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Slice filtration of certain C_{pq}-spectra

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ABSTRACT. For p, q distinct odd primes and a virtual C_{pq}-representation α, we compute the slices of the C_{pq}-spectrum $S^\alpha \wedge H\underline{\mathbb{Z}}$, and prove the existence of *spherical slice* for this spectrum.

CONTENTS

1. Introduction

For a fixed finite group G, a central computational tool in G-equivariant stable homotopy theory is the $RO(G)$-graded homotopy groups of a G-spectrum, but there are difficult to compute.

It was first observed by Voevodsky ([17]) that the stable motivic homotopy theory can be understood by their slice filtration. This slice filtration yields a spectral sequence, a tool computing various cohomology theories in algebraic geometry context. In equivariant homotopy theory, for any G-spectrum X, we have an analogous equivariant *slice tower*

$$\cdots P^{n+1}X \to P^nX \to P^{n-1}X \to \cdots. \tag{1}$$

In this tower, P^nX is the localization with respect to the localizing category of G-spectra generated by $\Sigma^\infty_G G_+ \wedge_H S^{k\rho_H}$ where ρ_H is the regular representation of $H \leq G$ and $k|H| \geq n$. The homotopy fiber P^n_nX of $P^nX \to P^{n-1}X$ is called the *n-slice* of X. An n-slice is called *spherical* if it is of the form $S^V \wedge H\underline{\mathbb{Z}}$, where V is a (non-virtual) representation of G of dimension n.

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The homotopy spectral sequence of (1) is an essential tool in modern homotopy theory due to Hill-Hopkins-Ravenel in [10]. This is called the *slice spectral sequence* and it eventually computes the $RO(G)$-graded homotopy groups of spectra.

The homotopy groups of the C_{2^n}-spectrum $N_2^{2^n} MU_{\mathbb{R}}$ played a crucial role in solving the Kervaire invariant one problem. In [10, Theorem 6.5], Hill, Hopkins, and Ravenel proved that the 0-th slice of this spectrum is the Eilenberg-Mac Lane spectrum $H\underline{\mathbb{Z}}$. This yields that in the slice spectral sequence of any $N_2^{2^n} MU_{\mathbb{R}}$-module, all the differentials and extensions live in $H\underline{\mathbb{Z}}$-modules. This motivates us to compute the slice filtration of the certain $H\underline{\mathbb{Z}}$-modules. The computation of the slice tower for $S^n \wedge H\underline{\mathbb{Z}}$ was made in [20] and for $S^{n\xi} \wedge H\underline{\mathbb{Z}}$ in [11] for the group C_{p^n}. Recently, in [8], Guilliou and Yarnall computed the $C_2 \times C_2$-slice tower for the spectrum $S^n \wedge H\underline{\mathbb{F}_2}$.

In this article, for a C_{pq}-representation V, we investigate the spectrum $S^V \wedge H\underline{\mathbb{Z}}$ and provide a description of the slices of it as follows:

Theorem A. *For any d-dimensional C_{pq}-representation V, the d-slice of the C_{pq}-spectrum $S^V \wedge H\underline{\mathbb{Z}}$ is spherical.*

See Theorem 5.7.

As a direct consequence, we derive:

Corollary B. *For $\alpha \in RO(C_{pq})$ there exists $\beta \in RO(C_{pq})$ such that $S^\alpha \wedge H\underline{\mathbb{Z}}$ has a* $\dim(\alpha)$*-slice $S^\beta \wedge H\underline{\mathbb{Z}}$. The other slices are suspensions of $H\mathcal{K}_p\langle \mathbb{Z}/p\rangle$ or $H\mathcal{K}_q\langle \mathbb{Z}/q\rangle$ or wedges of them.*

See Corollary 5.8.

Organization. Section 2 provides a short tour of equivariant homotopy theory. A few useful Mackey functors both in the context of C_p and C_{pq} have been described, and as a consequence, we prove certain equivalences of spectra in Section 3. Section 4 is dedicated to explaining the slice filtration of the C_p-spectrum $S^W \wedge H\underline{\mathbb{Z}}$. Finally, in Section 5, we provide a careful analysis of the C_{pq}-spectrum $S^V \wedge H\underline{\mathbb{Z}}$ with a result concerning the existence of a *spherical slice*.

Notation 1.1. *For an orthogonal G-representation V, $S(V)$ denotes the unit sphere, $D(V)$ the unit disk, and S^V the one-point compactification $\cong D(V)/S(V)$.*

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2. Preliminaries

In this section, we recall specific fundamental ideas and techniques in Bredon cohomology. Throughout the paper, G will be a finite group.

