that $\{C_s : s \in S\}$ has the infinite finite intersection property and pick $q \in S^*$ such that $\{C_s : s \in S\} \subseteq q$. Given $s \in S$, λ_s is equal to the identity on a member of q so $sq = q$. Then ρ_q is constantly equal to q on S, so for all $p \in \beta S$, $pq = q$. □

Theorem 2.7. *Let S be an infinite discrete semigroup. The following statements are equivalent.*

(a) *The minimal left ideals of βS are singletons.*
(b) *For all $p \in \beta S$ and all $q \in K(\beta S)$, $pq = q$.*
(c) *$K(\beta S)$ is a right zero semigroup.*

Proof. Assume that the minimal left ideals of βS are singletons and let $q \in K(\beta S)$. Then $\{q\}$ is a left ideal of βS so $\beta S q = \{q\}$. Thus, (a) implies (b). It is trivial that (b) implies (c).

Now assume that $K(\beta S)$ is a right zero semigroup, let L be a minimal left ideal of βS, and let $q \in L$. Then $L = Lq \subseteq K(\beta S)q = \{q\}$. □

The next theorem provides a class of examples of semigroups S such that $K(\beta S) \subseteq S^*$ and the minimal left ideals are singletons.

Theorem 2.8. *Let κ be a discrete infinite cardinal. For $\alpha, \delta \in \kappa$, let $\alpha \vee \delta = \max\{\alpha, \delta\}$ and let $S = (\kappa, \vee)$. Let $U = \{p \in \beta S : (\forall B \in p)(B \text{ is cofinal in } \kappa)\}$. Then $K(\beta S) = U$, for all $q \in \beta S$ and all $p \in U$, $q \vee p = p$, and if $q \in \beta S \setminus U$ and $p \in U$, then $p \vee q = p$.*

Proof. First assume that $q \in \beta S$ and $p \in U$. For $s \in S$, $\{t \in \kappa : s < t\} \in p$ so $\lambda_s(t) = t$ for all t in a member of p and thus $\lambda_s(p) = p$. Thus, ρ_p is constantly equal to p on S so for all $q \in \beta S$, $q \vee p = p$.

Now assume that $q \in \beta S \setminus U$ and $p \in U$. Since $q \notin U$, pick $A \in q$ such that A is not cofinal in κ and let t be the supremum of A. Let $B = \{s \in S : t < s\}$. Then $B \in p$. For $x \in A$ and $s \in B$, $s \vee x = s$ so λ_s is constantly equal to s on A and so $s \vee q = s$. Then ρ_q is the identity on B, so $p \vee q = p$. The fact that $K(\beta S) = U$ follows immediately. □

We note next that given any $n \in \mathbb{N}$, there exists a semigroup S such that $K(\beta S) \subseteq S^*$ and the minimal left ideals of βS have n elements. A special case of this result was established by Will Brian in [3].

Theorem 2.9. *Let $(T, \cdot)$ be an infinite discrete semigroup, let $(F, *)$ be a finite semigroup, and let $S = T \times F$. Then βS is topologically isomorphic to $\beta S \times F$ and the minimal left ideals of βS are the sets of the form $L \times M$ where L is a minimal left ideal of βT and M is a minimal left ideal of F.*

Proof. Let ι be the identity function on $T \times F$ and let $\widetilde{\tau} : \beta S \to \beta T \times F$ be the continuous extension of ι. Noting that $T \times F$ is contained in the topological center of $\beta T \times F$, we have by [8, Corollary 4.22] that $\widetilde{\tau}$ is a homomorphism. Since $T \times F$ is dense in $\beta T \times F$, we have that $\widetilde{\tau}$ is surjective. To see that $\widetilde{\tau}$ is an isomorphism and a homeomorphism, it suffices to show that it is injective.

Since $S = \bigcup_{x \in F} (T \times \{x\})$, for each $p \in \beta S$, there is a unique $\psi(p) \in F$ such that $T \times \{\psi(p)\} \in p$. Define $\varphi(p) = \{A \subseteq T : A \times \{\psi(p)\} \in p\}$. It is a routine exercise to show that $\varphi(p)$ is an ultrafilter on T and $\widetilde{\tau}(p) = \big(\varphi(p), \psi(p)\big)$. It then follows that if p and q are in βS and $\widetilde{\tau}(p) = \widetilde{\tau}(q)$, then $p = q$.

Letting $S_1 = \beta T$ and $S_2 = F$, the assertion about minimal left ideals of βS follows from Lemma 2.2. □

By Theorem 2.8, if κ is an infinite discrete cardinal and $T = (\kappa, \vee)$, then $K(\beta T) \subseteq T^*$ and the minimal left ideals of βT are singletons. Thus, if one lets $n \in \mathbb{N}$ and $F = \mathbb{Z}_n$ in Theorem 2.9, then the minimal left ideals of βS all have n elements.

Lemma 2.10. *Let S be an infinite discrete semigroup and let L be a minimal left ideal of βS. For $x \in L$, let $B_x = \{s \in S : (\forall q \in E(L))(sq = x)\}$.*

(1) *For all $x, y \in L$, $B_x B_y \subseteq B_{xy}$.*
(2) *If x is isolated in L and $E(L)$ is finite, then $B_x \in x$ and in particular, $B_x \neq \emptyset$.*

Proof. (1) Let $x, y \in L$. Let R be the minimal right ideal of βS such that $y \in R$, let r be the identity of $R \cap L$, and note that $ry = y$. Now let $s \in B_x$ and let $t \in B_y$. To see that $st \in B_{xy}$, let $q \in E(L)$. Then $stq = sy = sry = xy$.

(2) Assume that x is isolated in L and $E(L)$ is finite. Pick $A \subseteq S$ such that $\overline{A} \cap L = \{x\}$. For each $q \in E(L)$, $xq = x$ so pick $C_q \in x$ such that $\overline{C_q}q \subseteq \overline{A}$. If $s \in C_q$, then $sq \in \overline{A} \cap L$ so $sq = x$. Therefore $\bigcap_{q \in E(L)} C_q \subseteq B_x$. □

Of course, Lemma 2.10(1) is trivial if $B_x = \emptyset$ or $B_y = \emptyset$. In Theorem 2.14 we will produce a semigroup S and a minimal left ideal L for which (1) $B_x \neq \emptyset$ if and only if x is isolated in L and (2) $|E(L)| \geq 2^{\mathfrak{c}}$. So, the sufficient condition of Lemma 2.10 is not necessary.

We show next that the existence of finite minimal left ideals guarantees substantial structure.

Theorem 2.11. *Let S be an infinite discrete semigroup and let L be a finite minimal left ideal of βS. For $x \in L$, let $B_x = \{s \in S : (\forall q \in E(L))(sq = x)\}$. Let $T = \bigcup_{x \in L} B_x$. Then $\{B_x : x \in L\}$ is a partition of T,*

$$T = \{s \in S : (\forall q, r \in E(L))(sq = sr)\},$$

T is a left ideal of S, $L \subseteq \overline{T}$, L is a minimal left ideal of $\overline{T}$, and for each $q \in E(L)$, the restriction of ρ_q to $\overline{T}$ is a homomorphism of $\overline{T}$ onto L.

Proof. We have by Lemma 2.10(2) that for each $x \in L$, $B_x \neq \emptyset$ and trivially if $x \neq y$, then $B_x \cap B_y = \emptyset$, so $\{B_x : x \in L\}$ is a partition of T. Trivially $T \subseteq \{s \in S : (\forall q, r \in E(L))(sq = sr)\}$. If $s \in S$ and for all $q, r \in E(L)$, $sq = sr$, then $s \in B_x$ where $x = sq$ for all $q \in E(L)$. We thus have that T is a left ideal of S. Given $x \in L$, $B_x \in x$, so $x \in \overline{T}$. Since $L \subseteq \overline{T}$, $\overline{T} \cap K(\beta S) \neq \emptyset$ so, by [8, Theorem 1.65(2)], L is a mimimal left ideal of $\overline{T}$. Finally, let $q \in E(L)$. For $x \in L$, ρ_q is constantly equal to x on B_x so $\rho_q[\overline{B_x}] = \{x\}$ and thus $\rho_q[\overline{T}] = L$. To see that the restriction of ρ_q to $\overline{T}$ is a homomorphism, let $u, v \in \overline{T}$. Pick x and y in L such that $B_x \in u$ and $B_y \in v$. Then $B_x B_y \in uv$ so by Lemma 2.10(1), $B_{xy} \in uv$ and so $\rho_q(uv) = xy = \rho_q(u)\rho_q(v)$. □

We note that if in Theorem 2.11, L is a group, equivalently βS has only one minimal right ideal, then $T = S$. We see now that in general one cannot require that $T = S$.

