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The bicategory of groupoid correspondences

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ABSTRACT. We define a bicategory with étale, locally compact groupoids as objects and suitable correspondences, that is, spaces with two commuting actions as arrows; the 2-arrows are injective, equivariant continuous maps. We prove that the usual recipe for composition makes this a bicategory, carefully treating also non-Hausdorff groupoids and correspondences. We extend the groupoid C*-algebra construction to a homomorphism from this bicategory to that of C*-algebra correspondences. We describe the C*-algebras of self-similar groups, higher-rank graphs, and discrete Conduché fibrations in our setup.

CONTENTS

1. Introduction

Many interesting C^*-algebras may be realised as C^*-algebras of étale, locally compact groupoids. Examples are the C^*-algebras associated to group actions on spaces, (higher-rank) graphs, self-similar groups, and many C^*-algebras associated to semigroups. A (higher-rank) graph is interpreted in [2] as a generalised dynamical system. A self-similarity of a group may also be interpreted in this way, namely, as a generalised endomorphism of a group. This suggests a way to put various constructions of groupoids and their C^*-algebras under a common roof, starting with a rather general kind of dynamical system.

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This programme is worked out to a large extent in the dissertation of Albandik [1]. His results have not yet been published in journal articles. This article is concerned with the most basic part of the programme. Namely, we define groupoid correspondences, which are the "generalised maps" between étale, locally compact groupoids; we show that they form a bicategory, and that taking groupoid C^*-algebras is a homomorphism to the C^*-correspondence bicategory $\mathfrak{Corr}$ introduced in [9].

A homomorphism from a category to this bicategory $\mathfrak{Corr}$ is identified with a product system over $\mathcal{C}$ in [3]. If the product system is proper, then it gives rise to an "absolute" Cuntz–Pimsner algebra, where the Cuntz–Pimsner covariance condition is asked for all elements. We show that many constructions of C^*-algebras from combinatorial or dynamical data are examples of such absolute Cuntz–Pimsner algebras of product systems obtained from homomorphisms to the bicategory of groupoid correspondences. This contains the C^*-algebras of regular topological graphs, self-similar groups, row-finite higher-rank graphs, higher-rank self-similar groups, and even the rather general discrete Conduché fibrations of Brown and Yetter [6]. Thus the theory developed here offers a unified approach to several important constructions of C^*-algebras. In this article, we only set up the bicategories and the homomorphism to $\mathfrak{Corr}$ and identify the resulting Cuntz–Pimsner algebras in some examples. In following papers and in the thesis [1], it is shown how to realise these Cuntz–Pimsner algebras as groupoid C^*-algebras, provided the underlying category satisfies Ore conditions. The relevance of the Ore conditions is also noticed in the theory of discrete Conduché fibrations in [6].

While most results that we prove here are rather basic, there are some technical difficulties that warrant a careful treatment. It is well known that mapping a groupoid to its groupoid C^*-algebra cannot be functorial when we use functors as arrows between groupoids. The problem is manifest if we look at the subclasses of spaces and groups: the group C^*-algebra is a covariant functor for group homomorphisms, whereas the map $X \mapsto C_0(X) = C^*(X)$ for locally compact spaces is a contravariant functor for proper continuous maps. Buneci and Stachura [7] found a way around this: they define suitable arrows between groupoids that do induce morphisms between the groupoid C^*-algebras. Our aim are C^*-algebra correspondences instead of morphisms of C^*-algebras. Our theorem that there is a homomorphism of bicategories from the bicategory of groupoid correspondences to that of C^*-correspondences makes precise that the groupoid C^*-algebra is "functorial" for these two types of correspondences. In the dissertation of Holkar (see [15–17]), a similar homomorphism is constructed in the realm of Hausdorff, locally compact groupoids with Haar system. Holkar must decorate a groupoid correspondence with an analogue of a Haar system, which makes his theory much more difficult. To reduce the technicalities, Holkar assumes his groupoids to be Hausdorff. We cannot do this, however, because the groupoids associated to self-similar groups may fail to be Hausdorff, and we want our theory to cover this case.

A groupoid correspondence is a space with commuting actions of the two groupoids involved, which satisfy some extra conditions. Asking for more conditions, we get Morita equivalences of groupoids. It is well known that these form a bicategory and that taking groupoid C^*-algebras is a homomorphism from this bicategory to the bicategory of C^*-algebras and Morita–Rieffel equivalences; this goes back already to the seminal work of Muhly–Renault–Williams in [27], except that they do not use the language of bicategories and allow the more general case of locally compact groupoids with Haar systems. Another variant of groupoid correspondences was studied by Hilsum–Skandalis [14] to construct wrong-way functoriality maps between the K-theory groups of groupoid C^*-algebras. These, however, usually fail to induce C^*-correspondences.

Groupoid correspondences, Morita equivalences, and the morphisms of Hilsum–Skandalis differ only in the technical details, that is, in the extra conditions asked for the commuting actions of the two groupoids. The composition is defined in the same way in all three cases. What is different, of course, is the proof that the composite again satisfies the relevant extra conditions. Since the bicategory of étale groupoid correspondences is a critical ingredient in a larger programme, we find it useful to prove its expected properties from scratch.

This article is structured as follows. In Section 2, we define étale groupoids and their actions, and classes like free, proper, and basic actions. We also prove again that an action is free and proper if and only if it is basic and its orbit space is Hausdorff. In Section 3, we define étale groupoid correspondences. Section 4 illustrates them by several examples, which are related to topological graphs, self-similar groups and self-similar graphs. This justifies viewing groupoid correspondences as generalised maps between groupoids. In Section 5, we define the composition of groupoid correspondences. In Section 6, we build a bicategory that has groupoids as objects and groupoid correspondences as arrows. We also briefly recall the analogous bicategory of C^*-correspondences. In Section 7, we build C^*-correspondences from groupoid correspondences and show that this is part of a homomorphism of bicategories. For topological graphs and self-similar groups and graphs, we recover C^*-correspondences that were used before to describe their C^*-algebras as Cuntz–Pimsner algebras. More generally, a homomorphism from a monoid to the groupoid correspondence bicategory gives rise to a product system over that monoid. This suggests a way to associate a C^*-algebra to any such homomorphism. We examine such homomorphisms and the resulting product systems at the end of this article. In Section 8, we identify discrete Conduché fibrations with bicategory homomorphisms into the bicategory of groupoid correspondences, and identify the Cuntz–Pimsner algebra of the resulting product system with the C^*-algebra of the discrete Conduché fibration as defined previously. We also briefly discuss the self-similar k-graphs of Li and Yang.

2. Étale groupoids and groupoid actions

Here we define (locally compact) étale groupoids and their actions on topological spaces. We prove that an action is free and proper if and only if it is "basic" and has Hausdorff orbit space. Most of this is standard, and the last result is shown in [8, Proposition A.7].

Definition 2.1. An *étale* (*topological*) *groupoid* is a groupoid $\mathcal{G}$ with topologies on the arrow and object spaces $\mathcal{G}$ and $\mathcal{G}^0$ such that the range and source maps $r, s\colon \mathcal{G} \rightrightarrows \mathcal{G}^0$ are local homeomorphisms and the multiplication and inverse maps are continuous. An étale groupoid is *locally compact* if the object space $\mathcal{G}^0$ is Hausdorff and locally compact.

We usually view $\mathcal{G}^0$ as a subset of $\mathcal{G}$ by the unit map, that is, we identify an object $x \in \mathcal{G}^0$ with the unit arrow on x.

Remark 2.2. We assume étale groupoids to be locally compact in order to pass to C^*-algebras later on. The bicategory of groupoid correspondences may also be defined more generally, to have all étale groupoids as objects. The reader interested in this will note that local compactness only becomes relevant in Section 7 when we turn to C^*-algebras. Since all groupoids in this article shall be étale and locally compact, we usually drop these adjectives. The more general setting will, however, be used in [20].

Definition 2.3. Let $\mathcal{G}$ be a groupoid. A *right $\mathcal{G}$-space* is a topological space $\mathcal{X}$ with a continuous map $s\colon \mathcal{X} \to \mathcal{G}^0$, the *anchor map*, and a continuous map

$$\mathrm{mult}\colon \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{X}, \qquad \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} := \{(x, g) \in \mathcal{X} \times \mathcal{G} : s(x) = r(g)\},$$

which we denote multiplicatively as $\cdot$, such that

(1) $s(x \cdot g) = s(g)$ for $x \in \mathcal{X}$, $g \in \mathcal{G}$ with $s(x) = r(g)$;
(2) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 \cdot g_2)$ for $x \in \mathcal{X}$, $g_1, g_2 \in \mathcal{G}$ with $s(x) = r(g_1)$, $s(g_1) = r(g_2)$;
(3) $x \cdot s(x) = x$ for all $x \in \mathcal{X}$.

Definition 2.4. The *orbit space* $\mathcal{X}/\mathcal{G}$ is the quotient $\mathcal{X}/{\sim_{\mathcal{G}}}$ with the quotient topology, where $x \sim_{\mathcal{G}} y$ if there is an element $g \in \mathcal{G}$ with $s(x) = r(g)$ and $x \cdot g = y$. We always write $p\colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ for the orbit space projection.

Left $\mathcal{G}$-spaces are defined similarly. We always write $s\colon \mathcal{X} \to \mathcal{G}^0$ for the anchor map in a right action and $r\colon \mathcal{X} \to \mathcal{G}^0$ for the anchor map in a left action.

Definition 2.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be right $\mathcal{G}$-spaces. A continuous map $f\colon \mathcal{X} \to \mathcal{Y}$ is *$\mathcal{G}$-equivariant* if $s(f(x)) = s(x)$ for all $x \in \mathcal{X}$ and $f(x \cdot g) = f(x) \cdot g$ for all $x \in \mathcal{X}$, $g \in \mathcal{G}$ with $s(x) = r(g)$.

Definition 2.6. Let $\mathcal{X}$ be a right $\mathcal{G}$-space and $\mathcal{Z}$ a space. A continuous map $f\colon \mathcal{X} \to \mathcal{Z}$ is *$\mathcal{G}$-invariant* if $f(x \cdot g) = f(x)$ for all $x \in \mathcal{X}$, $g \in \mathcal{G}$ with $s(x) = r(g)$.

Definition 2.7. A right $\mathcal{G}$-space $\mathcal{X}$ is *basic* if the following map is a homeomorphism onto its image with the subspace topology from $\mathcal{X} \times \mathcal{X}$:

$$f\colon \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{X} \times \mathcal{X}, \qquad (x, g) \mapsto (x \cdot g, x). \tag{2.1}$$

The following useful lemma is used repeatedly throughout this article:

Lemma 2.8. *A pullback of a local homeomorphism is again a local homeomorphism.*

Proof. Consider the following pullback diagram with a continuous map $\alpha\colon A \to C$ and a local homeomorphism $\beta\colon B \to C$:

$$\begin{tikzcd} A \times_C B \arrow[r, "\mathrm{pr}_B"] \arrow[d, "\mathrm{pr}_A"'] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & C \end{tikzcd} \tag{2.2}$$

Let $(a, b) \in A \times_C B$. By the definition of the product topology, any neighbourhood N of (a, b) in $A \times_C B$ contains a neighbourhood of the form $U_a \times_C U_b$ with open neighbourhoods U_a and U_b of a and b in A and B, respectively. Since β is a local homeomorphism, we may shrink U_b so that $\beta(U_b)$ is open and β restricts to a homeomorphism on U_b. Now

$$\begin{aligned} \mathrm{pr}_A(U_a \times_C U_b) &= \{x \in U_a : \text{there is } y \in U_b \text{ with } \alpha(x) = \beta(y)\} \\ &= \alpha^{-1}(\beta(U_b)) \cap U_a. \end{aligned}$$

Since α is continuous, $\alpha^{-1}(\beta(U_b)) \cap U_a$ is open in A. It follows that $\mathrm{pr}_A(N)$ is a neighbourhood of $\mathrm{pr}_A(a, b) = a$. Therefore, pr_A is open. Let $(a_1, b_1), (a_2, b_2) \in A \times_C U_b$ satisfy $\mathrm{pr}_A(a_1, b_1) = \mathrm{pr}_A(a_2, b_2)$. Then $a_1 = a_2$ and $\beta(b_1) = \alpha(a_1) = \alpha(a_2) = \beta(b_2)$. Since $\beta|_{U_b}$ is injective, this implies $b_1 = b_2$. So $\mathrm{pr}_A|_{A\times_C U_b}$ is a homeomorphism onto an open subset of A. □

Lemma 2.9. *Let $\mathcal{X}$ be a right $\mathcal{G}$-space. The action* $\mathrm{mult}\colon \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{X}$ *is a surjective local homeomorphism.*

Proof. The map

$$(\mathrm{mult}, \mathrm{id}_{\mathcal{G}})\colon \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{X} \times_{s,\mathcal{G}^0,s} \mathcal{G}, \qquad (x, g) \mapsto (x \cdot g, g),$$

is continuous because mult is continuous. It has an inverse map

$$(\mathrm{mult}', \mathrm{id}_{\mathcal{G}})\colon \mathcal{X} \times_{s,\mathcal{G}^0,s} \mathcal{G} \to \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G}, \qquad (y, g) \mapsto (y \cdot g^{-1}, g),$$

which is also continuous. So $(\mathrm{mult}, \mathrm{id}_{\mathcal{G}})$ is a homeomorphism. Since $s\colon \mathcal{G} \to \mathcal{G}^0$ is a local homeomorphism, so is $\mathrm{pr}_{\mathcal{X}}\colon \mathcal{X} \times_{s,\mathcal{G}^0,s} \mathcal{G} \to \mathcal{X}$ by Lemma 2.8. Then $\mathrm{pr}_{\mathcal{X}} \circ (\mathrm{mult}, \mathrm{id}_{\mathcal{G}}) = \mathrm{mult}$ is a local homeomorphism as the composite of two local homeomorphisms. It is surjective because of the section $\mathcal{X} \to \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G}$, $x \mapsto (x, s(x))$. □

Lemma 2.10. *The orbit space projection $p\colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ for a basic $\mathcal{G}$-action is a surjective local homeomorphism.*

Proof. Let

$$\begin{aligned} \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} &:= \{(x, g) \in \mathcal{X} \times \mathcal{G} : s(x) = r(g)\}, \\ \mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X} &:= \{(x_1, x_2) \in \mathcal{X} \times \mathcal{X} : p(x_1) = p(x_2)\}. \end{aligned}$$

Since the right $\mathcal{G}$-action is basic, the following map is a homeomorphism:

$$f\colon \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}, \qquad (x,g) \mapsto (x\cdot g, x).$$

The set of identity arrows $\mathcal{G}^0 \subseteq \mathcal{G}^1$ is open because $\mathcal{G}$ is étale. Then $\mathcal{X} \times \mathcal{G}^0$ is open in $\mathcal{X} \times \mathcal{G}$. Hence $I := (\mathcal{X} \times \mathcal{G}^0) \cap (\mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G})$ is open in $\mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G}$. If $(x,g) \in I$, then $x \cdot g = x \cdot s(x) = x$. Therefore, $f(I) = \{(x,x) : x \in X\}$. Since f is a homeomorphism, $f(I)$ is open in $\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}$. Thus there is an open subset $V \subseteq \mathcal{X} \times \mathcal{X}$ such that $f(I) = V \cap \mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}$. For each $x \in \mathcal{X}$, there is an open neighbourhood $U \subseteq \mathcal{X}$ with $U \times U \subseteq V$. Then

$$(U \times U) \cap (\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}) = \{(u,u) : u \in U\}.$$

Suppose $p(u_1) = p(u_2)$ for some $u_1, u_2 \in U$. Then $u_1 = u_2 \cdot g$ for some $g \in \mathcal{G}$. Then $(u_2 \cdot g, u_2) \in (U \times U) \cap (\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X})$ and hence $u_1 = u_2$. That is, p is injective on U. Next we show that p is open. Let $W \subseteq \mathcal{X}$ be open. Then

$$p^{-1}(p(W)) = \{w \cdot g : w \in W,\ g \in \mathcal{G},\ s(w) = r(g)\} = \operatorname{mult}(W \times_{s,\mathcal{G}^0,r} \mathcal{G})$$

is open by Lemma 2.9. Since p is a quotient map, it follows that $p(W)$ is open. □

Definition 2.11. A right $\mathcal{G}$-space is *free* if the map in (2.1) is injective; equivalently, $x \cdot g = x$ for $x \in \mathcal{X}$, $g \in \mathcal{G}$ with $s(x) = r(g)$ implies $g = s(x)$.

Definition 2.12. A continuous map f is *proper* if the map $f \times \mathrm{id}_Z$ is closed for any topological space Z. A right $\mathcal{G}$-space $\mathcal{X}$ is *proper* if the map in (2.1) is proper. An étale groupoid is proper if its canonical action on $\mathcal{G}^0$ is proper. Equivalently, the following map is proper:

$$(r,s)\colon \mathcal{G} \to \mathcal{G}^0 \times \mathcal{G}^0, \qquad g \mapsto (r(g), s(g)).$$

Our next goal is to relate free and proper actions to basic actions.

Lemma 2.13. *For a basic $\mathcal{G}$-action, $\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X} \subseteq \mathcal{X} \times \mathcal{X}$ is closed if and only if $\mathcal{X}/\mathcal{G}$ is Hausdorff.*

Proof. Assume first that $\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X} \subseteq \mathcal{X} \times \mathcal{X}$ is closed. Choose $y_1 \neq y_2$ in $\mathcal{X}/\mathcal{G}$. There are $x_1, x_2 \in \mathcal{X}$ with $p(x_1) = y_1$ and $p(x_2) = y_2$. Then $(x_1, x_2) \notin \mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}$ because x_1 and x_2 are not in the same orbit. Since $\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}$ is closed, its complement is open. This gives open neighbourhoods $U_1 \ni x_1$ and $U_2 \ni x_2$ with $(U_1 \times U_2) \cap (\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}) = \emptyset$. Then $p(U_1) \cap p(U_2) = \emptyset$ by definition of $\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}$. Since the $\mathcal{G}$-action is basic, p is open by Lemma 2.10. Hence $p(U_1)$ and $p(U_2)$ are open neighbourhoods that separate y_1 and y_2. This shows that $\mathcal{X}/\mathcal{G}$ is Hausdorff.

