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Variational inequalities for the differences of averages over lacunary sequences

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ABSTRACT. Let f be a locally integrable function defined on $\mathbb{R}$, and let (n_k) be a lacunary sequence. Define the operator A_{n_k} by

$$A_{n_k}f(x) = \frac{1}{n_k}\int_0^{n_k} f(x-t)\,dt.$$

We prove various types of new inequalities for the variation operator

$$\mathcal{V}_s f(x) = \left(\sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)|^s\right)^{1/s}$$

when $2 \leq s < \infty$.

An increasing sequence (n_k) of real numbers is called lacunary if there exists a constant $\beta > 1$ such that $n_{k+1}/n_k \geq \beta$ for all $k = 0, 1, 2, \dots$.

Let f be a locally integrable function defined on $\mathbb{R}$. Let (n_k) be a lacunary sequence and define the operator A_{n_k} by

$$A_{n_k}f(x) = \frac{1}{n_k}\int_0^{n_k} f(x-t)\,dt.$$

It is clear that

$$A_{n_k}f(x) = \frac{1}{n_k}\chi_{(0,n_k)} * f(x)$$

where $*$ stands for convolution. Consider the variation operator

$$\mathcal{V}_s f(x) = \left(\sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)|^s\right)^{1/s}$$

for $2 \leq s < \infty$. The boundedness of the variation operator $\mathcal{V}_s f$ provides an estimate on the speed (or rate) of convergence of the sequence $\{A_{n_k}f\}$.

Various types of inequalities for the two-sided variation operator

$$\mathcal{V}'_s f(x) = \left(\sum_{-\infty}^{\infty}\left|\frac{1}{2^n}\int_x^{x+2^n} f(t)\,dt - \frac{1}{2^{n-1}}\int_x^{x+2^{n-1}} f(t)\,dt\right|^s\right)^{1/s}$$

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when $2 \leq s < \infty$ have been proven by the author in Demir, S. [1]. In this research we prove that same types of inequalities are also true for any lacunary sequence (n_k) for the one-sided variation operator $\mathcal{V}_s f(x)$ for $2 \leq s < \infty$.

Lemma 1. *Let (n_k) be a lacunary sequence with the lacunarity constant β, i.e., $n_{k+1}/n_k \geq \beta > 1$ for all $k = 0, 1, 2, \ldots$. If $1 \leq s < \infty$, then there exists a sequence (m_j) such that*

$$\beta^2 \geq \frac{m_{j+1}}{m_j} \geq \beta > 1$$

for all j and

$$\left(\sum_{k=1}^{\infty} |A_{n_k} f(x) - A_{n_{k-1}} f(x)|^s\right)^{1/s} \leq \left(\sum_{j=1}^{\infty} |A_{m_j} f(x) - A_{m_{j-1}} f(x)|^s\right)^{1/s}.$$

Proof. Let us start our construction by first choosing $m_0 = n_0$. If

$$\beta^2 \geq \frac{n_1}{n_0} \geq \beta,$$

define $m_1 = n_1$. If $n_1/n_0 > \beta^2$, let $m_1 = \beta n_0$. Then we have

$$\beta^2 \geq \frac{m_1}{m_0} = \frac{\beta n_0}{n_0} = \beta \geq \beta.$$

Also,

$$\frac{n_1}{m_1} \geq \frac{\beta^2 n_0}{\beta n_0} = \beta.$$

Again, if $n_1/m_1 \leq \beta^2$, then choose $m_2 = n_1$. If this is not the case, choose $m_2 = \beta^2 n_0 \leq n_1$. By the same calculation as before, m_0, m_1, m_2 are part of a lacunary sequence satisfying

$$\beta^2 \geq \frac{m_{k+1}}{m_k} \geq \beta > 1.$$

To continue the sequence, either $m_3 = n_1$ if $n_1/m_2 \leq \beta^2$ or $m_3 = \beta^3 n_0$ if $n_1/m_2 > \beta^2$.

