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Amenable covers and integral foliated simplicial volume

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ABSTRACT. In analogy with ordinary simplicial volume, we show that integral foliated simplicial volume of oriented closed connected aspherical n-manifolds that admit an open amenable cover of multiplicity at most n is zero. This implies that the fundamental groups of such manifolds have fixed price and are cheap as well as reproves some statements about homology growth.

CONTENTS

1. Introduction

An open problem by Gromov [22, p. 232; 24, p. 769 3.1 (e)] asks whether oriented closed connected manifolds with vanishing simplicial volume have vanishing Euler characteristic. For the integral foliated simplicial volume such an implication holds [42] (as suggested by Gromov [23, p. 305ff]). However, the corresponding follow-up question is open:

Question 1.1. *Let M be an oriented closed connected aspherical manifold with vanishing simplicial volume. Does then also the integral foliated simplicial volume of M vanish?*

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This question is known to have an affirmative answer in many cases [8, 14, 15, 18]. In this article, we extend the class of positive examples by aspherical manifolds that admit amenable open covers of small multiplicity: If M admits an open amenable cover of multiplicity at most n, we write $\mathrm{cat}_{\mathrm{Am}} M \leq n$. Usually the amenable category $\mathrm{cat}_{\mathrm{Am}}$ of M is defined in terms of the cardinality of amenable covers. However, in the case of CW-complexes the two definitions are equivalent [10, Remark 3.13; 13, Lemma A.4].

Theorem 1.2. *Let M be an oriented closed connected smooth aspherical manifold with $\mathrm{cat}_{\mathrm{Am}} M \leq \dim M$. Let $\alpha\colon \pi_1(M) \curvearrowright (X,\mu)$ be an essentially free standard $\pi_1(M)$-space. Then*

$$|M|^\alpha = 0.$$

In particular, $|M| = 0$.

This is an extension of Gromov's vanishing result: Every oriented closed connected n-manifold M with $\mathrm{cat}_{\mathrm{Am}} M \leq \dim M$ has zero simplicial volume [21]. The bound on the amenable category is optimal because every smooth manifold M satisfies $\mathrm{cat}_{\mathrm{Am}} M \leq \dim M + 1$ [10, Remark 2.8].

A similar statement was already known for the weightless version of integral foliated simplicial volume [40] and in the presence of macroscopic scalar curvature conditions [5]. Our proof of Theorem 1.2 also relies on a Rokhlin lemma argument, but avoids delicate volume estimates in cubical nerves.

In the context of Gromov's question, we arrive at the following problem:

Question 1.3. *Do there exist oriented closed connected aspherical manifolds M with $\|M\| = 0$ but $\mathrm{cat}_{\mathrm{Am}} M = \dim M + 1$?*

We expect the answer to this question to be positive. However, exhibiting such an example probably requires finding new obstructions against the existence of open amenable covers of small multiplicity.

Further information on Gromov's problem and its ramifications can be found in the literature [33].

1.1. Examples. Oriented closed connected smooth manifolds M that satisfy at least one of the following conditions have $\mathrm{cat}_{\mathrm{Am}} M \leq \dim M$:

- Manifolds with amenable fundamental group;
- Graph 3-manifolds [20];
- Manifolds that are the total space of a fibre bundle $M \to B$ with oriented closed connected fibre N and $\mathrm{cat}_{\mathrm{Am}} N \leq \dim(M)/(\dim(B) + 1)$ [32];
- Manifolds of dimension $n \geq 4$ whose fundamental group Γ contains an amenable normal subgroup A with $\mathrm{cd}_{\mathbb{Z}}(\Gamma/A) < n$ (Lemma 7.2);
- Manifolds that admit a smooth S^1-action without fixed points (Corollary 7.4);
- Manifolds that admit a regular smooth circle foliation with finite holonomy groups (Proposition 7.3);
- Manifolds that admit an F-structure (of possibly zero rank) [4, Corollary 2.8].

In view of Theorem 1.2, *aspherical* manifolds of this type have vanishing integral foliated simplicial volume; more precisely, we have vanishing for all essentially free parameter spaces. In most of these cases, the corresponding vanishing results already had been proved by alternative means [8, 14, 16, 18]. Theorem 1.2 gives a uniform perspective.

1.2. Application to manifolds. Theorem 1.2 gives new examples of manifolds that satisfy integral approximation for simplicial volume [15; 16; 31; 36, Section 6]:

Corollary 1.4. *Let M be an oriented closed connected smooth aspherical manifold with* $\operatorname{cat}_{\mathrm{Am}} M \leq \dim M$ *and residually finite fundamental group. Then*

$$\|M\|_{\mathbb{Z}}^{\infty} = 0 = \|M\|.$$

More precisely, in this situation, the corresponding statement also holds for all residual chains in the fundamental group.

Proof. As $\pi_1(M)$ is residually finite, the profinite completion $\widehat{\pi_1(M)}$ endowed with its normalized Haar measure and the action by left translations constitutes a free standard $\pi_1(M)$-space, and we have [18, Theorem 2.6; 34, Theorem 6.6 and Remark 6.7]

$$\|M\|_{\mathbb{Z}}^{\infty} = \left| M \right|^{\widehat{\pi_1(M)}}.$$

We can now apply Theorem 1.2 and the general estimate $\|M\| \leq \|M\|_{\mathbb{Z}}^{\infty}$.

This also works in the situation of general residual chains [18, Theorem 2.6]. □

1.3. Applications to homology growth and groups. From Theorem 1.2, we obtain corresponding vanishing results for the L^2-Betti numbers [23, p. 307; 42, Corollary 5.28] (and whence the Euler characteristic), logarithmic torsion growth [18, Theorem 1.6; 41], cost, and the rank gradient [29].

The cost of a group [19] is a dynamical version of the rank gradient, which is the gradient invariant associated with the minimal number of generators [27].

Corollary 1.5. *Let M be an oriented closed connected aspherical manifold with* $\operatorname{cat}_{\mathrm{Am}} M \leq \dim M \neq 0$.

(1) *Then $\pi_1(M)$ is of fixed price and cheap. Thus,* $\operatorname{cost} \pi_1(M) = 1$.
(2) *If, in addition, $\pi_1(M)$ is residually finite, then* $\operatorname{rg}(\pi_1(M), \Gamma_*) = 0$ *for all Farber chains Γ_* of $\pi_1(M)$. In particular,* $\operatorname{rg} \pi_1(M) = 0$.

Proof. The first part follows from Theorem 1.2 and the cost estimate via integral foliated simplicial volume [30]. The second part follows from the first part and the computation of the rank gradient via cost [1]. □

We can view Corollary 1.5 as an extension of the vanishing results for cost and rank gradients of amalgamated free products of amenable groups [19, 39].

Moreover, we obtain a new proof of vanishing for homology growth and logarithmic torsion growth [41, Theorem 1.6]. Note that the statement for $\mathbb{F}_p$-coefficients is equivalent to the statement for all principal ideal domains. Via the bounds of homology growth by the stable integral simplicial volume [18, Theorem 1.6] and Theorem 1.2 we obtain:

Corollary 1.6. *Let M be an oriented closed connected aspherical manifold with* $\operatorname{cat}_{\mathrm{Am}} M \leq \dim M \neq 0$ *and residually finite fundamental group. Let* $(\Gamma_j)_{j \in \mathbb{N}}$ *be a residual chain in* $\Gamma := \pi_1(M)$*. Moreover, let* $k \in \mathbb{N}$*. Then:*

(1) *If R is a principal ideal domain, then*

$$\limsup_{j \to \infty} \frac{\operatorname{rk}_R H_k(\Gamma_j; R)}{[\Gamma : \Gamma_j]} = 0.$$

(2) *We have*

$$\limsup_{j \to \infty} \frac{\log \left|\operatorname{tors} H_k(\Gamma_j; \mathbb{Z})\right|}{[\Gamma : \Gamma_j]} = 0.$$

Organisation of this article. Section 2 introduces basic terminology. An outline of the proof of Theorem 1.2 is given in Section 3. The Rokhlin chain maps are developed in Section 4. The topological steps are carried out in Section 5. Finally, Section 6 wraps up the proof of Theorem 1.2. Section 7 contains examples of manifolds with small $\operatorname{cat}_{\mathrm{Am}}$.

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2. Preliminaries

We recall basic notions on integral foliated simplicial volume, amenable open covers, and resolutions.

2.1. Simplicial volume. The simplicial volume is a homotopy invariant of closed manifolds introduced by Gromov [21]. Let M be an oriented closed connected n-manifold and let R be a normed ring with unit, e.g., $\mathbb{R}$ or $\mathbb{Z}$. For a singular R-chain $c = \sum_{i=1}^{k} a_i \sigma_i \in C_n(M; R)$, one defines the *$\ell^1$-norm* by

$$|c|_1 := \sum_{i=1}^{k} |a_i|.$$

This norm induces a seminorm on singular homology: Given $\alpha \in H_n(M; R)$, the *ℓ^1-seminorm* of α is defined as

$$\|\alpha\|_1 := \inf\bigl\{|c|_1 \bigm| c \in C_n(M; R) \text{ is a cycle representing } \alpha\bigr\}.$$

By taking the ℓ^1-seminorm of a preferred class in homology we obtain the definition of simplicial volume:

Definition 2.1 (R-simplicial volume)**.** Let M be an oriented closed connected n-manifold. The *R-simplicial volume* of M is defined as

$$\|M\|_R := \left\|[M]_R\right\|_1,$$

where $[M]_R \in H_n(M;R)$ denotes the R-fundamental class of M.