Theorem 2.12. *Let $F = \{e, a, b\}$ where $\{a, b\}$ is a left zero semigroup and e is an identity adjoined to $\{a, b\}$. Let $S = (\mathbb{N}, \vee) \times F$, let L be a minimal left ideal of βS, and let $q \in E(L)$. Then $K(\beta S) \subseteq S^*$ and ρ_q is not a homomorphism on βS.*

Proof. By Theorem 2.8, the minimal left ideals of $(\beta\mathbb{N}, \vee)$ are the sets $\{p\}$ for $p \in \mathbb{N}^*$. By Theorem 2.9, we may presume that $L = \{(p,a),(p,b)\}$ for some $p \in \mathbb{N}^*$. Assume without loss of generality that $q = (p,a)$. Then $\rho_q\big((1,e)(1,b)\big) = (1,b)(p,a) = (p,b)$ while $\rho_q(1,e)\rho_q(1,b) = (p,a)(p,b) = (p,a)$. □

We would like the properties of Theorem 2.11 to characterize finite minimal left ideals. We see now that we would also have to add the requirement that the structure group of βS is finite.

Theorem 2.13. *Let S be an infinite discrete semigroup, let L be a minimal left ideal of βS, and assume that there exists $t \in S$ such that for all q and r in $E(L)$, $tq = tr$. Let $T = \{s \in S : \big(\forall q, r \in E(L)\big)(sq = sr)\}$. Then T is a left ideal of S, the structure group of βS is finite, and for every $t \in T$, tL is a minimal right ideal of L so tL is a copy of the structure group of βS.*

Proof. Since $T \neq \emptyset$ it is trivially a left ideal of S.

We show now that for each $q \in E(L)$, each $t \in T$, and each $x \in L$, $tqx = tx$. To see this, let q, t, and x be given. Let R be the minimal right ideal to which x belongs, and let r be the identity of $L \cap R$ so that $rx = x$. Then $tqx = trx = tx$. Thus for $q \in E(L)$ and $t \in T$, the restriction of λ_{tq} to L equals λ_t so the restriction of λ_{tq} to L is continuous.

Next we show that for $t \in T$ and $q \in E(L)$, $tq\beta S \cap L = tqL$. Trivially, $tqL \subseteq tq\beta S \cap L$. Now let $x \in tq\beta S \cap L$ and pick $y \in \beta S$ such that $x = tqy$. Since $x \in L$, $xq = x$, so $x = xq = tqyq \in tqL$.

Let $P = tq\beta S \cap L$. Since $tq\beta S$ is a minimal right ideal of βS, P is a copy of the structure group of βS. Since $P = tqL = \lambda_{tq}[L]$ and the restriction of λ_{tq} to L is continuous, P is compact. Since P is compact and homogeneous, by [8, Theorem 6.38], P is finite.

Since $P = tq\beta S \cap L$, by [8, Theorem 1.65(2)], P is a minimal right ideal of βS so all minimal right ideals of L are copies of the structure group of βS.

To complete the proof we let $t \in T$. Pick $q \in E(L)$. Then since for $x \in L$, $tx = tqx$, $tL = tqL = tq\beta S \cap L$ and we have seen that $tq\beta S \cap L$ is a minimal right ideal of L. □

By Theorem 2.11, if L is finite, then the hypothesis of Theorem 2.13 holds. We see next that the properties of Theorems 2.11 and 2.13 do not characterize the existence of finite minimal left ideals. If X and Y are discrete sets, $p \in \beta X$, and $q \in \beta Y$, then the *tensor product* of p and q is the ultrafilter on $X \times Y$ defined by

$$p \otimes q = \{C \subseteq X \times Y : \{x \in X : \{y \in Y : (x,y) \in C\} \in q\} \in p\}.$$

Theorem 2.14. *Let A be an infinite left zero semigroup and let $S = A \times (\mathbb{N}, \vee)$. Let L be a minimal left ideal of βS, for $x \in L$, let*

$$B_x = \{s \in S : \big(\forall q \in E(L)\big)(sq = x)\},$$

and let $T = \{s \in S : \big(\forall q, r \in E(L)\big)(sq = sr)\}$. Then $K(\beta S) \subseteq S^*$, for $x \in L$, $B_x \neq \emptyset$ if and only if x is isolated in L, $T = \bigcup\{B_x : x \text{ is isolated in } L\}$, $T = S$, for

all $q \in E(L)$, ρ_q is a homomorphism from βS onto L, $|L| = 2^{2^\kappa}$ where $\kappa = |A|$, and the structure group of βS is a singleton. In particular, the mininimal left ideals of βS are topologically isomorphic.

Proof. We use [10, Theorem 4.4]. In the statement of that theorem, $B = \mathbb{N}$, $U = \mathbb{N}^*$, and $V = \{p \in \beta S : \widetilde{\pi_2}(p) \in \mathbb{N}^*\}$. Numbers in parentheses in the argument below refer to statements in [10, Theorem 4.4].

By (8), $K(\beta S) \subseteq S^*$. By (10) and (12), we may pick $y \in \mathbb{N}^*$ such that $\widetilde{\pi}_2(x) = y$ for all $x \in L$ and $L = \beta A \otimes y = \{p \otimes y : p \in \beta A\}$ so that $|L| = 2^{2^\kappa}$ by [8, Theorem 3.62]. By (11), $\{x \in L : x \text{ is isolated in } L\} = A \otimes y$.

Given $s = (a, n) \in S$ and $q \in E(L)$, by (5) $sq = a \otimes y$, so $s \in T$ and sq is isolated in L so $s \in B_{sq}$. Thus, $T = S = \bigcup\{B_x : x \text{ is isolated in } L\}$. If $x \in L$ and $B_x \neq \emptyset$, pick $s = (a, n) \in B_x$. As above, $x = a \otimes y$ so x is isolated in L.

By (8), $K(\beta S)$ consists of idempotents so the structure group of βS is a singleton. It remains to show that ρ_q is a homomorphism onto L for each $q \in E(L)$, so let $q \in E(L)$ be given. Then $\rho_q(pr) = prq = pq$ by (7) and $\rho_q(p)\rho_q(r) = pqrq = pq$, again by (7).

For the final assertion, note that if L' is a minimal left ideal of βS, and q is any element of L, then the restriction of ρ_q to L' is a homeomorphism from L' onto L. □

3. Cartesian products

In this section, we obtain several results involving the smallest ideal of a Cartesian product of semigroups and the Cartesian product of smallest ideals or of minimal left ideals.

Lemma 3.1. *Let $\upsilon \in \mathbb{N}$ and for $i \in \{1, 2, \ldots, \upsilon\}$, let S_i be an infinite discrete semigroup such that $K(\beta S_i)$ is finite. Then*

$$K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big) = \times_{i=1}^{\upsilon} K(\beta S_i) \subseteq \times_{i=1}^{\upsilon} S_i\,.$$

Proof. Let $\widetilde{\iota} : \beta(\times_{i=1}^{\upsilon} S_i) \to \times_{i=1}^{\upsilon} \beta S_i$ be the continuous extension of the identity function. Since $\widetilde{\iota}[\beta(\times_{i=1}^{\upsilon} S_i)]$ is a compact set containing $\times_{i=1}^{\upsilon} S_i$, $\widetilde{\iota}$ is surjective.

We note that if $p \in \beta(\times_{i=1}^{\upsilon} S_i)$ and $\widetilde{\iota}(p) \in \times_{i=1}^{\upsilon} S_i$, then $p \in \times_{i=1}^{\upsilon} S_i$. To see this, assume that $\widetilde{\iota}(p) = \vec{x} \in \times_{i=1}^{\upsilon} S_i$. Then $\times_{i=1}^{\upsilon}\{x_i\} = \{\vec{x}\}$ is a neighborhood of $\widetilde{\iota}(p)$ so pick $B \in p$ such that $\widetilde{\iota}[\overline{B}\,] \subseteq \{\vec{x}\}$. Then $B = \iota[B] \subseteq \{\vec{x}\}$ so $B = \{\vec{x}\}$.

By Theorem 2.3, each $K(\beta S_i) \subseteq S_i$. By [8, Exercise 1.7.3],

$$\widetilde{\iota}\big[K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)\big] = K(\times_{i=1}^{\upsilon} \beta S_i)$$

and by [8, Theorem 2.23], $K(\times_{i=1}^{\upsilon} \beta S_i) = \times_{i=1}^{\upsilon} K(\beta S_i) \subseteq \times_{i=1}^{\upsilon} S_i$. So, as we observed above, $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big) \subseteq \times_{i=1}^{\upsilon} S_i$. Thus,

$$\times_{i=1}^{\upsilon} K(\beta S_i) = \widetilde{\iota}\big[K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)\big] = K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)\,.$$

□

Theorem 3.2. *Let S and T be infinite discrete semigroups and let $u, v \in \mathbb{N}$ with $u > 1$. Statements (a) and (b) are equivalent and imply statements (c) and (d).*

(a) *$c\ell K\big(\beta(S^v)\big)$ and $c\ell K\big((\beta T)^u\big)$ are homeomorphic.*
(b) *$K(\beta S)$ and $K(\beta T)$ are finite and $|K(\beta S)|^v = |K(\beta T)|^u$.*
(c) *$K\big(\beta(S^v)\big)$ and $K\big((\beta T)^u\big)$ are homeomorphic.*
(d) *If $S = T$, then either $|K(\beta S)| = 1$ or $K\big(\beta(S^v)\big) = K\big((\beta S)^u\big)$.*

Proof. We show first that (b) implies (a) and (b) implies (c), so assume that (b) holds. By Theorem 2.3, $K(\beta S) \subseteq S$ and $K(\beta T) \subseteq T$. By Lemma 3.1 $K\big(\beta(S^v)\big) = \big(K(\beta S)\big)^v$. By [8, Theorem 2.23], $K\big(\beta(T)^u\big) = \big(K(\beta T)\big)^u$. As discrete spaces of the same cardinality, $K\big(\beta(S^v)\big)$ and $K\big((\beta T)^u\big)$ are homeomorphic. Since these spaces are finite, they equal their closures.