Since $\beta > 1$, this process will end at some k_0 such that $m_{k_0} = n_1$. The remaining elements m_k are constructed in the same manner as the original n_k, with necessary terms added between two consecutive n_k to obtain the inequality

$$\beta^2 \geq \frac{m_{k+1}}{m_k} \geq \beta > 1.$$

Let now

$$J(k) = \{j : n_{k-1} < m_j \leq n_k\}.$$

Then we have

$$A_{n_k} f(x) - A_{n_{k-1}} f(x) = \sum_{j \in J(k)} (A_{m_j} f(x) - A_{m_{j-1}} f(x))$$

and thus we get

$$\begin{aligned}|A_{n_k}f(x) - A_{n_{k-1}}f(x)| &= \left|\sum_{j\in J(k)} (A_{m_j}f(x) - A_{m_{j-1}}f(x))\right| \\ &\leq \sum_{j\in J(k)} |A_{m_j}f(x) - A_{m_{j-1}}f(x)|.\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)| &\leq \sum_{k=1}^{\infty}\sum_{j\in J(k)} |A_{m_j}f(x) - A_{m_{j-1}}f(x)|. \\ &= \sum_{j=1}^{\infty} |A_{m_j}f(x) - A_{m_{j-1}}f(x)|.\end{aligned}$$

Thus, we have

$$\left(\sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)|^s\right)^{1/s} \leq \left(\sum_{j=1}^{\infty} |A_{m_j}f(x) - A_{m_{j-1}}f(x)|^s\right)^{1/s}.$$

and this completes the proof. □

Remark 2. We know from Lemma 1 that

$$\left(\sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)|^s\right)^{1/s} \leq \left(\sum_{j=1}^{\infty} |A_{m_j}f(x) - A_{m_{j-1}}f(x)|^s\right)^{1/s}.$$

and the new sequence (m_j) satisfies

$$\beta^2 \geq \frac{m_{j+1}}{m_j} \geq \beta > 1$$

for all $j \in \mathbb{Z}^+$. Therefore, we can assume without loss of generality that

$$\beta^2 \geq \frac{n_{k+1}}{n_k} \geq \beta > 1$$

for all $k \in \mathbb{Z}^+$ when we are proving any result for $\mathcal{V}_s(x)$.
Since

$$\frac{1}{n_k} = \frac{n_1}{n_2}\cdot\frac{n_2}{n_3}\cdot\frac{n_3}{n_4}\cdot\dots\cdot\frac{n_{k-1}}{n_k},$$

we can also assume that

$$\frac{1}{n_k} \leq \frac{1}{\beta^{2(k-1)}}$$

for all $k = 0, 1, 2, \dots$.

Lemma 3. *Let (n_k) be a lacunary sequence, and let γ denote the smallest positive integer satisfying*

$$\frac{1}{\beta} + \frac{1}{\beta^\gamma} \leq 1.$$

If $i \geq j+\gamma$, $0<y\leq n_j$ *and* $n_j<x<n_{i+1}$, *then*

$$\chi_{(y,y+n_k)}(x)-\chi_{(0,n_k)}(x)=0$$

unless $k=i$ *in which case*

$$\chi_{(y,y+n_k)}(x)-\chi_{(0,n_k)}(x)=\chi_{(n_i,y+n_i)}.$$

Proof. Since (n_k) is a lacunary sequence, there exists a constant $\beta>1$ such that $n_{k+1}/n_k\geq\beta$ for all k. We can assume that

$$\beta^2\geq\frac{n_{k+1}}{n_k}\geq\beta \tag{1}$$

for all k by Remark 2. Since we have

$$\frac{n_l}{n_k}=\frac{n_l}{n_{l+1}}\cdot\frac{n_{l+1}}{n_{l+2}}\cdot\dots\cdot\frac{n_{k-1}}{n_k}$$