When $R = \mathbb{R}$ we will simply write $\|M\| := \|M\|_{\mathbb{R}}$ and we will refer to it simply as the *simplicial volume of M*.

Remark 2.2. By definition, we have $\|M\| \leq \|M\|_{\mathbb{Z}}$ and $1 \leq \|M\|_{\mathbb{Z}}$ (if M is non-empty).

Simplicial volume is known to be positive in many cases: e.g. hyperbolic manifolds [21, 44], locally symmetric spaces of non-compact type [7, 28] and manifolds with sufficiently negative curvature [12, 26]. Two major sources of vanishing of simplicial volume are [21]:

- If M is an oriented closed connected n-manifold with $\operatorname{cat}_{\mathrm{Am}} M \leq \dim M$, then $\|M\| = 0$.
- If M is an oriented closed connected n-manifold that admits a self-map f with $|\deg f| \geq 2$, then $\|M\| = 0$.

A more comprehensive list of examples can be found in the literature [33].

In general integral simplicial volume and simplicial volume are far from being equal (Remark 2.2). The situation becomes more interesting after stabilisation [17, 23]:

Definition 2.3 (stable integral simplicial volume)**.** Let M be an oriented closed connected n-manifold. The *stable integral simplicial volume of M* is defined as

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf\left\{\frac{\|N\|_{\mathbb{Z}}}{d} \;\middle|\; d \in \mathbb{N},\ N \text{ is a } d\text{-sheeted covering space of } M\right\}.$$

Similarly to Question 1.1, working with stable integral simplicial volume the leading question is the following:

Question 2.4. *Let M be an oriented closed connected aspherical n-manifold with $\|M\| = 0$ and residually finite fundamental group. Then, do we have $\|M\|_{\mathbb{Z}}^{\infty} = 0$?*

The question is known to have a positive answer in the the case of aspherical surfaces [21], aspherical 3-manifolds [15, 16], smooth aspherical manifolds admitting a smooth S^1-action without fixed points [14], smooth aspherical manifolds admitting a regular smooth circle foliation with additional properties [8], generalised graph manifolds [15], and closed aspherical manifolds with vanishing minimal volume [5].

2.2. Integral foliated simplicial volume. Integral foliated simplicial volume is defined via simplicial volumes with twisted coefficients in $\mathrm{L}^\infty(X, \mathbb{Z})$. The basic idea is that $\mathrm{L}^\infty(X, \mathbb{Z})$ provides enough rigidity through the constraint that functions are integer-valued, but also enough flexibility through partitions of X. We recall the terminology in more detail:

Let Γ be a countable group. A *standard* Γ*-space* is a measure preserving action $\Gamma \curvearrowright (X,\mu)$ of Γ on a standard Borel probability space (X,μ). We then equip $\mathrm{L}^\infty(X,\mathbb{Z})$ with the $\mathbb{Z}\Gamma$-module structure given by

$$\gamma \cdot f := \bigl(x \mapsto f(\gamma^{-1} \cdot x)\bigr)$$

for all $\gamma \in \Gamma$ and all $f \in \mathrm{L}^\infty(X,\mathbb{Z})$. We consider homology of spaces with fundamental group Γ with twisted coefficients in $\mathrm{L}^\infty(X,\mathbb{Z})$.

Definition 2.5. Let M be a connected CW-complex (or a path-connected space that admits a universal covering) with countable fundamental group Γ and let $\alpha\colon \Gamma \curvearrowright (X,\mu)$ be a standard Γ-space.

Let $n \in \mathbb{N}$ and let $x \in H_n(M;\mathbb{Z})$. We write $x^\alpha \in H_n\bigl(M;\mathrm{L}^\infty(X,\mathbb{Z})\bigr)$ for the image of x under the change of coefficients map $\mathbb{Z} \hookrightarrow \mathrm{L}^\infty(X,\mathbb{Z})$.

We define the *α-parametrised norm of x* by

$$|x|^\alpha := \inf\bigl\{|z|_1 \bigm| z \in \mathrm{L}^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z}) \text{ is a cycle representing } x^\alpha\bigr\},$$

where

$$\Bigl|\sum_{i\in I} f_i \otimes \sigma_i\Bigr|_1 := \sum_{i\in I} \int_X |f_i| \, d\mu$$

for all chains $\sum_{i\in I} f_i \otimes \sigma_i \in \mathrm{L}^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z})$ in reduced form.

Definition 2.6 (integral foliated simplicial volume [23, 42])**.** Let M be an oriented closed connected n-manifold with fundamental group Γ.

- If $\alpha\colon \Gamma \curvearrowright (X,\mu)$ is a standard Γ-space, then the *α-parametrised simplicial volume* of M is defined as

$$|M|^\alpha := |[M]_\mathbb{Z}|^\alpha.$$

- The *integral foliated simplicial volume* $|M|$ *of* M is defined as the infimum of $|M|^\alpha$ over all standard Γ-spaces α.

For further background on the integral foliated simplicial volume, we refer to the literature [31, 42].

2.3. Amenable open covers. We recall amenable open covers; more information on amenable groups and their dynamical properties can be found in the literature.

Definition 2.7 (amenable subset, amenable open cover)**.** Let M be a topological space.

- A subset $U \subset M$ is called *amenable* if the subgroup $\operatorname{im}(\pi_1(U,x) \to \pi_1(M,x))$ of $\pi_1(M,x)$ is amenable for every $x \in U$.
- An open cover of M is called *amenable* if each member is amenable in the sense above.

2.4. Resolutions from free actions. We fix basic notation concerning resolutions and recall the fundamental theorem of homological algebra.

Definition 2.8. Let Γ be a group and A, B be $\mathbb{Z}\Gamma$-modules. Let $\varepsilon : C_* \to A$ and $\eta : D_* \to B$ be augmented $\mathbb{Z}\Gamma$-chain complexes. We say that a $\mathbb{Z}\Gamma$-chain map $\varphi : C_* \to D_*$ *extends* a $\mathbb{Z}\Gamma$-homomorphism $f : A \to B$ if $\eta \circ \varphi_0 = f \circ \varepsilon$.

Lemma 2.9 (Fundamental theorem of homological algebra)**.** *Let Γ be a group, let B be a $\mathbb{Z}\Gamma$-module, let $\varepsilon : C_* \to \mathbb{Z}$ be a projective $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$, and let $\eta : D_* \to B$ be a $\mathbb{Z}\Gamma$-resolution of B (not necessarily projective). If $\varphi, \varphi' : C_* \to D_*$ extend the same $\mathbb{Z}\Gamma$-homomorphism $\mathbb{Z} \to B$, then we have $\varphi \simeq_{\mathbb{Z}\Gamma} \varphi'$.*

Example 2.10. Let Γ be a group and let $\Gamma \curvearrowright E$ be a free Γ-action on a set E. For each $n \in \mathbb{N}$, the product E^{n+1} carries the diagonal Γ-action. Then $\mathbb{Z}[E^{*+1}]$ with the simplicial boundary operator given by

$$\partial_n(e_0, \dots, e_n) := \sum_{j=0}^{n} (-1)^j \cdot (e_0, \dots, \widehat{e}_j, \dots, e_n)$$

for all $n \in \mathbb{N}$, $(e_0, \dots, e_n) \in E^{n+1}$, together with the augmentation

$$\begin{aligned} \varepsilon_E : \mathbb{Z}[E^{0+1}] &\to \mathbb{Z} \\ E \ni e_0 &\mapsto 1, \end{aligned}$$

is a free $\mathbb{Z}\Gamma$-resolution of the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$.

Example 2.11. Let M be a path-connected aspherical space with fundamental group Γ. Then the singular chain complex $C_*(\widetilde{M}; \mathbb{Z})$ of the universal covering $\widetilde{M}$ of M, together with the augmentation $C_0(\widetilde{M}; \mathbb{Z}) \to \mathbb{Z}$ mapping singular 0-simplices to 1, is a free $\mathbb{Z}\Gamma$-resolution of the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$.

Definition 2.12 (essential chains, essential norm)**.** Let Γ be a group, let E be a free Γ-set, and let $n \in \mathbb{N}$.

- A tuple $(x_0, \dots, x_n) \in E^{n+1}$ is *degenerate* if there are $k, l \in \{0, \dots, n\}$ with $k \neq l$ and $x_k = x_l$. Non-degenerate tuples are called *essential*.
- We write $E^{\langle n+1 \rangle}$ for the set of all essential tuples in E^{n+1}.
- For a normed $\mathbb{Z}$-module A and a reduced chain $c = \sum_{x \in E^{n+1}} a_x \otimes x \in A \otimes_{\mathbb{Z}} \mathbb{Z}[E^{n+1}]$, we set

$$\begin{aligned} |c|_{1,\mathrm{ess}} &:= \sum_{x \in E^{\langle n+1 \rangle}} |a_x|, \\ |c|_1 &:= \sum_{x \in E^{n+1}} |a_x|. \end{aligned}$$

Note that $|\cdot|_{1,\mathrm{ess}}$ is a semi-norm on $A \otimes_{\mathbb{Z}} \mathbb{Z}[E^{n+1}]$ whose restriction to essential chains is a norm.