Conversely, let $\mathcal{X}/\mathcal{G}$ be Hausdorff. Let $(x_1, x_2) \in (\mathcal{X} \times \mathcal{X}) \setminus (\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X})$. Then $p(x_1) \neq p(x_2)$. Since $\mathcal{X}/\mathcal{G}$ is Hausdorff, there are open neighbourhoods $V_1 \ni p(x_1)$ and $V_2 \ni p(x_2)$ with $V_1 \cap V_2 = \emptyset$. Then $p^{-1}(V_1) \times p^{-1}(V_2)$ is an open neighbourhood of (x_1, x_2) that does not meet $\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}$. This shows that $\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}$ is closed in $\mathcal{X} \times \mathcal{X}$. □

Lemma 2.14. *For a free and proper action, $\mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X} \subseteq \mathcal{X} \times \mathcal{X}$ is closed.*

Proof. The composite map

$$F\colon \mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{G} \xrightarrow{f} \mathcal{X}\times_{p,\mathcal{X}/\mathcal{G},p}\mathcal{X} \hookrightarrow \mathcal{X}\times\mathcal{X} \tag{2.3}$$

is proper. Hence $F(\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{G})$ is closed. This is equal to $\mathcal{X}\times_{p,\mathcal{X}/\mathcal{G},p}\mathcal{X}$ because f is bijective. □

Lemma 2.15. *For a basic action, if $\mathcal{X}\times_{p,\mathcal{X}/\mathcal{G},p}\mathcal{X}\subseteq\mathcal{X}\times\mathcal{X}$ is closed, then the map F defined in (2.3) is proper.*

Proof. By definition, F is proper if and only if the map $F\times\mathrm{id}_Z$ is closed for any topological space Z. Since the action is basic, F is a homeomorphism onto its image, which is closed by assumption. The property of being a homeomorphism onto a closed subset is preserved by taking products with any space Z. Thus $F\times\mathrm{id}_Z$ is a closed map. □

Proposition 2.16. *Let $\mathcal{G}$ be an étale groupoid and $\mathcal{X}$ a right $\mathcal{G}$-space. The following are equivalent:*

(1) *the action of $\mathcal{G}$ on $\mathcal{X}$ is basic and the orbit space $\mathcal{X}/\mathcal{G}$ is Hausdorff;*
(2) *the action of $\mathcal{G}$ on $\mathcal{X}$ is free and proper.*

Proof. Suppose first that the $\mathcal{G}$-action is basic and $\mathcal{X}/\mathcal{G}$ is Hausdorff. Since $f\colon \mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{G}\to\mathcal{X}\times_{p,\mathcal{X}/\mathcal{G},p}\mathcal{X}$ is a homeomorphism, it must be injective. That is, the $\mathcal{G}$-action is free. By Lemma 2.13, $\mathcal{X}\times_{p,\mathcal{X}/\mathcal{G},p}\mathcal{X}\subseteq\mathcal{X}\times\mathcal{X}$ is closed. Then F is proper by Lemma 2.15. That is, the $\mathcal{G}$-action is proper.

Conversely, suppose the $\mathcal{G}$-action to be free and proper. We first show that the $\mathcal{G}$-action is basic. Since f is clearly bijective, it remains to show that f^{-1} is continuous. Let U be open in $\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{G}$. Then $(f^{-1})^{-1}(U)=f(U)$ is open since f is closed and bijective, so that f is open. By Lemma 2.14, $\mathcal{X}\times_{p,\mathcal{X}/\mathcal{G},p}\mathcal{X}\subseteq\mathcal{X}\times\mathcal{X}$ is closed. Hence $\mathcal{X}/\mathcal{G}$ is Hausdorff by Lemma 2.13. □

3. Groupoid correspondences

Definition 3.1. Let $\mathcal{H}$ and $\mathcal{G}$ be étale groupoids. An (étale) *groupoid correspondence* from $\mathcal{G}$ to $\mathcal{H}$, denoted $\mathcal{X}\colon \mathcal{H}\leftarrow\mathcal{G}$, is a space $\mathcal{X}$ with commuting actions of $\mathcal{H}$ on the left and $\mathcal{G}$ on the right, such that the right anchor map $s\colon \mathcal{X}\to\mathcal{G}^0$ is a local homeomorphism and the right $\mathcal{G}$-action is free and proper.

That the actions of $\mathcal{H}$ and $\mathcal{G}$ commute means that $s(h\cdot x)=s(x)$, $r(x\cdot g)=r(x)$, and $(h\cdot x)\cdot g=h\cdot(x\cdot g)$ for all $g\in\mathcal{G}$, $x\in\mathcal{X}$, $h\in\mathcal{H}$ with $s(h)=r(x)$ and $s(x)=r(g)$, where $s\colon \mathcal{X}\to\mathcal{G}^0$ and $r\colon \mathcal{X}\to\mathcal{H}^0$ are the anchor maps.

Remark 3.2. Since $\mathcal{G}^0$ is locally compact and s is a local homeomorphism, the underlying space $\mathcal{X}$ of a groupoid correspondence is locally compact as well. The space $\mathcal{X}$ itself need not be Hausdorff. Proposition 2.16 implies, instead, that the space $\mathcal{X}/\mathcal{G}$ is Hausdorff.

Definition 3.3. A correspondence $\mathcal{X}\colon \mathcal{H}\leftarrow\mathcal{G}$ is *proper* if the map $r_*\colon \mathcal{X}/\mathcal{G}\to\mathcal{H}^0$ induced by r is proper. It is *tight* if r_* is a homeomorphism.

Definition and Lemma 3.4. *Let $\mathcal{X}$ be a space with a basic right $\mathcal{G}$-action. Let $p\colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ be the orbit space projection. The image of the map (2.1) is the subset $\mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X} = \mathcal{X} \times_{p,\mathcal{X}/\mathcal{G},p} \mathcal{X}$ of all $(x_1, x_2) \in \mathcal{X} \times \mathcal{X}$ with $p(x_1) = p(x_2)$. The inverse to the map in (2.1) induces a continuous map*

$$\mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X} \xrightarrow{\sim} \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} \xrightarrow{\mathrm{pr}_{\mathcal{G}}} \mathcal{G}, \qquad (x_1, x_2) \mapsto \langle x_2 \mid x_1 \rangle. \tag{3.1}$$

That is, $\langle x_1 \mid x_2 \rangle$ is defined for $x_1, x_2 \in \mathcal{X}$ with $p(x_1) = p(x_2)$ in $\mathcal{X}/\mathcal{G}$, and it is the unique $g \in \mathcal{G}$ with $s(x_1) = r(g)$ and $x_2 = x_1 g$. Conversely, if $g \in \mathcal{G}$ with $x_2 = x_1 g$ for $x_1, x_2 \in \mathcal{X}$ with $p(x_1) = p(x_2)$ is unique and depends continuously on $(x_1, x_2) \in \mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X}$, then the right $\mathcal{G}$-action on $\mathcal{X}$ is basic.

Proof. The inverse to the map in (2.1) is of the form

$$\mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X} \to \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G}, \qquad (x_1, x_2) \mapsto (x_2, \langle x_2 \mid x_1 \rangle).$$

This is continuous if and only if the map in (3.1) is continuous. □

The following proposition says that $\langle x_1 \mid x_2 \rangle$ has properties analogous to those of rank-one operators on Hilbert modules, which justifies our notation.

Proposition 3.5. *Let $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence. The map in (3.1) is a local homeomorphism. It has the following properties:*

1. *$r(\langle x_1 \mid x_2 \rangle) = s(x_1)$, $s(\langle x_1 \mid x_2 \rangle) = s(x_2)$, and $x_2 = x_1 \cdot \langle x_1 \mid x_2 \rangle$ for all $x_1, x_2 \in \mathcal{X}$ with $p(x_1) = p(x_2)$;*
2. *$\langle x \mid x \rangle = s(x)$ for all $x \in \mathcal{X}$;*
3. *$\langle x_1 \mid x_2 \rangle = \langle x_2 \mid x_1 \rangle^{-1}$ for all $x_1, x_2 \in \mathcal{X}$ with $p(x_1) = p(x_2)$;*
4. *$\langle h x_1 g_1 \mid h x_2 g_2 \rangle = g_1^{-1} \langle x_1 \mid x_2 \rangle g_2$ for all $h \in \mathcal{H}$, $x_1, x_2 \in \mathcal{X}$, $g_1, g_2 \in \mathcal{G}$ with $s(h) = r(x_1) = r(x_2)$, $s(x_1) = r(g_1)$, $s(x_2) = r(g_2)$, $p(x_1) = p(x_2)$.*

Proof. Since s is a local homeomorphism, so is

$$\mathrm{pr}_{\mathcal{G}}\colon \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{G}$$

by Lemma 2.8. This map composed with the homeomorphism $\mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X} \cong \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{G}$ is the bracket map. The properties of $\langle x_1 \mid x_2 \rangle$ are checked by direct computations, using that for any $(x_1, x_2) \in \mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X}$, there is only one $g \in \mathcal{G}$ with $x_1 \cdot g = x_2$, namely, $g = \langle x_1 \mid x_2 \rangle$. This equation forces $r(g) = s(x_1)$ and $s(g) = s(x_1 g) = s(x_2)$, which gives (1). Since $x = x s(x)$, it implies (2). Since $x_1 = x_2 g$ if and only if $x_1 g^{-1} = x_2$, it implies (3). Since $(h x_1 g_1) \cdot (g_1^{-1} \langle x_1 \mid x_2 \rangle g_2) = h x_1 \langle x_1 \mid x_2 \rangle g_2 = h x_2 g_2$, it implies (4). □

4. Some examples of groupoid correspondences

In this section, we examine our definition of a groupoid correspondence when both groupoids $\mathcal{G}$ and $\mathcal{H}$ are locally compact spaces, discrete groups, or transformation groups. We get topological graphs, self-similarities of groups, and self-similarities of graphs in these three cases, respectively; these objects have been used before in order to define C^*-algebras. More precisely, the self-similarities of groups and graphs correspond to *proper* groupoid correspondences on groups and transformation groups, respectively.

Any locally compact space gives a groupoid with only identity arrows. We first describe groupoid correspondences between such groupoids.

Example 4.1. Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact spaces, viewed as groupoids with only identity arrows. A groupoid action of $\mathcal{G}$ or $\mathcal{H}$ is simply a continuous map to these two spaces. The orbit space of an action on a space $\mathcal{X}$ is again $\mathcal{X}$. Any such action is basic. By Proposition 2.16, the underlying space of a groupoid correspondence must be locally compact and Hausdorff. Summing up, a groupoid correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ is the same as a locally compact, Hausdorff space $\mathcal{X}$ with a continuous map $r\colon \mathcal{X} \to \mathcal{H}$ and a local homeomorphism $s\colon \mathcal{X} \to \mathcal{G}$. The correspondence is proper if and only if $r\colon \mathcal{X} \to \mathcal{H}$ is proper, and tight if and only if $r\colon \mathcal{X} \to \mathcal{H}$ is a homeomorphism. In the tight case, we may use r to identify $\mathcal{X}$ with $\mathcal{H}$. This gives an isomorphic groupoid correspondence with $r = \mathrm{id}_{\mathcal{H}}$. Thus a tight groupoid correspondence $\mathcal{H} \leftarrow \mathcal{G}$ is equivalent to a local homeomorphism $\mathcal{H} \to \mathcal{G}$.

A groupoid correspondence $(\mathcal{X}, r, s)$ as above with locally compact, Hausdorff $\mathcal{X}$, $\mathcal{G}$ and $\mathcal{H}$ is called a topological correspondence in [2]. If, in addition, $\mathcal{H} = \mathcal{G}$, then it is called a topological graph in [18]; this is the data from which topological graph C^*-algebras are built. Similar notions are also introduced by Deaconu [11] and Nekrashevych [29], under the names “continuous graph” or “topological automaton.” These notions are meant to generalise non-invertible dynamical systems.

There are two different ways to turn a local homeomorphism $f\colon \mathcal{G} \to \mathcal{G}$ into a groupoid correspondence, namely, $r = f$ and $s = \mathrm{id}_{\mathcal{G}}$ or $r = \mathrm{id}_{\mathcal{G}}$ and $s = f$. Unless f is a homeomorphism, the resulting topological graph C^*-algebras are not closely related. So these two constructions must be distinguished carefully.

Next we describe groupoid correspondences between groups.

Example 4.2. Let $\mathcal{H}$ and $\mathcal{G}$ be discrete groups. A groupoid correspondence $\mathcal{H} \leftarrow \mathcal{G}$ is a space $\mathcal{X}$ with commuting actions of $\mathcal{H}$ on the left and $\mathcal{G}$ on the right, such that the right action is basic with Hausdorff orbit space and the right anchor map is a local homeomorphism. Since $\mathcal{G}^0$ is the one-point set, the anchor map $s\colon \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism if and only if $\mathcal{X}$ is discrete. In this case, the right $\mathcal{G}$-action is basic if and only if it is free. Thus a groupoid correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ is a set $\mathcal{X}$ with the discrete topology and with commuting actions of $\mathcal{H}$ on the left and $\mathcal{G}$ on the right, where the right action is free.

Let $A := \mathcal{X}/\mathcal{G}$. Since the right action is free, there is a bijection $A \times \mathcal{G} \cong \mathcal{X}$ such that the right $\mathcal{G}$-action on $\mathcal{X}$ becomes $(x, g_1) \cdot g_2 = (x, g_1 g_2)$ on $A \times \mathcal{G}$. We transfer the left $\mathcal{H}$-action to $A \times \mathcal{G}$ using this bijection. Thus $A \times \mathcal{G}$ becomes a groupoid correspondence $\mathcal{H} \leftarrow \mathcal{G}$ that is isomorphic to $\mathcal{X}$. The left action of

$h \in \mathcal{H}$ on $A \times \mathcal{G}$ must be of the form

$$h \cdot (x, g) = (\pi_h(x), h|_x \cdot g)$$

for some map $\pi_h : A \to A$ and some $h|_x \in \mathcal{G}$ because it commutes with the right $\mathcal{G}$-action. The property $(h_1 \cdot h_2) \cdot (x, g) = h_1 \cdot (h_2 \cdot (x, g))$ of a left action is equivalent to the following two conditions. First, $h \mapsto \pi_h$ must be a group action of $\mathcal{H}$ on A. Secondly, the map $\varphi : \mathcal{H} \times A \to \mathcal{G}$, $(h, x) \mapsto h|_x$, must be a 1-cocycle, that is,

$$(h_1 h_2)|_x = h_1|_{h_2 x} \cdot h_2|_x$$

for all $h_1, h_2 \in \mathcal{H}$, $x \in A$.

The bijection $\mathcal{X} \cong A \times \mathcal{G}$ is unique only up to a map of the form $(x, g) \mapsto (x, \psi(x) \cdot g)$ for some map $\psi : A \to \mathcal{G}$. This does not change the action π and replaces the cocycle φ by

$$\varphi^\psi(h, x) := \psi(\pi_h(x))^{-1} \cdot \varphi(h, x) \cdot \psi(x) \tag{4.1}$$

because $(\pi_h(x), h|_x \cdot \psi(x) \cdot g) = (\pi_h(x), \psi(\pi_h(x)) \cdot \varphi^\psi(h, x) \cdot g)$. The map $\psi \mapsto \varphi^\psi$ defines a right action of the group of maps $\psi : A \to \mathcal{G}$ on the set of 1-cocycles $\varphi : \mathcal{H} \times A \to \mathcal{G}$. The $\mathcal{H}$-space A and the orbit of φ under this action are uniquely determined by the isomorphism class of the groupoid correspondence $\mathcal{H} \leftarrow \mathcal{G}$, and any such pair (A, φ) comes from a groupoid correspondence $\mathcal{H} \leftarrow \mathcal{G}$.

Since $\mathcal{H}^0$ is the one-point set as well, a groupoid correspondence is proper if and only if the set $A \cong \mathcal{X}/\mathcal{G}$ is finite, and tight if and only if $|A| = 1$ or, equivalently, the right $\mathcal{G}$-action on $\mathcal{X}$ is transitive. In the tight case, the construction above identifies $\mathcal{X} \cong \mathcal{G}$ as a right $\mathcal{G}$-space by picking a base point in $\mathcal{X}$. The left action must be of the form $h \cdot g = \varphi(h)g$ for a group homomorphism $\varphi : \mathcal{H} \to \mathcal{G}$ in order to commute with the right $\mathcal{G}$-action. If we pick another base point $g \in \mathcal{G} \cong \mathcal{X}$, then the homomorphism φ is replaced by $\mathrm{Ad}_g^{-1} \circ \varphi$ by (4.1). Thus isomorphism classes of tight groupoid correspondences $\mathcal{H} \leftarrow \mathcal{G}$ are canonically in bijection with equivalence classes of group homomorphisms $\mathcal{H} \to \mathcal{G}$, where we consider two homomorphisms equivalent if they differ by an inner automorphism of $\mathcal{G}$.

An injective group homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$ gives a tight groupoid correspondence $\mathcal{G} \leftarrow \mathcal{H}$. But it also gives a groupoid correspondence $\mathcal{H} \leftarrow \mathcal{G}$ by taking $\mathcal{H}$ with the left $\mathcal{H}$-action by translation and the right $\mathcal{G}$-action $h \cdot g := h\varphi(g)$. Such groupoid correspondences for $\mathcal{G} = \mathcal{H}$ are implicitly used by Stammeier [31]. If $\mathcal{G}$ and $\mathcal{H}$ are Abelian discrete groups, an injective group homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$ is equivalent to a surjective group homomorphism $\hat{\varphi} : \hat{\mathcal{H}} \to \hat{\mathcal{G}}$. These are used by Cuntz and Vershik [10].

The analysis in Example 4.2 shows that a proper groupoid correspondence $\mathcal{G} \leftarrow \mathcal{G}$ for a group $\mathcal{G}$ is the same as a "covering permutational bimodule" over $\mathcal{G}$ in the notation of [28, Section 2]. These covering permutational bimodules are another way to describe self-similarities of groups. It is customary, however, to assume a certain faithfulness property for self-similarities (see [28, Definition 2.1]). To formulate it, let A^* be the set of finite words over A. The 1-cocycle allows to extend the action of $\mathcal{G}$ on A to an action on A^* by the recursive formula

$$g \cdot (xw) = g(x)(g|_x \cdot w) \tag{4.2}$$

for $g \in \mathcal{G}$, $x \in A$, $w \in A^*$. The triple consisting of the group $\mathcal{G}$, the $\mathcal{G}$-set A, and the 1-cocycle $\mathcal{G} \times A \to \mathcal{G}$ is called a *self-similar group* if A is finite and the action of $\mathcal{G}$ on A^* defined above is faithful. The latter condition ensures that a self-similarity is sufficiently nontrivial. We shall not use it in this article.

The relationship to self-similar groups suggests to view a proper correspondence $A\colon \mathcal{G} \leftarrow \mathcal{G}$ for a groupoid $\mathcal{G}$ as a self-similarity of $\mathcal{G}$. What does this mean for a transformation groupoid $\Gamma \ltimes V$, where Γ is a group and V is a left Γ-set?