and

$$\frac{1}{\beta}\leq\frac{n_k}{n_{k+1}}\leq\frac{1}{\beta^{k-l}}$$

for all k, we see that

$$\frac{1}{\beta^{2(k-l)}}\leq\frac{n_l}{n_k}\leq\frac{1}{\beta^{k-l}} \tag{2}$$

for all $k>l$. Let γ denote the smallest positive integer satisfying

$$\frac{1}{\beta}+\frac{1}{\beta^{\gamma}}\leq 1.$$

We see from (2) that

$$n_j+n_k\leq n_{k+1} \tag{3}$$

for all $k\geq j+\gamma-1$. It is easy to see that for $k>i$,

$$0<y\leq n_j\leq n_i<x<n_{i+1}\leq n_k<y+n_k,$$

and this implies that

$$\left[\chi_{(y,y+n_k)}(x)-\chi_{(0,n_k)}(x)\right]\cdot\chi_{(n_i,n_{i+1})}(x)=0.$$

For $k\leq i-1$, we see by (3) that

$$n_k<y+n_k\leq n_j+n_{i-1}\leq n_i.$$

Then we have

$$\chi_{(y,y+n_k)}(x)\cdot\chi_{(n_i,n_{i+1})}(x)=\chi_{(0,n_k)}(x)\cdot\chi_{(n_i,n_{i+1})}=0.$$

Suppose now that $k=i$; by (3), we have

$$y<n_i<y+n_i\leq n_j+n_i\leq n_{i+1}$$

and this implies that

$$\chi_{(y,y+n_i)}(x)-\chi_{(0,n_i)}(x)=\chi_{(y,y+n_i)}\cdot\chi_{(n_i,n_{i+1})}(x)=\chi_{(n_i,y+n_i)}(x). \quad \square$$

Let

$$\phi_k(x) = \frac{1}{n_k}\chi_{(0,n_k)}(x)$$

and define the kernel operator $K : \mathbb{R} \to \ell^s(\mathbb{Z}^+)$ as

$$K(x) = \{\phi_k(x) - \phi_{k-1}(x)\}_{k\in\mathbb{Z}^+}.$$

It is clear that

$$\begin{aligned}\mathcal{V}_s f(x) &= \|K * f(x)\|_{\ell^s(\mathbb{Z}^+)}\\ &= \left(\sum_{k=1}^{\infty} |\phi_k * f(x) - \phi_{k-1} * f(x)|^s\right)^{1/s}\\ &= \left(\sum_{k=1}^{\infty} |A_{n_k} f(x) - A_{n_{k-1}} f(x)|^s\right)^{1/s}\end{aligned}$$

where $*$ denotes convolution, i.e.,

$$K * f(x) = \int K(x-y)\cdot f(y)\,dy.$$

Let B be a Banach space. We say that the B-valued kernel K satisfies the D_r condition, for $1 \le r < \infty$, and write $K \in D_r$, if there exists a sequence $\{c_l\}_{l=1}^{\infty}$ of positive numbers such that $\sum_l c_l < \infty$ and such that

$$\left(\int_{S_l(|y|)} \|K(x-y) - K(x)\|_B^r\,dx\right)^{1/r} \le c_l |S_l(|y|)|^{-1/r'},$$

for all $l \ge 1$ and all $y > 0$, where $S_l(|y|)$ denotes the spherical shell $2^l|y| < |x| < 2^{l+1}y$ and $\frac{1}{r} + \frac{1}{r'} = 1$.

When $K \in D_1$ we have the Hörmander condition:

$$\int_{|x|>2|y|} \|K(x-y) - K(x)\|_B\,dx \le C$$

where C is a positive constant which does not depend on $y > 0$.

Lemma 4. *Let γ denote the smallest positive integer satisfying*

$$\frac{1}{\beta} + \frac{1}{\beta^\gamma} \le 1.$$

and let $1 \le r, s < \infty$, $i \ge j + \gamma$, and $0 < y \le n_j$. Then

$$\left(\int_{n_i}^{n_{i+1}} \|K(x-y) - K(x)\|_{\ell^s(\mathbb{Z}^+)}^r\,dx\right)^{1/r} \le C_i n_i^{1/r-1},$$

i.e., K satisfies the D_r condition for $1 \le r < \infty$.