3. Outline of the proof of Theorem 1.2

The proof of Theorem 1.2 is a dynamical version of the amenable reduction lemma [2, 3, 24]. As in Theorem 1.2, let M be an oriented closed connected smooth aspherical manifold with $\operatorname{cat}_{\mathrm{Am}} M \leq \dim M$ and let $\alpha : \pi_1(M) \curvearrowright (X, \mu)$ be an essentially free standard $\pi_1(M)$-space. Let $(U_i)_{i\in I}$ be a finite amenable open cover of M with multiplicity at most $n := \dim M$. Let $(\Gamma_i)_{i\in I}$ be the (amenable) subgroups of $\Gamma := \pi_1(M)$ associated with $(U_i)_{i\in I}$.

To prove that $|M|^{\alpha} = 0$, we proceed in the following steps: Let $E := \Gamma \times I$, with the Γ-translation action on the first factor. Because M is aspherical, we can replace the $\mathbb{Z}\Gamma$-chain complex $C_*(\widetilde{M}; \mathbb{Z})$ by $\mathbb{Z}[E^{*+1}]$.

- Refining the open cover $(U_i)_{i\in I}$ and using a subdivision process, we obtain a fundamental cycle
$$z' = \sum_{\sigma \in E^{n+1}} a_\sigma \otimes \sigma \in \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}[E^{n+1}]$$
of M with the following property: If $\sigma = ((\gamma_0, i_0), \dots, (\gamma_n, i_n))$ is a simplex with $a_\sigma \neq 0$, then σ satisfies the "colouring condition"
$$\exists_{j,k\in\{0,\dots,n\}} \quad j \neq k \quad \text{and} \quad i_j = i_k \quad \text{and} \quad \gamma_j^{-1} \cdot \gamma_k \in \Gamma_{i_j} = \Gamma_{i_k}.$$
This is a pigeon-hole principle argument based on $\operatorname{mult} U \leq n$.
- In the complex $\mathbb{Z}[E^{*+1}]$, we have an additional degree of freedom. Using Rokhlin partitions of X coming from the amenable groups Γ_i, we find good equivariant Borel partitions of the Γ-space $X \times E$; such partitions lead to the Rokhlin chain map $\mathbb{Z}[E^{*+1}] \to \mathrm{L}^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\dots^{*+1}]$. We apply these Rokhlin chain maps to z'. Because z' satisfies the colouring condition, each simplex in z' has one edge that is susceptible to amenability of the Γ_i; thus, for each $\delta \in \mathbb{R}_{>0}$ we find a fundamental cycle z'_δ that admits a decomposition
$$\begin{aligned} z'_\delta = {} & \text{a chain consisting of degenerate simplices} \\ & + \text{a chain of norm in } o(\delta). \end{aligned}$$
Here, the "almost invariance" property of the amenable groups Γ_i is reflected in the fact that almost everything cancels, except for degenerate terms.

As degenerate simplices can be neglected when considering the ℓ^1-semi-norm on homology, taking $\delta \to 0$ shows that $|M|^{\alpha} = 0$. This strategy can be subsumed in the following diagram:

$$\begin{tikzcd}
C_*(\widetilde{M}; \mathbb{Z}) \arrow[r, "\text{equivariant subdivision}"] \arrow[d, dashed, "\simeq \text{ incl}"'] & \mathbb{Z}[E^{*+1}] \arrow[d, "\text{Rokhlin}"] \\
\mathrm{L}^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} C_*(\widetilde{M}; \mathbb{Z}) & \mathrm{L}^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\dots^{*+1}] \arrow[l, "\text{filling}"] \arrow[loop right, dashed, "\text{essentialness}"]
\end{tikzcd}$$

Section 4 is devoted to the Rokhlin map; Section 5 contains the subdivision argument. The actual proof of Theorem 1.2 is given in Section 6.

4. The Rokhlin chain map

We construct the Rokhlin chain map, we determine its effect on homology, and – most importantly – establish the key norm estimates.

We first explain how Borel partitions lead to chain maps. The Rokhlin chain maps then are the chain maps associated with Rokhlin partitions.

4.1. Borel chains. Let $\Gamma \curvearrowright (X,\mu)$ be a standard Γ-space and let S be a countable free Γ-set. We endow $X \times S^{n+1}$ with the diagonal Γ-action and the product measure of μ and the counting measure on S. We write

$$\mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{n+1}, \mathbb{Z})$$

for the submodule of $\mathrm{L}^\infty(X \times S^{n+1}, \mathbb{Z})$ consisting of those (equivalence classes of) essentially bounded functions $X \times S^{n+1} \to \mathbb{Z}$ whose support is contained in a set of the form $X \times F$, where $F \subset S^{n+1}$ is finite. For $n \in \mathbb{N}_{>0}$, we let

$$\begin{aligned} \partial_n \colon \ & \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{n+1}, \mathbb{Z}) \to \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{n}, \mathbb{Z}) \\ & f \mapsto \Bigl((x,s) \mapsto \sum_{j=0}^{n} (-1)^j \cdot \sum_{t \in S} f(x, s_0, \dots, s_{j-1}, t, s_j, \dots, s_{n-1}) \Bigr). \end{aligned}$$

This turns $\mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1}, \mathbb{Z})$ into a $\mathbb{Z}\Gamma$-chain complex. We consider the augmentation map

$$\begin{aligned} \widetilde{\varepsilon}_S \colon \ & \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{0+1}, \mathbb{Z}) \to \mathrm{L}^\infty(X, \mathbb{Z}) \\ & f \mapsto \sum_{s \in S} f(\,\cdot\,, s). \end{aligned}$$

Furthermore, we equip $\mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1}, \mathbb{Z})$ with the *essential norm* given by

$$|f|_{1,\mathrm{ess}} := \sum_{s \in S^{\langle n+1 \rangle}} \int_X |f(\,\cdot\,, s)| \, d\mu(x)$$

for all $f \in \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{n+1}, \mathbb{Z})$.

Lemma 4.1. *Let Γ be a countable group, let S be a free Γ-set, and let $\Gamma \curvearrowright (X,\mu)$ be a standard Γ-space. Then*

$$\begin{aligned} \xi_S \colon \ & \mathrm{L}^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[S^{*+1}] \to \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1}, \mathbb{Z}) \\ & f \otimes s \mapsto \left((x,t) \mapsto \begin{cases} f(x) & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases} \right) \end{aligned}$$

and

$$\begin{aligned} \zeta_S \colon \ & \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1}, \mathbb{Z}) \to \mathrm{L}^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[S^{*+1}] \\ & f \mapsto \sum_{s \in S^{*+1}} f(\,\cdot\,, s) \otimes s \end{aligned}$$

are well-defined mutually inverse $\mathbb{Z}\Gamma$*-chain isomorphisms, where the left hand side is endowed with the diagonal* Γ*-action.*

These chain maps are compatible with the augmentations $\mathrm{id}_{\mathrm{L}^\infty(X,\mathbb{Z})} \otimes_{\mathbb{Z}} \varepsilon_S$ *and* $\widetilde{\varepsilon}_S$ *as well as with the essential norms* $|\cdot|_{1,\mathrm{ess}}$.

Proof. This is a straightforward computation. □

Corollary 4.2. *In the situation of Lemma* 4.1*,* $\widetilde{\varepsilon}_S\colon\ \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1}, \mathbb{Z}) \to \mathrm{L}^\infty(X, \mathbb{Z})$ *is a* $\mathbb{Z}\Gamma$*-resolution of* $\mathrm{L}^\infty(X, \mathbb{Z})$*.*

Proof. The chain complex $\mathbb{Z}[S^{*+1}]$ with augmentation $\varepsilon_S\colon\ \mathbb{Z}[S^{*+1}] \to \mathbb{Z}$ is a $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$. The $\mathbb{Z}$-module $\mathrm{L}^\infty(X, \mathbb{Z})$ is torsionfree, thus flat. So

$$\mathrm{L}^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[S^{*+1}] \to \mathrm{L}^\infty(X, \mathbb{Z})$$

is exact. With the diagonal action on $\mathrm{L}^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[S^{*+1}]$ it becomes a $\mathbb{Z}[\Gamma]$-resolution of $\mathrm{L}^\infty(X, \mathbb{Z})$. Lemma 4.1 concludes the proof. □

4.2. The chain map associated with a Borel partition. Borel partitions lead to corresponding equivariant partition chain maps.

Definition 4.3. Let Γ be a countable group, let E be a countable free Γ-set, and let $\Gamma \curvearrowright (X, \mu)$ be a free standard Γ-space. A Γ*-equivariant* μ*-partition of* $X \times E$ is a family $P := (W_s)_{s\in S}$ of Borel subsets of $X \times E$ with the following properties:

- The sets $(W_s)_{s\in S}$ are pairwise disjoint.
- The union $\bigcup_{s\in S} W_s$ is $\mu \otimes \delta$-conull in $X \times E$, where δ denotes the counting measure on E.
- The set S is equipped with a free Γ-action and for all $\gamma \in \Gamma$, $s \in S$, we have
$$\gamma \cdot W_s = W_{\gamma\cdot s}.$$

We say that P is *of finite type* if the following holds: For each $e \in E$, there are only finitely many $s \in S$ with $W_s \cap X \times \{e\} \neq \emptyset$.

Remark 4.4 (induced partitions)*.* Let Γ be a countable group, let E be a countable free Γ-set, let $\Gamma \curvearrowright (X, \mu)$ be a free standard Γ-space, and let $P = (W_s)_{s\in S}$ be a Γ-equivariant μ-partition of $X \times E$.