To see that (a) implies (b), assume that (a) holds. By Theorem 1.3 (1) and (2), $c\ell K\big(\beta(S^v)\big)$ is an F-space. But $c\ell K\big((\beta T)^u\big) = \big(c\ell K(\beta T)\big)^u$, so by Theorem 1.3(4), $c\ell K(\beta T)$ is a P-space, and by Theorem 1.3(5), $c\ell K(\beta T)$ is finite. Since $c\ell K\big(\beta(S^v)\big)$ and $c\ell K\big((\beta T)^u\big)$ are homeomorphic, they are the same size and both are finite, so $K\big(\beta(S^v)\big)$ and $K\big((\beta T)^u\big)$ are homeomorphic. By Lemma 3.1, $K\big(\beta(S^v)\big) = \big(K(\beta S)\big)^v$. Finally, by [8, Theorem 2.23], $K\big((\beta T)^u\big) = \big(K(\beta T)\big)^u$.

To conclude the proof, we assume that (b) holds and that $S = T$. If $|K(\beta S)| \neq 1$, then $u = v$ so Lemma 3.1 applies. □

Lemma 3.3. *Let I be a set and for $i \in I$, let S_i be an infinite discrete semigroup. For $i \in I$, let $\pi_i : \times_{j\in I} S_j \to S_i$ be the projection and let $\widetilde{\pi}_i : \beta(\times_{j\in I} S_j) \to \beta S_i$ be its continuous extension. Let*

$$M = \big\{p \in \beta(\times_{j\in I} S_j) : (\forall i \in I)\big(\widetilde{\pi}_i(p) \in K(\beta S_i)\big)\big\}.$$

Then M is an ideal of $\beta(\times_{j\in I} S_j)$.

Proof. Let $p \in M$ and $q \in \beta(\times_{j\in I} S_j)$. By [8, Corollary 4.22] given $i \in I$, $\widetilde{\pi}_i$ is a homomorphism so $\widetilde{\pi}_i(pq) \in K(\beta S_i)$ and $\widetilde{\pi}_i(qp) \in K(\beta S_i)$. □

Theorem 3.4. *Let $n \in \mathbb{N}$ and let $S = (\mathbb{N}, \vee)$. Let*

$$M = \{p \in \beta(S^n) : (\forall i \in \{1, 2, \ldots, n\})(\widetilde{\pi}_i(p) \in \mathbb{N}^*)\}.$$

Then $M = K\big(\beta(S^n)\big)$ and for all $p \in M$ and all $q \in \beta(S^n)$, $q \vee p = p$.

Proof. By Lemma 3.3, M is an ideal of $\beta(S^n)$ and so $K(\beta(S^n)\big) \subseteq M$. It then suffices to show that for all $p \in M$ and all $q \in \beta(S^n)$, $q \vee p = p$, because then for each $p \in M$, $\{p\}$ is a minimal left ideal of $\beta(S^n)$.

Let $p \in M$. We claim that for all $\vec{x} \in S^n$, $\vec{x} \vee p = p$. This will suffice since then ρ_p is constantly equal to p on S^n, hence on $c\ell(S^n) = \beta(S^n)$. So let $\vec{x} \in S^n$ be given. Given $i \in \{1, 2, \ldots, n\}$, we have $\{m \in \mathbb{N} : m > x_i\} \in \widetilde{\pi}_i(p)$ so we can pick $D_i \in p$ such that $\widetilde{\pi}_i[\,\overline{D_i}\,] \subseteq \overline{\{m \in \mathbb{N} : m > x_i\}}$.

If $\vec{y} \in \bigcap_{i=1}^n D_i$, then $\vec{x} \vee \vec{y} = \vec{y}$ so $\lambda_{\vec{x}}$ is the identity on a member of p, so $\lambda_{\vec{x}}(p) = p$ as required. □

Notice that if $S = T = (\mathbb{N}, \vee)$, and $u, v \in \mathbb{N}$, the assertion that $K\big(\beta(S^v)\big)$ and $K\big((\beta S)^u\big)$ are isomorphic does not imply any of the other statements of Theorem 3.2. Indeed, $K\big(\beta(S^v)\big)$ and $K\big((\beta S)^u\big)$ are both right zero semigroups of cardinality $2^{\mathfrak{c}}$ and are therefore isomorphic.

Lemma 3.5. *Let S be an infinite discrete semigroup, let L be a finite minimal left ideal of βS, and let q be an idempotent in L. For each $x \in L$, pick $a_x \in S$ such that $a_x q = x$. For each $b \in S$, $\{t \in S : bt = a_{bq}t\} \in q$.*

Proof. Note that by Lemma 2.10(2), for each $x \in L$, a_x exists. Pick $B \subseteq S$ such that $\overline{B} \cap L = \{q\}$. Since q is an idempotent, $Q = \{s \in S : s^{-1}B \in q\} \in q$. If $s \in Q$, then $sq \in \overline{B} \cap L$ so $Q \subseteq \{s \in S : sq = q\}$. Then given $s \in Q$, $\lambda_s(q) = q$ so by [8, Theorem 3.35], $Q_s = \{t \in S : st = t\} \in q$.

Now let $b \in S$ and let $x = bq$. Then $bq = a_x q$, $bQ \in bq$, and $a_x Q \in a_x q$. Therefore, $bQ \cap a_x Q \neq \emptyset$ so pick s and s' in Q such that $bs = a_x s'$. Given $t \in Q_s \cap Q_{s'}$, we have $bt = bst = a_x s't = a_x t$ so $Q_s \cap Q_{s'} \subseteq \{t \in S : bt = a_{bq}t\}$. □

Theorem 3.6. *Let $v \in \mathbb{N}$ and for $i \in \{1, 2, \ldots, v\}$, let S_i be an infinite discrete semigroup such that the minimal left ideals of βS_i are finite. Let $\widetilde{\tau} : \beta(\times_{i=1}^{v} S_i) \to \times_{i=1}^{v} \beta S_i$ be the continuous extension of the identity. Let M be a minimal left ideal of $\beta(\times_{i=1}^{v} S_i)$. Then $\widetilde{\tau}$ is injective on M. For each $i \in \{1, 2, \ldots, v\}$, there is a minimal left ideal L_i of βS_i such that the restriction of $\widetilde{\tau}$ to M is an isomorphism (and a homeomorphism) onto $\times_{i=1}^{v} L_i$.*

Proof. We put "and a homeomorphism" in parentheses because once we know that the restriction of $\widetilde{\tau}$ is an isomorphism, it is a bijection between discrete spaces of the same size.

Pick an idempotent $u \in M$ and let $\vec{q} = \widetilde{\tau}(u)$. By [8, Exercise 1.7.3], $\vec{q}$ is a minimal idempotent of $\times_{i=1}^{v} \beta S_i$. For $i \in \{1, 2, \ldots, v\}$, let $L_i = (\beta S_i) q_i$. By [8, Theorem 2.23], $\vec{q} \in \times_{i=1}^{v} K(\beta S_i)$ so each L_i is a minimal left ideal of βS_i. By Lemma 2.2, $\times_{i=1}^{v} L_i$ is a minimal left ideal of $\times_{i=1}^{v} \beta S_i$. By [8, Exercise 1.7.3] again, $\widetilde{\tau}[M]$ is a minimal left ideal of $\times_{i=1}^{v} \beta S_i$. Since $\widetilde{\tau}(u) \in \times_{i=1}^{v} L_i \cap \widetilde{\tau}[M]$, we have that $\times_{i=1}^{v} L_i = \widetilde{\tau}[M]$. Consequently, $|M| \geq |\times_{i=1}^{v} L_i|$.

For each $i \in \{1, 2, \ldots, v\}$ and each $x \in L_i$ pick by Lemma 2.10(2), $a_{i,x} \in S_i$ such that $a_{i,x} q_i = x$. For $i \in \{1, 2, \ldots, v\}$ and $b \in S_i$, let $Q_{i,b} = \{t \in S_i : bt = a_{i,bq_i}t\}$. Then by Lemma 3.5, $Q_{i,b} \in q_i$.