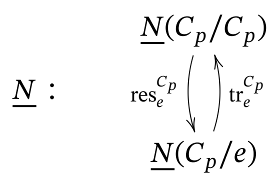
Definition 2.1. The *Burnside category* of G is the category $\mathsf{Burn_G}$ of finite G-sets, with $\mathrm{Hom}(S,T)$ the group completion of the set of correspondences $S \leftarrow U \rightarrow T$.

A functor $\underline{M} : \mathsf{Burn}_\mathsf{G}^{\mathsf{op}} \rightarrow \mathsf{Ab}$ from the Burnside category into abelian groups is called *Mackey functor*.

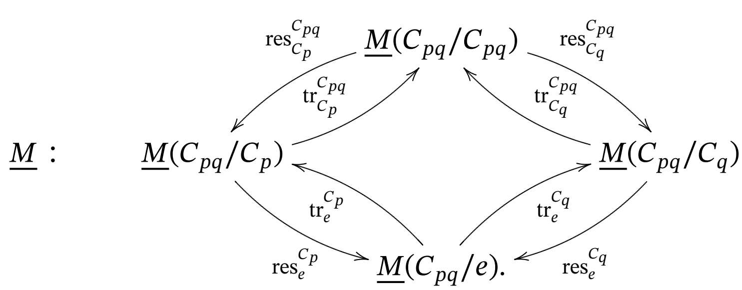
Explicitly, a G-Mackey functor $\underline{M}$ is a collection of commutative $W_G(H)$-groups $\underline{M}(G/H)$ one for each subgroup $H \le G$, each accompanied by *transfer* $\mathrm{tr}_K^H : \underline{M}(G/K) \rightarrow \underline{M}(G/H)$ and *restriction* $\mathrm{res}_K^H : \underline{M}(G/H) \rightarrow \underline{M}(G/K)$ for $K \le H \le G$ such that

(1) $\mathrm{tr}_J^H = \mathrm{tr}_K^H \,\mathrm{tr}_J^K$ and $\mathrm{res}_J^H = \mathrm{res}_J^K \,\mathrm{res}_K^H$ for all $J \le K \le H$.
(2) $\mathrm{tr}_K^H(\gamma.x) = \mathrm{tr}_K^H(x)$ for all $x \in \underline{M}(G/K)$ and $\gamma \in W_H(K)$.
(3) $\gamma.\,\mathrm{res}_K^H(x) = \mathrm{res}_K^H(x)$ for all $x \in \underline{M}(G/H)$ and $\gamma \in W_H(K)$.
(4) $\mathrm{res}_K^H \,\mathrm{tr}_K^J(x) = \sum_{\gamma \in W_H(K)} \gamma.\,\mathrm{tr}_{J\cap K}^K(x)$ for all subgroups $J, H \le K$.

Usually, we describe a Mackey functor by a *Lewis diagram* (see [13]). For a C_p-Mackey functor $\underline{N}$, we shall use the diagram:



and a C_{pq}-Mackey functor $\underline{M}$, we shall associate the following diagram.



We denote the category of G-Mackey functors by $\mathsf{Mack_G}$.

Example 2.2. For an abelian group C, the *constant Mackey functor* $\underline{C}$ is given by the assignment $\underline{C}(G/H) = C$ with $\mathrm{res}_K^H = \mathrm{Id}_C$ and $\mathrm{tr}_K^H =$ multiplication by the index $[H : K]$. One can define the *dual constant Mackey functor* $\underline{C}^*$ by interchanging the role of tr_K^H and res_K^H for $K \le H \le G$. In particular, for $G = C_p$, cyclic group of order p, the constant and dual constant C_p-Mackey functors are listed as follows.

$$\underline{\mathbb{Z}}:\ \begin{array}{c}\mathbb{Z}\\ 1\big\downarrow\ \big\uparrow p\\ \mathbb{Z}\end{array} \qquad\qquad \underline{\mathbb{Z}}^*:\ \begin{array}{c}\mathbb{Z}\\ p\big\downarrow\ \big\uparrow 1\\ \mathbb{Z}\end{array}$$

Let Sp^{G} be the category of orthogonal G-spectra ([14]). It has a symmetric model category structure $(\mathsf{Sp}^{\mathsf{G}}, \wedge, \mathbb{S}^0)$ as in [10, Appendix A, B], and we use Sp^{G} as our model for the G-equivariant stable homotopy theory.