If $n \in \mathbb{N}$ and $s_0, \dots, s_n \in S$, we write

$$\begin{aligned} W_{(s_0,\dots,s_n)} &:= W_{s_0} \times_X \cdots \times_X W_{s_n} \\ &:= \big\{(x, e_0, \dots, e_n) \bigm| \forall_{r\in\{0,\dots,n\}} \quad (x, e_r) \in W_{s_r}\big\} \subset X \times E^{n+1}. \end{aligned}$$

Then, for each $n \in \mathbb{N}$, the family $(W_s)_{s\in S^{n+1}}$ is a Γ-equivariant μ-partition of $X \times E^{n+1}$. If P is of finite type, then also $(W_s)_{s\in S^{n+1}}$ is so.

Proposition 4.5. *Let* Γ *be a countable group, let* E *be a free* Γ*-space, and let* $\Gamma \curvearrowright (X, \mu)$ *be a free standard* Γ*-space. Let* $P := (W_s)_{s\in S}$ *be a* Γ*-equivariant* μ*-partition of* $X \times E$ *of finite type. Then*

$$\begin{aligned} \Phi^P\colon\ & \mathbb{Z}[E^{*+1}] \to \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1}, \mathbb{Z}) \\ & (e_0, \dots, e_n) \mapsto \big((x, s_0, \dots, s_n) \mapsto \chi_{W_{(s_0,\dots,s_n)}}(x, e_0, \dots, e_n)\big) \end{aligned}$$

is a $\mathbb{Z}\Gamma$*-chain map that extends* $\mathbb{Z} \hookrightarrow \mathrm{L}^\infty(X, \mathbb{Z})$.

Proof. Let $n \in \mathbb{N}$, let $e \in E^{n+1}$, and let $f := \Phi^P(e)$.

We first show that f indeed lies in $\mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1}, \mathbb{Z})$. Clearly, f is measurable and essentially bounded by 1 because $(W_s)_{s \in S^{n+1}}$ is a μ-partition of $X \times E^{n+1}$ (Remark 4.4). Moreover, f has finite support as P is of finite type.

For Γ-equivariance, we compute for all $(x, s) \in X \times S^{n+1}$ that (using that P is Γ-equivariant)

$$\begin{aligned} \big(\gamma \cdot (\Phi^P(e))\big)(x, s) &= \chi_{W_{\gamma^{-1} \cdot s}}(\gamma^{-1} \cdot x, e) \\ &= \chi_{\gamma^{-1} \cdot W_s}(\gamma^{-1} \cdot x, e) \\ &= \chi_{W_s}(x, \gamma \cdot e) \\ &= \big(\Phi^P(\gamma \cdot e)\big)(x, s). \end{aligned}$$

For the chain map property, we compute for all $j \in \{0, \dots, n\}$ and all $s \in S^n$ that (using that P induces a partition on $X \times E^{n+1}$)

$$\begin{aligned} \sum_{t \in S} \Phi^P(e)(x, s_0, \dots, s_{j-1}, t, s_j, \dots, s_{n-1}) &= \sum_{t \in S} \chi_{W_{(s_0, \dots, s_{j-1}, t, s_j, \dots, s_{n-1})}}(x, e) \\ &= \chi_{\bigcup_{t \in S} W_{(s_0, \dots, s_{j-1}, t, s_j, \dots, s_{n-1})}}(x, e) \\ &= \chi_{W_{(s_0, \dots, s_{n-1})}}(x, e_0, \dots, \widehat{e}_j, \dots, e_n). \end{aligned}$$

Therefore, $\partial\big(\Phi^P(e)\big) = \Phi^P(\partial e)$.

Finally, we consider degree 0: As $\widetilde{\varepsilon}_S \colon \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1}, \mathbb{Z}) \to \mathrm{L}^\infty(X, \mathbb{Z})$ is a $\mathbb{Z}\Gamma$-resolution (Corollary 4.2), the notion of "extends" from Definition 2.8 is available. To see that the extension condition is satisfied, it suffices to show that $\widetilde{\varepsilon}_S \circ \Phi^P(e_0)$ is the constant function 1 for each $e_0 \in E^{0+1}$: By construction,

$$\widetilde{\varepsilon}_S \circ \Phi^P(e_0) = \sum_{s \in S} \big(x \mapsto \chi_{W_s}(x, e_0)\big).$$

Because $(W_s)_{s \in S}$ is a μ-partition of $X \times E$, this sum equals 1. □

4.3. The Rokhlin chain map. The Rokhlin chain maps are chain maps obtained by applying the construction of the previous section to Rokhlin partitions.

We use the following version of the Ornstein–Weiss Rokhlin lemma, which allows us to perform divisions on the level of Borel sets:

Theorem 4.6 (Rokhlin lemma [11, Theorem 3.6]). *Suppose* Γ *is a countable amenable group. Let* $\Gamma \curvearrowright (X, \mu)$ *be a free standard* Γ*-action, let* $F \subset \Gamma$ *be a finite set, and let* $\delta \in \mathbb{R}_{>0}$.

Then, there exists a μ*-conull* Γ*-invariant Borel set* $X' \subset X$*, a finite set* J*, a family* $(A_j)_{j \in J}$ *of Borel subsets of* X'*, and a family* $(T_j)_{j \in J}$ *of* (F, δ)*-invariant nonempty finite subsets of* Γ *such that* $(T_j \cdot x)_{j \in J, x \in A_j}$ *partitions* X'.

Here, a non-empty finite subset $T \subset \Gamma$ is (F, δ)-*invariant* if

$$\frac{|F \cdot T \triangle T|}{|T|} \leq \delta.$$

Setup 4.7. *Let Γ be a countable group, let $(\Gamma_i)_{i \in I}$ be a non-empty finite family of amenable subgroups of Γ, and let $F \subset \Gamma$ be a finite set.*

Moreover, let $\Gamma \curvearrowright (X, \mu)$ be an essentially free standard Γ-space.

Let $\delta \in \mathbb{R}_{>0}$; for each $i \in I$, let $((A_{i,j})_{j \in J_i}, (T_{i,j})_{j \in J_i})$ be a $(F \cap \Gamma_i, \delta)$-Rokhlin partition of a Γ_i-invariant μ-conull subset X_i' of X with respect to $F \cap \Gamma_i$, as provided by Theorem 4.6. *Without loss of generality we also assume that each $T_{i,j}$ contains the neutral element of Γ.*

Let $E := \Gamma \times I$, with the Γ-translation action on the first factor.

Remark 4.8. For later use, we record that in the situation of Setup 4.7, we have

$$\mu(A_{i,j}) = \frac{\mu(T_{i,j} \cdot A_{i,j})}{|T_{i,j}|}$$

for all $i \in I$ and all $j \in J_i$, by the partition condition.

Moreover, given $i \in I$ and $j_0, j_1 \in J_i$ with $j_0 \neq j_1$, we have $A_{i,j_0} \cap A_{i,j_1} = \emptyset$ because T_{i,j_0} and T_{i,j_1} contain the neutral element.

Definition 4.9 (the Rokhlin chain map)**.** In the situation of Setup 4.7, the Rokhlin partition gives a Γ-equivariant μ-partition of $X \times \Gamma \times I$ which has finite type, namely

$$P^\delta := \bigl((\gamma \cdot A_{i,j}) \times (\gamma \cdot T_{i,j}{}^{-1}) \times \{i\}\bigr)_{\gamma \in \Gamma, i \in I, j \in J_i}.$$

We write $S := \{(\gamma, i, j) \mid \gamma \in \Gamma, i \in I, j \in J_i\}$, equipped with the Γ-translation action on the first factor. For $s = (\gamma, i, j) \in S$, we set

$$W_s := (\gamma \cdot A_{i,j}) \times (\gamma \cdot T_{i,j}^{-1}) \times \{i\}.$$

Finally, we abbreviate $\Phi^\delta := \Phi^{P^\delta} : \mathbb{Z}[E^{*+1}] \to \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1}, \mathbb{Z})$ (Proposition 4.5).

Definition 4.10 (colouring condition)**.** In the situation of Setup 4.7, we say that a tuple $z = (\lambda_r, t_r)_{r \in [n]} \in E^{n+1}$ satisfies the F-*colouring condition* if there exists an $i \in I$ and $k, l \in [n]$ with $k < l$ such that $t_k = t_l = i$ and

$$\lambda_k^{-1} \cdot \lambda_l \in F \cap \Gamma_i.$$

Here, $[n] := \{0, \dots, n\}$.

In the situation of Setup 4.7, we always consider S to be the set described in Definition 4.9.

Theorem 4.11 (properties of the Rokhlin chain map)**.** *Assuming the situation of Setup* 4.7, *the Rokhlin chain map $\Phi^\delta : \mathbb{Z}[E^{*+1}] \to \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1}, \mathbb{Z})$ has the following properties:*

(1) *The map Φ^δ is a $\mathbb{Z}\Gamma$-chain map that extends $\mathbb{Z} \hookrightarrow \mathrm{L}^\infty(X, \mathbb{Z})$.*

(2) *Let $z \in \mathbb{Z}[E^{*+1}]$ be a chain that satisfies the F-colouring condition. Then $|\Phi^\delta(z)|_{1,\mathrm{ess}} \le \delta \cdot |z|_1$; in particular,*

$$\lim_{\delta \to 0} \left|\Phi^\delta(z)\right|_{1,\mathrm{ess}} = 0.$$

Proof. *Ad 1.* This is a general property for chain maps associated with Borel partitions (Proposition 4.5).