For $i \in \{1, 2, \ldots, v\}$ let $A_i = \{a_{i,x} : x \in L_i\}$ and note that $|A_i| = |L_i|$. Let $A = \times_{i=1}^{v} A_i$. We claim that for each $\vec{b} \in \times_{i=1}^{v} S_i$, there exists $\vec{a} \in A$ such that $\vec{b}u = \vec{a}u$. So let $\vec{b} \in \times_{i=1}^{v} S_i$ and let $\vec{a} = \langle a_{1,b_1 q_1}, a_{2,b_2 q_2}, \ldots, a_{v,b_v q_v} \rangle$. Then $\vec{a} \in A$. Now $\times_{i=1}^{v} \overline{Q_{i,b_i}}$ is a neighborhood of $\vec{q}$ so pick $B \in u$ such that $\widetilde{\tau}[\,\overline{B}\,] \subseteq \times_{i=1}^{v} \overline{Q_{i,b_i}}$. Then $B \subseteq \times_{i=1}^{v} Q_{i,b_i}$ so $\times_{i=1}^{v} Q_{i,b_i} \in u$. For $\vec{t} \in \times_{i=1}^{v} Q_{i,b_i}$, $\vec{b}\,\vec{t} = \vec{a}\,\vec{t}$ so $\lambda_{\vec{b}}$ and $\lambda_{\vec{a}}$ agree on a member of u so $\vec{b}u = \vec{a}u$ as required.

Now $M = c\ell\big((\times_{i=1}^{v} S_i)u\big)$ and $(\times_{i=1}^{v} S_i)u \subseteq \{\vec{a}u : \vec{a} \in A\}$. Since the latter set is finite, it is closed so $M \subseteq \{\vec{a}u : \vec{a} \in A\}$ so $|M| \leq |\{\vec{a}u : \vec{a} \in A\}| \leq |A| =$

$|\times_{i=1}^{\upsilon} L_i|$ and thus $|M| = |\times_{i=1}^{\upsilon} L_i|$. Since $\widetilde{\iota}[M] = \times_{i=1}^{\upsilon} L_i$ which is finite, $\widetilde{\iota}$ must be injective on M. By [8, Corollary 4.22], $\widetilde{\iota}$ is a homomorphism. □

Lemma 3.7. *Let κ be a cardinal and for $\sigma < \kappa$ let S_σ be an infinite discrete semigroup with the property that the minimal left ideals of βS_σ are singletons. Then the minimal left ideals of $\beta(\times_{\sigma<\kappa} S_\sigma)$ are singletons.*

Proof. For $\sigma < \kappa$, pick a minimal idempotent q_σ in βS_σ. Then $|(\beta S_\sigma)q_\sigma| = 1$ so for all $s \in S_\sigma$, $sq_\sigma = q_\sigma$. Thus, by [8, Theorem 3.35], for each $s \in S_\sigma$, $Q_{\sigma,s} = \{t \in S_\sigma : st = t\} \in q_\sigma$.

Let $S = \times_{\sigma<\kappa} S_\sigma$. For $\vec{s} \in S$, let $T_{\vec{s}} = \{\vec{t} \in S : \vec{s}\,\vec{t} = \vec{t}\}$. We claim that $\{T_{\vec{s}} : \vec{s} \in S\}$ has the finite intersection property. To see this, let $F \in \mathcal{P}_f(S)$. For $\sigma < \kappa$, pick $t_\sigma \in \bigcap\{Q_{\sigma,s_\sigma} : \vec{s} \in F\}$. We claim that $\vec{t} \in \bigcap_{\vec{s}\in F} T_{\vec{s}}$. Indeed, given $\vec{s} \in F$ and $\sigma < \kappa$, $t_\sigma \in Q_{\sigma,s_\sigma}$ so $s_\sigma t_\sigma = t_\sigma$ and thus $\vec{s}\,\vec{t} = \vec{t}$.

Pick $u \in \beta S$ such that $\{T_{\vec{s}} : \vec{s} \in S\} \subseteq u$. Then given $\vec{s} \in S$, $\lambda_{\vec{s}}$ is the identity on $T_{\vec{s}}$ so $\vec{s}u = u$. Then ρ_u is constantly equal to u on S, so for all $p \in \beta S$, $pu = u$. That is $|(\beta S)u| = 1$. □

Theorem 3.8. *Let κ be a cardinal and for $\sigma < \kappa$ let S_σ be an infinite discrete semigroup with the property that the minimal left ideals of βS_σ are finite and let L_σ be a minimal left ideal of βS_σ. The following statements are equivalent.*

(a) *The minimal left ideals of $\beta(\times_{\sigma<\kappa} S_\sigma)$ are topologically isomorphic to $\times_{\sigma<\kappa} L_\sigma$.*
(b) *The minimal left ideals of $\beta(\times_{\sigma<\kappa} S_\sigma)$ are homeomorphic to $\times_{\sigma<\kappa} L_\sigma$.*
(c) *$\{\sigma < \kappa : |L_\sigma| > 1\}$ is finite.*

Proof. That (a) implies (b) is trivial.

To see that (b) implies (c), assume that (b) holds, let $F = \{\sigma < \kappa : |L_\sigma| > 1\}$, and suppose that F is infinite. The minimal left ideals of $\beta(\times_{\sigma<\kappa} S_\sigma)$ are F-spaces so $\times_{\sigma<\kappa} L_\sigma$ is an F-space so by Theorem 1.3(3), $\times_{\sigma\in F} L_\sigma$ is an F-space. Let G be an infinite subset of F such that $F \setminus G$ is infinite. Then

$$(\times_{\sigma\in G} L_\sigma) \times (\times_{\sigma\in F\setminus G} L_\sigma)$$

is an F-space so by Theorem 1.3(4) without loss of generality $\times_{\sigma\in G} L_\sigma$ is a P-space. But $\times_{\sigma\in G} L_\sigma$ is infinite and compact, contradicting Theorem 1.3(5).

To see that (c) implies (a), let $F = \{\sigma < \kappa : |L_\sigma| > 1\}$ and assume that F is finite. If κ is finite, the conclusion follows from Theorem 3.6, so assume that κ is infinite. If $F = \emptyset$, the conclusion follows from Lemma 3.7, so assume that $F \neq \emptyset$.

Let I be a minimal left ideal of $\beta(\times_{\sigma\in F} S_\sigma)$. By Theorem 3.6, I is topologically isomorphic to $\times_{\sigma\in F} L_\sigma$. Let M be a minimal left ideal of $\beta(\times_{\sigma\in\kappa\setminus F} S_\sigma)$. By Lemma 3.7, $|M| = 1$.

Now $\times_{\sigma<\kappa} S_\sigma$ is isomorphic to $(\times_{\sigma\in F} S_\sigma)\times(\times_{\sigma\in\kappa\setminus F} S_\sigma)$ so $\beta(\times_{\sigma<\kappa} S_\sigma)$ is topologically isomorphic to $\beta\big((\times_{\sigma\in F} S_\sigma) \times (\times_{\sigma\in\kappa\setminus F} S_\sigma)\big)$. By Theorem 3.6, the minimal left ideals of $\beta\big((\times_{\sigma\in F} S_\sigma) \times (\times_{\sigma\in\kappa\setminus F} S_\sigma)\big)$ are topologically isomorphic to

$I \times M$. Since $|M| = 1$, they are topologically isomorphic to I which is topologically isomorphic to $\times_{\sigma\in F} L_\sigma$. □

We saw in Lemma 3.1 that if $K(\beta S_i)$ is finite for each $i \in \{1, 2, \ldots, \upsilon\}$, then $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big) = \times_{i=1}^{\upsilon} K(\beta S_i)$. We obtain now substantial information about $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$ from the much weaker assumption that the minimal left ideals are finite.

Theorem 3.9. *Let $\upsilon \in \mathbb{N} \setminus \{1\}$ and for $i \in \{1, 2, \ldots, \upsilon\}$, let S_i be an infinite discrete semigroup such that the minimal left ideals of βS_i are finite. Let $\widetilde{\tau} : \beta(\times_{i=1}^{\upsilon} S_i) \to \times_{i=1}^{\upsilon} \beta S_i$ be the continuous extension of the identity. The following statements are equivalent.*

(a) *$\widetilde{\tau}$ is injective on $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$.*
(b) *$\widetilde{\tau}$ is a topological isomorphism from $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$ onto $\times_{i=1}^{\upsilon} K(\beta S_i)$.*
(c) *$K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$ and $\times_{i=1}^{\upsilon} K(\beta S_i)$ are homeomorphic.*
(d) *For all but at most one $i \in \{1, 2, \ldots, \upsilon\}$, $K(\beta S_i)$ is finite.*

Proof. Given $i \in \{1, 2, \ldots, \upsilon\}$, we have noted that βS_i has finitely many minimal right ideals so by [10, Theorem 3.2], $K(\beta S_i)$ is compact. By Lemma 2.2, the minimal left ideals of $\beta(\times_{i=1}^{\upsilon} S_i)$ are finite so, again by [10, Theorem 3.2], $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$ is compact. By [8, Corollary 4.22], $\widetilde{\tau}$ is a homomorphism so it is trivial that (a) implies (b) and that (b) implies (c).