Proposition 4.3. *Let Γ be a discrete group and let V be a left Γ-set. Let $\mathcal{G} := \Gamma \ltimes V$ be the transformation groupoid. Let E be a left Γ-set with a 1-cocycle $\varphi\colon \Gamma \times E \to \Gamma$, $(h, e) \mapsto h|_e$, that is, $(gh)|_e = g|_{h\cdot e} \cdot h|_e$ for all $g, h \in \Gamma$, $e \in E$ and with maps $r, s\colon E \rightrightarrows V$ that satisfy*

$$s(g \cdot e) = (g|_e) \cdot s(e) \quad \textit{and} \quad r(g \cdot e) = g \cdot r(e) \tag{4.3}$$

for all $g \in \Gamma$, $e \in E$. Then $\mathcal{X} := E \times \Gamma$ with the discrete topology, the anchor maps $r, s\colon \mathcal{X} \rightrightarrows V$, $r(e, g) = r(e)$, $s(e, g) = g^{-1} \cdot s(e)$, with the obvious right Γ-action, $(e, g) \cdot g_2 = (e, g \cdot g_2)$, and the left Γ-action $h \cdot (e, g) = (h \cdot e, h|_e \cdot g)$ is a groupoid correspondence $\mathcal{G} \leftarrow \mathcal{G}$. Any groupoid correspondence $\mathcal{G} \leftarrow \mathcal{G}$ is isomorphic to one of this form, where (E, r, s) is unique up to isomorphism, and φ is unique up to the action of the group of maps $\psi\colon E \to \Gamma$ by

$$\varphi^{\psi}(h, e) := \psi(\pi_h(e))^{-1} \cdot \varphi(h, e) \cdot \psi(e).$$

The correspondence $\mathcal{X}$ is proper if and only if the map $r\colon E \to V$ is finite-to-one, and tight if and only if the map $r\colon E \to V$ is bijective.

Proof. An action of $\Gamma \ltimes V$ on $\mathcal{X}$ is equivalent to a pair consisting of a Γ-action on $\mathcal{X}$ and a Γ-equivariant map $\mathcal{X} \to V$. Thus a groupoid correspondence $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ is a space with commuting left and right actions of Γ and with anchor maps $r, s\colon \mathcal{X} \rightrightarrows V$, with some extra properties. Since V is discrete and $s\colon \mathcal{X} \to V$ is a local homeomorphism, $\mathcal{X}$ must be discrete. Then the right $\Gamma \ltimes V$-action is basic if and only if the right Γ-action is free. Choose a fundamental domain $E \subseteq \mathcal{X}$ for it. Then the map $E \times \Gamma \to \mathcal{X}$, $(e, g) \mapsto e \cdot g$, is a homeomorphism. We use it to identify $\mathcal{X}$ with $E \times \Gamma$. Then the right Γ-action becomes $(e, g_1) \cdot g_2 = (e, g_1 \cdot g_2)$. The anchor maps satisfy $s(e, g) = s(e \cdot g) = g^{-1} \cdot s(e)$ and $r(e, g) = r(e \cdot g) = r(e)$ for all $e \in E$, $g \in \Gamma$ because s is equivariant and r is invariant for the right Γ-action. Thus s and r are determined by their restrictions to E, which we also denote by s and r.

For $e \in E$, we may write $h \cdot (e, 1) = (h \cdot e, h|_e)$ with $h \cdot e \in E$, $h|_e \in \Gamma$. As in Example 4.2, the left action of Γ on $\mathcal{X}$ must be of the form $h \cdot (e, g) = (h \cdot e, h|_e \cdot g)$ because it commutes with the right Γ-action; and this gives a left Γ-action on $E \times \Gamma$ if and only if $(h_1 \cdot h_2) \cdot e = h_1 \cdot (h_2 \cdot e)$ and $(h_1 h_2)|_e = h_1|_{h_2 \cdot e} \cdot h_2|_e$ for all $h_1, h_2 \in \Gamma$, $e \in E$. That is, E is a left Γ-space and the map $\varphi\colon \Gamma \times E \to \Gamma$, $(h, e) \mapsto h|_e$, is a 1-cocycle.

The map r is equivariant for the left Γ-action. Hence

$$r(h\cdot e) = r(h\cdot e, h|_e\cdot g) = r(h\cdot(e,g)) = h\cdot r(e,g) = h\cdot r(e)$$

for all $g,h\in\Gamma$, $e\in E$. The map s is invariant for the left Γ-actions. Hence

$$g^{-1}\cdot(h|_e)^{-1}\cdot s(h\cdot e) = s(h\cdot e, h|_e\cdot g) = s(h\cdot(e,g)) = s(e,g) = g^{-1}\cdot s(e)$$

for all $h,g\in\Gamma$, $e\in E$. Equivalently, $s(h\cdot e) = h|_e\cdot s(e)$ for all $h\in\Gamma$, $e\in E$. So r and s satisfy the two conditions in (4.3).

By now, we have seen that any correspondence is of the asserted form. The only choice in the construction is that of a fundamental domain for the free right Γ-action on $\mathcal{X}$. Two such choices differ by right multiplication with a map $E\to\Gamma$. Hence isomorphisms of groupoid correspondences are in bijection with pairs (f,ψ), where f is a bijection $f\colon E \xrightarrow{\sim} E'$ that intertwines the range and source maps and $\psi\colon E\to\Gamma$ is such that $\varphi'(h,f(e)) = \varphi^{\psi}(h,e)$ for all $h\in\Gamma$, $e\in E$. Since $E\cong\mathcal{X}/\mathcal{G}$ and both E and V are discrete, a groupoid correspondence is tight or proper if and only if the corresponding map $r\colon E\to V$ is bijective or finite-to-one, respectively. □

It is possible, though not recommended, to view the space E in Proposition 4.3 as a directed graph with vertex set V and range and source maps r,s. The group Γ acts both on the vertices and the edges in this graph, and r is equivariant. But the map $s\colon E\to V$ is not Γ-equivariant, so Γ does not act by graph automorphisms. The extra condition $s(g\cdot e) = g\cdot s(e)$, which says that the Γ-action preserves the graph structure, is equivalent to $g|_e\cdot s(e) = g\cdot s(e)$ for all $g\in\Gamma$, $e\in E$. Exel and Pardo [13] define a self-similar graph as a graph with an action of Γ by graph automorphisms, such that $g|_e\cdot x = g\cdot x$ for all $g\in\Gamma$, $e\in E$, $x\in V$ (see [13, Equation (2.3.1)]). Thus a self-similar graph as in [13] gives a groupoid correspondence on the transformation groupoid $\Gamma\ltimes V$. The converse is not true, however.

We suggest that the right common generalisation of graph C*-algebras and Nekrashevych's C*-algebras is a proper groupoid correspondence $\mathcal{G}\leftarrow\mathcal{G}$ for a discrete groupoid $\mathcal{G}$. The idea of [13] to look at self-similar graphs restricts $\mathcal{G}$ to be a transformation groupoid $\Gamma\ltimes V$ for a group action on a discrete set – the vertices of the graph – and it leads to the unnecessary condition $s(g\cdot e) = g\cdot s(e)$ on the source map $s\colon E\to V$ in order for Γ to act by graph automorphisms. We now show that the setting in [22] is almost equivalent to that of a groupoid correspondence on a discrete groupoid, except for an extra faithfulness condition in [22], which is analogous to the faithfulness condition in the definition of a self-similar group.

Example 4.4. Let $\mathcal{G}$ be a (discrete) groupoid with object set V. Let $\mathcal{X}\colon\mathcal{G}\leftarrow\mathcal{G}$ be a groupoid correspondence. As above, the space $\mathcal{X}$ is discrete and there is a fundamental domain E with a map $s\colon E\to V$ for the right $\mathcal{G}$-action on it such that $\mathcal{X}\cong E\times_{s,V,r}\mathcal{G}$ as a right $\mathcal{G}$-set. The left action of $\mathcal{G}$ induces an action on $E\cong\mathcal{X}/\mathcal{G}$. Let $r\colon E\to V$ denote its anchor map. There is a map $\varphi\colon\mathcal{G}\times_{s,V,r}E\to\mathcal{G}$, $(g,e)\mapsto g|_e$, such that $s(g|_e) = s(e)$ and $g\cdot(e,h) = (g\cdot e, g|_e\cdot h)$

for all $g, h \in \mathcal{G}$, $e \in E$ with $s(g) = r(e)$, $s(e) = r(h)$. The maps above must satisfy $s(g|_e) = s(e)$, $r(g \cdot e) = r(g)$, $s(g \cdot e) = r(g|_e)$, and the cocycle condition $(g \cdot h)|_e = g|_{h \cdot e} \cdot h|_e$ for all $g, h \in \mathcal{G}$, $e \in E$ with $s(g) = r(h)$, $s(h) = r(e)$ in order for $g \cdot (e, h) = (g \cdot e, g|_e \cdot h)$ to define a groupoid action on $E \times_{s,V,r} \mathcal{G}$. Conversely, if we are given a $\mathcal{G}$-set E with a cocycle satisfying these conditions, then it comes from a unique groupoid correspondence. So the only difference between a groupoid correspondence on $\mathcal{G}$ and a self-similar action of $\mathcal{G}$ as in [22, Definition 3.3] is the assumption in [22] that the induced action of $\mathcal{G}$ on the space of finite paths is faithful. In particular, it is already noticed in [22] how to remove the assumption $s(g \cdot e) = g \cdot s(e)$ in [13].

5. Composition of groupoid correspondences

We are ready to define the composition of groupoid correspondences and see that these form a bicategory. Let $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{K}$ be étale groupoids and let $\mathcal{X} : \mathcal{H} \leftarrow \mathcal{G}$ and $\mathcal{Y} : \mathcal{G} \leftarrow \mathcal{K}$ be groupoid correspondences. Let

$$\mathcal{X} \times_{\mathcal{G}^0} \mathcal{Y} := \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{Y} := \{(x, y) \in \mathcal{X} \times \mathcal{Y} : s(x) = r(y)\}.$$

Let $\mathcal{G}$ act on $\mathcal{X} \times_{\mathcal{G}^0} \mathcal{Y}$ by the *diagonal action*

$$g \cdot (x, y) := (x \cdot g^{-1}, g \cdot y)$$

for $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $g \in \mathcal{G}$ with $s(g) = r(y) = s(x)$. Let $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ be the orbit space of this action. The image of $(x, y) \in \mathcal{X} \times_{\mathcal{G}^0} \mathcal{Y}$ in $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ is usually denoted by $[x, y]$.

The maps $r(x, y) := r(x)$ and $s(x, y) := s(y)$ on $\mathcal{X} \times_{\mathcal{G}^0} \mathcal{Y}$ are invariant for this action and thus induce maps $r : \mathcal{X} \circ_{\mathcal{G}} \mathcal{Y} \to \mathcal{H}^0$ and $s : \mathcal{X} \circ_{\mathcal{G}} \mathcal{Y} \to \mathcal{K}^0$. These are the anchor maps for the commuting actions of $\mathcal{H}$ on the left and $\mathcal{K}$ on the right, which we define by

$$h \cdot [x, y] := [h \cdot x, y], \qquad [x, y] \cdot k := [x, y \cdot k]$$

for all $h \in \mathcal{H}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $k \in \mathcal{K}$ with $s(h) = r(x)$, $s(x) = r(y)$, and $s(y) = r(k)$. This is well defined because $[h \cdot x \cdot g^{-1}, g \cdot y] = [h \cdot x, y]$ and $[x \cdot g^{-1}, g \cdot y \cdot k] = [x, y \cdot k]$ for $g \in \mathcal{G}$ with $s(g) = s(x) = r(y)$.

We are going to prove that $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ with these two actions is again a groupoid correspondence. The following lemmas are needed for this. In some of the statements, we use the construction of $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ also when $\mathcal{Y}$ is merely a left $\mathcal{G}$-space, without a groupoid $\mathcal{K}$ that acts on $\mathcal{Y}$ on the right. Then $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ is still a left $\mathcal{H}$-space.

Lemma 5.1. *The pullback of a proper map is also a proper map.*

Proof. We form the pullback of two continuous maps $\alpha : A \to C$ and $\beta : B \to C$ as in (2.2). We assume β to be proper, that is, stably closed, and want to prove the same for the pullback map $\mathrm{pr}_A : A \times_C B \to A$. Let X be another space. Then the map $\mathrm{id}_{X \times A} \times \beta : X \times A \times B \to X \times A \times C$ is closed because β is proper. The space $X \times A$ is homeomorphic to a subspace of $X \times A \times C$ by the embedding $j : X \times A \to X \times A \times C$, $(x, a) \mapsto (x, a, \alpha(a))$; this is an embedding because the projection to the first two coordinates is a one-sided inverse. The space $X \times A \times_{\alpha,C,\beta} B$ of all (x,a,b) with $\alpha(a)=\beta(b)$ is the preimage of $j(X\times A)$ under $\mathrm{id}_{X\times A}\times\beta$, and $\mathrm{id}_X \times \mathrm{pr}_A\colon X\times A\times_{\alpha,C,\beta} B \to X\times A$ is the restriction of $\mathrm{id}_{X\times A}\times\beta$ to this preimage, composed with j^{-1}. A subset of $X\times A\times_{\alpha,C,\beta} B$ is closed if and only if it is of the form $D\cap(X\times A\times_{\alpha,C,\beta} B)$ for a closed subset D of $X\times A\times B$. Then $\mathrm{id}_X\times\mathrm{pr}_A$ maps it to the j-preimage of $(\mathrm{id}_{X\times A}\times\beta)(D)$, which is closed because $\mathrm{id}_{X\times A}\times\beta$ is closed and j is continuous. Thus $\mathrm{id}_X\times\mathrm{pr}_A$ is closed. □

Lemma 5.2. *Let A be a right $\mathcal{G}$-space and B a left $\mathcal{G}$-space. If the $\mathcal{G}$-action on A is proper and B is Hausdorff, then the diagonal $\mathcal{G}$-action on $A\times_{s,\mathcal{G}^0,r} B$ defined by $g\cdot(a,b) := (a\cdot g^{-1}, g\cdot b)$ is proper.*

Proof. Since B is Hausdorff, the diagonal inclusion

$$\Delta\colon B\to B\times B,\qquad b\mapsto (b,b),$$

is a closed map. Since Δ is also injective, it is even a proper map. By Lemma 5.1, the pullback of Δ along any map into $B\times B$ is again proper. It is useful to generalise this result a bit. Consider maps $\alpha\colon A\to C$ and maps $f\colon B_1\to B_2$ and $\beta\colon B_2\to C$. In the diagram

$$\begin{tikzcd}
A\times_C B_1 \arrow[r, "\mathrm{pr}_{B_1}"] \arrow[d, "\mathrm{id}_A\times_C f"'] \arrow[dr, phantom, "\lrcorner"] & B_1 \arrow[d, "f"] \\
A\times_C B_2 \arrow[r, "\mathrm{pr}_{B_2}"] \arrow[d, "\mathrm{pr}_A"'] \arrow[dr, phantom, "\lrcorner"] & B_2 \arrow[d, "\beta"] \\
A \arrow[r, "\alpha"'] & C
\end{tikzcd}$$

the lower square and the whole rectangle are pullbacks, and this implies that the top square is a pullback square as well. Therefore, the map $\mathrm{id}_A\times_C f$ is a pullback of f and inherits the property of being proper from f.

We now form this kind of pullback of Δ along the maps $B\times B\to\mathcal{G}^0\times\mathcal{G}^0$, $(b_1,b_2)\mapsto(r(b_1),r(b_2))$, and $\mathcal{G}\times_{s,\mathcal{G}^0,s} A\to\mathcal{G}^0\times\mathcal{G}^0$, $(g,a)\mapsto(s(g),s(a)) = (s(g),s(g))$. This gives a map from the space of triples $(g,a,b)\in\mathcal{G}\times A\times B$ with $s(g)=s(a)=r(b)$ to the space of quadruples (g,a,b_1,b_2) with $s(g)=s(a)=r(b_1)=r(b_2)$. The formula $(g,a,b_1,b_2)\mapsto(g,a,g\cdot b_1,b_2)$ defines a homeomorphism from the target of this map to the space of quadruples $(g,a,b_1,b_2)\in\mathcal{G}\times A\times B\times B$ with $s(g)=s(a)=r(b_2)$ and $r(g)=r(b_1)$; the inverse is defined by $(g,a,b_1,b_2)\mapsto(g,a,g^{-1}\cdot b_1,b_2)$.

Since the $\mathcal{G}$-action on A is proper, the following map is proper:

$$\varphi\colon \mathcal{G}\times_{s,\mathcal{G}^0,s} A\to A\times A,\qquad (g,a)\mapsto(a\cdot g^{-1},a).$$

We map $A\times A\to\mathcal{G}^0\times\mathcal{G}^0$, $(a_1,a_2)\mapsto(s(a_1),s(a_2))$, and $B\times B\to(\mathcal{G}^0,\mathcal{G}^0)$, $(b_1,b_2)\mapsto(r(b_1),r(b_2))$. Then the pullback of the proper map φ along these maps becomes the map $(g,a,b_1,b_2)\mapsto(a\cdot g^{-1},a,b_1,b_2)$ from the space of all quadruples $(g,a,b_1,b_2) \in \mathcal{G} \times A \times B \times B$ with $s(g) = s(a) = r(b_2)$ and $s(a\cdot g^{-1}) = r(b_1)$ to the space of all quadruples $(a_1,a_2,b_1,b_2) \in A \times A \times B \times B$ with $s(a_1) = r(b_1)$ and $s(a_2) = r(b_2)$. This map is again proper as a pullback of a proper map. Now use $s(a\cdot g^{-1}) = r(g)$ to identify the domain of this map with the codomain of the proper map that we constructed above from Δ. Composing the two proper maps above gives the map $(g,a,b) \mapsto (a\cdot g^{-1}, a, g\cdot b, b)$ from the space of triples $(g,a,b) \in \mathcal{G} \times A \times B$ with $s(g) = s(a) = r(b)$ to the space of quadruples (a_1,a_2,b_1,b_2) with $s(a_1) = r(b_1)$ and $s(a_2) = r(b_2)$. Exchanging the order of a_2 and b_1, this becomes the map that witnesses that the diagonal $\mathcal{G}$-action on $A \times_{s,\mathcal{G}^0,r} B$ is proper. □

Example 5.3. By Proposition 2.16, the trivial action of the trivial group on a space B is proper if and only if B is Hausdorff. This shows that Lemma 5.2 becomes false if we do not assume B to be Hausdorff.

Lemma 5.4. *We have* $(\mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{Y})/\mathcal{K} \cong \mathcal{X} \times_{s,\mathcal{G}^0,r_{\mathcal{Y}_*}} (\mathcal{Y}/\mathcal{K})$.