Ad 2. The proof is given in Section 4.4. □

4.4. The norm estimate for the Rokhlin chain map. We now prove the second part of Theorem 4.11. In view of the triangle inequality, it suffices to prove that

$$\left|\Phi^\delta(z)\right|_{1,\mathrm{ess}} \le \delta$$

holds for all tuples $z \in E^{*+1}$ that satisfy the F-colouring condition. Since all the involved terms are sufficiently Γ-equivariant and symmetric in the coordinates of $z = (\lambda_r, t_r)_{r \in [n]}$, we may assume without loss of generality that $\lambda_0 = 1$ and $\lambda_1 \in F \cap \Gamma_i$, where $i = t_0 = t_1$.

Setup 4.12. *We assume Setup* 4.7. *In addition, we let $n \in \mathbb{N}$ and we let*

$$z = \big((\lambda_0, t_0), \dots, (\lambda_n, t_n)\big) \in E^{n+1}$$

be a tuple that satisfies the F-colouring condition in the first two coordinates. More specifically, let $\lambda_0 = 1$, and let $i \in I$ with $i = t_0 = t_1$ and

$$\lambda_1 \in F \cap \Gamma_i.$$

Furthermore, we set

$$S_z^{\langle n+1\rangle} := \big\{(\gamma_r, i_r, j_r)_{r \in [n]} \in S^{\langle n+1\rangle} \bigm| \forall_{r \in [n]} \quad i_r = t_r,\ \forall_{r \in [n]} \quad \gamma_r \in \lambda_r \cdot T_{t_r, j_r}\big\}.$$

By definition,

$$\left|\Phi^\delta(z)\right|_{1,\mathrm{ess}} \le \sum_{s \in S^{\langle n+1\rangle}} \int_X \chi_{W_s}(x, z)\, d\mu(x).$$

As a first reduction, we show that only summands in $S_z^{\langle n+1\rangle}$ contribute to this sum:

Lemma 4.13. *In the situation of Setup* 4.12, *let $s \in S^{\langle n+1\rangle} \setminus S_z^{\langle n+1\rangle}$. Then*

$$\int_X \chi_{W_s}(x, z)\, d\mu(x) = 0.$$

Proof. We write $s = (\gamma_r, i_r, j_r)_{r \in [n]}$ and compute

$$\begin{aligned}\int_X \chi_{W_s}(x, z)\, d\mu(x) &= \mu\big(\{x \in X \bigm| (x, z) \in W_s\}\big) \\ &= \mu\big(\{x \in X \bigm| \forall_{r \in [n]} \quad (x, \lambda_r, t_r) \in W_{\gamma_r, i_r, j_r}\}\big).\end{aligned}$$

As $s \notin S_z^{\langle n+1\rangle}$, we can distinguish two cases:

- there exists an $\overline{r} \in [n]$ with $i_{\overline{r}} \ne t_{\overline{r}}$;

- for all $r \in [n]$, we have $i_r = t_r$ but there exists an $R \in [n]$ with $\gamma_R \notin \lambda_R \cdot T_{t_R,j_R}$.

In the first case, $(x, \lambda_{\overline{r}}, t_{\overline{r}}) \notin W_{\gamma_{\overline{r}},i_{\overline{r}},j_{\overline{r}}}$ in view of the I-coordinate, and so the set in question is empty.

In the second case, we obtain

$$\int_X \chi_{W_s}(x,z)\, d\mu(x) = \mu\big(\big\{x \in X \bigm| \forall_{r\in[n]} \quad (x,\lambda_r) \in \gamma_r \cdot A_{t_r,j_r} \times \gamma_r \cdot T^{-1}_{t_r,j_r}\big\}\big);$$

this latter set is empty because of the R-th term.

Thus, in both cases, the integral is zero. □

Lemma 4.14. *In the situation of Setup* 4.12*, we have*

$$\left|\Phi^\delta(z)\right|_{1,\mathrm{ess}} \le \delta.$$

Proof. By the reduction in Lemma 4.13 and the definition of $S_z^{\langle n+1\rangle}$ and W_s, we have

$$\begin{aligned}\left|\Phi^\delta(z)\right|_{1,\mathrm{ess}} &\le \sum_{s\in S_z^{\langle n+1\rangle}} \int_X \chi_{W_s}(x,z)\, d\mu(x)\\ &= \sum_{(\gamma_r,i_r,j_r)_r\in S_z^{\langle n+1\rangle}} \mu\Big(\bigcap_{r\in[n]} \gamma_r \cdot A_{t_r,j_r}\Big)\\ &= \sum_{(\gamma_r,i_r,j_r)_r\in S_z^{\langle n+1\rangle}} \mu\Big(\gamma_0 \cdot A_{i,j_0} \cap \gamma_1 \cdot A_{i,j_1} \cap \bigcap_{r=2}^{n} \gamma_r \cdot A_{t_r,j_r}\Big).\end{aligned}$$

By the Rokhlin partition condition, for each $r \in [n]$, the sets in the family $(\gamma_r \cdot A_{t_r,j_r})_{j_r\in J_{t_r},\gamma_r\in\lambda_r T_{t_r,j_r}}$ are pairwise disjoint. Therefore, inductively (over $r \ge 2$), we can simplify to

$$\begin{aligned}\left|\Phi^\delta(z)\right|_{1,\mathrm{ess}} \le &\sum_{j_0\in J_i}\sum_{\gamma_0\in T_{i,j_0}}\sum_{j_1\in J_i}\sum_{\gamma_1\in\lambda_1\cdot T_{i,j_1}\setminus\{\gamma_0\}} \mu(\gamma_0 \cdot A_{i,j_0} \cap \gamma_1 \cdot A_{i,j_1})\\ &+ \sum_{j_0\in J_i}\sum_{\gamma_0\in T_{i,j_0}}\sum_{j_1\in J_i\setminus\{j_0\}} \mu(\gamma_0 \cdot A_{i,j_0} \cap \gamma_0 \cdot A_{i,j_1}).\end{aligned}$$

In the first sum, if $\gamma_1 \in T_{i,j_1}$, then the intersection term is empty because we have a Rokhlin partition. Similarly, in the second sum, only empty sets appear (here we use the fact that in our setup A_{i,j_0} and A_{i,j_1} are disjoint; Remark 4.8). So, we obtain

$$\left|\Phi^\delta(z)\right|_{1,\mathrm{ess}} \le \sum_{j_0\in J_i}\sum_{\gamma_0\in T_{i,j_0}}\sum_{j_1\in J_i}\sum_{\gamma_1\in\lambda_1\cdot T_{i,j_1}\setminus T_{i,j_1}} \mu(\gamma_0 \cdot A_{i,j_0} \cap \gamma_1 \cdot A_{i,j_1}).$$

Again, using the Rokhlin partition, we see that the sets $\gamma_0 \cdot A_{i,j_0}$ are all pairwise disjoint for $j_0 \in J_i, \gamma_0 \in T_{i,j_0}$. Thus, the sum reduces to

$$\begin{aligned}\left|\Phi^\delta(z)\right|_{1,\mathrm{ess}} &\le \sum_{j_1\in J_i} \sum_{\gamma_1 \in \lambda_1 \cdot T_{i,j_1} \setminus T_{i,j_1}} \mu(\gamma_1 \cdot A_{i,j_1}) \\ &= \sum_{j_1\in J_i} \sum_{\gamma_1 \in \lambda_1 \cdot T_{i,j_1} \setminus T_{i,j_1}} \mu(A_{i,j_1}).\end{aligned}$$

Finally, we use the $(F \cap \Gamma_i, \delta)$-invariance of T_{i,j_1} (and Remark 4.8):

$$\begin{aligned}\left|\Phi^\delta(z)\right|_{1,\mathrm{ess}} &\le \sum_{j_1\in J_i} \mu(A_{i,j_1}) \cdot |\lambda_1 \cdot T_{i,j_1} \setminus T_{i,j_1}| \\ &\le \sum_{j_1\in J_i} \frac{\mu(T_{i,j_1} \cdot A_{i,j_1})}{|T_{i,j_1}|} \cdot \delta \cdot |T_{i,j_1}| \\ &= \delta \cdot \sum_{j_1\in J_i} \mu(T_{i,j_1} \cdot A_{i,j_1}) \\ &= \delta \cdot 1.\end{aligned}$$

This is the claimed estimate. □

This completes the proof of Theorem 4.11.

5. Fundamental cycles subordinate to open covers

To apply the Rokhlin chain map to obtain fundamental cycles of small norm, we subdivide a fundamental cycle such that the resulting cycle is subordinate to the given open cover. We formulate this subdivision in the language of equivariant chain maps. Moreover, we show how these subdivisions can be integrated into the norm estimates for simplicial volume.

Setup 5.1. *Let $n \in \mathbb{N}$. We consider the following situation*

- *Let M be a path-connected finite CW-complex with fundamental group $\Gamma := \pi_1(M)$;*
- *Let $\pi : \widetilde{M} \to M$ be the universal covering map;*
- *Let $(U_i)_{i\in I}$ be a finite open amenable cover of M such that each U_i is path-connected. For every $i \in I$, let $\Gamma_i \le \Gamma$ be $\operatorname{im}(\pi_1(U_i, x_i) \to \pi_1(M, x_i))$ for some choice of $x_i \in U_i$ (up to conjugacy, this choice will have no effect).*
- *Let $E := \Gamma \times I$, equipped with the Γ-action on the first factor.*

5.1. From open covers to small equivariant open covers.

As a first step, we refine the given open cover and pass to an equivariant setting.