To see that (c) implies (d), assume that $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$ and $\times_{i=1}^{\upsilon} K(\beta S_i)$ are homeomorphic. If there is no i such that $K(\beta S_i)$ is infinite, we are done, so assume without loss of generality that $K(\beta S_1)$ is infinite. Since $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$ is a compact subset of $\beta(\times_{i=1}^{\upsilon} S_i)$, we have by Theorem 1.3 (1) and (2) that $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$ is an F-space, so $\times_{i=1}^{\upsilon} K(\beta S_i)$ is an F-space. Since $K(\beta S_1)$ is infinite and compact, by Theorem 1.3 (4) and (5), $\times_{i=2}^{\upsilon} K(\beta S_i)$ is finite.

To see that (d) implies (a), assume that for each $i \in \{2, 3, \ldots, \upsilon\}$, $K(\beta S_i)$ is finite. By Theorem 2.3, for $i \in \{2, 3, \ldots, \upsilon\}$, $K(\beta S_i) \subseteq S_i$.

Let $M = \{p \in \beta(\times_{i=1}^{\upsilon} S_i) : (\forall i \in \{1, 2, \ldots, \upsilon\})(\widetilde{\pi}_i(p) \in K(\beta S_i)\}$. By Lemma 3.3, M is an ideal of $\beta(\times_{i=1}^{\upsilon} S_i)$ so it suffices to show that $\widetilde{\tau}$ is injective on M. For $i \in \{1, 2, \ldots, \upsilon\}$, let $\widetilde{\pi}_i : \beta(\times_{j=1}^{\upsilon} S_j) \to \beta S_i$ be the continuous extension of the projection function and let $\mu_i : \times_{j=1}^{\upsilon} \beta S_j \to \beta S_i$ be the projection function. Then $\mu_i \circ \widetilde{\tau}$ and $\widetilde{\pi}_i$ are continuous functions agreeing on $\times_{j=1}^{\upsilon} S_j$, so they are equal. For any $p \in M$ and $i \in \{2, 3, \ldots, \upsilon\}$ there is some $x_{p,i} \in S_i$ such that $\widetilde{\pi}_i(p) = x_{p,i}$ so that, since $\mu_i \circ \widetilde{\tau} = \widetilde{\pi}_i$, $\widetilde{\tau}(p) = \langle \widetilde{\pi_1}(p), x_{p,2}, \ldots, x_{p,\upsilon}\rangle$. Now assume that $p, q \in M$ and $\widetilde{\tau}(p) = \widetilde{\tau}(q)$. Then

$$\langle \widetilde{\pi_1}(p), x_{p,2}, \ldots, x_{p,\upsilon}\rangle = \langle \widetilde{\pi_1}(q), x_{q,2}, \ldots, x_{q,\upsilon}\rangle.$$

For $i \in \{2, 3, \ldots, \upsilon\}$, let $x_i = x_{p,i}$. Since π_1 is injective on $S_1 \times \times_{i=2}^{\upsilon} \{x_i\}$, $\widetilde{\pi_1}$ is injective on $c\ell(S_1 \times \times_{i=2}^{\upsilon} \{x_i\}) = \beta S_1 \times \times_{i=2}^{\upsilon} \{x_i\}$. Since $\widetilde{\pi_1}(p) = \widetilde{\pi_1}(q)$, we have that $p = q$. □

For any infinite discrete semigroups $\langle S_i\rangle_{i=1}^{\upsilon}$, we know by [8, Exercise 1.7.3] that $\widetilde{\tau}[K(\beta(\times_{i=1}^{\upsilon}S_i))] = K(\times_{i=1}^{\upsilon}\beta S_i)$. We investigate now the preimage under $\widetilde{\tau}$ of $K(\times_{i=1}^{\upsilon}\beta S_i)$ in the event that the minimal left ideals of βS_i are finite.

Theorem 3.10. *Let $\upsilon \in \mathbb{N}$ and for $i \in \{1, 2, \ldots, \upsilon\}$, let S_i be an infinite discrete semigroup such that the minimal left ideals of βS_i are finite. Let $\vec{x} \in K(\times_{i=1}^{\upsilon}\beta S_i)$. There is a minimal idempotent u of $\beta(\times_{i=1}^{\upsilon}S_i)$ such that, if $y \in \beta(\times_{i=1}^{\upsilon}S_i)$ and $\widetilde{\tau}(y) = \vec{x}$, then $uy = y$.*

Proof. For $i \in \{1, 2, \ldots, \upsilon\}$, let $L_i = (\beta S_i)x_i$ and note that L_i is a minimal left ideal of βS_i. For $i \in \{1, 2, \ldots, \upsilon\}$, let q_i be the identity of $L_i \cap x_i(\beta S_i)$ so that $q_i x_i = x_i$.

Let $\vec{q} = \langle q_1, q_2, \ldots, q_\upsilon\rangle$. We claim that for each $i \in \{1, 2, \ldots, \upsilon\}$, there exists $Q_i \in q_i$ such that for all $s \in Q_i$, $sx_i = x_i$. To see this, let $i \in \{1, 2, \ldots, \upsilon\}$. Since x_i is isolated in L_i, pick $A \subseteq S_i$ such that $\overline{A} \cap L_i = \{x_i\}$. Since $q_i x_i = x_i$, pick $Q_i \in q_i$ such that $\overline{Q_i}x_i \subseteq \overline{A}$. Given $s \in Q_i$, $sx_i \in \overline{A} \cap L_i$ so $sx_i = x_i$. For $s \in Q_i$, let $X_{i,s} = \{t \in S_i : st = t\}$. Then by [8, Theorem 3.35], $X_{i,s} \in x_i$.

By [8, Exercise 1.7.3(3)] we may pick a minimal idempotent $u \in \beta(\times_{i=1}^{\upsilon}S_i)$ such that $\widetilde{\tau}(u) = \vec{q}$. Let $Q = \times_{i=1}^{\upsilon}Q_i$. We claim that $Q \in u$. To see this, note that $\times_{i=1}^{\upsilon}\overline{Q_i}$ is a neighborhood of $\vec{q}$ so pick $B \in u$ such that $\widetilde{\tau}[\overline{B}] \subseteq \times_{i=1}^{\upsilon}\overline{Q_i}$. Then $B \subseteq Q$. For $\vec{s} \in Q$, let $X_{\vec{s}} = \times_{i=1}^{\upsilon}X_{i,s_i}$.

Now assume we have $y \in \beta(\times_{i=1}^{\upsilon}S_i)$ such that $\widetilde{\tau}(y) = \vec{x}$. We claim that for $\vec{s} \in Q$, $X_{\vec{s}} \in y$, so let $\vec{s} \in Q$. For $i \in \{1, 2, \ldots, \upsilon\}$, $s_i \in Q_i$ so $X_{i,s_i} \in x_i$ so $\times_{i=1}^{\upsilon}\overline{X_{i,s_i}}$ is a neighborhood of $\vec{x} = \widetilde{\tau}(y)$ so pick $C \in y$ such that $\widetilde{\tau}[\overline{C}] \subseteq \times_{i=1}^{\upsilon}\overline{X_{i,s_i}}$. Then $C \subseteq \times_{i=1}^{\upsilon}X_{i,s_i}$.

To see that $uy = y$, it suffices to show that ρ_y is constantly equal to y on Q, so let $\vec{s} \in Q$. To see that $\vec{s}y = y$, it suffices that $\lambda_{\vec{s}}$ is the identity on $X_{\vec{s}}$, so let $\vec{t} \in X_{\vec{s}}$. Then for $i \in \{1, 2, \ldots, \upsilon\}$, $t_i \in X_{i,s_i}$ so $s_i t_i = t_i$. □

Corollary 3.11. *Let $\upsilon \in \mathbb{N}$ and for $i \in \{1, 2, \ldots, \upsilon\}$, let S_i be an infinite discrete semigroup such that the minimal left ideals of βS_i are finite. Then*

$$\widetilde{\tau}^{-1}[K(\times_{i=1}^{\upsilon}\beta S_i)] = K(\beta(\times_{i=1}^{\upsilon}S_i)).$$

Proof. Trivially $K(\beta(\times_{i=1}^{\upsilon}S_i)) \subseteq \widetilde{\tau}^{-1}[K(\times_{i=1}^{\upsilon}\beta S_i)]$. To see that

$$\widetilde{\tau}^{-1}[K(\times_{i=1}^{\upsilon}\beta S_i)] \subseteq K(\beta(\times_{i=1}^{\upsilon}S_i))$$

let $y \in \widetilde{\tau}^{-1}[K(\times_{i=1}^{\upsilon}\beta S_i)]$. Let $\vec{x} = \widetilde{\tau}(y)$ and pick u as guaranteed by Theorem 3.10. Since $uy = y$, $y \in K(\beta(\times_{i=1}^{\upsilon}S_i))$. □

Corollary 3.12. *Let $\upsilon \in \mathbb{N}$ and for $i \in \{1, 2, \ldots, \upsilon\}$, let S_i be an infinite discrete semigroup such that the minimal left ideals of βS_i are finite. Let y and z be in $K(\beta(\times_{i=1}^{\upsilon}S_i))$ and assume that $\widetilde{\tau}(y) = \widetilde{\tau}(z)$. Then y and z lie in the same minimal right ideal of $\beta(\times_{i=1}^{\upsilon}S_i)$.*

Proof. Let $\vec{x} = \widetilde{\tau}(y)$ and pick u as guaranteed by Theorem 3.10. Then $uy = y$ and $uz = z$ so y and z are both in the minimal right ideal $u\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$. □

We conclude this section with a consideration of the number of minimal left ideals of $\beta(\times_{i=1}^{\upsilon} S_i)$ that $\widetilde{\tau}$ takes to a given minimal left ideal of $\times_{i=1}^{\upsilon} \beta S_i$.