Proof. The orbit space projection $p_{\mathcal{Y}} \colon \mathcal{Y} \twoheadrightarrow \mathcal{Y}/\mathcal{K}$ is a local homeomorphism by Lemma 2.10. Then so is any pullback of it by Lemma 2.8. As in the proof of Lemma 5.2, this applies to the map

$$\mathrm{id}_{\mathcal{X}} \times_{\mathcal{G}^0} p_{\mathcal{Y}} \colon \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{Y} \to \mathcal{X} \times_{s,\mathcal{G}^0,r_{\mathcal{Y}_*}} (\mathcal{Y}/\mathcal{K}),$$

which is equivalent to the pullback along the maps $r_{\mathcal{Y}_*} \colon \mathcal{Y}/\mathcal{K} \to \mathcal{G}^0$ and $s \colon \mathcal{X} \to \mathcal{G}^0$ because $r_{\mathcal{Y}_*} \circ p_{\mathcal{Y}} = r$. The quotient map $p \colon \mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{Y} \to (\mathcal{X} \times_{s,\mathcal{G}^0,r_{\mathcal{Y}_*}} \mathcal{Y})/\mathcal{K}$ is a local homeomorphism as well by Lemma 5.2. By the universal property of the orbit space, we get a commuting diagram of continuous maps

$$\begin{tikzcd}
\mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{Y} \arrow[d, "p"'] \arrow[dr, "\mathrm{id}_{\mathcal{X}} \times_{\mathcal{G}^0} p_{\mathcal{Y}}"] & \\
(\mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{Y})/\mathcal{K} \arrow[r, "(\mathrm{id}_{\mathcal{X}} \times_{\mathcal{G}^0} p_{\mathcal{Y}})_*"'] & \mathcal{X} \times_{s,\mathcal{G}^0,r_{\mathcal{Y}_*}} (\mathcal{Y}/\mathcal{K}).
\end{tikzcd}$$

The map in the bottom is easily seen to be bijective. Since both maps that go down are surjective and local homeomorphisms, it follows that the bottom map is a local homeomorphism as well. Being bijective, it is a homeomorphism. □

Since the map $\mathrm{pr}'_{\mathcal{X}} \colon (\mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{Y})/\mathcal{K} \cong \mathcal{X} \times_{s,\mathcal{G}^0,r} (\mathcal{Y}/\mathcal{K}) \to \mathcal{X}$ is $\mathcal{G}$-equivariant, it induces a map $(\mathrm{pr}'_{\mathcal{X}})_* \colon (\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y})/\mathcal{K} \to \mathcal{X}/\mathcal{G}$ on the $\mathcal{G}$-orbit spaces.

Lemma 5.5. *If* $\mathrm{pr}'_{\mathcal{X}}$ *is proper, then so is* $(\mathrm{pr}'_{\mathcal{X}})_*$.

Proof. Let Z be a topological space. Lemma 5.4 implies $(\mathcal{X}/\mathcal{G}) \times Z \cong (\mathcal{X} \times Z)/\mathcal{G}$ and $(\mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{Y})/\mathcal{K} \times Z \cong (\mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{Y} \times Z)/\mathcal{K}$. Consider the following diagram:

$$\begin{tikzcd}
(\mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{Y} \times Z)/\mathcal{K} \arrow[r, "\mathrm{pr}'_{\mathcal{X}} \times \mathrm{id}_Z"] \arrow[d, "p"'] & \mathcal{X} \times Z \arrow[d, "p_{\mathcal{X}}"] \\
(\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y} \times Z)/\mathcal{K} \arrow[r, "(\mathrm{pr}'_{\mathcal{X}})_* \times \mathrm{id}_Z"'] & (\mathcal{X}/\mathcal{G}) \times Z
\end{tikzcd}$$

We abbreviate $f := \mathrm{pr}'_{\mathcal{X}} \times \mathrm{id}_Z$. Let A be a closed subset in $(\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}\times Z)/\mathcal{K}$. Since p is continuous and f is proper, $f(p^{-1}(A))$ is closed. So $(\mathcal{X}\times Z)\setminus f(p^{-1}(A))$ is open in $\mathcal{X}\times Z$. It consists of those (x,z) whose $\mathcal{G}$-orbit is disjoint from $f(A)$. The map $p_{\mathcal{X}}$ is open by Lemma 2.10. So $p_{\mathcal{X}}(\mathcal{X}\times Z\setminus f(p^{-1}(A))) = (\mathcal{X}/\mathcal{G}\times Z)\setminus p_{\mathcal{X}}(f(p^{-1}(A)))$ is open in $\mathcal{X}/\mathcal{G}\times Z$. Thus $p_{\mathcal{X}}(f(p^{-1}(A))) = (\mathrm{pr}'_{\mathcal{X}})_* \times \mathrm{id}_Z(A)$ is closed. □

Lemma 5.6. *There is a canonical homeomorphism $(\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y})/\mathcal{K} \cong \mathcal{X}\circ_{\mathcal{G}}(\mathcal{Y}/\mathcal{K})$.*

Proof. Let $p_0\colon (\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y})/\mathcal{K} \to (\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y})/(\mathcal{G}\times\mathcal{K})$ be the orbit space projection. By definition, $(\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y})/(\mathcal{G}\times\mathcal{K}) \cong (\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y})/\mathcal{K}$. Thus we may define $p_1\colon (\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y})/\mathcal{K} \to (\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y})/\mathcal{K}$. Let $p_2\colon \mathcal{X}\times_{s,\mathcal{G}^0,r_{\mathcal{Y}_*}}(\mathcal{Y}/\mathcal{K}) \to \mathcal{X}\circ_{\mathcal{G}}(\mathcal{Y}/\mathcal{K})$ be the orbit space projection. There is a commutative diagram

$$\begin{tikzcd}
(\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y})/\mathcal{K} \arrow[r, "\cong"] \arrow[d, "p_1"'] & \mathcal{X}\times_{s,\mathcal{G}^0,r_{\mathcal{Y}_*}}(\mathcal{Y}/\mathcal{K}) \arrow[d, "p_2"] \\
(\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y})/\mathcal{K} \arrow[r, "h"'] & \mathcal{X}\circ_{\mathcal{G}}(\mathcal{Y}/\mathcal{K}).
\end{tikzcd}$$

The map h is clearly bijective, and the homeomorphism in the top row is shown in Lemma 5.4. By Lemma 2.10, p_1 and p_2 are surjective local homeomorphisms, so h is a homeomorphism. □

Proposition 5.7. *The actions of $\mathcal{H}$ and $\mathcal{K}$ on $\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}$ are well defined and continuous and turn this into a groupoid correspondence $\mathcal{H} \leftarrow \mathcal{K}$.*

If both correspondences $\mathcal{X}$ and $\mathcal{Y}$ are proper or tight, then so is $\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}$.

Proof. To show that the actions are well defined, let $[x',y'] = [x,y]$ be two representatives for the same element in $\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}$. Then $x' = x\cdot g^{-1}$ and $y' = g\cdot y$ for some $g\in\mathcal{G}$. Then $[h\cdot x', y'] = [(h\cdot x)\cdot g^{-1}, g\cdot y] = [h\cdot x, y]$ and $[x', y'\cdot k] = [x\cdot g^{-1}, g\cdot(y\cdot k)] = [x, y\cdot k]$.

The actions on $\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}$ are continuous because they are continuous on $\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y}$ and $\mathcal{H}\times_{s,\mathcal{H}^0,r}(\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}) \cong (\mathcal{H}\times_{s,\mathcal{H}^0,r}\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y})/\mathcal{G}$ by Lemma 5.6, and similarly for $\mathcal{K}$.

To show that $\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}$ is a groupoid correspondence, we first check that the actions commute. Indeed, $s(h\cdot[x,y]) = s(y) = s([x,y])$, $r([x,y]\cdot k) = r(x) = r([x,y])$, and $(h\cdot[x,y])\cdot k = [h\cdot x, y\cdot k] = h\cdot([x,y]\cdot k)$.

Next we will show that $s\colon \mathcal{X}\circ_{\mathcal{G}}\mathcal{Y} \to \mathcal{K}^0$ is a local homeomorphism. Since $s_{\mathcal{X}}\colon \mathcal{X}\to\mathcal{G}^0$ is a local homeomorphism, Lemma 2.8 implies that $\mathrm{pr}_{\mathcal{Y}}\colon \mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y}\to\mathcal{Y}$ is a local homeomorphism as well. Since $s_{\mathcal{Y}}$ is a local homeomorphism, too, the composite map in the top of the following diagram is a local homeomorphism:

$$\begin{tikzcd}
\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y} \arrow[r, "\mathrm{pr}_{\mathcal{Y}}"] \arrow[d, "p"'] & \mathcal{Y} \arrow[r, "s_{\mathcal{Y}}"] & \mathcal{K}^0 \\
\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y} \arrow[urr, "s"'] & &
\end{tikzcd}$$

The map $s\colon \mathcal{X}\circ_{\mathcal{G}}\mathcal{Y} \to \mathcal{K}^0$ is defined so as to make this diagram commute, and the vertical map is a surjective local homeomorphism by Lemma 2.10. Then it follows that s is a local homeomorphism as well.

Next, we show that the $\mathcal{K}$-action on $\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}$ is basic. It is easy to check that this action is free. Therefore, if $[x_1, y_1], [x_2, y_2] \in \mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}$ are in the same $\mathcal{K}$-orbit, there is a unique $k \in \mathcal{K}$ with $s(y_1) = s[x_1, y_1] = r(k)$ and $[x_2, y_2] = [x_1, y_1]\cdot k$. We must show that k depends continuously on the pair $[x_1, y_1], [x_2, y_2]$ – subject to the condition that they lie in the same $\mathcal{K}$-orbit. First, $[x_1, y_1]\cdot k = [x_1, y_1\cdot k]$, and this is equal to $[x_2, y_2]$ if and only if there is $g \in \mathcal{G}$ with $s(x_1) = r(g)$ and $(x_2, y_2) = (x_1\cdot g, g^{-1}\cdot y_1\cdot k)$. Then Lemma 3.4 implies $g = \langle x_1 \mid x_2\rangle$ and

$$k = \langle g^{-1}y_1 \mid y_2\rangle = \big\langle \langle x_2 \mid x_1\rangle \cdot y_1 \,\big|\, y_2\big\rangle.$$

Since the bracket maps for $\mathcal{X}$ and $\mathcal{Y}$ are continuous, it follows that k depends continuously on $(x_1, y_1), (x_2, y_2) \in \mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y}$. Since the orbit space projection from $\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y}$ to $\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}$ is a local homeomorphism by Lemma 2.10, it follows that k still depends continuously on $[x_1, y_1], [x_2, y_2] \in \mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}$.

To prove that $(\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y})/\mathcal{K}$ is a groupoid correspondence, it only remains to show that the orbit space $(\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y})/\mathcal{K}$ is Hausdorff; then Proposition 2.16 shows that the right $\mathcal{K}$-action on $\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}$ is free and proper. Since the right $\mathcal{K}$-action on $\mathcal{Y}$ is free and proper, $\mathcal{Y}/\mathcal{K}$ is Hausdorff by Proposition 2.16. Then Lemma 5.2 shows that the diagonal $\mathcal{G}$-action on $\mathcal{X}\times_{s,\mathcal{G}^0,r_{\mathcal{Y}_*}}(\mathcal{Y}/\mathcal{K})$ is proper. By Proposition 2.16, its orbit space $\mathcal{X}\circ_{\mathcal{G}}(\mathcal{Y}/\mathcal{K})$ is Hausdorff. Then $(\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y})/\mathcal{K}$ is Hausdorff by Lemma 5.6.

Now assume that both correspondences $\mathcal{X}$ and $\mathcal{Y}$ are proper. That is, the maps $r_{\mathcal{X}_*}\colon \mathcal{X}/\mathcal{G} \to \mathcal{H}^0$ and $r_{\mathcal{Y}_*}\colon \mathcal{Y}/\mathcal{K} \to \mathcal{G}^0$ are proper. Lemma 5.1 applied to the pullback diagram

$$\begin{tikzcd}
\mathcal{X}\times_{s,\mathcal{G}^0,r_{\mathcal{Y}_*}}(\mathcal{Y}/\mathcal{K}) \arrow[r, "\mathrm{pr}_{\mathcal{Y}/\mathcal{K}}"] \arrow[d, "\mathrm{pr}_{\mathcal{X}}"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y}/\mathcal{K} \arrow[d, "r_{\mathcal{Y}_*}"] \\
\mathcal{X} \arrow[r, "s_{\mathcal{X}}"'] & \mathcal{G}^0
\end{tikzcd}$$

shows that $\mathrm{pr}_{\mathcal{X}}$ is proper. Thus the map $\mathrm{pr}'_{\mathcal{X}}\colon (\mathcal{X}\times_{s,\mathcal{G}^0,r}\mathcal{Y})/\mathcal{K} \to \mathcal{X}$ defined through the homeomorphism in Lemma 5.4 is also proper. Then the map $(\mathrm{pr}'_{\mathcal{X}})_*$ in Lemma 5.5 is proper. Then $r_{\mathcal{X}_*}\circ(\mathrm{pr}'_{\mathcal{X}})_*$ is proper, and this map is equal to $r_{\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}_*}$.

Finally, suppose that both correspondences $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ and $\mathcal{Y}\colon \mathcal{G} \leftarrow \mathcal{K}$ are tight. Then we follow the argument in the proper case and observe instead that each of the maps $r_{\mathcal{X}_*}$, $\mathrm{pr}_{\mathcal{X}}$, $\mathrm{pr}'_{\mathcal{X}}$ and $(\mathrm{pr}'_{\mathcal{X}})_*$, $r_{\mathcal{Y}_*}$ is a homeomorphism, and then so is the composite map $r_{\mathcal{X}\circ_{\mathcal{G}}\mathcal{Y}_*} = r_{\mathcal{X}_*}\circ(\mathrm{pr}'_{\mathcal{X}})_*$. □

6. The bicategory of groupoid correspondences

In this section, we define the bicategory $\mathfrak{Gr}$ of groupoid correspondences. Its objects are étale, locally compact groupoids with Hausdorff object space, which we continue to call just "groupoids" (see Definition 2.1). Let $\mathcal{H}$ and $\mathcal{G}$ be two such groupoids. An arrow $\mathcal{H} \leftarrow \mathcal{G}$ or $\mathcal{G} \to \mathcal{H}$ is a groupoid correspondence $\mathcal{X} : \mathcal{H} \leftarrow \mathcal{G}$ as in Definition 3.1; beware that the source of such a correspondence is on the right, as in our notation for the anchor maps. Let $\mathcal{X}, \mathcal{Y} : \mathcal{H} \leftleftarrows \mathcal{G}$ be two such groupoid correspondences. A 2-arrow $\mathcal{X} \Rightarrow \mathcal{Y}$ is an injective, $\mathcal{H}, \mathcal{G}$-equivariant, continuous map $\alpha : \mathcal{X} \to \mathcal{Y}$. The following lemma shows that such a 2-arrow is automatically open or, equivalently, a homeomorphism onto an open subset of $\mathcal{Y}$:

Lemma 6.1. *Let $\mathcal{X}, \mathcal{Y} : \mathcal{H} \leftleftarrows \mathcal{G}$ be groupoid correspondences. Then any $\mathcal{H}, \mathcal{G}$-equivariant, continuous map $\alpha : \mathcal{X} \to \mathcal{Y}$ is a local homeomorphism. Therefore, a 2-arrow $\mathcal{X} \Rightarrow \mathcal{Y}$ is a homeomorphism from $\mathcal{X}$ onto an open subset of $\mathcal{Y}$. It is a homeomorphism onto $\mathcal{Y}$ if it is also surjective.*

Proof. Let $x \in \mathcal{X}$. By assumption, both source maps are local homeomorphisms and $s_{\mathcal{Y}} \circ \alpha = s_{\mathcal{X}}$. Let $U_{\mathcal{Y}} \subseteq \mathcal{Y}$ be an open neighbourhood of $\alpha(x)$ on which $s_{\mathcal{Y}}$ is injective. Since α is continuous, there is an open neighbourhood $U_{\mathcal{X}} \subseteq \mathcal{X}$ with $\alpha(U_{\mathcal{X}}) \subseteq U_{\mathcal{Y}}$. Shrinking $U_{\mathcal{X}}$ further if necessary, we may arrange that $s_{\mathcal{X}}|_{U_{\mathcal{X}}}$ is injective. Then $\alpha|_{U_{\mathcal{X}}} : U_{\mathcal{X}} \to U_{\mathcal{Y}}$ is equal to the map $s_{\mathcal{Y}}|_{U_{\mathcal{Y}}}^{-1} \circ s_{\mathcal{X}}$, and this is a homeomorphism from $U_{\mathcal{X}}$ onto an open subset of $U_{\mathcal{Y}}$. This implies that α is a local homeomorphism. An injective local homeomorphism must be a homeomorphism onto an open subset of its codomain. □

The composition of 2-arrows is the obvious composition of maps. This is clearly associative, and the identity maps on groupoid correspondences are units for this composition. Thus there is a category $\mathfrak{Gr}(\mathcal{G}, \mathcal{H})$ with arrows $\mathcal{H} \leftarrow \mathcal{G}$ as objects and the 2-arrows between them as arrows.

Remark 6.2. We would still get a bicategory in the same way if we allow all $\mathcal{H}, \mathcal{G}$-equivariant, continuous maps as 2-arrows. The injectivity assumption is only needed for the homomorphism to C*-algebras in Section 7.

The composition of arrows $\mathcal{H} \leftarrow \mathcal{G} \leftarrow \mathcal{K}$ in $\mathfrak{Gr}$ is the construction $\circ_{\mathcal{G}}$. Let $\mathcal{X}_1, \mathcal{X}_2 : \mathcal{H} \leftleftarrows \mathcal{G}$ and $\mathcal{Y}_1, \mathcal{Y}_2 : \mathcal{G} \leftleftarrows \mathcal{K}$ be groupoid correspondences and let $\alpha : \mathcal{X}_1 \Rightarrow \mathcal{X}_2$ and $\beta : \mathcal{Y}_1 \Rightarrow \mathcal{Y}_2$ be 2-arrows. These induce a map

$$\alpha \circ_{\mathcal{G}} \beta : \mathcal{X}_1 \circ_{\mathcal{G}} \mathcal{Y}_1 \Rightarrow \mathcal{X}_2 \circ_{\mathcal{G}} \mathcal{Y}_2, \qquad [(x, y)] \mapsto [(\alpha(x), \beta(y)],$$

which inherits the properties of being injective, $\mathcal{H}, \mathcal{K}$-equivariant and continuous. In addition, this construction is "functorial", that is, the composition is a bifunctor

$$\circ_{\mathcal{G}} : \mathfrak{Gr}(\mathcal{G}, \mathcal{H}) \times \mathfrak{Gr}(\mathcal{K}, \mathcal{G}) \to \mathfrak{Gr}(\mathcal{K}, \mathcal{H}).$$

For each groupoid $\mathcal{G}$, the identity groupoid correspondence $1_{\mathcal{G}}$ on $\mathcal{G}$ is the arrow space of $\mathcal{G}$ – which we also denote by $\mathcal{G}$ – with the obvious left and right actions of $\mathcal{G}$ by multiplication; its anchor maps are the range and source maps $r, s\colon \mathcal{G} \rightrightarrows \mathcal{G}^0$. The following two easy lemmas describe the natural 2-arrows that complete the bicategory structure of $\mathfrak{Gr}$:

Lemma 6.3. *Let $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence. The maps*

$$\begin{aligned} \mathcal{H}\circ_{\mathcal{H}}\mathcal{X} \to \mathcal{X}, &\qquad [h, x] \mapsto h \cdot x,\\ \mathcal{X}\circ_{\mathcal{G}}\mathcal{G} \to \mathcal{X}, &\qquad [x, g] \mapsto x \cdot g, \end{aligned}$$

are $\mathcal{H}, \mathcal{G}$-equivariant homeomorphisms, which are natural for $\mathcal{H}, \mathcal{G}$-equivariant continuous maps $\mathcal{X} \to \mathcal{X}'$.