Definition 5.2. In the situation of Setup 5.1, a *small equivariant open cover associated with* $(U_i)_{i\in I}$ is an open cover $(\gamma \cdot K_i)_{(\gamma,i)\in\Gamma\times I}$ of $\widetilde{M}$ with the following properties:

- For all $i \in I$, the set $K_i \subset \widetilde{M}$ is relatively compact, open, and path-connected.

- For every $i \in I$, the sets $\gamma \cdot K_i$ with $\gamma \in \Gamma$ are pairwise distinct.
- For all $i \in I$, the set $\Gamma_i \cdot K_i$ is a path-component of $\pi^{-1}(U_i)$.
- For all $i \in I$ and all $\gamma, \lambda \in \Gamma$, we have $\gamma \cdot K_i \cap \lambda \cdot K_i \neq \emptyset \Longrightarrow \gamma^{-1} \cdot \lambda \in \Gamma_i$.

Lemma 5.3. *In the situation of Setup* 5.1*, there exists a small equivariant open cover associated with* $(U_i)_{i \in I}$.

Proof. Let $i \in I$. We choose a path-connected component V_i of $\pi^{-1}(U_i)$; we make this choice in such a way that the deck transformation action of Γ on $\widetilde{M}$ restricts to an action of Γ_i to V_i. The restriction $\pi|_{V_i} \colon V_i \to U_i$ is a covering map and we may choose a relatively compact, path-connected fundamental domain $D_i \subset V_i$ for the deck transformation action $\Gamma_i \curvearrowright V_i$. We enlarge D_i to a relatively compact, open, path-connected subset $K_i \subset V_i$ with $D_i \subset K_i$ and

$$\forall_{\gamma \in \Gamma \setminus \{e\}} \quad \gamma \cdot K_i \neq K_i.$$

Such a set can, for instance, be obtained by selecting a point $x_0 \in D_i^\circ$ and then looking at relatively compact, open, path-connected neighbourhoods of all points in $\overline{D_i}$ that avoid the discrete set $\Gamma \cdot x_0 \setminus \{x_0\}$; as D_i is relatively compact, finitely many such neighbourhoods suffice.

By construction, K_i is relatively compact, open, and path-connected. Moreover, $\gamma \cdot K_i \neq \lambda \cdot K_i$ for distinct group elements $\gamma, \lambda \in \Gamma$.

We have $\Gamma_i \cdot K_i = V_i$ because

$$V_i = \bigcup_{\gamma \in \Gamma_i} \gamma \cdot D_i \subset \bigcup_{\gamma \in \Gamma_i} \gamma \cdot K_i \subset \bigcup_{\gamma \in \Gamma_i} \gamma \cdot V_i = V_i.$$

The last property is automatically satisfied because $\Gamma_i \cdot K_i$ coincides with the path-component V_i of $\pi^{-1}(U_i)$ and Γ_i is $\operatorname{im}(\pi_1(U_i, x_i) \to \pi(M, x_i))$. □

5.2. Equivariant subdivision. In the situation of Setup 5.1, let Γ be torsion-free (e.g., this holds if M is a closed aspherical manifold), and we let N denote the nerve of a small equivariant open cover $(\gamma \cdot K_i)_{(\gamma,i) \in \Gamma \times I}$ associated with $(U_i)_{i \in I}$. Thus, N is a Γ-simplicial complex and the obvious simplicial Γ-action on N is free on the set of all simplices (since Γ is torsion-free).

Moreover, we denote the geometric realisation of a simplicial complex X by $|X|$ and we write $C_*^s(X;\mathbb{Z})$ for the ordered simplicial chain complex [43, Chapter 4.3].

Lemma 5.4. *In this situation, there exists a continuous* Γ*-map* $\widetilde{M} \to |N|$.

Proof. Nerve maps associated with equivariant Γ-partitions of unity are continuous Γ-maps [35, proof of Lemma 4.8]. Therefore, it suffices to find a Γ-partition of unity of $\widetilde{M}$ subordinate to $(\gamma \cdot K_i)_{(\gamma,i) \in \Gamma \times I}$.

Starting with an open cover of $\widetilde{M}$ that refines $(\gamma \cdot K_i)_{(\gamma,i) \in \Gamma \times I}$ and such that the universal covering map π of M is a homeomorphism on each member, we find a finite open cover V of M that refines $(U_i)_{i \in I}$ with the following property: For each member W of V, there exists an open set $\widetilde{W} \subset \widetilde{M}$ such that $\pi|_{\widetilde{W}} \colon \widetilde{W} \to W$ is a homeomorphism and such that there exists an $i(W) \in I$ with $\widetilde{W} \subset K_{i(W)}$.

Let $(\varphi_W)_{W\in V}$ be a partition of unity of M subordinate to V. Then, for $(\gamma, i) \in \Gamma \times I$, we set

$$\widetilde{\varphi}_{(\gamma,i)} := \sum_{W\in V, i(W)=i} \chi_{\gamma\cdot\widetilde{W}} \cdot \varphi_W \circ \pi.$$

A straightforward computation shows that each $\widetilde{\varphi}_{(\gamma,i)}$ is continuous and that $(\widetilde{\varphi}_{(\gamma,i)})_{(\gamma,i)\in\Gamma\times I}$ is a partition of unity that is subordinate to $(\gamma \cdot K_i)_{(\gamma,i)\in\Gamma\times I}$. Moreover, we have

$$\forall_{x\in\widetilde{M}} \quad \widetilde{\varphi}_{(\gamma,i)}(\lambda \cdot x) = \widetilde{\varphi}_{\lambda^{-1}\cdot(\gamma,i)}(x)$$

for all $(\gamma, i) \in \Gamma\times I$ and all $\lambda \in \Gamma$. This shows that $(\widetilde{\varphi}_{(\gamma,i)})_{(\gamma,i)\in\Gamma\times I}$ has the desired properties. □

Lemma 5.5. *Let Γ be a group and let X be a simplicial complex with a simplicial Γ-action that is free on the set of all simplices. Then the canonical chain map $C^s_*(X;\mathbb{Z}) \to C_*(|X|;\mathbb{Z})$* [43, Chapter 4.4] *is a $\mathbb{Z}\Gamma$-chain homotopy equivalence.*

Proof. The canonical chain map $i\colon C^s_*(X;\mathbb{Z}) \to C_*(|X|;\mathbb{Z})$, given by affine linear parametrisation [43, Chapter 4.4 on p. 173], induces an isomorphism on homology [43, Theorem 8 in Chapter 4.6 on p. 191]. Moreover, $C^s_*(X;\mathbb{Z})$ and $C_*(|X|;\mathbb{Z})$ are free $\mathbb{Z}\Gamma$-chain complexes and i is a $\mathbb{Z}\Gamma$-chain map. Therefore, i is a $\mathbb{Z}\Gamma$-chain homotopy equivalence [6, Theorem (8.4) on p. 29]. □

Let $\nu\colon \widetilde{M} \to |N|$ be a Γ-map as provided by Lemma 5.4. We then write

$$\beta\colon C_*(\widetilde{M};\mathbb{Z}) \to C^s_*(N;\mathbb{Z})$$

for the composition of $C_*(\nu;\mathbb{Z})\colon C_*(\widetilde{M};\mathbb{Z}) \to C_*(|N|;\mathbb{Z})$ and of a $\mathbb{Z}\Gamma$-chain homotopy equivalence $C_*(|N|;\mathbb{Z}) \to C^s_*(N;\mathbb{Z})$ (Lemma 5.5). It should be noted that in general β will not be a degree-wise bounded linear map because the number of subdivisions cannot be uniformly controlled.

Moreover, let $\alpha\colon C^s_*(N;\mathbb{Z}) \to \mathbb{Z}[E^{*+1}]$ be the canonical $\mathbb{Z}\Gamma$-chain map, given by viewing $C^s_*(N;\mathbb{Z})$ as a subcomplex of $\mathbb{Z}[E^{*+1}]$.

Finally, we set

$$\varphi^s := \alpha\circ\beta\colon C_*(\widetilde{M};\mathbb{Z}) \to \mathbb{Z}[E^{*+1}]$$

and call this an *equivariant subdivision*. By construction, φ^s is a $\mathbb{Z}\Gamma$-chain map.

Proposition 5.6. *In the situation of Setup* 5.1*, let $(\gamma \cdot K_i)_{(\gamma,i)\in\Gamma\times I}$ be a small equivariant open cover associated with $U = (U_i)_{i\in I}$ and let φ^s be an equivariant subdivision as above. We set*

$$F := \bigcup_{i\in I}\{\gamma \in \Gamma \mid \gamma \cdot K_i \cap K_i \neq \emptyset\} \subset \Gamma.$$

Then:

(1) *The set F is finite.*
(2) *If $\mathrm{mult}(U) \le n$, then for each singular n-simplex $\sigma\colon \Delta^n \to \widetilde{M}$ and every simplex occuring in $\varphi^s(\sigma)$, at least two entries have the same F-colour. Here, $(\gamma, i), (\lambda, j) \in E$ have the* same F-colour *if $i = j$ and $\gamma^{-1}\cdot\lambda \in F\cap\Gamma_i$.*

Proof. *Ad 1.* As the deck transformation action $\Gamma \curvearrowright \widetilde{M}$ is properly discontinuous, as I is finite, and as each K_i is relatively compact, the set F is finite.