Theorem 3.13. *Let $\upsilon \in \mathbb{N}\setminus\{1\}$ and for $i \in \{1, 2, \ldots, \upsilon\}$, let S_i be an infinite discrete semigroup such that the minimal left ideals of βS_i are finite. If*

$$|\{i \in \{1, 2, \ldots, \upsilon\} : K(\beta S_i) \text{ is infinite}\}| \geq 2\,,$$

then there exist distinct minimal left ideals M_1 and M_2 of $\beta(\times_{i=1}^{\upsilon} S_i)$ such that $\widetilde{\tau}[M_1] = \widetilde{\tau}[M_2]$.

Proof. Suppose that there do not exist distinct minimal left ideals M_1 and M_2 of $\beta(\times_{i=1}^{\upsilon} S_i)$ such that $\widetilde{\tau}[M_1] = \widetilde{\tau}[M_2]$. We shall show that $\widetilde{\tau}$ is injective on $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$ so that, by Theorem 3.9 there is at most one $i \in \{1, 2, \ldots, \upsilon\}$ such that $K(\beta S_i)$ is infinite.

To this end, let p and q be distinct members of $K\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)$. If

$$\big(\beta(\times_{i=1}^{\upsilon} S_i)\big)p = \big(\beta(\times_{i=1}^{\upsilon} S_i)\big)q$$

then by Theorem 3.6, $\widetilde{\tau}(p) \neq \widetilde{\tau}(q)$. So we assume that $M_1 = \big(\beta(\times_{i=1}^{\upsilon} S_i)\big)p \neq \big(\beta(\times_{i=1}^{\upsilon} S_i)\big)q = M_2$. Since $\widetilde{\tau}[M_1] \neq \widetilde{\tau}[M_2]$, they are distinct minimal left ideals of $\times_{i=1}^{\upsilon} K(\beta S_i)$ and are therefore disjoint so again $\widetilde{\tau}(p) \neq \widetilde{\tau}(q)$. □

Given that for any discrete semigroup S all minimal left ideals of βS are homeomorphic and isomorphic, we (or at least one of us) would have thought that the number of minimal left ideals taken to a given minimal left ideal of $\times_{i=1}^{\upsilon} \beta S_i$ by $\widetilde{\tau}$ must be independent of the given minimal left ideal. It turns out that, at least consistently, this is not true.

Theorem 3.14. *Let $S = (\mathbb{N}, \vee)$. For minimal left ideals L_1 and L_2 of βS, let $\mathcal{M}(L_1, L_2) = \{M : M \text{ is a minimal left ideal of } \beta(S \times S) \text{ and } \widetilde{\tau}[M] = L_1 \times L_2\}$.*

1. *There exists a minimal left ideal L_1 of βS such that for every minimal left ideal L_2 of βS, $|\mathcal{M}(L_1, L_2)| = 2^{\mathfrak{c}}$.*
2. *Assume the continuum hypothesis. For each $n \in \mathbb{N} \setminus \{1\}$, there exist minimal left ideals L_1 and L_2 of βS such that $|\mathcal{M}(L_1, L_2)| = n$.*

Proof. The minimal left ideals of βS are the sets $\{p\}$ for $p \in \mathbb{N}^*$ while by Theorem 3.4, the minimal left ideals of $\beta(S \times S)$ are the sets of the form $\{p\}$ such that $\widetilde{\pi_1}(p) \in \mathbb{N}^*$ and $\widetilde{\pi_2}(p) \in \mathbb{N}^*$. As we saw in the proof of Theorem 3.9, if $p \in \beta(S \times S)$, then $\widetilde{\tau}(p) = \langle \widetilde{\pi_1}(p), \widetilde{\pi_2}(p) \rangle$. Thus, given p and q in $\mathbb{N}^*$, $\mathcal{M}(\{p\}, \{q\}) = \big\{\{r\} : r \in \beta(S \times S) \text{ and } \widetilde{\tau}(r) = (p, q)\big\}$ so that $|\mathcal{M}(\{p\}, \{q\})| = |\widetilde{\tau}^{-1}[\{(p, q)\}]|$.

(1) It was shown in [6] that there exists $p \in \mathbb{N}^*$ such that for all $q \in \mathbb{N}^*$, $|\widetilde{\tau}^{-1}[\{\langle p, q \rangle\}]| = 2^{\mathfrak{c}}$.

(2) It was shown in [1] that, assuming the continuum hypothesis, for each $n \in \mathbb{N} \setminus \{1\}$ there exist p and q in $\mathbb{N}^*$ such that $|\widetilde{\tau}^{-1}[\{\langle p, q \rangle\}]| = n$. □

M. Daguenet established in [4] that if there exist p and q in $\mathbb{N}^*$ such that $\widetilde{\iota}^{-1}[\{\langle p, q\rangle\}]$ is finite, then there exist P-points in $\beta\mathbb{N}$, so this cannot be established in ZFC. (An elementary proof of this assertion is in [2, Theorem 14].)

Question 3.15. *Can one produce in ZFC an infinite discrete semigroup S and minimal left ideals L_1, L_2, L_3, and L_4 of βS such that $|\mathcal{M}(L_1, L_2)| \neq |\mathcal{M}(L_3, L_4)|$?*

4. Semiprincipal left ideals

Our first theorem of this section, Theorem 4.4, extends [8, Theorem 6.56], by replacing the assumption that S can be embedded in a group by the weaker assumption that S is cancellative, and by proving the result for any countable set of elements $\{p_i : i \in \mathbb{N}\}$ in $\beta S \setminus K(\beta S)$, instead of a finite set.

We shall show that, if S is any countably infinite cancellative semigroup, every non-minimal semiprincipal left ideal in βS contains many semiprincipal left ideals defined by right cancelable elements of βS, and we shall explore some of the properties of these left ideals. Semiprincipal left ideals defined by right cancelable elements are of interest, because their topology is known. They are homeomorphic to βS if S is an arbitrary semigroup. If S is a group or $(\mathbb{N}, +)$, they are the only semiprincipal left ideals of βS which are homeomorphic to βS. In the case in which S is a countably infinite semigroup which can be embedded in a group, the semiprincipal left ideals of βS defined by right cancelable elements, have rich algebraic properties. It is only because of the fact that, in this case, every non-minimal semiprincipal left ideal L of βS contains semiprincipal left ideals defined by right cancelable elements of βS, that we know that L contains many infinite decreasing chains of idempotents.

Lemma 4.1. *Let S be an infinite discrete space. If A and B are countable subsets of βS for which $\overline{A} \cap \overline{B} \neq \emptyset$, then $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.*

Proof. [8, Theorem 3.40]. □

A subset A of a semigroup S is *piecewise syndetic* in S if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$.

Lemma 4.2. *Let S be a countably infinite discrete cancellative semigroup, let $\langle p_n\rangle_{n=1}^{\infty}$ be a sequence in $\beta S \setminus K(\beta S)$, and let Q be a piecewise syndetic subset of S. There is an infinite set $R \subseteq Q$ such that*

(1) *for every $x \in R^*$, every i and j in $\mathbb{N}$, and every $a \in S$, $ap_i \notin \beta S x p_j$,*
(2) *for every distinct a and b in S, $aR^* \cap bR^* = \emptyset$, and there is a cofinite subset V of R such that $as \neq bt$ for all s and t in V.*

Proof. Pick $q \in K(\beta S) \cap \overline{Q}$. We claim that, if $a \in S$ and $i, j \in \mathbb{N}$, then $ap_i \notin \beta S q p_j$. If we assume the contrary, then $ap_i \in K(\beta S)$ and so $ap_i u = ap_i$ for some minimal idempotent $u \in \beta S$. Hence, by [8, Lemma 8.1], $p_i u = p_i$, contradicting the assumption that $p_i \notin K(\beta S)$. Therefore by the continuity of λ_a, we may choose a member $D_{i,j,a}$ of p_i for which $a\overline{D_{i,j,a}} \cap \beta S q p_j = \emptyset$. Again

by the continuity of λ_a, we have that $a\overline{D_{i,j,a}} = \overline{aD_{i,j,a}}$. Then given any $b \in S$, $bqp_j \notin \overline{aD_{i,j,a}}$ so we may choose $E_{i,j,a,b} \in q$ such that $E_{i,j,a,b} \subseteq \{s \in S : bsp_j \notin \overline{aD_{i,j,a}}\}$.