It seems that the equivariant homotopy groups are more naturally graded on $RO(G)$, the free abelian group generated by the irreducible representations of G. For any $\alpha = V - W \in RO(G)$, we define

$$\pi_\alpha(X) := [S^V, S^W \wedge X]^G,$$

where the right hand side denotes the set of homotopy classes of maps in Sp^{G}. Thus, any orthogonal G-spectrum X has $RO(G)$-graded homotopy groups, denoted by $\pi_\bigstar(X)$. It induces a Mackey functor $\underline{\pi}_\alpha(X)$ such that

$$\underline{\pi}_\alpha(X)(G/H) := \pi_\alpha(\Sigma^\infty_G G/H_+ \wedge X).$$

For any G-Mackey functor $\underline{M}$ there is an equivariant Eilenberg-Mac Lane spectrum $H\underline{M}$ [7, Theorem 5.3] with

$$\underline{\pi}_k(H\underline{M}) = \begin{cases} \underline{M} & \text{for } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The equivariant Eilenberg-Mac Lane spectrum $H\underline{M}$ gives rise to a reduced cohomology theory given by

$$\tilde{H}^\alpha_G(X; \underline{M}) \cong [X, \Sigma^\alpha H\underline{M}]^G.$$

This cohomology is called the $RO(G)$-graded Bredon cohomology with coefficients in the Mackey functor $\underline{M}$.

It is interesting to note that the suspension isomorphism for $RO(G)$-graded cohomology theory $\tilde{H}^\alpha_G(-; \underline{M})$ takes the form $\tilde{H}^\alpha_G(X; \underline{M}) \cong \tilde{H}^{\alpha+V}_G(S^V \wedge X; \underline{M})$ for every based G-space X and representation V.

Notation 2.3. *For each odd cyclic group C_n, the representation ring $RO(C_n)$ is generated by the trivial representation* 1*, and the* 2*-dimension representation ξ^j given by the rotation by the angle $\frac{2\pi j}{n}$ for $j = 1, \cdots, \frac{n-1}{2}$.*

Proposition 2.4 ([22], Proposition 4.25)**.** *Let p be an odd prime. In $\mathsf{Sp}^{\mathsf{C_p}}$, there is an equivalence $H\underline{\mathbb{Z}} \wedge S^\xi \simeq H\underline{\mathbb{Z}} \wedge S^{\xi^j}$ whenever $p \nmid j$.*

We recall that there are change of groups functors on equivariant spectra. The restriction functor $i^*_H : \mathsf{Sp}^{\mathsf{G}} \to \mathsf{Sp}^{\mathsf{H}}$ from G-spectra to H-spectra has a left adjoint given by smashing with $G_+ \wedge_H (-)$. For a G-CW complex X, this also induces an isomorphism for cohomology with Mackey functor coefficients

$$\tilde{H}^{\alpha}_{G}(G_+ \wedge_H X; \underline{M}) \cong \tilde{H}^{\alpha}_{H}(i_H^* X; \downarrow^G_H (\underline{M})).$$

Here $\downarrow^G_H (\underline{M})$ is the H-Mackey functor defined by $\downarrow^G_H (\underline{M})(H/L) := \underline{M}(G/H \times H/L)$.

The $RO(G)$-graded cohomology theories may also be assumed to be Mackey functor-valued as in the definition below.

Definition 2.5. Let X be a pointed G-space, $\underline{M}$ be any G-Mackey functor, and $\alpha \in RO(G)$. Then the Mackey functor valued cohomology $\underline{H}^{\alpha}_{G}(X; \underline{M})$ ([16, §2.3]) is defined:

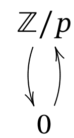
$$\underline{H}^{\alpha}_{G}(X; \underline{M})(G/K) = \tilde{H}^{\alpha}_{G}((G/K)_+ \wedge X; \underline{M}).$$

The restriction and transfer maps are induced by the appropriate maps of G-spectra.

3. Some important Mackey functors

We start by recalling the C_p-Mackey functors from Example 2.2 along with the following.

$\langle \mathbb{Z}/p \rangle$:



Now observe that the Burnside category $\mathsf{Burn}_{\mathsf{C_{pq}}}$ is isomorphic to $\mathsf{Burn}_{\mathsf{C_p}} \otimes \mathsf{Burn}_{\mathsf{C_q}}$ formed as the product set of objects and tensor product set of morphisms. Thus, we may define a C_{pq}-Mackey functor by tensoring Mackey functors on C_p and C_q. This is defined by a functor $\boxtimes : \mathsf{Mack}_{\mathsf{C_p}} \times \mathsf{Mack}_{\mathsf{C_q}} \to \mathsf{Mack}_{\mathsf{C_{pq}}}$ denoted by $(\underline{M}, \underline{N}) \mapsto \underline{M} \boxtimes \underline{N}$ such that on objects $G/H = C_p/H_1 \times C_q/H_2$,