Proof. It is easy to see that these multiplication maps are bijective, continuous and $\mathcal{H}, \mathcal{G}$-equivariant. Then they are isomorphisms of correspondences by Lemma 6.1. The naturality statement is obvious. □

Lemma 6.4. *Let $\mathcal{G}_i$ for $1 \le i \le 4$ be étale groupoids. Let $\mathcal{X}_i\colon \mathcal{G}_i \leftarrow \mathcal{G}_{i+1}$ for $1 \le i \le 3$ be correspondences. The map*

$$\mathrm{assoc}\colon \mathcal{X}_1\circ_{\mathcal{G}_2}(\mathcal{X}_2\circ_{\mathcal{G}_3}\mathcal{X}_3) \to (\mathcal{X}_1\circ_{\mathcal{G}_2}\mathcal{X}_2)\circ_{\mathcal{G}_3}\mathcal{X}_3, \qquad [x_1, [x_2, x_3]] \mapsto [[x_1, x_2], x_3],$$

is a $\mathcal{G}_1, \mathcal{G}_4$-equivariant homeomorphism, which is natural with respect to $\mathcal{G}_i, \mathcal{G}_{i+1}$-equivariant continuous maps $\alpha_i\colon \mathcal{X}_i \to \mathcal{X}_i'$ for $1 \le i \le 3$; that is, the following square commutes:

$$\begin{tikzcd} \mathcal{X}_1\circ_{\mathcal{G}_2}(\mathcal{X}_2\circ_{\mathcal{G}_3}\mathcal{X}_3) \arrow[r, Rightarrow, "\alpha_1\circ_{\mathcal{G}_2}(\alpha_2\circ_{\mathcal{G}_3}\alpha_3)"] \arrow[d, Rightarrow, "\mathrm{assoc}"] & \mathcal{X}_1'\circ_{\mathcal{G}_2}(\mathcal{X}_2'\circ_{\mathcal{G}_3}\mathcal{X}_3') \arrow[d, Rightarrow, "\mathrm{assoc}"] \\ (\mathcal{X}_1\circ_{\mathcal{G}_2}\mathcal{X}_2)\circ_{\mathcal{G}_3}\mathcal{X}_3 \arrow[r, Rightarrow, "(\alpha_1\circ_{\mathcal{G}_2}\alpha_2)\circ_{\mathcal{G}_3}\alpha_3"] & (\mathcal{X}_1'\circ_{\mathcal{G}_2}\mathcal{X}_2')\circ_{\mathcal{G}_3}\mathcal{X}_3'. \end{tikzcd}$$

Proposition 6.5. *The data above defines a bicategory $\mathfrak{Gr}$.*

Proof. It is trivial to check that the coherence diagrams for a bicategory commute (these diagrams are shown, for instance, in [5, 23]). □

Remark 6.6. General properties of groupoids, groupoid actions and groupoid principal bundles are shown in [4, 26] in a rather abstract setting. When we apply the definitions and results in [4] to the category of locally compact, topological spaces with local homeomorphisms as partial covers, we also get a construction of a bicategory of étale groupoids and étale groupoid correspondences. Here we only require for an étale groupoid correspondence that the right action should be basic and that its anchor map should be a local homeomorphism. In this article, we have added the assumptions that the object spaces of our groupoids and the orbit spaces of our groupoid correspondences for the right actions should be Hausdorff. Both assumptions are needed for the homomorphism to C^*-algebras in Section 7. The fact that these Hausdorffness assumptions define a subbicategory does not follow from the general theory in [26], unless we also require all arrow spaces of groupoids to be Hausdorff. This, however, is too much for certain applications.

To prepare for the next section, we briefly recall how to define the bicategory $\mathfrak{Corr}$ of C^*-correspondences (see [9]). Its objects are C^*-algebras. Its arrows $A \leftarrow B$ for two C^*-algebras A and B are A-B-correspondences, that is, Hilbert B-modules with a nondegenerate *-representation of A by adjointable operators. If $\mathcal{E}, \mathcal{F} : A \leftleftarrows B$ are two such C^*-correspondences, then a 2-arrow $\alpha : \mathcal{E} \Rightarrow \mathcal{F}$ is an A,B-bimodule map $\alpha : \mathcal{E} \to \mathcal{F}$ that is isometric in the sense that $\langle \alpha(x) \mid \alpha(y) \rangle = \langle x \mid y \rangle$ for all $x, y \in \mathcal{E}$. These isometric bimodule maps are composed as maps. This makes the arrows $A \leftarrow B$ and their 2-arrows into a category. The composition of arrows is the usual completed tensor product of C^*-correspondences. This is clearly a bifunctor for isometric bimodule maps, as required for a bicategory.

The unit arrow 1_A for a C^*-algebra is A itself, viewed as a correspondence $A \leftarrow A$ in the obvious way, using the bimodule structure by left and right multiplication and the A-valued inner product $\langle x \mid y \rangle := x^* y$ for $x, y \in A$. There are natural isomorphisms of C^*-correspondences

$$A \otimes_A \mathcal{E} \cong \mathcal{E}, \qquad \mathcal{E} \otimes_B B \cong \mathcal{E}, \qquad (\mathcal{E} \otimes_B \mathcal{F}) \otimes_C \mathcal{G} \cong \mathcal{E} \otimes_B (\mathcal{F} \otimes_C \mathcal{G})$$

for three composable C^*-correspondences $\mathcal{E} : A \leftarrow B$, $\mathcal{F} : B \leftarrow C$, $\mathcal{G} : C \leftarrow D$. It is easy to check that these are natural with respect to the 2-arrows above and make the diagrams required for a bicategory commute (see [5, 23]).

$\mathfrak{Corr}$ is defined in [9] using only isomorphisms of C^*-correspondences as 2-arrows. Here we allow isometries that are not invertible, not even adjointable. This is useful in some situations. In particular, non-invertible isometries of C^*-correspondences are crucial in [25] or to treat partial actions in bicategorical terms. In this article, we allow 2-arrows that are not invertible because this does not cause any extra problems and it allows us to prove a stronger statement.

A C^*-correspondence $\mathcal{E} : A \leftarrow B$ is called *proper* if the left action factors through the ideal of compact operators, $A \to \mathbb{K}(\mathcal{E})$. The collection of proper C^*-correspondences is a subbicategory $\mathfrak{Corr}_{\mathrm{prop}}$, that is, identity correspondences are proper and the composite of two proper correspondences is again proper.

7. The homomorphism to C*-algebras

In this section, we first recall how to define the C^*-algebra of an (étale) groupoid. Then we turn a groupoid correspondence into a C^*-correspondence between the groupoid C^*-algebras. We prove that this is part of a homomorphism of bicategories $\mathfrak{Gr} \to \mathfrak{Corr}$ between the bicategories of groupoid and C^*-correspondences.

Both $\mathrm{C}^*(\mathcal{G})$ for a groupoid $\mathcal{G}$ and $\mathrm{C}^*(\mathcal{X})$ for a groupoid correspondence $\mathcal{X}$ are defined using a certain space of "quasi-continuous" functions. We define this in the generality of a locally compact, locally Hausdorff space X to treat both cases simultaneously. If $V \subseteq X$ is open and Hausdorff, then we extend a function $f \in \mathrm{C}_c(V)$ to X by letting $f(g) = 0$ for all $g \in X \setminus V$. Let $\mathfrak{S}(X)$ be the linear span of functions on X of this form. If X is Hausdorff, then $\mathfrak{S}(X)$ is equal to the space $\mathrm{C_c}(X)$ of continuous, compactly supported functions on X. If X is not Hausdorff, then there may be too few continuous functions on it, and we have to use the space $\mathfrak{S}(X)$ instead. The following proposition allows us to build $\mathfrak{S}(X)$ using a specific covering by Hausdorff open subsets:

Proposition 7.1 ([12, Proposition 3.10])**.** *Let X be a topological space. Let U_i for $i \in I$ be open subsets that are locally compact and Hausdorff. Assume $X = \bigcup_{i\in I} U_i$. Then $\mathfrak{S}(X)$ is equal to the linear span of the subspaces $\mathrm{C_c}(U_i)$ for $i \in I$.*

Proof. Let V be a locally compact Hausdorff open subset of X and $f \in \mathrm{C_c}(V)$. Since the support of f is compact, it is covered by finitely many of the open subsets U_i. In addition, this covering has a finite partition of unity. Then we may write $f = \sum_{j=1}^{n} f_j$ with $f_j \in \mathrm{C_c}(U_{i_j})$. □

Definition 7.2. A *slice* of a groupoid $\mathcal{G}$ is an open subset $V \subseteq \mathcal{G}$ such that $s|_V$ and $r|_V$ are injective. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A *slice* of a groupoid correspondence $\mathcal{X} : \mathcal{H} \leftarrow \mathcal{G}$ is an open subset $V \subseteq \mathcal{X}$ such that $s|_V : V \to \mathcal{G}^0$ and $p|_V : V \to \mathcal{X}/\mathcal{G}$ are injective.

The name "slice" comes from [12]. We find this name more friendly than the more common name "bisection".

When we view a groupoid $\mathcal{G}$ as the identity correspondence over itself, then the range map induces a homeomorphism $\mathcal{G}/\mathcal{G} \cong \mathcal{G}^0$. Therefore, $p : \mathcal{G} \to \mathcal{G}/\mathcal{G}$ is equivalent to the range map and $\mathcal{G}$ as a groupoid and as a groupoid correspondence have the same slices. If $\mathcal{X}$ is a groupoid correspondence, then $s : \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism by assumption and $p : \mathcal{X} \to \mathcal{X}/\mathcal{G}$ is one by Lemma 2.10. Therefore, any point in $\mathcal{X}$ has an open neighbourhood that is a slice. In other words, the slices cover $\mathcal{X}$. Then Proposition 7.1 allows us to write any element of $\mathfrak{S}(\mathcal{X})$ as a finite sum $\sum_{i=1}^{n} f_i$ for functions $f_i \in \mathrm{C_c}(V_i)$ and slices V_i for $i = 1, \dots, n$. The special case when $\mathcal{X} = 1_{\mathcal{G}} \cong \mathcal{G}$ is itself a groupoid is already well known.

We define a *-algebra structure on $\mathfrak{S}(\mathcal{G})$ as in [19]. If $\xi, \eta \in \mathfrak{S}(\mathcal{G})$, let

$$\xi * \eta(g) = \sum_{h \in \mathcal{G}^{r(g)}} \xi(h)\eta(h^{-1}g), \qquad \xi^*(g) = \overline{\xi(g^{-1})}.$$

We recall why this is well defined. Assume $\xi \in \mathrm{C_c}(V)$ and $\eta \in \mathrm{C_c}(W)$ for slices V and W. Then $V \cdot W := \{g \cdot h : g \in V,\ h \in W,\ s(g) = r(h)\}$ is a slice as well, and so is $V^* := \{g^{-1} : g \in V\}$. The formula for $\xi * \eta$ above simplifies to $\xi * \eta(g) = \xi(h)\eta(k)$ if there are $h \in V$ and $k \in W$ with $h \cdot k = g$, and $\xi * \eta(g) = 0$ otherwise. As a result, $\xi * \eta \in \mathrm{C_c}(V \cdot W) \subseteq \mathfrak{S}(\mathcal{G})$. Since $\mathfrak{S}(\mathcal{G})$ is spanned by functions in $\mathrm{C_c}(V)$ for slices V, this implies that $\xi * \eta \in \mathfrak{S}(\mathcal{G})$ for all $\xi, \eta \in \mathfrak{S}(\mathcal{G})$. Similarly, $\xi^* \in \mathrm{C_c}(V^{-1})$ if $\xi \in \mathrm{C_c}(V)$, and then $\xi^* \in \mathfrak{S}(\mathcal{G})$ for all $\xi \in \mathfrak{S}(\mathcal{G})$.

Routine computations show that the convolution above is bilinear and associative and that $\xi \mapsto \xi^*$ is a conjugate-linear involution with $(\xi * \eta)^* = \eta^* * \xi^*$. Thus $\mathfrak{S}(\mathcal{G})$ is a *-algebra.

Next, we recall why a maximal C^*-seminorm on $\mathfrak{S}(\mathcal{G})$ exists. The subset $\mathcal{G}^0 \subseteq \mathcal{G}$ is a slice, called the unit slice. So $C_c(\mathcal{G}^0) \subseteq \mathfrak{S}(\mathcal{G})$. The convolution and involution on $\mathfrak{S}(\mathcal{G})$ restrict to the usual pointwise multiplication and pointwise involution on $C_c(\mathcal{G}^0)$. Since $C_c(\mathcal{G}^0)$ is the union of the C^*-subalgebras $C_0(U)$ for relatively compact, open subsets $U \subseteq \mathcal{G}^0$, any *-representation of $C_c(\mathcal{G}^0)$ on a Hilbert space is bounded by the supremum norm. Therefore, any C^*-seminorm on $C_c(\mathcal{G}^0)$ is bounded by the usual supremum norm. If $\xi \in C_c(V)$ for a slice V, then $\xi * \xi^* \in C_c(V \cdot V^{-1}) \subseteq C_c(\mathcal{G}^0)$, and so

$$\|\xi\| = \|\xi * \xi^*\|^{1/2} \leq \|\xi * \xi^*\|_\infty^{1/2} = \|\xi\|_\infty.$$

Any element $\xi \in \mathfrak{S}(\mathcal{G})$ is a finite linear combination of such functions on slices. Therefore, there is $C > 0$ with $\|\xi\| \leq C$ for all C^*-seminorms on $\mathfrak{S}(\mathcal{G})$. Therefore, the supremum of the set of C^*-seminorms on $\mathfrak{S}(\mathcal{G})$ exists. This is again a C^*-seminorm, and clearly the maximal such. Actually, it is a C^*-norm and not just a C^*-seminorm because of the regular representations, but this will not be crucial in what follows.

Definition 7.3. The groupoid C^*-algebra $C^*(\mathcal{G})$ of $\mathcal{G}$ is the completion of $\mathfrak{S}(\mathcal{G})$ in the largest C^*-norm.

Now let $\mathcal{G}$ and $\mathcal{H}$ be groupoids and let $\mathcal{X} : \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence. We are going to build a C^*-correspondence $C^*(\mathcal{X}) \colon C^*(\mathcal{H}) \leftarrow C^*(\mathcal{G})$. That is, $C^*(\mathcal{X})$ is a Hilbert $C^*(\mathcal{G})$-module with a nondegenerate left action of $C^*(\mathcal{H})$. The first result of this kind for Morita equivalences of Hausdorff locally compact groupoids with Haar systems was proven by Muhly–Renault–Williams in [27]. The Hausdorffness assumption was soon removed by Renault (see [30, Corollaire 5.4]). More general groupoid correspondences between Hausdorff locally compact groupoids with Haar systems were treated by Holkar [15]. But we are not aware that this construction has been carried over to groupoid correspondences between possibly non-Hausdorff étale groupoids as we need it. Therefore, we give the details of this construction. Since we only work in the étale case, this is much easier than the constructions in the papers mentioned above.

We will define $C^*(\mathcal{X})$ as a completion of $\mathfrak{S}(\mathcal{X})$ in a suitable norm. The algebraic structure of the correspondence is easy to write down on the dense subspaces of $\mathfrak{S}$-functions. Namely, let $\xi, \eta \in \mathfrak{S}(\mathcal{X})$, $\gamma \in \mathfrak{S}(\mathcal{G})$, $\zeta \in \mathfrak{S}(\mathcal{H})$ and $x \in \mathcal{X}$, $g \in \mathcal{G}$. Then we define

$$\xi * \gamma(x) = \sum_{g \in \mathcal{G}^{s(x)}} \xi(x \cdot g)\gamma(g^{-1}), \tag{7.1}$$

$$\langle \xi \mid \eta \rangle(g) = \sum_{\{x \in \mathcal{X} \,:\, s(x) = r(g)\}} \overline{\xi(x)}\eta(x \cdot g), \tag{7.2}$$

$$\zeta * \xi(x) = \sum_{h \in \mathcal{H}^{r(x)}} \zeta(h)\xi(h^{-1}x). \tag{7.3}$$

We first check that this is well defined, that is, that these functions are finite linear combinations of $\mathrm{C_c}$-functions on slices.

Lemma 7.4. *Let $\xi \in \mathrm{C_c}(V_1)$, $\eta \in \mathrm{C_c}(V_2)$, $\gamma \in \mathrm{C_c}(W)$, and $\zeta \in \mathrm{C_c}(Z)$ for slices $V_1, V_2 \subseteq \mathcal{X}$, $W \subseteq \mathcal{G}$ and $Z \subseteq \mathcal{H}$. The following subsets are also slices:*

$$\begin{aligned} V_1W &:= \{xg : x \in V_1,\ g \in W,\ s(x) = r(g)\} \subseteq \mathcal{X}, \\ \langle V_1 \mid V_2\rangle &:= \{\langle x_1 \mid x_2\rangle : x_1 \in V_1,\ x_2 \in V_2,\ p(x_1) = p(x_2)\} \subseteq \mathcal{G}, \\ ZV_1 &:= \{hx : h \in Z,\ x \in V_1,\ s(h) = r(x)\} \subseteq \mathcal{X}. \end{aligned}$$

For $y \in V_1W$, there are unique $x \in V_1$, $g \in W$ with $xg = y$, and $\xi * \gamma \in \mathrm{C_c}(V_1W)$ with $(\xi * \gamma)(y) = \xi(x)\gamma(g)$. For $g \in \langle V_1 \mid V_2\rangle$, there are unique $x_1 \in V_1$, $x_2 \in V_2$ with $g = \langle x_1 \mid x_2\rangle$, and $\langle \xi \mid \eta\rangle \in \mathrm{C_c}(\langle V_1 \mid V_2\rangle)$ with $\langle \xi \mid \eta\rangle(g) = \overline{\xi(x_1)}\eta(x_2)$. For $y \in ZV_1$, there are unique $h \in Z$, $x \in V_1$ with $y = hx$, and $\zeta * \xi \in \mathrm{C_c}(ZV_1)$ with $(\zeta * \xi)(y) = \zeta(h)\xi(x)$.

Proof. All three assertions are proven similarly, and the claims about V_1W and ZV_1 are special cases of Lemma 7.11 and Proposition 7.12 about the composition of correspondences later on. Therefore, we only write down the proof for $\langle V_1 \mid V_2\rangle$.