For every distinct $a, b \in S$, $aq \neq bq$ by [8, Corollary 8.2] so we may choose $H_{a,b} \in q$ such that $a\overline{H_{a,b}} \cap b\overline{H_{a,b}} = \emptyset$. By [8, Theorem 4.36], $K(\beta S) \subseteq S^*$ so Q is infinite. Let $C = Q^* \cap \bigcap\{\overline{E_{i,j,a,b}} : i, j \in \mathbb{N} \text{ and } a, b \in S\} \cap \bigcap\{\overline{H_{a,b}} : a, b \in S \text{ and } a \neq b\}$. Then C is a G_δ subset of S^* which is non-empty, because $q \in C$. So it follows from [8, Theorem 3.36], that C has a non-empty interior in S^*. We can therefore choose an infinite subset R of S for which $R^* \subseteq C$, because the sets of the form R^* provide a base for the topology of S^*. Since $R^* \subseteq Q^*$, $R \setminus Q$ is finite, so we may presume that $R \subseteq Q$.

(1) Now let $x \in R^*$, let $i, j \in \mathbb{N}$, let $a \in S$, and suppose that $ap_i \in \beta S x p_j = c\ell\{bxp_j : b \in S\}$. Now $aD_{i,j,a} \in ap_i$ so one may pick $b \in S$ such that $bxp_j \in \overline{aD_{i,j,a}}$. Pick $B \in x$ such that $b\overline{B}p_j \subseteq \overline{aD_{i,j,a}}$ and pick $s \in B \cap E_{i,j,a,b}$. Then $bsp_j \in \overline{aD_{i,j,a}}$, a contradiction.

(2) Assume that a and b are distinct elements of S. Then $aR^* \cap bR^* = \emptyset$ because $R^* \subseteq \overline{H_{a,b}}$. Now let $A = \{s \in R : (\exists t \in R)(as = bt)\}$ and let $B = \{s \in R : (\exists t \in R)(at = bs)\}$. We claim that $A \cup B$ is finite so that we can let $V = R \setminus (A \cup B)$. Suppose instead without loss of generality that A is infinite and let $\langle s_n \rangle_{n=1}^\infty$ enumerate A. For each $n \in \mathbb{N}$, let t_n be the unique member of R such that $as_n = bt_n$. Let $p \in \mathbb{N}^*$, let $u = p\text{-}\lim_{n\in\mathbb{N}} s_n$ and let $v = p\text{-}\lim_{n\in\mathbb{N}} t_n$. Then u and v are in R^* and $au = bv$, a contradiction. □

Lemma 4.3. *Let S be a countably infinite discrete cancellative semigroup, let $\langle p_n \rangle_{n=1}^\infty$ be a sequence in $\beta S \setminus K(\beta S)$, and let R be an infinite subset of S as guaranteed by Lemma 4.2 for $Q = S$. Let $i, j \in \mathbb{N}$, let $y, z \in \beta S$, let $w, x \in R^*$, let $Y \in y$, let $Z \in z$, let $W \in w$, let $X \in x$, and assume that $ywp_i = zxp_j$. There exist $a \in Y$, $b \in Z$, $u \in W^*$, and $v \in X^*$ such that $aup_i = bvp_j$.*

Proof. Since $ywp_i \in c\ell(Ywp_i)$ and $zxp_j \in c\ell(Zxp_j)$, we may apply Lemma 4.1 and, essentially without loss of generality, assume we have $b \in Z$ such that $bxp_j \in c\ell(Ywp_i)$. (The other choice of $a \in Y$ with $awp_i \in c\ell(Zxp_j)$ would end up letting $v = x$ and picking $u \in W^*$ during the argument.) Since $bxp_j \in c\ell(bXp_j)$, applying Lemma 4.1 again we either get some $d \in X$ such that $bdp_j \in c\ell(Ywp_i)$ or some $a \in Y$ such that $awp_i \in c\ell(bXp_j) = b\overline{X}p_j$. Since $c\ell(Ywp_i) \subseteq \beta S wp_i$, we can't have $bdp_j \in c\ell(Ywp_i)$ by Lemma 4.2(1) so we have $awp_i \in b\overline{X}p_j$. Since $awp_i \notin bXp_j$ by Lemma 4.2(1), we must have some $v \in X^*$ such that $awp_i = bxp_j$. Let $u = w$. □

Theorem 4.4. *Let S be a countably infinite discrete cancellative semigroup, let $\langle p_n \rangle_{n=1}^\infty$ be a sequence in $\beta S \setminus K(\beta S)$, and let R be an infinite subset of S as guaranteed by Lemma 4.2 for $Q = S$. Then:*

(1) *For every $i \in \mathbb{N}$ and every $x \in R^*$, xp_i is right cancelable in βS;*

(2) *For every distinct* $i, j \in \mathbb{N}$ *for which* $Sp_i \cap Sp_j = \emptyset$*, and for every* $w, x \in R^*$*, the left ideals* βSwp_i *and* βSxp_j *of* βS *are disjoint;*
(3) *For every* $i \in \mathbb{N}$ *and every distinct* $w, x \in R^*$*, the left ideals* βSwp_i *and* βSxp_i *of* βS *are disjoint.*

Proof. (1) Suppose that y and z are distinct elements of βSxp_i for which $yxp_i = zxp_i$. We can choose disjoint subsets Y and Z of S which are members of y and z respectively. By Lemma 4.3 with $i = j$, $w = x$, and $W = X = R$, pick $a \in Y$, $b \in Z$, and $u, v \in R^*$ such that $aup_i = bvp_i$.

Since $a \neq b$, pick by Lemma 4.2(2) a cofinite subset V of R such that for all $s, t \in V$, $as \neq bt$. Since u and v are in R^*, $V \in u$ and $V \in v$. Then $c\ell(aVp_i) \cap c\ell(bVp_i) \neq \emptyset$, so by another application of Lemma 4.1 we can assume without loss of generality that we have $s \in V$ such that $asp_i \in c\ell(bVp_i) = b\overline{V}p_i$ so we can pick $t \in \overline{V}$ such that $asp_i = btp_i$. We can't have $t \in V^*$ by Lemma 4.2(1), so we must have $t \in V$. But then by [8, Corollary 8.2], we must have $as = bt$, contradicting our choice of V.

(2) Assume that $ywp_i = zxp_j$, where $y, z \in \beta S$ and $w, x \in R^*$. Let $Y = Z = S$ and $W = X = R$ and pick by Lemma 4.3, $a, b \in S$ and $u, v \in R^*$ such that $aup_i = bvp_j$. Now $c\ell(aRp_i) \cap c\ell(bRp_j) \neq \emptyset$ so by an application of Lemma 4.1 we may presume that we have $s \in R$ such that $asp_i \in b\overline{R}p_j$ so we may pick $t \in \overline{R}$ such that $asp_i = btp_j$. By Lemma 4.2(1) we cannot have $t \in R^*$, so $t \in R$ and thus $Sp_i \cap Sp_j \neq \emptyset$.

(3) Assume that $ywp_i = zxp_i$ for some $y, z \in \beta S$. We can choose disjoint subsets W and X of R which are members of w and x respectively. By Lemma 4.3 with $i = j$ and $Y = V = S$, we may choose $a, b \in S$, $u \in W^*$, and $v \in X^*$ such that $aup_i = bvp_i$. If $a \neq b$ we reach a contradiction as in the second paragraph of the proof of (1). So assume that $a = b$. Then by [8, Lemma 8.1] we have that $up_i = vp_i$. Then $c\ell(Wp_i) \cap c\ell(Xp_i) \neq \emptyset$, so applying Lemma 4.1 we may assume we have $s \in W$ such that $sp_i \in \overline{X}p_i$ so pick $t \in \overline{X}$ such that $sp_i = tp_i$. By [8, Corollary 8.2] we can't have $t \in X$ and by Lemma 4.2(1) we can't have $t \in X^*$. □

Corollary 4.5. *Let* S *be a countably infinite discrete cancellative semigroup and let* L *be a left ideal of* βS*. Then* $L \subseteq K(\beta S)$ *if and only if* L *is a union of pairwise disjoint groups.*

Proof. Necessity. Assume that $L \subseteq K(\beta S)$. Let $\mathcal{M} = \{M : M \text{ is a minimal left ideal of } \beta S \text{ and } L \cap M \neq \emptyset\}$. Since $L \neq \emptyset$, $\mathcal{M} \neq \emptyset$ and if $M \in \mathcal{M}$, then $M \subseteq L$ and by [8, Theorem 1.61], M is the union of pairwise disjoint groups.

Sufficiency. Assume that L is the union of groups and suppose $p \in L \setminus K(\beta S)$. By Theorem 4.4(1) pick $x \in S^*$ such that xp is right cancelable in βS. Then $xp \in L$ so pick a group $G \subseteq L$ such that $xp \in G$ and let e be the identity of G. Let r be the inverse of xp in G. Then $e = xpr \in S^*$, so the fact that $exp = xp$ shows that $xp \in S^*xp$, contradicting [8, Lemma 8.15]. □

Note that if $p \in K(\beta S)$, then all idempotents in βSp are minimal so βSp does not have a chain of idempotents of length 2.