Ad 2. Let $((\gamma_0, i_0), \dots, (\gamma_n, i_n))$ be an n-simplex that appears in $\varphi^s(\sigma)$. By definition of N and $C_*^s(N;\mathbb{Z})$, this means that

$$\bigcap_{r=0}^{n} \gamma_r \cdot K_{i_r} \neq \emptyset.$$

In particular, also $\bigcap_{r=0}^{n} \pi^{-1}(U_{i_r}) \neq \emptyset$. The multplicity of $(\pi^{-1}(U_i))_{i\in I}$ equals the multiplicity of $(U_i)_{i\in I}$, which is $\leq n$. In particular, at least two of the $n+1$ elements $i_0, \dots, i_r$ must be equal, say $i_0 = i_1 =: i$. By definition of small equivariant open covers associated with $(U_i)_{i\in I}$ (Definition 5.2), we have that $\gamma_0 \cdot K_i \cap \gamma_1 \cdot K_i \neq \emptyset$ implies $\gamma_0^{-1} \cdot \gamma_1 \in \Gamma_i$. Moreover, we also have by definition of F that $\gamma_0^{-1} \cdot \gamma_1 \in F$. Thus, (γ_0, i_0) and (γ_1, i_1) have the same F-colour. □

5.3. The simplicial volume estimate. We combine the previously developed subdivision tools and take advantage of asphericity to get back from the combinatorial resolutions to the singular chain complex:

Lemma 5.7. *Let M be a connected aspherical CW-complex with fundamental group Γ and let S be a free Γ-set. Then there exists a $\mathbb{Z}\Gamma$-chain map*

$$\psi\colon\ \mathbb{Z}[S^{*+1}] \to C_*(\widetilde{M};\mathbb{Z})$$

that extends the identity $\mathbb{Z} \to \mathbb{Z}$ and satisfies $\|\psi\| \leq 1$ with respect to $|\cdot|_1$.

Proof. Using asphericity of M, inductively filling simplices, one can construct a $\mathbb{Z}\Gamma$-chain map $\psi\colon\ \mathbb{Z}[S^{*+1}] \to C_*(\widetilde{M};\mathbb{Z})$ that extends $\mathrm{id}_{\mathbb{Z}}$.

This filling construction satisfies $\|\psi\| \leq 1$ because each tuple is mapped to a single singular simplex. □

Proposition 5.8. *In the situation of Setup* 5.1*, let M be aspherical, let φ^s be an equivariant subdivision as in Section* 5.2*, and let $z \in \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z})$ be a cycle. Moreover, let S be a free Γ-set, let $\alpha\colon\ \Gamma \curvearrowright (X,\mu)$ be a free standard Γ-space, and let $\Phi\colon\ \mathbb{Z}[E^{*+1}] \to \mathrm{L}^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[S^{*+1}]$ be a $\mathbb{Z}\Gamma$-chain map that extends the inclusion $\mathbb{Z} \hookrightarrow \mathrm{L}^\infty(X,\mathbb{Z})$. Then*

$$\big\|[z]\big\|^\alpha \leq \Big|(\Phi\circ\varphi^s)_\Gamma(z)\Big|_1.$$

Proof. The methods of Section 5.2 and Section 4 apply because the fundamental group Γ of the finite aspherical CW-complex M is torsion-free and countable. Let $\psi\colon\ \mathbb{Z}[S^{*+1}] \to C_*(\widetilde{M};\mathbb{Z})$ be a $\mathbb{Z}\Gamma$-chain map that extends the identity $\mathbb{Z} \to \mathbb{Z}$ and satisfies $\|\psi\| \leq 1$; such a chain map exists by Lemma 5.7. Then

$$\Psi := (\mathrm{id}_{\mathrm{L}^\infty(X,\mathbb{Z})} \otimes_{\mathbb{Z}} \psi)\circ\Phi\circ\varphi^s$$

is a $\mathbb{Z}\Gamma$-chain map $C_*(\widetilde{M};\mathbb{Z}) \to \mathrm{L}^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_*(\widetilde{M};\mathbb{Z})$ that extends the inclusion $\mathbb{Z} \hookrightarrow \mathrm{L}^\infty(X,\mathbb{Z})$; because M is aspherical, Ψ is thus $\mathbb{Z}\Gamma$-chain homotopic to the change of coefficients map (Lemma 2.9 and Example 2.11) and the induced map

$$\Psi_\Gamma\colon\ \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z}) \to \mathrm{L}^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z})$$

is chain homotopic to the change of coefficients map. In particular, $\Psi_\Gamma(z)$ is a cycle representing $[z]^\alpha$; we thus obtain

$$\begin{aligned} \|[z]\|^\alpha &\leq \left|\Psi_\Gamma(z)\right|_1 \leq \|\psi\| \cdot \left|(\Phi\circ\varphi^s)_\Gamma(z)\right|_1 \\ &\leq \left|(\Phi\circ\varphi^s)_\Gamma(z)\right|_1, \end{aligned}$$

as claimed. □

5.4. The simplicial volume estimate, essential simplices. We refine the simplicial volume estimate of Proposition 5.8 to a bound that only counts essential simplices and ignores degenerate simplices.

Proposition 5.9. *Let Γ be a torsion-free group and let S be a non-empty free Γ-set. Then there exists a Γ-equivariant chain map $\eta\colon \mathbb{Z}[S^{*+1}] \to \mathbb{Z}[S^{*+1}]$ extending $\mathrm{id}_\mathbb{Z}\colon \mathbb{Z} \to \mathbb{Z}$ (whence $\eta \simeq_{\mathbb{Z}\Gamma} \mathrm{id}_{\mathbb{Z}[S^{*+1}]}$) with following property: For all chains $c \in \mathbb{Z}[S^{k+1}]$, we have*

$$\left|\eta_k(c)\right|_1 \leq (k+1)! \cdot |c|_{1,\mathrm{ess}}.$$

Proof. We argue similarly to case of integral singular homology [9] through a barycentric subdivision: As S is non-empty, there exists a $\sigma_0 \in S$.

We write D for the set of all finite non-empty subsets of S. Because Γ is torsion-free, the induced action on D is also free. Hence, $\mathbb{Z}[D^{*+1}]$ and $\mathbb{Z}[S^{*+1}]$ are free $\mathbb{Z}\Gamma$-resolutions of $\mathbb{Z}$.

First, choosing a Γ-fundamental domain $F \subset D$, we have the $\mathbb{Z}\Gamma$-chain map $\varrho\colon \mathbb{Z}[D^{*+1}] \to \mathbb{Z}[S^{*+1}]$ defined as follows: For $\gamma_0, \dots, \gamma_k \in \Gamma$ and $x_0, \dots, x_k \in F$, we set

$$\varrho(\gamma_0 \cdot x_0, \dots, \gamma_k \cdot x_k) := (\gamma_0 \cdot \sigma_0, \dots, \gamma_k \cdot \sigma_0).$$

Conversely, we construct the barycentric subdivision map $\delta\colon \mathbb{Z}[S^{*+1}] \to \mathbb{Z}[D^{*+1}]$ inductively as follows: In degree 0, we set $\delta(x_0) := \{x_0\}$ for all $x_0 \in S$. For $z \in D$ let $(\_, z)\colon \mathbb{Z}[D^k] \to \mathbb{Z}[D^{k+1}]$ denote the linear extension of the map $D^k \to D^{k+1}$, $(y_1, \dots, y_k) \mapsto (y_1, \dots, y_k, z)$. Inductively, for $k \in \mathbb{N}_{>0}$, we set

$$\delta(x_0, \dots, x_k) := \sum_{j=0}^{k} (-1)^{j+k} \cdot \big(\delta(x_0, \dots, \widehat{x}_j, \dots, x_k), \{x_0, \dots, x_k\}\big)$$

for all $x_0, \dots, x_k \in S$. Note that in the previous formula $\delta(x_0, \dots, \widehat{x}_j, \dots, x_k)$ inductively is a linear combination of elements in D^k, of norm at most $k!$; moreover, we inductively see that δ is compatible with the boundary operator. Therefore, δ is a $\mathbb{Z}\Gamma$-chain map with

$$\|\delta_k\| \leq (k+1)!$$

for all $k \in \mathbb{N}$. Moreover, a straightforward inductive computation shows that δ maps $(\tau(x_0), \dots, \tau(x_k))$ to $-\delta(x_0, \dots, x_k)$ for every transposition τ. In particular, δ maps degenerate tuples to 0.