The subset $\langle V_1 \mid V_2\rangle$ is open by Proposition 3.5. Let $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$ satisfy $p(x_1) = p(x_2)$ and $p(y_1) = p(y_2)$, so that $\langle x_1 \mid x_2\rangle$ and $\langle y_1 \mid y_2\rangle$ are defined. Assume first that $s(\langle x_1 \mid x_2\rangle) = s(\langle y_1 \mid y_2\rangle)$. Then Lemma 3.4 implies $s(x_2) = s(\langle x_1 \mid x_2\rangle) = s(\langle y_1 \mid y_2\rangle) = s(y_2)$. Then $x_2 = y_2$ because V_2 is a slice of $\mathcal{X}$. Then $p(x_1) = p(x_2) = p(y_2) = p(y_1)$. Then $x_1 = y_1$ because V_1 is a slice of $\mathcal{X}$. This proves both that s is injective on $\langle V_1 \mid V_2\rangle$ and that the elements $x_1 \in V_1$, $x_2 \in V_2$ with $g = \langle x_1 \mid x_2\rangle$ for $g \in \langle V_1 \mid V_2\rangle$ are unique. A similar argument shows that $x_1 = y_1$ and $x_2 = y_2$ hold if $r(\langle x_1 \mid x_2\rangle) = r(\langle y_1 \mid y_2\rangle)$. Therefore, $\langle V_1 \mid V_2\rangle$ is a slice. By definition, $\langle \xi \mid \eta\rangle$ is only nonzero in $\langle V_1 \mid V_2\rangle$, that is, for $g \in G$ for which there are $x_1 \in V_1$, $x_2 \in V_2$ with $x_1 g = x_2$. Since x_1 and x_2 are unique, we compute $\langle \xi \mid \eta\rangle(g) = \overline{\xi(x_1)}\eta(x_2)$ as asserted. Thus $\langle \xi \mid \eta\rangle \in \mathrm{C_c}(\langle V_1 \mid V_2\rangle)$. □

Lemma 7.5. *$\mathfrak{S}(\mathcal{X})$ becomes a $\mathfrak{S}(\mathcal{H})$-$\mathfrak{S}(\mathcal{G})$-bimodule with the multiplication maps above, that is, these maps are bilinear and $\xi * (\gamma_1 * \gamma_2) = (\xi * \gamma_1) * \gamma_2$, $(\zeta_1 * \zeta_2) * \xi = \zeta_1 * (\zeta_2 * \xi)$, $(\zeta * \xi) * \gamma = \zeta * (\xi * \gamma)$ for $\xi \in \mathfrak{S}(\mathcal{X})$, $\gamma, \gamma_1, \gamma_2 \in \mathfrak{S}(\mathcal{G})$, $\zeta, \zeta_1, \zeta_2 \in \mathfrak{S}(\mathcal{H})$. The inner product is linear in the second variable and satisfies $(\langle \xi \mid \eta\rangle)^* = \langle \eta \mid \xi\rangle$, $\langle \xi \mid \eta * \gamma\rangle = \langle \xi \mid \eta\rangle * \gamma$, and $\langle \zeta * \xi \mid \eta\rangle = \langle \xi \mid \zeta^* * \eta\rangle$ for $\xi, \eta \in \mathfrak{S}(\mathcal{X})$, $\gamma \in \mathfrak{S}(\mathcal{G})$, $\zeta \in \mathfrak{S}(\mathcal{H})$. It follows that the inner product is conjugate-linear in the first variable and satisfies $\langle \xi * \gamma \mid \eta\rangle = \gamma^* * \langle \xi \mid \eta\rangle$ for $\xi, \eta \in \mathfrak{S}(\mathcal{X})$, $\gamma \in \mathfrak{S}(\mathcal{G})$.*

Proof. All claims for functions in $\mathfrak{S}$ follow if they hold for compactly supported continuous functions on slices. In this special case, they follow from the associativity of our products or from the properties of the bracket operation in Lemma 3.4. □

Lemma 7.6. *Let $\xi \in \mathfrak{S}(\mathcal{X})$. Then there are finitely many elements $a_i \in \mathfrak{S}(\mathcal{G})$ for $i = 1, \dots, n$ with $\langle \xi \mid \xi \rangle = \sum_{i=1}^{n} a_i * a_i^*$; roughly speaking $\langle \xi \mid \xi \rangle \geq 0$ in $\mathfrak{S}(\mathcal{X})$. In addition, if $\xi \neq 0$, then $\langle \xi \mid \xi \rangle \neq 0$.*

Proof. Write $\xi = \sum_{k=1}^{n} \xi_k$ with $\xi_k \in \mathrm{C_c}(V_k)$ for slices $V_k \subseteq \mathcal{X}$ for $k = 1, \dots, n$. Let $p \colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ be the quotient map. The space $\mathcal{X}/\mathcal{G}$ is locally compact and Hausdorff by Proposition 2.16. The subset $K := \bigcup_{k=1}^{n} p(\operatorname{supp} \xi_k) \subseteq \mathcal{X}/\mathcal{G}$ is compact, hence paracompact. Then there are functions $\varphi_i' \in \mathrm{C_c}(p(V_i))$ for $i = 1, \dots, n$ with $\sum_{i=1}^{n} |\varphi_i'(y)|^2 = 1$ for all $y \in K$. Since each V_i is a slice, $p|_{V_i} \colon V_i \to p(V_i)$ is a homeomorphism. Let $\varphi_i := \varphi_i' \circ p|_{V_i}^{-1} \in \mathrm{C_c}(V_i)$. We claim that $a_i := \langle \xi \mid \varphi_i \rangle$ for $i = 1, \dots, n$ will do the job.

To prove the claim, we first compute $\varphi_i * \langle \varphi_i \mid \xi_k \rangle \in \mathfrak{S}(\mathcal{X})$. Since all functions involved are continuous with compact support on certain slices of $\mathcal{X}$, so is this combination by the proof of Lemma 7.4. Its support is $V_i \cdot \langle V_i \mid V_k \rangle \subseteq \mathcal{X}$. Any element of this slice is of the form $v_1 \cdot \langle v_2 \mid v_3 \rangle$ for some $v_1, v_2 \in V_i$, $v_3 \in V_k$, for which this expression is defined; that is, $p(v_2) = p(v_3)$ and $s(v_1) = r(\langle v_2 \mid v_3 \rangle) = s(v_2)$. And then

$$\varphi_i * \langle \varphi_i \mid \xi_k \rangle (v_1 \cdot \langle v_2 \mid v_3 \rangle) = \varphi_i(v_1) \overline{\varphi_i(v_2)} \xi_k(v_3).$$

Since $v_1, v_2 \in V_i$ and V_i is a slice of $\mathcal{X}$, $s(v_1) = s(v_2)$ implies $v_1 = v_2$. Then $v_1 \cdot \langle v_2 \mid v_3 \rangle = v_3$ by Lemma 3.4. As a result, $\varphi_i * \langle \varphi_i \mid \xi_k \rangle \in \mathrm{C_c}(V_k)$ and if $x \in V_k$, then

$$\varphi_i * \langle \varphi_i \mid \xi_k \rangle (x) = |\varphi_i(a)|^2 \xi_k(x)$$

if there is $a \in V_i$ with $p(a) = p(x)$, and 0 otherwise. Then $|\varphi_i(a)|^2 = |\varphi_i'(p(x))|^2$. Since $\sum |\varphi_i'(p(x))|^2 = 1$ for all $x \in V_k$ with $\xi_k(x) \neq 0$, this implies

$$\sum_{i=1}^{n} \varphi_i * \langle \varphi_i \mid \xi_k \rangle = \xi_k.$$

Since this holds for all k, summing over k gives $\sum_{i=1}^{n} \varphi_i * \langle \varphi_i \mid \xi \rangle = \xi$. Therefore,

$$\langle \xi \mid \xi \rangle = \left\langle \xi \,\middle|\, \sum_{i=1}^{n} \varphi_i * \langle \varphi_i \mid \xi \rangle \right\rangle = \sum_{i=1}^{n} \langle \xi \mid \varphi_i \rangle * \langle \varphi_i \mid \xi \rangle = \sum_{i=1}^{n} a_i * a_i^*$$

as desired.

If $y \in \mathcal{G}^0$, then we compute

$$\langle \xi \mid \xi \rangle (y) = \sum_{\{x \in \mathcal{X} \,:\, s(x) = y\}} \overline{\xi(x)} \xi(x) = \sum_{\{x \in \mathcal{X} \,:\, s(x) = y\}} |\xi(x)|^2.$$

If this vanishes for all $y \in \mathcal{G}^0$, then $\xi(x) = 0$ for all $x \in \mathcal{X}$, and then $\xi = 0$. $\square$

Lemma 7.6 implies the following:

- $\|\xi\| := \|\langle \xi \mid \xi \rangle\|_{\mathrm{C}^*(\mathcal{G})}^{1/2}$ is a norm on $\mathfrak{S}(\mathcal{X})$;

- the given inner product and right $\mathfrak{S}(\mathcal{G})$-module structure on $\mathfrak{S}(\mathcal{X})$ extend to a Hilbert $\mathrm{C}^*(\mathcal{G})$-module structure on the norm completion of $\mathfrak{S}(\mathcal{X})$.

We denote this Hilbert $\mathrm{C}^*(\mathcal{G})$-module by $\mathrm{C}^*(\mathcal{X})$.

Lemma 7.7. *If $\zeta \in \mathfrak{S}(\mathcal{H})$, then the map $\xi \mapsto \zeta * \xi$ is bounded for the norm on $\mathfrak{S}(\mathcal{X})$ defined above.*

Proof. First, let $\zeta \in \mathrm{C_c}(\mathcal{H}^0) \subseteq \mathfrak{S}(\mathcal{H})$. Then $(\zeta * \xi)(x) = \zeta(r(x)) \cdot \xi(x)$ for all $\xi \in \mathfrak{S}(\mathcal{X})$. Let M be the maximum of $|\zeta|$ and let $\tau(y) := \sqrt{M^2 - |\zeta(y)|^2}$ for all $y \in \mathcal{H}^0$. This defines a bounded function on $\mathcal{H}^0$ with $\zeta^* * \zeta + \tau^* * \tau = M^2$. Therefore, we may estimate

$$\begin{aligned} M^2 \cdot \langle \xi \mid \xi \rangle &= \langle \xi \mid (\zeta^* * \zeta + \tau^* * \tau) * \xi \rangle \\ &= \langle \zeta * \xi \mid \zeta * \xi \rangle + \langle \tau * \xi \mid \tau * \xi \rangle \geq \langle \zeta * \xi \mid \zeta * \xi \rangle. \end{aligned}$$

This inequality in $\mathfrak{S}(\mathcal{G}) \subseteq \mathrm{C}^*(\mathcal{G})$ implies $M \cdot \|\xi\| \geq \|\zeta * \xi\|$. Equivalently, the operator norm of $\zeta \in \mathrm{C}(\mathcal{H}^0)$ is at most $\|\zeta\|_\infty$.

Next, if $\zeta \in \mathrm{C_c}(Z)$ for a slice $Z \subseteq \mathcal{H}$, then $\|\zeta * \xi\|^2 = \|\langle \xi \mid \zeta^* * \zeta * \xi \rangle\|$, and $\zeta^* * \zeta \in \mathrm{C_c}(\mathcal{G}^0)$. Since the latter has operator norm at most $\|\zeta\|_\infty^2 < \infty$, it follows that the operator norm of left convolution with ζ is at most $\|\zeta\|_\infty$. Finally, Proposition 7.1 reduces the case of general $\zeta \in \mathfrak{S}(\mathcal{H})$ to this special case. □

Lemma 7.7 implies that the representation of $\mathfrak{S}(\mathcal{H})$ on $\mathfrak{S}(\mathcal{X})$ by left convolution extends to a representation by bounded linear operators on $\mathrm{C}^*(\mathcal{X})$. This representation inherits the algebraic properties in Lemma 7.5 by continuity. Therefore, we get a *-homomorphism from $\mathfrak{S}(\mathcal{H})$ to the C^*-algebra of adjointable operators on $\mathrm{C}^*(\mathcal{X})$. This extends uniquely to a *-homomorphism on $\mathrm{C}^*(\mathcal{H})$ by the universal property of the C^*-completion. The computations above also show that the left $\mathrm{C_c}(\mathcal{H}^0)$-module structure on $\mathfrak{S}(\mathcal{X})$ is nondegenerate. This implies that the representation of $\mathrm{C}^*(\mathcal{H})$ on $\mathrm{C}^*(\mathcal{X})$ is nondegenerate. This completes the construction of the $\mathrm{C}^*(\mathcal{H})$-$\mathrm{C}^*(\mathcal{G})$-correspondence $\mathrm{C}^*(\mathcal{X})$ from a groupoid correspondence $\mathcal{X} : \mathcal{H} \leftarrow \mathcal{G}$.

Before we continue to build the homomorphism $\mathfrak{Gr} \to \mathfrak{Corr}$, we examine $\mathrm{C}^*(\mathcal{X})$ for groupoid correspondences between spaces, groups and transformation groups. We get the C^*-correspondences whose Cuntz–Pimsner algebras (relative to Katsura's ideal) are the C^*-algebras defined in these situations by Katsura, Nekrashevych and Exel–Pardo, respectively.

Example 7.8. Let $\mathcal{G} = \mathcal{H}$ be a locally compact space. In Example 4.1, we have identified a groupoid correspondence $\mathcal{X} : \mathcal{G} \leftarrow \mathcal{G}$ with a topological graph. When we pass to C^*-algebras, we get $\mathrm{C}^*(\mathcal{G}) = \mathrm{C}^*(\mathcal{H}) = \mathrm{C}_0(\mathcal{G})$. Since $\mathcal{X}$ is Hausdorff, $\mathfrak{S}(\mathcal{X}) = \mathrm{C_c}(\mathcal{X})$. The $\mathrm{C_c}(\mathcal{G}^0)$-bimodule structure on $\mathrm{C_c}(\mathcal{X})$ extends continuously to the $\mathrm{C}_0(\mathcal{G}^0)$-bimodule structure $(f \cdot \xi \cdot g)(x) = f(r(x)) \cdot \xi(x) \cdot g(s(x))$ for all $f, g \in \mathrm{C}_0(\mathcal{G})$, $\xi \in \mathrm{C_c}(\mathcal{X})$, $x \in \mathcal{X}$. The inner product of $\xi, \eta \in \mathrm{C_c}(\mathcal{X})$ simplifies to $\langle \xi \mid \eta \rangle(y) = \sum_{s(x)=y} \overline{\xi(x)} \eta(x)$ for all $y \in \mathcal{G}$. The norm completion

$C^*(\mathcal{X})$ of this is exactly the C^*-correspondence that is used by Katsura [18] to define topological graph C^*-algebras.

Example 7.9. We have identified a proper groupoid correspondence $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ for a (discrete) group $\mathcal{G}$ in Example 4.1 with the covering permutational bimodule of a self-similar group, minus a certain faithfulness property. The assumption that $\mathcal{X}$ is a proper correspondence says that the set $\mathcal{X}/\mathcal{G}$ is finite. Here $\mathfrak{S}(\mathcal{G})$ is simply the group ring $\mathbb{C}[\mathcal{G}]$, and $\mathfrak{S}(\mathcal{X})$ is the vector space $\mathbb{C}[\mathcal{X}]$ with basis $\mathcal{X}$. The bimodule structure and the inner product on $\mathbb{C}[\mathcal{X}]$ above are the same as defined by Nekrashevych in [28, Section 3.1]. The Cuntz–Pimsner algebra of the C^*-correspondence $C^*(\mathcal{X})\colon C^*(\mathcal{G}) \leftarrow C^*(\mathcal{G})$ is Nekrashevych's universal Cuntz–Pimsner algebra $\mathcal{O}_{(\mathcal{G},\mathcal{X})}$ of the self-similar group as defined in [28].

Example 7.10. Now let $\mathcal{G} = \mathcal{H} = V \rtimes \Gamma$ for a group Γ and a discrete set V. We have described groupoid correspondences $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ in Proposition 4.3, relating them to the self-similar graphs of Exel and Pardo [13]. Exel and Pardo associate a C^*-algebra to such a self-similar graph and identify this C^*-algebra with the Cuntz–Pimsner algebra of a C^*-correspondence on $C^*(V \ltimes \Gamma)$. We claim that this C^*-correspondence is exactly our $C^*(\mathcal{X})$. To begin with, $\mathfrak{S}(\mathcal{G})$ is the algebraic crossed product algebra $\mathbb{C}[V] \rtimes \Gamma$, which is spanned by the characteristic functions $\delta_{v,g}$ for $v \in V$ and $g \in \Gamma$. Similarly, $\mathfrak{S}(\mathcal{X})$ is the vector space $\mathbb{C}[E \times \Gamma]$ with basis $E \times \Gamma$. The bimodule structure and inner product above may easily be expressed in terms of these bases, and the map $\delta_{e,g} \mapsto t_e v_g$ gives an isomorphism from $C^*(\mathcal{X})$ to the C^*-correspondence that is denoted M in [13, Section 10].

After these examples, we resume the construction of a homomorphism $\mathfrak{Gr} \to \mathfrak{Corr}$ and turn to the action of 2-arrows. Let $\mathcal{H}$ and $\mathcal{G}$ be groupoids and let $\mathcal{X}, \mathcal{Y}\colon \mathcal{H} \leftarrow \mathcal{G}$ be groupoid correspondences. A 2-arrow $\alpha\colon \mathcal{X} \Rightarrow \mathcal{Y}$ is, by definition, an injective, $\mathcal{H}$-$\mathcal{G}$-equivariant, continuous map $\alpha\colon \mathcal{X} \to \mathcal{Y}$. By Lemma 6.1, it follows that α is a homeomorphism from $\mathcal{X}$ onto an open subset of $\mathcal{Y}$. Then any slice of $\mathcal{X}$ is also a slice of $\mathcal{Y}$, and so extension by zero defines an injective map $\mathfrak{S}(\mathcal{X}) \hookrightarrow \mathfrak{S}(\mathcal{Y})$. This map preserves both the bimodule structure and the inner product. Therefore, it is an isometry for the Hilbert module norms. Then it extends uniquely to an isometric map on the completions. This extension remains a $C^*(\mathcal{H})$-$C^*(\mathcal{G})$-bimodule map, which we denote by $C^*(\alpha)\colon C^*(\mathcal{X}) \Rightarrow C^*(\mathcal{Y})$.

It is easy to see that $\alpha \mapsto C^*(\alpha)$ is functorial, that is, the identity map $\mathcal{X} \xrightarrow{\sim} \mathcal{X}$ goes to the identity map $C^*(\mathcal{X}) \xrightarrow{\sim} C^*(\mathcal{X})$, and $C^*(\alpha)\circ C^*(\beta) = C^*(\alpha\circ\beta)$ for composable 2-arrows α and β.