Theorem 4.6. *Let S be a countably infinite discrete semigroup which can be embedded in a group. For every $p \in S^* \setminus K(\beta S)$, the left ideal βSp of βS contains infinite decreasing chains of idempotents.*

Proof. Since S can be embedded in a group, it generates a countable discrete group G. Let $p \in S^* \setminus K(\beta S)$. By [8, Theorem 6.56] we may pick $r \in S^*$ such that rp is right cancelable in βG. By [8, Corollary 8.54], C_{rp} contains an infinite decreasing chain of idempotents, where $C_{rp} = \bigcap\{D \subseteq \beta G : D \text{ is a compact subsemigroup of } \beta G \text{ and } rp \in D\}$. Since βSp is a compact subsemigroup of βG and $rp \in \beta Sp$, $C_{rp} \subseteq \beta Sp$. □

If βS is an arbitrary discrete semigroup, it is obvious that every semiprincipal left ideal of βS is compact. This illustrates a striking contrast between left ideals and right ideals. For example, we know that the semiprincipal right ideals of $\beta\mathbb{N}$ defined by elements of $\mathbb{N}^*$ are not Borel [9, Theorem 2.9]. We shall discuss some of the topological properties of left ideals in βS.

Recall that a *Stonean* space is a compact Hausdorff extremally disconnected space. Recall also that a space satisfies the *countable chain condition* if and only if any collection of pairwise disjoint open sets is countable. We shall list some conditions which imply that a left ideal of βS is Stonean.

Theorem 4.7. *Let S be an arbitrary discrete semigroup.*

(1) *If p is an idempotent in βS, the left ideal βSp of βS is Stonean.*
(2) *Every minimal left ideal of βS is Stonean.*
(3) *If p is a right cancelable element of βS, the left ideal βSp is Stonean and Sp consists of points isolated in βSp.*
(4) *If S is countable, every semiprincipal left ideal of βS is Stonean.*
(5) *The support of any probability measure μ defined on βS, with the property that $\mu(s^{-1}(B)) = \mu(B)$ for every Borel subset of βS and every $s \in S$, is a Stonean left ideal in βS.*

Proof. (1) This follows immediately from [7, Lemma 2].

(2) Every minimal left ideal L of βS has an idempotent p and $L = \beta Sp$ so (1) applies.

(3) Since ρ_p is a continuous bijective map from βS onto βSp, it is a homeomorphism.

(4) If S is countable, every semiprincipal left ideal of βS is separable. It was shown in the Proposition on page 19 of [11] that every compact F-space which satisfies the countable chain condition is Stonean.

(5) It is obvious that the support of μ is a compact left ideal in βS. Since it is a compact F-space which satisfies the countable chain condition, it is Stonean. □

In the proof of the next lemma, we will use the fact from [8, Theorem 6.54] that the center of $(\beta\mathbb{Z}, +)$ is $\mathbb{Z}$.

Lemma 4.8. *Let S be a group or $(\mathbb{N},+)$, let $p \in S^*$, and let $L = \beta Sp$. If there is a point of Sp which is isolated in L, then all points of Sp are isolated in L.*

Proof. Assume that $s \in S$ and sp is isolated in L. Assume first that S is a group. Given $u \in S$, pick $t \in S$ such that $tu = s$. Then the restriction of λ_t to L is a homeomorphism from L to itself and $sp = \lambda_t(up)$ so up is isolated in L.

Now assume that $S = (\mathbb{N},+)$. Pick $A \subseteq \mathbb{N}$ such that $\overline{A} \cap L = \{s+p\}$. We show that for all $t \in \mathbb{N}$ such that $t < s$, $s - t + p$ is isolated in L, and for all $t \in \mathbb{N}$, $s + t + p$ is isolated in L.

Assume first that $t < s$. Then $s - t + p \in \overline{A - t} \cap L$. Now let $r \in \overline{A - t} \cap L$ and pick $q \in \beta\mathbb{N}$ such that $r = q + p$. Then $r + t = q + t + p \in \overline{A} \cap L$, so $r + t = s + p$ and so $r = s - t + p$.

Now let $t \in \mathbb{N}$. Then $s + t + p \in \overline{A + t} \cap L$. Suppose that $s + t + p$ is not isolated in L and let $B = (\overline{A + t} \cap L) \setminus \{s + t + p\}$. Then $B \neq \emptyset$. We claim B is infinite. For each $r \in B$, pick $D_r \in (s + t + p) \setminus r$. If B is finite, then $\overline{(A + t) \cap \bigcap_{r \in B} D_r} \cap L = \{s + t + p\}$ so that $s + t + p$ is isolated in L.

Since B is infinite, pick $r \in (\overline{A + t} \cap L) \setminus \big(\{s + t + p\} \cup (\{1, 2, \ldots, t\} + p)\big)$. Pick $q \in \beta\mathbb{N}$ such that $r = q + p$. Then $\{1, 2, \ldots, t\} \notin q$ so $q - t \in \beta\mathbb{N}$ and $q - t + p \in \overline{A} \cap L$ so $q - t + p = s + p$ and thus $r = q + p = s + t + p$. □

Theorem 4.9. *Let S be a group or $(\mathbb{N},+)$ and let $p \in S^*$. The following statements are equivalent.*

(a) *βSp is Stonean and Sp consists of points isolated in βSp.*
(b) *βSp is Stonean and there is a point of βSp which is isolated in βSp.*
(c) *p is right cancelable in βS.*

Proof. It is trivial that (a) implies (b) and the fact that (c) implies (a) is Theorem 4.7(3).

To see that (b) implies (c), let $L = \beta Sp$ and assume that L is Stonean and has an isolted point x. Pick $y \in \beta S$ such that $x = yp$. Pick $A \subseteq S$ such that $\overline{A} \cap L = \{x\}$. Then $\{s \in S : s^{-1}A \in p\} \in y$. If $s^{-1}A \in p$, then $sp \in \overline{A} \cap L$ so $sp = x$. Thus $\{s \in S : s^{-1}A \in p\}$ is a singleton. Recalling that we have identified the points of S with the principal ultrafilters on S, we have that $y \in S$. By Lemma 4.8, every point of Sp is isolated in L.

Note that the function $sp \mapsto s$ from Sp to S is well defined. Since L is extremally disconnected we have by [5, Exercise 6M(2)] that L is a copy of $\beta(Sp)$, so we may pick a continuous function $f : \beta Sp \to \beta S$ such that $f(sp) = s$ for all $s \in S$.

Given any $q \in \beta S$, we claim that $f(qp) = q$. To see this, pick a net $\langle s_\iota \rangle_{\iota \in I}$ in S converging to q. Then $\langle f(s_\iota p) \rangle_{\iota \in I}$ converges to $f(qp)$ while for each $\iota \in I$, $f(s_\iota p) = s_\iota$ so $f(qp) = q$. Therefore if $q, r \in \beta S$ and $qp = rp$, then we have $q = f(qp) = f(rp) = r$ so p is right cancelable in βS. □

Recall that $\mathbb{H} = \bigcap_{n=1}^{\infty} cl_{\beta\mathbb{N}}(2^n\mathbb{N})$. By [8, Lemma 6.8] $\mathbb{H}$ is a compact subsemigroup of $\beta\mathbb{N}$ which contains all of the idempotents of $\beta\mathbb{N}$.

Theorem 4.10. *Let G be a countably infinite discrete group which can be embedded algebraically in a compact metrizable topological group.*

(1) *If q is an idempotent in βG, then there is an idempotent $p \in \beta\omega$ such that $\beta G p$ and $\beta\omega + q$ are homeomorphic.*
(2) *Every minimal left ideal of βG is homeomorphic to a minimal left ideal of $\beta\omega$.*

Proof. (1) By [8, Theorem 7.28], there is a bijective map $\psi : \omega \to G$, with the following properties: If $m, k, n \in \omega$, $m < 2^k$, and $n \in 2^{k+1}\mathbb{N}$, then $\psi(m+n) = \psi(m)\psi(n)$ and if $\widetilde{\psi} : \beta\omega \to \beta G$ denotes the continuous extension of ψ, then $\widetilde{\psi}$ is a homeomorphism which maps $\mathbb{H}$ isomorphically onto a subsemigroup of βG which contains all the idempotents of G^*.

If $n \in \omega$, $x \in \mathbb{H}$, and $n < 2^k$, then $\widetilde{\psi}\circ\lambda_n$ and $\lambda_{\psi(n)}\circ\widetilde{\psi}$ agree on $2^{k+1}\mathbb{N}$ so $\widetilde{\psi}(n+x) = \psi(n)\widetilde{\psi}(x)$. If q is an idempotent in βG, we can choose an idempotent p in $\beta\omega$ for which $\widetilde{\psi}(p) = q$. Since $\widetilde{\psi}(n+p) = \psi(n)q$ for every $n \in \omega$, $\widetilde{\psi}(\beta\omega + p)$ contains the dense subspace Gq of βGq. So $\widetilde{\psi}(\beta\omega + p) = \beta Gq$.

(2) This follows from (1) since if q is minimal in βG one can show that p is minimal in $\beta\omega$. We omit the verification since this conclusion also follows from [8, Theorem 7.32]. □

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