Proof. Let

$$\Phi_k(x,y) = \phi_k(x-y) - \phi_k(x).$$

Then it is easy to check that

$$K(x-y) - K(x) = \{\Phi_k(x,y) - \Phi_{k-1}(x,y)\}_{k\in\mathbb{Z}^+}.$$

On the other hand, because of a property of the norm we have

$$\begin{aligned}\|K(x-y) - K(x)\|_{\ell^s(\mathbb{Z}^+)} &= \|\Phi_k(x,y) - \Phi_{k-1}(x,y)\|_{\ell^s(\mathbb{Z}^+)}\\ &\le \|\Phi_k(x,y)\|_{\ell^s(\mathbb{Z}^+)} + \|\Phi_{k-1}(x,y)\|_{\ell^s(\mathbb{Z}^+)}\\ &\le 2\|\Phi_{k-1}(x,y)\|_{\ell^s(\mathbb{Z}^+)},\end{aligned}$$

where x and y are fixed and $\|\Phi_{k-1}(x,y)\|_{\ell^s(\mathbb{Z}^+)}$ is the $\ell^s(\mathbb{Z}^+)$-norm of the sequence whose k^{th}-entry is $\Phi_k(x,y)$.
We now have

$$\begin{aligned}&\left(\int_{n_i}^{n_{i+1}} \|K(x-y) - K(x)\|^r_{\ell^s(\mathbb{Z}^+)}\,dx\right)^{1/r}\\ &\le 2\left(\int_{n_i}^{n_{i+1}} \|\Phi_{k-1}(x,y)\|^r_{\ell^s(\mathbb{Z}^+)}\,dx\right)^{1/r}\\ &\le 2\left(\int_{n_i}^{n_{i+1}} \|\Phi_{k-1}(x,y)\|^r_{\ell^1(\mathbb{Z}^+)}\,dx\right)^{1/r}\\ &= 2\left(\int_{n_i}^{n_{i+1}} \left(\sum_{n_i<n_{k-1}} \frac{1}{n_{k-1}}\chi_{(n_i,y+n_i)}(x)\right)^r dx\right)^{1/r}\\ &= 2\left(\int_{n_i}^{n_{i+1}} \left(\sum_{n_i<n_{k-1}} \frac{1}{\beta^{2(k-2)}}\chi_{(n_i,y+n_i)}(x)\right)^r dx\right)^{1/r}\\ &\le 2\left(\beta^2 + \frac{1}{1-\beta^2}\right)\cdot\frac{1}{n_i}\cdot\left(\int_{n_i}^{n_{i+1}} \chi_{(n_i,y+n_i)}(x)\,dx\right)^{1/r}\\ &= 2\left(\beta^2 + \frac{1}{1-\beta^2}\right)\cdot\frac{1}{n_i}\cdot y^{1/r}\\ &\le 2\left(\beta^2 + \frac{1}{1-\beta^2}\right)\frac{1}{\beta^{(i-j)/r}}n_i^{1/r-1}\end{aligned}$$

where in the last inequality we used

$$y \le n_j \le \frac{n_i}{\beta^{i-j}}$$

by (2), and this completes our proof with

$$C_i = 2\left(\beta^2 + \frac{1}{1-\beta^2}\right)\frac{1}{\beta^{(i-j)/r}}. \quad \square$$

Lemma 5. *Let $\{n_k\}$ be a lacunary sequence. Then there exists a constant $C > 0$ such that*

$$\sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| < C$$

for all $x \in \mathbb{R}$, where $\phi_k(x) = \frac{1}{n_k}\chi_{(0,n_k)}(x)$, and $\hat{\phi}_k$ is its Fourier transform.