The unit correspondence $1_{\mathcal{G}}$ on a groupoid $\mathcal{G}$ is $\mathcal{G}$ with the actions of $\mathcal{G}$ by left and right multiplication. Inspection shows that the resulting $\mathfrak{S}(\mathcal{G})$-bimodule structure and inner product on $\mathfrak{S}(\mathcal{G})$ are given by left and right convolution and the usual inner product formula $\langle \xi \mid \eta \rangle := \xi^* * \eta$. Therefore, the completion $C^*(1_{\mathcal{G}})$ is equal to $C^*(\mathcal{G})$ with the usual Hilbert bimodule structure over itself. Thus the construction $\mathcal{X} \mapsto C^*(\mathcal{X})$ maps the unit groupoid correspondence to the unit C^*-correspondence. We briefly say that our construction is *strictly unital*.

Next, we turn to the compatibility with composition of groupoid correspondences. We first prove a preparatory lemma about the composition of slices, which is analogous to Lemma 7.4.

Lemma 7.11. *Let $\mathcal{X}\colon \mathcal{K} \leftarrow \mathcal{H}$ and $\mathcal{Y}\colon \mathcal{H} \leftarrow \mathcal{G}$ be composable groupoid correspondences. Recall that $\mathcal{X} \circ_{\mathcal{H}} \mathcal{Y}$ is the orbit space of the diagonal $\mathcal{H}$-action on $\mathcal{X} \times_{s,\mathcal{H}^0,r} \mathcal{Y}$, with a canonical structure of groupoid correspondence $\mathcal{K} \leftarrow \mathcal{G}$. Denote elements of $\mathcal{X} \circ_{\mathcal{H}} \mathcal{Y}$ as $[x, y]$ for $x \in \mathcal{X}$, $y \in \mathcal{Y}$ with $s(x) = r(y)$. Let $U \subseteq \mathcal{X}$ and $V \subseteq \mathcal{Y}$ be slices. Define*

$$U \cdot V := \{[x, y] : x \in U,\ y \in V,\ s(x) = r(y)\} \subseteq \mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}.$$

This subset is a slice. For each point in $z \in U \cdot V$, the elements $x \in U$ and $y \in V$ with $z = [x, y]$ are unique.

Proof. The subset $U \times_{\mathcal{G}^0} V$ is open in $\mathcal{X} \times_{s,\mathcal{G}^0,r} \mathcal{H}$ by definition. Then its image $U \cdot V$ in $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ is open by Lemma 2.10. Let $x_1, x_2 \in U$ and $y_1, y_2 \in V$ be such that $s(x_1) = r(y_1)$ and $s(x_2) = r(y_2)$. Assume first that $s[x_1, y_1] = s[x_2, y_2]$. This means that $s(y_1) = s(y_2)$. Then $y_1 = y_2$ because V is a slice. Then $s(x_1) = r(y_1) = r(y_2) = s(x_2)$. This implies $x_1 = x_2$ because U is a slice. This proves that the elements $x \in U$ and $y \in V$ with $z = [x, y]$ are unique and that $s|_{U \cdot V}$ is injective. Next, we assume instead that $p[x_1, y_1] = p[x_2, y_2]$. This means that there is $g \in \mathcal{G}$ with $s[x_1, y_1] = r(g)$ and $[x_1, y_1] \cdot g = [x_2, y_2]$. Equivalently, $s(y_1) = r(g)$ and $[x_1, y_1 \cdot g] = [x_2, y_2]$. This means that there is $h \in \mathcal{H}$ with $r(h) = s(x_1) = r(y_1)$ such that $(x_1 \cdot h, h^{-1} \cdot y_1 \cdot g) = (x_2, y_2)$. Then $x_1 \cdot h = x_2$, so that $p(x_1) = p(x_2)$ in $\mathcal{X}/\mathcal{H}$. This implies $x_1 = x_2$ because U is a slice. Since the right $\mathcal{H}$-action on $\mathcal{X}$ is free, it follows that $h = s(x_1)$. Then $y_1 \cdot g = y_2$, and an analogous argument for the slice V in the groupoid correspondence $\mathcal{Y}$ shows that $y_1 = y_2$. Therefore, the orbit space projection is injective on $U \cdot V$. This finishes the proof that $U \cdot V$ is a slice in $\mathcal{X} \circ_{\mathcal{H}} \mathcal{Y}$. $\square$

Proposition 7.12. *Let $\mathcal{X}\colon \mathcal{K} \leftarrow \mathcal{H}$ and $\mathcal{Y}\colon \mathcal{H} \leftarrow \mathcal{G}$ be composable groupoid correspondences. There is a well defined map*

$$\begin{aligned} &\mu^0_{\mathcal{X},\mathcal{Y}}\colon \mathfrak{S}(\mathcal{X}) \otimes \mathfrak{S}(\mathcal{Y}) \to \mathfrak{S}(\mathcal{X} \circ_{\mathcal{H}} \mathcal{Y}),\\ &\mu^0_{\mathcal{X},\mathcal{Y}}(f_1 \otimes f_2)([x, y]) = \sum_{\{h \in \mathcal{H}^0 \,:\, r(h) = s(x)\}} f_1(xh) \cdot f_2(h^{-1}y). \end{aligned}$$

It extends to an isomorphism of $C^*(\mathcal{K})$-$C^*(\mathcal{G})$-correspondences

$$\mu_{\mathcal{X},\mathcal{Y}}\colon C^*(\mathcal{X}) \otimes_{C^*(\mathcal{H})} C^*(\mathcal{Y}) \to C^*(\mathcal{X} \circ_{\mathcal{H}} \mathcal{Y}).$$

This isomorphism is natural with respect to the 2-arrows in $\mathfrak{Gr}$.

Proof. First examine $\mu^0_{\mathcal{X},\mathcal{Y}}$ on $f_1 \otimes f_2$ with $f_1 \in C_c(U)$ and $f_2 \in C_c(V)$ for slices $U \subseteq \mathcal{X}$ and $V \subseteq \mathcal{Y}$. Then Lemma 7.11 implies that $U \cdot V$ is a slice and that $\mu^0_{\mathcal{X},\mathcal{Y}}(f_1 \otimes f_2) \in C_c(U \cdot V)$, with the function given by $[x, y] \mapsto f_1(x) f_2(y)$

for $x \in U$, $y \in V$. It follows that $\mu^0_{\mathcal{X},\mathcal{Y}}$ maps $\mathfrak{S}(\mathcal{X}) \otimes \mathfrak{S}(\mathcal{Y})$ to $\mathfrak{S}(\mathcal{X} \circ_{\mathcal{H}} \mathcal{Y})$. This map is surjective because slices of the form $U \cdot V$ cover $\mathcal{X} \circ_{\mathcal{H}} \mathcal{Y}$ and all functions in $\mathrm{C_c}(U \cdot V)$ may be written in the form $\mu^0_{\mathcal{X},\mathcal{Y}}(f_1 \otimes f_2)$.

Give $\mathfrak{S}(\mathcal{X}) \otimes \mathfrak{S}(\mathcal{Y})$ the usual $\mathfrak{S}(\mathcal{K})$-$\mathfrak{S}(\mathcal{G})$-bimodule structure and the usual inner product, defined by $\langle f_1 \otimes f_2 \mid f_3 \otimes f_4 \rangle = \langle f_2 \mid \langle f_1 \mid f_3 \rangle * f_4 \rangle$. Lemma 7.4 implies that $\mu^0_{\mathcal{X},\mathcal{Y}}$ is an $\mathfrak{S}(\mathcal{K})$-$\mathfrak{S}(\mathcal{G})$-bimodule map with $\langle \mu^0_{\mathcal{X},\mathcal{Y}}(F_1) \mid \mu^0_{\mathcal{X},\mathcal{Y}}(F_2) \rangle = \langle F_1 \mid F_2 \rangle$ for all $F_1, F_2 \in \mathfrak{S}(\mathcal{X}) \otimes \mathfrak{S}(\mathcal{Y})$. Then $\mu^0_{\mathcal{X},\mathcal{Y}}$ extends uniquely to an isometry $\mu_{\mathcal{X},\mathcal{Y}}$ between the norm completions. The extension remains an $\mathfrak{S}(\mathcal{K})$-$\mathfrak{S}(\mathcal{G})$-bimodule map. The bimodule property extends by continuity to $\mathrm{C}^*(\mathcal{K})$ and $\mathrm{C}^*(\mathcal{G})$. Since $\mu^0_{\mathcal{X},\mathcal{Y}}$ is surjective, so is $\mu_{\mathcal{X},\mathcal{Y}}$. This makes it unitary. It is easy to check that $\mu_{\mathcal{X},\mathcal{Y}}$ is natural for the 2-arrows in $\mathfrak{Gr}$. □

Theorem 7.13. *The maps $\mathcal{G} \mapsto \mathrm{C}^*(\mathcal{G})$ on objects, $\mathcal{X} \mapsto \mathrm{C}^*(\mathcal{X})$ on arrows, and $\alpha \mapsto \mathrm{C}^*(\alpha)$ on 2-arrows together with the identity maps $\mathrm{C}^*(1_{\mathcal{G}}) = 1_{\mathrm{C}^*(\mathcal{G})}$ and the maps $\mu_{\mathcal{X},\mathcal{Y}} \colon \mathrm{C}^*(\mathcal{X}) \otimes_{\mathrm{C}^*(\mathcal{H})} \mathrm{C}^*(\mathcal{Y}) \to \mathrm{C}^*(\mathcal{X} \circ_{\mathcal{H}} \mathcal{Y})$ form a strictly unital homomorphism of bicategories $\mathfrak{Gr} \to \mathfrak{Corr}$.*

Proof. The only reason to call this a theorem is that it summarises all the constructions in this section. It remains to prove that the identity maps $\mathrm{C}^*(1_{\mathcal{G}}) = 1_{\mathrm{C}^*(\mathcal{G})}$ and the maps $\mu_{\mathcal{X},\mathcal{Y}}$ in Proposition 7.12 make the three diagrams commute that are required for homomorphisms of bicategories (see [5, 23]). This is checked by plugging in elementary tensors where each factor is supported on a slice. □

Let P be a monoid with unit element $1 \in P$. We view P as a bicategory with only one object, set of arrows P, and only identity 2-arrows. A bicategory homomorphism from P to $\mathfrak{Gr}$ as defined in [5, 23] is described by the following data:

- a groupoid $\mathcal{G}$;
- groupoid correspondences $\mathcal{X}_a \colon \mathcal{G} \leftarrow \mathcal{G}$ for $a \in P \setminus \{1\}$;
- biequivariant homeomorphisms $\sigma_{a,b} \colon \mathcal{X}_a \circ_{\mathcal{G}} \mathcal{X}_b \to \mathcal{X}_{ab}$ for $a, b \in P \setminus \{1\}$;

here we let $\mathcal{X}_1 := 1_{\mathcal{G}}$. Let $\sigma_{1,b} \colon 1_{\mathcal{G}} \circ_{\mathcal{G}} \mathcal{X}_b \to \mathcal{X}_b$ and $\sigma_{a,1} \colon \mathcal{X}_a \circ_{\mathcal{G}} 1_{\mathcal{G}} \to \mathcal{X}_a$ be the canonical maps in Lemma 6.3. The above data gives a bicategory homomorphism if and only if the following diagram commutes for all $a, b, c \in P$:

$$\begin{tikzcd}
\mathcal{X}_a \circ_{\mathcal{G}} \mathcal{X}_b \circ_{\mathcal{G}} \mathcal{X}_c \arrow[r, "\mathrm{id}_{\mathcal{X}_a} \circ_{\mathcal{G}} \sigma_{b,c}"] \arrow[d, "\sigma_{a,b} \circ_{\mathcal{G}} \mathrm{id}_{\mathcal{X}_c}"'] & \mathcal{X}_a \circ_{\mathcal{G}} \mathcal{X}_{bc} \arrow[d, "\sigma_{a,bc}"] \\
\mathcal{X}_{ab} \circ_{\mathcal{G}} \mathcal{X}_c \arrow[r, "\sigma_{ab,c}"'] & \mathcal{X}_{abc}.
\end{tikzcd} \tag{7.4}$$

More generally, we may replace a monoid by a category. Then homomorphisms are defined in the same way. The only change is that we now have one groupoid $\mathcal{G}_x$ for each object $x \in \mathcal{C}^0$, and the groupoid correspondence $\mathcal{X}_a$ for $a \colon x \to y$ becomes a correspondence $\mathcal{G}_y \leftarrow \mathcal{G}_x$.

Two homomorphisms of bicategories $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ may be composed in a canonical way to a homomorphism $\mathcal{C} \to \mathcal{E}$ (see [5]). Therefore, we may compose the homomorphism described by the data above with the homomorphism in Theorem 7.13 to get a homomorphism $P \to \mathfrak{Corr}$. Such homomorphisms are identified in [3] with product systems over the monoid P. The product system resulting from this composition has the following data:

- the fibres are the C^*-algebra $\mathrm{C}^*(\mathcal{G})$ at $1 \in P$ and the underlying Banach spaces of the correspondences $\mathrm{C}^*(\mathcal{X}_a)\colon\ \mathrm{C}^*(\mathcal{G}) \leftarrow \mathrm{C}^*(\mathcal{G})$ for $a \in P\backslash\{1\}$;
- the multiplication with $\mathrm{C}^*(\mathcal{G})$ and the inner product maps on the fibres of the product system come from the correspondence structure on $\mathrm{C}^*(\mathcal{X}_a)$;
- the multiplication map $\mathrm{C}^*(\mathcal{X}_a) \otimes_{\mathrm{C}^*(\mathcal{G})} \mathrm{C}^*(\mathcal{X}_b) \to \mathrm{C}^*(\mathcal{X}_{ab})$ for $a, b \in P \setminus \{1\}$ is the isomorphism of correspondences

$$\mathrm{C}^*(\mathcal{X}_a) \otimes_{\mathrm{C}^*(\mathcal{G})} \mathrm{C}^*(\mathcal{X}_b) \xrightarrow{\mu_{\mathcal{X}_a,\mathcal{X}_b}} \mathrm{C}^*(\mathcal{X}_a \circ_{\mathcal{G}} \mathcal{X}_b) \xrightarrow{\mathrm{C}^*(\sigma_{a,b})} \mathrm{C}^*(\mathcal{X}_{ab}).$$

This multiplication is associative because of the commuting diagrams (7.4) and the general theory of bicategories. In particular, it uses the commuting diagrams in the definition of a bicategory homomorphism (see [5, 23]), which involve the maps $\mu_{\mathcal{X},\mathcal{Y}}$ in Proposition 7.12. Thus the construction of a product system from a homomorphism $P \to \mathfrak{Gr}$ uses Theorem 7.13.

Once we have got a product system, we may take its Cuntz–Pimsner algebra to associate a C^*-algebra to a homomorphism $P \to \mathfrak{Gr}$.

The further study of this Cuntz–Pimsner algebra becomes much easier if the composite homomorphism $P \to \mathfrak{Corr}$ lands in the subbicategory of proper correspondences, that is, correspondences where the left action is by compact operators. The following theorem characterises when this happens:

Theorem 7.14. *Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A groupoid correspondence $\mathcal{X}\colon\ \mathcal{H} \leftarrow \mathcal{G}$ is proper if and only if the associated C^*-correspondence $\mathrm{C}^*(\mathcal{X})$ is proper.*

Proof. Since $\mathrm{C}_0(\mathcal{H}^0)$ is embedded nondegenerately into $\mathrm{C}^*(\mathcal{H})$, a correspondence $\mathrm{C}^*(\mathcal{H}) \leftarrow B$ for a C^*-algebra B is proper if and only if $\mathrm{C}_0(\mathcal{H}^0)$ acts by compact operators. In the proof of Lemma 7.6, we have shown that the operators of pointwise multiplication by functions in $\mathrm{C}_\mathrm{c}(\mathcal{X}/\mathcal{G})$ on $\mathrm{C}^*(\mathcal{X})$ are compact. More precisely, if $V \subseteq \mathcal{X}$ is a slice and $\varphi \in \mathrm{C}_\mathrm{c}(p(V))$, then we identified the operator of pointwise multiplication by $|\varphi|^2$ with a rank-one operator. Then the C^*-completion $\mathrm{C}_0(\mathcal{X}/\mathcal{G})$ of $\mathrm{C}_\mathrm{c}(\mathcal{X}/\mathcal{G})$ also acts by compact operators. Since pointwise multiplication by $\mathrm{C}_0(\mathcal{X}/\mathcal{G})$ is nondegenerate on $\mathfrak{S}(\mathcal{X})$, it remains nondegenerate on $\mathrm{C}^*(\mathcal{X})$. This implies that $\mathrm{C}_0(\mathcal{X}/\mathcal{G})$ embeds nondegenerately into the C^*-algebra of compact operators on $\mathrm{C}^*(\mathcal{X})$. Therefore, pointwise multiplication with a function in $\mathrm{C}_\mathrm{b}(\mathcal{X}/\mathcal{G})$ is only compact if the function belongs to $\mathrm{C}_0(\mathcal{X}/\mathcal{G})$. The action of $\mathrm{C}_0(\mathcal{H}^0)$ on $\mathrm{C}^*(\mathcal{X})$ factors through the homomorphism $\mathrm{C}_0(\mathcal{H}^0) \to \mathrm{C}_\mathrm{b}(\mathcal{X}/\mathcal{G})$ induced by $(r_\mathcal{X})_*\colon\ \mathcal{X}/\mathcal{G} \to \mathcal{H}^0$. This homomorphism factors through $\mathrm{C}_0(\mathcal{X}/\mathcal{G})$ if and only if $(r_\mathcal{X})_*$ is proper. This finishes the proof. $\square$

Tight groupoid correspondences are proper, of course, and therefore induce proper C^*-correspondences. It is unclear, however, which further extra properties $\mathrm{C}^*(\mathcal{X})$ has if $\mathcal{X}$ is a tight correspondence.

Example 7.15. If $\mathcal{G}$ is Hausdorff, then the arrow space of the groupoid $\mathcal{G}$ with the usual left action and the source map as right anchor map is a groupoid correspondence $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}^0$; here we view $\mathcal{G}^0$ as a groupoid with only identity arrows. The resulting C^*-correspondence $\mathrm{C}^*(\mathcal{X})\colon \mathrm{C}^*(\mathcal{G}) \leftarrow \mathrm{C}^*(\mathcal{G}^0) \cong \mathrm{C}_0(\mathcal{G}^0)$ is equivalent to a continuous family of representations of $\mathrm{C}^*(\mathcal{G})$ on a continuous field of Hilbert spaces over $\mathcal{G}^0$. This is the family of regular representations of $\mathrm{C}^*(\mathcal{G})$ on the Hilbert spaces $\ell^2(\mathcal{G}_x)$ for $x \in \mathcal{G}^0$. The homomorphism $\mathrm{C}^*(\mathcal{G}) \to \mathbb{B}(\mathrm{C}^*(\mathcal{X}))$ descends to a faithful representation of the *reduced groupoid* C^**-algebra*. In particular, it need not be faithful on $\mathrm{C}^*(G)$. This example shows that it may be hard to characterise when the left action of $\mathrm{C}^*(\mathcal{G})$ on $\mathrm{C}^*(\mathcal{X})$ for a groupoid correspondence $\mathcal{X}$ is faithful.