We now consider the $\mathbb{Z}\Gamma$-chain map

$$\eta := \varrho\circ\delta\colon \mathbb{Z}[S^{*+1}] \to \mathbb{Z}[S^{*+1}]$$

By construction, η extends $\mathrm{id}_{\mathbb{Z}}$. Moreover, $\|\eta_k\| \le (k+1)!$ and η_k maps degenerate tuples to 0. The claim follows. □

Corollary 5.10. *In the situation of Setup* 5.1*, let M be aspherical with torsion-free and countable fundamental group Γ, let φ^s be an equivariant subdivision as in Section* 5.2*, and let $z \in \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z})$ be a cycle. Moreover, let S be a free Γ-set, let $\alpha\colon\ \Gamma \curvearrowright (X,\mu)$ be a free standard Γ-space, and let*

$$\Phi\colon\ \mathbb{Z}[E^{*+1}] \to \mathrm{L}^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[S^{*+1}]$$

be a $\mathbb{Z}\Gamma$-chain map that extends the inclusion $\mathbb{Z} \hookrightarrow \mathrm{L}^\infty(X,\mathbb{Z})$. Then

$$\big|[z]\big|^\alpha \le (n+1)! \cdot \Big|(\Phi\circ\varphi^s)_\Gamma(z)\Big|_{1,\mathrm{ess}}.$$

Proof. We apply Proposition 5.8 to the composition $(\mathrm{id}_{\mathrm{L}^\infty(X,\mathbb{Z})} \otimes_{\mathbb{Z}\Gamma} \eta)\circ\Phi$, where η is a $\mathbb{Z}\Gamma$-chain map as provided by Proposition 5.9 and combine the resulting estimate with the norm estimate of Proposition 5.9. □

6. Proof of Theorem 1.2

We prove Theorem 1.2 following the outline of Section 3, i.e., by combining the results from Section 4 and Section 5. More precisely, we prove the following slightly more general statement:

Theorem 6.1. *Let M be a finite connected aspherical CW-complex with fundamental group Γ, let $n \in \mathbb{N}$, and let $U = (U_i)_{i\in I}$ be an open amenable cover of M by path-connected subsets with* $\operatorname{mult} U \le n$. *Moreover, let $\Gamma \curvearrowright (X,\mu)$ be an essentially free standard Γ-space. Let $x \in H_n(M;\mathbb{Z})$. Then*

$$|x|^\alpha = 0.$$

Proof. By passing to a conull subspace, we may assume that $\Gamma \curvearrowright (X,\mu)$ is a free standard Γ-space.

By hypothesis, in particular, we are in the situation of Setup 5.1. As M is finite, M is compact and Γ is countable and torsion-free. We may assume that I is finite. Therefore, Lemma 5.3 and Section 5.2 guarantee the existence of an equivariant subdivision φ^s.

Let $z \in \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z})$ be a cycle representing x in $H_n(M;\mathbb{Z})$ and let

$$z' := (\varphi^s)_\Gamma(z) \in \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}[E^{n+1}]$$

be the corresponding subdivided cycle.

For each $\delta \in \mathbb{R}_{>0}$, we can construct a corresponding Rokhlin chain map $\Phi^\delta\colon\ \mathbb{Z}[E^{*+1}] \to \mathrm{L}^\infty_{\mathrm{fs}}(X \times S^{*+1},\mathbb{Z})$ (Lemma 4.1 and Definition 4.9). Let

$$\overline{\Phi}^\delta := \zeta_S\circ\Phi^\delta\colon\ \mathbb{Z}[E^{*+1}] \to \mathrm{L}^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[S^{*+1}].$$

Then $\overline{\Phi}^\delta$ is a $\mathbb{Z}\Gamma$-chain map that extends the inclusion $\mathbb{Z} \hookrightarrow \mathrm{L}^\infty(X,\mathbb{Z})$. Because M is aspherical, we therefore obtain

$$|x|^\alpha \le (n+1)! \cdot \Big|(\overline{\Phi}^\delta)_\Gamma(z')\Big|_{1,\mathrm{ess}}$$

from Corollary 5.10. By Proposition 5.6, every simplex in z' satisfies the colouring condition required in Theorem 4.11. Therefore, Theorem 4.11 shows that

$$\| x \|^{\alpha} \leq (n+1)! \cdot \delta \cdot |z'|_1.$$

Taking $\delta \to 0$ gives $\| x \|^{\alpha} = 0$. □

Proof of Theorem 1.2. We only need to note that every closed connected manifold has the homotopy type of a finite connected CW-complex. Therefore, we can apply Theorem 6.1 to $n := \dim M$ and $[M] \in H_n(M; \mathbb{Z})$. □

7. Examples of manifolds admitting small amenable covers

In this section, we recall standard techniques to produce small amenable covers and we apply them to the manifolds in Section 1.1.

One key ingredient is the following elementary fact:

Lemma 7.1. *Let X be a connected topological space and let Y be a connected CW-complex. Suppose that there exists a continuous map $f : X \to Y$ whose π_1-kernel* $\ker(\pi_1(f))$ *is amenable. Then, we have*

$$\operatorname{cat}_{\mathrm{Am}}(X) \leq \dim(Y) + 1.$$

Proof. The CW-complex Y is homotopy equivalent to a simplicial complex of the same dimension [25, Theorem 2C.5 on p. 182]. Thus we may assume that Y is already a simplicial complex. The space Y is covered by the open stars of its vertices. The multiplicity of this cover is $\dim(Y) + 1$. Each open star is contractible. In particular, we have $\operatorname{cat}_{\mathrm{Am}}(Y) \leq \dim(Y) + 1$. By taking the pullback along f of every open amenable cover of Y, the amenability of $\ker \pi_1(f)$ gives $\operatorname{cat}_{\mathrm{Am}}(X) \leq \operatorname{cat}_{\mathrm{Am}}(Y)$ [10, Remark 2.9]. □

This result allows us to compute the amenable category of a space in terms of the cohomological dimension of certain quotients:

Lemma 7.2. *Let $n \geq 4$. Let X be a connected CW-complex with fundamental group $\Gamma := \pi_1(X)$. Suppose Γ contains an amenable normal subgroup A such that $\Lambda := \Gamma/A$ has cohomological dimension* $\operatorname{cd}_{\mathbb{Z}} \Lambda < n$. *Then,* $\operatorname{cat}_{\mathrm{Am}}(X) \leq n$.

Proof. Because $n \geq 4$ and $\operatorname{cd}_{\mathbb{Z}} \Lambda < n$, there exists a model Y of the classifying space $B\Lambda$ with $\dim Y < n$ [6, Chapter VIII.7]. We now apply Lemma 7.1 to the composition $B\pi \circ c_X : X \to Y$, where $\pi : \Gamma \twoheadrightarrow \Lambda$ is the canonical projection and $c_X : X \to B\Gamma$ is the classifying map. □

In particular, Theorem 1.2 applies to oriented closed connected aspherical n-manifolds with $n \geq 4$, whose fundamental group Γ contains an amenable normal subgroup A with $\operatorname{cd}_{\mathbb{Z}}(\Gamma/A) < n$.

The approach of constructing amenable covers as pullbacks dates back to Gromov's proof of Yano's theorem [21, p. 41] and was recently generalized to the case of F-structures by Babenko and Sabourau [4, Corollary 2.8].

Similarly, we can handle S^1-foliations:

Proposition 7.3. *Let M be an oriented closed connected smooth manifold that admits a regular smooth circle foliation with finite holonomy groups. Then we have* $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$.

Proof. Let $\mathcal{F}$ be a regular smooth circle foliation of M with finite holonomy groups, and let $X := M/\mathcal{F}$ be the leaf space. Let $\pi : M \to X$ be the projection map. Then the leaf space X is an orbifold of dimension $\dim(X) = \dim(M) - 1$ and can be triangulated [37, Theorem 2.15 on p. 40; 38, Proposition 1.2.1].

We explain how the open stars covering of a subdivision of X leads to an amenable cover of M with small multiplicity:

Let $n := \dim(M)$, let $L \subset M$ be a leaf of $\mathcal{F}$ and let H be its holonomy group. Given $x \in L$, there exists a sufficiently small open disk D^{n-1} that is a transversal section of $\mathcal{F}$ at x [37, Section 2.3 and Remark on p. 31] and such that each element of H can be represented by a holonomy diffeormorphism of D^{n-1}. In this situation we can define the following: If $\overline{L} \to L$ is a finite covering of L corresponding to the finite group H, we denote by $\overline{L} \times_H D^{n-1}$ the quotient space of $\overline{L} \times D^{n-1}$ under the identification $(lh, d) \sim (l, hd)$ for all $l \in \overline{L}$, $h \in H$, and all $d \in D^{n-1}$. Since each leaf is a circle, by construction, $\overline{L} \times_H D^{n-1}$ has the structure of a disk bundle over S^1 [37, p. 17]. In particular, $\overline{L} \times_H D^{n-1}$ has an amenable fundamental group.

By the local Reeb stability theorem [37, Theorem 2.9], every leaf L of $\mathcal{F}$ admits a saturated open neighbourhood $V_L \subset M$ that is diffeomorphic to the set $\overline{L} \times_H D^{n-1}$ as above; a set V is *saturated* if for every $y \in V$, the leaf passing through y is entirely contained in V.

We now consider for each leaf L of $\mathcal{F}$ the projection $\pi(V_L) \subset M/\mathcal{F}$. By construction, the open sets $\pi(V_L) \cong D^{n-1}/H$ provide an atlas for the orbifold $M/\mathcal{F}$. Hence, passing to an iterated subdivision T of the triangulation of $X = M/\mathcal{F}$, we can assume that each open star at a vertex of T is entirely contained in a set of the form $\pi(V_L)$ with L a leaf of $\mathcal{F}$. Let U be the open cover of X corresponding to the open stars at the vertices of T and let U' be the pullback of U along π. By construction, we have $\mathrm{mult}(U') = \mathrm{mult}(U) = \dim(M/\mathcal{F}) + 1 = \dim(M)$. Moreover, the open cover U' is amenable because each member of U is entirely contained in some amenable set V_L. This shows that $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$. $\square$

As a corollary we deduce the case of smooth S^1-action without fixed points:

Corollary 7.4. *Let M be an oriented closed connected smooth manifold that admits a smooth S^1-action without fixed points. Then,* $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$.

Proof. It is sufficient to notice that every smooth S^1-action without fixed points gives rise to a regular smooth circle foliation with finite holonomy groups [37, p. 16]. Therefore, the result is a direct consequence of Proposition 7.3. $\square$

It should be noted that every smooth non-trivial S^1-action on a closed aspherical manifold has no fixed points [36, Corollary 1.43].

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