Proof. First, note that we have

$$I(x) = \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| = \sum_{k=1}^{\infty} \left| \frac{1 - e^{-ixn_k}}{xn_k} - \frac{1 - e^{-ixn_{k-1}}}{xn_{k-1}} \right|.$$

Let

$$I(x) = \sum_{\{k:|x|n_k \geq 1\}} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| + \sum_{\{k:|x|n_k<1\}} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| = I_1(x) + I_2(x).$$

Let us now fix $x \in \mathbb{R}$ and let k_0 be the first k such that $|x|n_k \geq 1$. Since $\hat{\phi}_k(x)$ is an even function, we can assume without the loss of generality that $x \geq 0$. We clearly have

$$I_1(x) \leq \sum_{\{k:|x|n_k \geq 1\}} \frac{4}{|x|n_k}.$$

Since the sequence $\{n_k\}$ is lacunary, there exists a constant $\beta > 1$ such that $n_{k+1}/n_k \geq \beta$ for all $k \in \mathbb{N}$. Also note that in the sum, I_1, the term with index n_{k_0} is the term with smallest index, since it is the first term that satisfies condition $|x|n_k \geq 1$ and the sequence $\{n_k\}$ is increasing. On the other hand, we have

$$\frac{n_{k_0}}{n_k} = \frac{n_{k_0}}{n_{k_0+1}} \cdot \frac{n_{k_0+1}}{n_{k_0+2}} \cdot \frac{n_{k_0+2}}{n_{k_0+3}} \cdots \frac{n_{k-1}}{n_k} \leq \frac{1}{\beta^k}.$$

We now have

$$\begin{aligned} I_1(x) &\leq \sum_{\{k:|x|n_k\}} \frac{4}{|x|n_k} \\ &= \sum_{\{k:|x|n_k \geq 1\}} \frac{4n_{k_0}}{|x|n_{k_0}n_k} \\ &= \frac{4}{|x|n_{k_0}} \sum_{\{k:|x|n_k \geq 1\}} \frac{n_{k_0}}{n_k} \\ &\leq 4 \sum_{\{k:|x|n_k \geq 1\}} \frac{1}{\beta^k} \end{aligned}$$

since $\frac{1}{|x|n_{k_0}} \leq 1$ and $\frac{n_{k_0}}{n_k} = \frac{1}{\beta^k}$. Also, since

$$\sum_{k=1}^{\infty} \frac{1}{\beta^k} = \frac{1}{1 - \frac{1}{\beta}},$$

we clearly see that $I_1(x) \leq C_1$ for some constant $C_1 > 0$.

To control the summation I_2 let us first define the function F as

$$F(r) = \frac{1 - e^{-ir}}{r}.$$

Then we have $\hat{\phi}_k(x) = F(xn_k)$. Now by the Mean Value Theorem, there exists a constant $\xi \in (xn_k, xn_{k+1})$ such that

$$|F(xn_{k+1}) - F(xn_k)| = |F'(\xi)||xn_{k+1} - xn_k|.$$

Also, it is easy to verify that

$$|F'(x)| \leq \frac{x+2}{x^2},$$

for $x > 0$.

Now we have

$$\begin{aligned} |F(xn_{k+1}) - F(xn_k)| &= |F'(\xi)||xn_{k+1} - xn_k| \\ &\leq \frac{\xi + 2}{\xi^2}|x|(n_{k+1} - n_k) \\ &\leq \frac{xn_{k+1} + 2}{x^2 n_k^2}|x|(n_{k+1} - n_k) \\ &= \frac{2n_{k+1}}{n_k^2}(n_{k+1} - n_k). \end{aligned}$$