If $\mathcal{G}$ is not Hausdorff, then the above groupoid correspondence no longer exists because we require $\mathcal{G}/\mathcal{G}_0 = \mathcal{G}$ to be Hausdorff for a groupoid correspondence. We may, however, pick $x \in \mathcal{G}^0$ and construct a representation of $\mathrm{C}^*(\mathcal{G})$ on the Hilbert space $\ell^2(\mathcal{G}_x)$. These representations, however, no longer fit together into a continuous field. We may view the representation of $\mathrm{C}^*(\mathcal{G})$ on $\ell^2(\mathcal{G}_x)$ as a C^*-correspondence $\mathrm{C}^*(\mathcal{G}) \leftarrow \mathbb{C} = \mathrm{C}^*(\mathrm{pt})$ for the trivial groupoid pt. This comes from $\mathcal{G}_x = s^{-1}(\{x\}) \subseteq \mathcal{G}$, viewed as a groupoid correspondence $\mathcal{G}_x\colon \mathcal{G} \leftarrow \mathrm{pt}$.

8. Conduché fibrations as diagrams of groupoid correspondences

Let $\mathcal{C}$ be a small category. In this section, we examine bicategory homomorphisms from $\mathcal{C}$ to $\mathfrak{Gr}$ that map each object of $\mathcal{C}$ to a locally compact groupoid with only identity arrows. If $\mathcal{G}_x$ and $\mathcal{G}_y$ are locally compact spaces, then a groupoid correspondence $\mathcal{G}_x \leftarrow \mathcal{G}_y$ is the same as a *topological correspondence* as defined in [2] (see Example 4.1). The Cuntz–Pimsner algebras of the product systems associated to diagrams of proper topological correspondences are already studied in [2], and we have nothing to add to this, except that it is easy to generalise from diagrams defined over monoids to diagrams defined over categories. We would like to point out, however, that the C^*-algebras of discrete Conduché fibrations studied by Brown and Yetter [6] are the special case of this where the locally compact spaces in the diagram are all discrete.

Definition 8.1 ([6])**.** Let $\mathcal{E}$ and $\mathcal{C}$ be two small categories. A functor $F\colon \mathcal{E} \to \mathcal{C}$ is called a *discrete Conduché fibration* if it has the following unique factorisation lifting property (also called the *Conduché condition*): if $\varphi\colon y \to x$ is an arrow in $\mathcal{E}$, then any factorisation

$$\begin{tikzcd} F(y) \arrow[r, "\lambda"] \arrow[rr, bend right, "F(\varphi)"'] & z \arrow[r, "\varrho"] & F(x) \end{tikzcd}$$

of $F(\varphi)$ in $\mathcal{C}$ lifts uniquely to a factorisation

$$\begin{tikzcd} y \arrow[r, "\tilde{\lambda}"] \arrow[rr, bend right, "\varphi"'] & \tilde{z} \arrow[r, "\tilde{\varrho}"] & x \end{tikzcd}$$

of φ in $\mathcal{E}$; here lifting means that F maps $\tilde{\lambda}$ to λ and $\tilde{\varrho}$ to ϱ. A discrete Conduché fibration is *row-finite* if for every object X in $\mathcal{E}$ and every morphism $\beta\colon y \to F(X)$ in $\mathcal{C}$, the class of morphisms with target X whose image under F is β is a finite set.

Definition 8.2. Suppose F is a row-finite discrete Conduché fibration. The C^*-algebra of F is the universal C^*-algebra $\mathrm{C}^*(F)$ generated by orthogonal projections P_X for $X \in \mathcal{E}^0$ and partial isometries S_α for morphisms $\alpha\colon Y \to X$ in $\mathcal{E}$ that satisfy the following relations:

(1) if $X \neq Y$, then $P_X P_Y = 0$;
(2) if α and β are composable, then $S_{\alpha\beta} = S_\alpha S_\beta$;
(3) $P_x = S_{\mathrm{id}_x}$ for all x;
(4) if $\alpha\colon y \to x$, then $S_\alpha^* S_\alpha = P_y$;
(5) if $F(\alpha) = F(\beta)$ and $\alpha \neq \beta$, then $S_\beta^* S_\alpha = 0$;
(6) if $X \in \mathcal{E}^0$ and $g\colon y \to F(X)$ is an arrow in $\mathcal{C}$, then
$$\sum_{F(\alpha)=g, r(\alpha)=X} S_\alpha S_\alpha^* = P_X.$$

These definitions are inspired by the usual definition of a (row-finite) higher-rank graph and its C^*-algebra: this is the special case when $\mathcal{C}$ is the monoid $(\mathbb{N}^k, +)$. We will not say more about the theory of discrete Conduché fibrations and refer to [6] for further discussion. The theorem below describes them in another way, using the bicategory of groupoid correspondences. We believe that this alternative description is much more informative. From our point of view, a discrete Conduché fibration is equivalent to an action of $\mathcal{C}$ on discrete sets by correspondences. In particular, a higher-rank graph is an action of $\mathbb{N}^k$ on a discrete set by correspondences.

To be precise, Brown and Yetter define $\mathrm{C}^*(F)$ only if F is row-finite and "strongly surjective". This says that the sum on the right in condition (6) is non-empty for all X and g. If this fails, the condition asks for $P_X = 0$, which may entail further generators to become 0 and may reduce the whole C^*-algebra to be 0. We allow this degenerate case, however, because the following theorem remains true.

The following theorem uses product systems over categories and their absolute Cuntz–Pimsner algebras. All this is usually considered only over monoids, but the definitions make perfect sense over a category instead. The quickest way for us to define a product system over the category $\mathcal{C}$ is as a bicategory homomorphism $\mathcal{C} \to \mathfrak{Corr}$. We call a product system proper if this homomorphism lands in the subbicategory of proper correspondences. The *absolute Cuntz–Pimsner algebra* of a proper product system is defined in [3, Definition 6.8].

Over a monoid, this definition is the usual one, asking the Cuntz–Pimsner covariance condition for all elements of the underlying C^*-algebra and not just on some ideal.

Theorem 8.3. *A discrete Conduché fibration $F\colon \mathcal{E} \to \mathcal{C}$ is equivalent to a bicategory homomorphism $\tilde{F}\colon \mathcal{C} \to \mathfrak{Gr}$ with the extra property that each groupoid $\mathcal{G}_x$ for $x \in \mathcal{C}^0$ is a discrete set with only identity arrows. Here "equivalent" means a bijection on isomorphism classes. The fibration F is row-finite if and only if the corresponding homomorphism $\tilde{F}$ is proper. If F is row-finite, then $\mathrm{C}^*(F)$ is naturally isomorphic to the absolute Cuntz–Pimsner algebra of the product system over $\mathcal{C}$ associated to $\tilde{F}$.*

Proof. For $x \in \mathcal{C}^0$ and $g \in \mathcal{C}$, let $\mathcal{G}_x := F^{-1}(x) \subseteq \mathcal{E}^0$ and $\mathcal{X}_g := F^{-1}(g) \subseteq \mathcal{E}$, respectively. So $\mathcal{E}^0 = \bigsqcup_{x\in\mathcal{C}^0} \mathcal{G}_x$ and $\mathcal{E} = \bigsqcup_{g\in\mathcal{C}} \mathcal{X}_g$. We view $\mathcal{G}_x$ for $x \in \mathcal{C}^0$ as an étale groupoid with only identity arrows and the discrete topology. The range and source maps $r, s\colon \mathcal{E} \rightrightarrows \mathcal{E}^0$ restrict to maps $r\colon \mathcal{X}_g \to \mathcal{G}_{r(g)}$ and $s\colon \mathcal{X}_g \to \mathcal{G}_{s(g)}$ because F is a functor. These make $\mathcal{X}_g$ a groupoid correspondence $\mathcal{G}_{r(g)} \leftarrow \mathcal{G}_{s(g)}$; the conditions for a groupoid correspondence are obviously satisfied because our groupoids have only identity arrows and $\mathcal{X}_g$ has the discrete topology. Since F is a functor, the composition in $\mathcal{E}$ restricts to maps $\mu_{g,h}\colon \mathcal{X}_g \times_{\mathcal{G}_{s(g)}} \mathcal{X}_h \to \mathcal{X}_{g\cdot h}$ for all $g, h \in \mathcal{C}$ with $s(g) = r(h)$. Since each $\mathcal{G}_x$ is a space, $\mathcal{X}_g \times_{\mathcal{G}_{s(g)}} \mathcal{X}_h = \mathcal{X}_g \circ_{\mathcal{G}_{s(g)}} \mathcal{X}_h$. The discrete Conduché fibration condition says exactly that each map $\mu_{g,h}$ is bijective, as required for a homomorphism of bicategories. The maps $\mu_{g,h}$ determine the multiplication in $\mathcal{E}$ because the set of composable pairs of arrows in $\mathcal{E}$ is the disjoint union of the fibre products $\mathcal{X}_g \times_{\mathcal{G}_{s(g)}} \mathcal{X}_h$ for $(g, h) \in \mathcal{C}^2$. The associativity of the multiplication in $\mathcal{E}$ is equivalent to the associativity condition for a bicategory homomorphism in (7.4).

If $x \in \mathcal{C}^0 \subseteq \mathcal{C}$, then $\mathcal{X}_x = \mathcal{G}_x$ is the set of all unit arrows on objects $\hat{x} \in \mathcal{E}^0$ with $F(\hat{x}) = x$ by [6, Lemma 2.3], as required for a groupoid homomorphism. Since all arrows in $\mathcal{G}_x$ are identities, the multiplication maps $\mu_{g,h}$ when g or h is a unit arrow in $\mathcal{C}$ contain no information. So a discrete Conduché fibration $\mathcal{E} \to \mathcal{C}$ gives a homomorphism of bicategories $\mathcal{C} \to \mathfrak{Gr}$ with the extra property that each $\mathcal{G}_x$ is a discrete set viewed as an étale groupoid with the discrete topology and only identity arrows. This construction is reversible, that is, any bicategory homomorphism with this extra property comes from a discrete Conduché fibration, which is unique up to isomorphism.

The correspondence $\mathcal{X}_g$ for an arrow g in $\mathcal{C}$ is proper if and only if for each object $x \in \mathcal{E}^0$, the set of $\gamma \in \mathcal{X}_g$ with $r(\gamma) = g$ is finite. This happens for all g if and only if the fibration F is row-finite.

Since the spaces $\mathcal{G}_x$ and $\mathcal{X}_g$ are discrete, the C^*-algebras and C^*-correspondences in our proper product system have obvious bases, namely, the delta-functions of elements in $\mathcal{G}_x$ and $\mathcal{X}_g$, respectively. Let $\pi_g\colon \mathcal{X}_g \to \mathbb{B}(\mathcal{H})$ for $g \in \mathcal{C}$ be a Cuntz–Pimsner covariant representation of the proper product system $\mathrm{C}^*(\mathcal{X}_g)_{g\in\mathcal{C}}$ over $\mathcal{C}$. Since the delta-functions span dense subspaces in $\mathrm{C}^*(\mathcal{G}_x)$ and $\mathrm{C}^*(\mathcal{X}_g)$, the representation $\pi = (\pi_g)_{g\in\mathcal{C}}$ of the product system is determined by its values on the delta-functions. Actually, since $\mathcal{X}_x = \mathcal{G}_x$ for a unit arrow, there is no need to list the generators for objects at all. So we only need the values $S_\alpha := \pi_f(\delta_\alpha)$ if $f \in \mathcal{C}(x,y)$, $x,y \in \mathcal{C}^0$, and $\alpha \in \mathcal{X}_f$. It is convenient to also define $P_X := S_X = \pi_x(\delta_X)$ if $x \in \mathcal{C}^0$ and $X \in \mathcal{G}_x^0$. Actually, $\alpha \in \bigsqcup_{g\in\mathcal{C}} \mathcal{X}_g = \mathcal{E}$ and $X \in \bigsqcup_{x\in\mathcal{C}^0} \mathcal{G}_x^0 = \mathcal{E}^0$. It suffices to check the conditions for a Cuntz–Pimsner covariant representation of the product system on the basis of delta-functions. We claim that a family $(S_\alpha)_{\alpha\in\mathcal{E}}$ as above defines a representation if and only if

(1) $P_X := S_X$ for $X \in \bigsqcup_{x\in\mathcal{C}^0} \mathcal{G}_x = \mathcal{E}^0$ are mutually orthogonal projections;
(2) each S_α is a partial isometry;
(3) for all $\alpha \in \mathcal{E}$, $S_\alpha^* S_\alpha = P_{s(\alpha)}$;
(4) if $\alpha, \beta \in \mathcal{E}$ are different, then $S_\alpha^* S_\beta = 0$;
(5) if α and β are composable in $\mathcal{E} := \bigsqcup_{g\in\mathcal{C}} \mathcal{X}_g$, then $S_\alpha S_\beta = S_{\alpha\beta}$.
(6) if $x, y \in \mathcal{C}^0$, $X \in \mathcal{G}_x^0$, $g \in \mathcal{C}(y,x)$, then $P_X = \sum_{\alpha\in\mathcal{X}_g, r(\alpha)=X} S_\alpha S_\alpha^*$.

Condition (1) says that the map $\delta_X \mapsto P_X$ extends to a *-homomorphism on $C_0(\bigsqcup_{x\in\mathcal{C}^0} \mathcal{G}_x^0) = C_0(\mathcal{E}^0)$. Condition (2) follows from (3) and (4), and these say that $\pi_{s(g)}(\langle \delta_\alpha \mid \delta_\beta \rangle) = \pi_g(\delta_\alpha)^* \pi_g(\delta_\beta)$ for all $g \in \mathcal{C}$, $\alpha, \beta \in \mathcal{X}_g$; this is the condition for the right inner product in a Toeplitz representation of a product system. Condition (5) says that $\pi_g(S_\alpha)\pi_h(S_\beta) = \pi_{gh}(S_{\alpha\beta})$ for all $(g,h) \in \mathcal{C}^2$ and $\alpha \in \mathcal{X}_g$, $\beta \in \mathcal{X}_h$; this is the condition for the multiplication in a Toeplitz representation of a product system for g, h not a unit. If g or h is a unit, then the conditions $P_X S_\alpha = \delta_{X,r(\alpha)} S_\alpha$ and $S_\alpha P_X = \delta_{X,s(\alpha)} S_\alpha$ follow from (1), (3) and (6). Condition (6) says that the representation of $C^*(\mathcal{X}_g)$ for an arrow $g \in \mathcal{C}(y,x)$ is Cuntz–Pimsner covariant on all elements of $C^*(\mathcal{G}_x)$; by assumption, the product system is proper, so that $C^*(\mathcal{G}_x)$ acts on $C^*(\mathcal{X}_g)$ by compact operators. Now it is easy to see that the resulting list of conditions is equivalent to the list in Definition 8.2. Therefore, both universal properties define the same C^*-algebra. □

We now specialise this to k-graph C^*-algebras (see [21]):

Corollary 8.4. *A k-graph is equivalent to a bicategory homomorphism from $(\mathbb{N}^k, +)$ viewed as a bicategory to $\mathfrak{Gr}$ that maps the unique object of the bicategory $\mathbb{N}^k$ to a discrete set. The k-graph is row-finite if and only if it corresponds to a homomorphism to $\mathfrak{Gr}_{\mathrm{prop}}$. The C^*-algebra of a row-finite k-graph without sources is naturally isomorphic to the absolute Cuntz–Pimsner algebra of the resulting proper product system over $\mathbb{N}^k$.*

Proof. This follows from Theorem 8.3 because discrete Conduché fibrations with $\mathcal{C} = (\mathbb{N}^k, +)$ are the same as rank-k graphs (see [6]). The condition of having no sources is a standing assumption in [21]. It says that none of the sums in (6) is empty, so that it is reasonable to impose this condition for each (X, g). □

We may also allow each $\mathcal{G}_x$ to be a locally compact space, made a groupoid with only identity arrows. We propose this as the right locally compact version of a Conduché fibration. Now the category $\mathcal{E} = \bigsqcup_{g\in\mathcal{C}} \mathcal{X}_g$ is a locally compact topological category with a continuous functor $F\colon \mathcal{E} \to \mathcal{C}$ that is a discrete Conduché fibration as defined above when we forget the topologies. In addition, the source map $s\colon \mathcal{E} \to \mathcal{E}^0$ and the multiplication in $\mathcal{E}$ are local homeomorphisms. These extra conditions are necessary and sufficient for a functor $F\colon \mathcal{E} \to \mathcal{C}$ to come from a bicategory homomorphism $\mathcal{C} \to \mathfrak{Gr}$ by the construction above.

If $\mathcal{G}_x$ are locally compact spaces, then a groupoid correspondence $\mathcal{G}_x \leftarrow \mathcal{G}_y$ is the same as a *topological correspondence* as defined in [2] (see Example 4.1). When $\mathcal{C}$ is a monoid, we get actions of that monoid on a topological space by topological correspondences exactly as in [2]. If $\mathcal{C} = (\mathbb{N}^k, +)$, we get the topological rank-k graphs by Yeend [32]. The topological analogue of row-finiteness says exactly that these topological correspondences are proper. If they are also all surjective, then the C^*-algebra of the topological rank-k graph is isomorphic to the absolute Cuntz–Pimsner algebra of the proper product system over $\mathbb{N}^k$ defined by the homomorphism $\mathbb{N}^k \to \mathfrak{Gr} \to \mathfrak{Corr}$.

We may also replace a discrete set by a discrete group G or a transformation group $V \rtimes \Gamma$ for an action of a group Γ on a discrete set V. We have related groupoid correspondences on $V \rtimes \Gamma$ to self-similar graphs in Example 7.10. In an analogous way, a bicategory homomorphism $\mathbb{N}^k \to \mathfrak{Gr}_{\mathrm{prop}}$ that maps the unique object of the category $\mathbb{N}^k$ to $V \rtimes \Gamma$ is equivalent to a self-similar (row-finite) k-graph as defined by Li and Yang [24]. They also describe the C^*-algebra of such an object as the Cuntz–Pimsner algebra of a proper product system over $\mathbb{N}^k$. For each $n \in \mathbb{N}^k$ and $\mu \in d^{-1}(n)$, the map $\delta_{\mu,g} \mapsto \xi_\mu j(g)$ implements an isomorphism from $\mathrm{C}^*(\mathcal{X})$ to the C^*-correspondence that is denoted by $X_{G,\Lambda,n}$ in [24]. Then the product system by Li and Yang is naturally isomorphic to the product system corresponding to the composite homomorphism $\mathbb{N}^k \to \mathfrak{Gr} \to \mathfrak{Corr}$.

Summing up, various constructions of C^*-algebras from combinatorial or dynamical data may be realised as Cuntz–Pimsner algebras of product systems associated to homomorphisms to the bicategory of groupoid correspondences. This interprets these C^*-algebras as covariance algebras of generalised dynamical systems, where a category acts on an étale groupoid by groupoid correspondences.

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