Thus, we have

$$\begin{aligned} I_2(x) &= \sum_{\{k \,:\, |x|n_k < 1\}} |F(xn_{k+1}) - F(xn_k)| \\ &\leq \sum_{\{k \,:\, |x|n_k < 1\}} \frac{2}{|x|n_k} \cdot \frac{2n_{k+1}}{n_k^2}(n_{k+1} - n_k) \\ &\leq \sum_{\{k \,:\, |x|n_k < 1\}} \frac{4n_{k+1}^2}{n_k^2|x|}\left(\frac{1}{n_k} - \frac{1}{n_{k+1}}\right) \\ &= \sum_{\{k \,:\, |x|n_k < 1\}} \frac{16}{|x|}\left(\frac{1}{n_k} - \frac{1}{n_{k+1}}\right) \\ &= \frac{16}{|x|}\left(\frac{1}{n_1} - \frac{1}{n_{k_0+1}}\right) \\ &\leq \frac{16}{|x|n_{k_0+1}} \\ &\leq 16. \end{aligned}$$

We thus conclude that

$$I(x) = I_1(x) + I_2(x) \leq C_1 + 16 := C$$

for all $x \in \mathbb{R}$ and this completes our proof. □

Lemma 6. *Let $s \geq 2$ and (n_k) be a lacunary sequence. Then there exits a constant $C > 0$ such that*

$$\|\mathcal{V}_s f\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$.

Proof. Since

$$\sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|^2 \leq \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|,$$

it is clear from Lemma 5 that there exists a constant $C > 0$ such that

$$\sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|^2 < C$$

for all $x \in \mathbb{R}$.

We now obtain

$$\begin{aligned}
\|\mathcal{V}_s f\|_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left(\sum_{k=1}^{\infty} \left|\phi_k * f(x) - \phi_{k-1} * f(x)\right|^{\rho} \right)^{2/\rho} dx \\
&\leq \int_{\mathbb{R}} \sum_{k=1}^{\infty} \left|\phi_k * f(x) - \phi_{k-1} * f(x)\right|^2 dx \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left|\phi_k * f(x) - \phi_{k-1} * f(x)\right|^2 dx \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left|(\phi_k - \phi_{k-1}) * f(x)\right|^2 dx \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left|\Delta_k * f(x)\right|^2 dx \qquad (\Delta_k(x) = \phi_k(x) - \phi_{k-1}(x)) \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |\widehat{\Delta_k * f}(x)|^2 \, dx \qquad \text{(by Plancherel's theorem)} \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |\widehat{\Delta_k}(x)|^2 \cdot |\hat{f}(x)|^2 \, dx \\
&= \int_{\mathbb{R}} \sum_{k=1}^{\infty} |\widehat{\Delta_k}(x)|^2 \cdot |\hat{f}(x)|^2 \, dx \\
&= \int_{\mathbb{R}} \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|^2 \cdot |\hat{f}(x)|^2 \, dx \\
&\leq C \int_{\mathbb{R}} |\hat{f}(x)|^2 \, dx
\end{aligned}$$

$$= C \int_{\mathbb{R}} |f(x)|^2 \, dx \qquad \text{(by Plancherel's theorem)}$$
$$= C\|f\|^2_{L^2(\mathbb{R})}$$

as desired. □

Remark 7. Since for $s \geq 2$, we have proved in Lemma 4 that the kernel operator $K(x) = \{\phi_k(x) - \phi_{k-1}(x)\}_{k \in \mathbb{Z}^+}$ satisfies the D_r condition for $1 \leq r < \infty$, it specifically satisfies D_1 condition. We also have proved in Lemma 6 that $Tf = \{(\phi_k - \phi_{k-1}) * f\}_{k \in \mathbb{Z}^+}$ is a bounded operator from $L^2(\mathbb{R})$ to $L^2_{\ell^s(\mathbb{Z}^+)}(\mathbb{R})$ since $\|K * f(x)\|_{\ell^s(\mathbb{Z}^+)} = \mathcal{V}_s f(x)$. Therefore, $Tf = \{(\phi_k - \phi_{k-1}) * f\}_{k \in \mathbb{Z}^+}$ is an ℓ^s-valued singular operator of convolution type for $s \geq 2$.

Lemma 8. *Let A and B be Banach spaces. A singular integral operator T mapping A-valued functions into B-valued functions can be extended to an operator defined in all L^p_A, $1 \leq p < \infty$, and satisfying*

(i) $\|Tf\|_{L^p_B} \leq C_p\|f\|_{L^p_A}, \quad 1 < p < \infty,$
(ii) $\|Tf\|_{WL^1_B} \leq C_1\|f\|_{L^1_A},$
(iii) $\|Tf\|_{L^1_B} \leq C_2\|f\|_{H^1_A},$
(iv) $\|Tf\|_{\mathrm{BMO}(B)} \leq C_3\|f\|_{L^\infty(A)}, \quad f \in L^\infty_c(A),$

where $C_p, C_1, C_2, C_3 > 0$, and $L^\infty_c(A)$ is the space of bounded functions with compact support.

Proof. This is Theorem 1.3 of Part II in Rubio de Francia, J. L. *et al* [5]. □

The following theorem is our first result:

Theorem 9. *Let $2 \leq s < \infty$, and let (n_k) be a lacunary sequence. Then there exits a constant $C > 0$ such that*

$$\|\mathcal{V}_s f\|_{L^1(\mathbb{R})} \leq C\|f\|_{H^1(\mathbb{R})}$$

for all $f \in H^1(\mathbb{R})$.

Proof. This follows from Remark 7 and Lemma 8 (iii) since $\|K * f(x)\|_{\ell^s(\mathbb{Z}^+)} = \mathcal{V}_s f(x)$. □

Remark 10. We have proved that $Tf = \{(\phi_k - \phi_{k-1}) * f\}_{k \in \mathbb{Z}^+}$ is an ℓ^s-valued singular operator of convolution type for $s \geq 2$. By applying Lemma 8 to this observation we also provide a different proof for the following known facts for $s \geq 2$ (see [4]) since $\|K * f(x)\|_{\ell^s(\mathbb{Z}^+)} = \mathcal{V}_s f(x)$.

(i) $\|\mathcal{V}_s f\|_{L^p(\mathbb{R})} \leq C_p\|f\|_{L^p(\mathbb{R})}, \quad 1 < p < \infty,$
(ii) $\|\mathcal{V}_s f\|_{WL^1(\mathbb{R})} \leq C_1\|f\|_{L^1(\mathbb{R})},$
(iii) $\|\mathcal{V}_s f\|_{\mathrm{BMO}(\mathbb{R})} \leq C_2\|f\|_{L^\infty(\mathbb{R})}, \quad f \in L^\infty_c(\mathbb{R}),$

where $C_p, C_1, C_2 > 0$.

Let $w \in L^1_{\text{loc}}(\mathbb{R})$ be a positive function. We say that w is an A_p weight for some $1 < p < \infty$ if the following condition is satisfied:

$$\sup_I \left(\frac{1}{|I|} \int_I w(x)\, dx \right) \left(\frac{1}{|I|} \int_I w(x)^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals I in $\mathbb{R}$.
We say that the function w is an A_∞ weight if there exist $\delta > 0$ and $\epsilon > 0$ such that given an interval I in $\mathbb{R}$, for any measurable $E \subset I$,

$$|E| < \delta \cdot |I| \implies w(E) < (1-\epsilon) \cdot w(I).$$

Here

$$w(E) = \int_E w.$$

It is well known and easy to see that $w \in A_p \implies w \in A_\infty$ if $1 < p < \infty$.
We say that $w \in A_1$ if given an interval I in $\mathbb{R}$ there is a positive constant C such that

$$\frac{1}{|I|} \int_I w(y)\, dy \le C w(x)$$

for a.e. $x \in I$.

Lemma 11. *Let A and B be Banach spaces, and T be a singular integral operator mapping A-valued functions into B-valued functions with kernel $K \in D_r$, where $1 < r < \infty$. Then, for all $1 < \rho < \infty$, and for all $(f_j) \in L^p_A(w) \cap L^p_A(\mathbb{R}^n)$, the weighted inequalities*

$$\left\| \left(\sum_j \|Tf_j\|_B^\rho \right)^{1/\rho} \right\|_{L^p(w)} \le C_{p,\rho}(w) \left\| \left(\sum_j \|f_j\|_A^\rho \right)^{1/\rho} \right\|_{L^p(w)}$$

hold if $w \in A_{p/r'}$ and $r' \le p < \infty$, or if $w \in A_p^{r'}$ and $1 < p \le r'$. Likewise, if $w(x)^{r'} \in A_1$, then the weak type inequality

$$w\left(\left\{ x : \left(\sum_j \|Tf_j(x)\|_B^\rho \right)^{1/\rho} > \lambda \right\} \right) \le C_\rho(w) \frac{1}{\lambda} \int \left(\sum_j \|f_j(x)\|_A^\rho \right)^{1/\rho} w(x)\, dx$$

holds for all $(f_j) \in L^1_A(w) \cap L^1_A(\mathbb{R}^n)$.

Proof. This is Theorem 1.6 of Part II in Rubio de Francia, J. L. *et al* [5]. □

Our next result is the following:

Theorem 12. *Let $2 \leq s < \infty$. Then, for all $1 < \rho < \infty$, and for all $(f_j) \in L^p(w) \cap L^p(\mathbb{R})$, the weighted inequalities*

$$\left\| \left(\sum_j (\mathcal{V}_s f_j)^\rho \right)^{1/\rho} \right\|_{L^p(w)} \leq C_{p,\rho}(w) \left\| \left(\sum_j |f_j|^\rho \right)^{1/\rho} \right\|_{L^p(w)}$$

hold if $w \in A_{p/r'}$ and $r' \leq p < \infty$, or if $w \in A_p^{r'}$ and $1 < p \leq r'$. Likewise, if $w(x)^{r'} \in A_1$, then the weak type inequality

$$w\left(\left\{ x : \left(\sum_j (\mathcal{V}_s f_j(x))^\rho \right)^{1/\rho} > \lambda \right\} \right) \leq C_\rho(w) \frac{1}{\lambda} \int \left(\sum_j |f_j(x)|^\rho \right)^{1/\rho} w(x)\, dx$$

holds for all $(f_j) \in L^1(w) \cap L^1(\mathbb{R})$.

Proof. We have proved for $2 \leq s < \infty$ that $Tf = \{(\phi_k - \phi_{k-1}) * f\}_{k \in \mathbb{Z}^+}$ is an ℓ^s-valued singular integral operator of convolution type and its kernel operator $K(x) = \{\phi_k(x) - \phi_{k-1}(x)\}_{k \in \mathbb{Z}^+}$ satisfies D_r condition for $1 \leq r < \infty$. Thus, the result follows from Lemma 11 and the fact that $\|K * f(x)\|_{\ell^s(\mathbb{Z}^+)} = \mathcal{V}_s f(x)$. □

In particular we have the following corollary:

Corollary 13. *Let $2 \leq s < \infty$. Then the weighted inequalities*

$$\|\mathcal{V}_s f\|_{L^p(w)} \leq C_{p,\rho}(w)\, \|f\|_{L^p(w)}$$

hold for all $(f_j) \in L^p_A(w) \cap L^p_A(\mathbb{R}^n)$ if $w \in A_{p/r'}$ and $r' \leq p < \infty$, or if $w \in A_p^{r'}$ and $1 < p \leq r'$. Likewise, if $w(x)^{r'} \in A_1$, then the weak type inequality

$$w\left(\{x : \mathcal{V}_s f(x) > \lambda\}\right) \leq C_\rho(w) \frac{1}{\lambda} \int |f(x)| w(x)\, dx$$

holds for all $(f_j) \in L^1(w) \cap L^1(\mathbb{R})$.

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