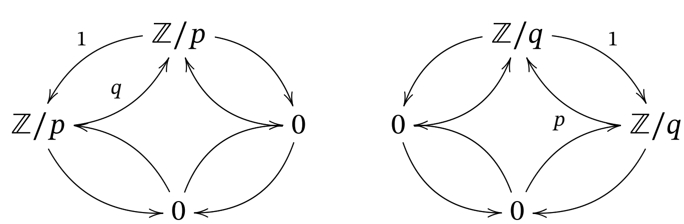
$$\underline{M} \boxtimes \underline{N}(G/H) = \underline{M}(C_p/H_1) \otimes \underline{N}(C_q/H_2).$$

The restrictions and transfers are given by tensoring the restrictions and transfers for $\underline{M}$ and $\underline{N}$ (§3, [1]). The following Mackey functors have particular importance in our case.

Definition 3.1. For a C_p-Mackey functor $\underline{M}$, define C_{pq}-Mackey functors

$$\mathcal{C}_p\underline{M} := \underline{M} \boxtimes \underline{\mathbb{Z}}^*, \ \mathcal{K}_p\underline{M} := \underline{M} \boxtimes \underline{\mathbb{Z}}.$$

The Lewis diagrams for the Mackey functors $\mathcal{K}_p\langle \mathbb{Z}/p \rangle$ and $\mathcal{K}_q\langle \mathbb{Z}/q \rangle$ are



(2)

$$\mathcal{K}_p\langle \mathbb{Z}/p\rangle \qquad\qquad \mathcal{K}_q\langle \mathbb{Z}/q\rangle$$

Also we denote the tensor products

$$\underline{\mathbb{Z}} \boxtimes \underline{\mathbb{Z}} \text{ and } \underline{\mathbb{Z}}^* \boxtimes \underline{\mathbb{Z}}^*$$

again by $\underline{\mathbb{Z}}$ and $\underline{\mathbb{Z}}^*$ respectively, and these are the constant and dual constant Mackey functors for the group C_{pq}. Recall [18, §7] that

Definition 3.2. A G-Mackey functor $\underline{M}$ is called *cohomological* Mackey functor if, for $K \le H \le G$, the composition $\mathrm{tr}_K^H \mathrm{res}_K^H : \underline{M}(G/H) \to \underline{M}(G/H)$ of the restriction $\mathrm{res}_K^H : \underline{M}(G/H) \to \underline{M}(G/K)$ with the transfer $\mathrm{tr}_K^H : \underline{M}(G/K) \to \underline{M}(G/H)$ is equal to the multiplication by index $[H : K]$.

Any module over the constant Mackey functor $\underline{\mathbb{Z}}$ is a cohomological Mackey functor (see [21]). For the group $G = C_{pq}$, any cohomological Mackey functor satisfies the following:

Proposition 3.3. *Let $\underline{M}$ be a cohomological C_{pq}-Mackey functor such that both the abelian groups $\underline{M}(C_{pq}/C_p)$ and $\underline{M}(C_{pq}/C_q)$ are trivial. Then, $\underline{M}(C_{pq}/C_{pq})$ is the trivial abelian group.*

Proof. Since $\underline{M}$ is cohomological, for $K \le H \le G$, the composition

$$\mathrm{tr}_K^H \mathrm{res}_K^H : \underline{M}(G/H) \to \underline{M}(G/H)$$

is given by the multiplication by index $[H : K]$. Let $x \in \underline{M}(C_{pq}/C_{pq})$; applying above map we get

$$px = 0 \text{ and } qx = 0.$$

Since, p and q are relatively prime, $x = 0$. The result follows. □

With the above notations, we recall from [1] the additive structure of the Mackey functor $\underline{H}^{\alpha}_{C_{pq}}(S^0; \underline{\mathbb{Z}})$ as follows:

Theorem 3.4 (Theorem 7.3, [1]). *Let $\alpha \in RO(C_{pq})$. Then the Mackey functor*

$$\underline{H}^{\alpha}_{C_{pq}}(S^0;\underline{\mathbb{Z}}) \cong \begin{cases} \mathcal{K}_p\langle \mathbb{Z}/p\rangle & \text{if } |\alpha| < 0, |\alpha^{C_p}| > 1, |\alpha^{C_q}| \le 1 \text{ odd} \\ \mathcal{K}_q\langle \mathbb{Z}/q\rangle & \text{if } |\alpha| < 0, |\alpha^{C_p}| \le 1, |\alpha^{C_q}| > 1 \text{ odd} \\ \mathcal{K}_p\langle \mathbb{Z}/p\rangle \oplus \mathcal{K}_q\langle \mathbb{Z}/q\rangle & \text{if } |\alpha| < 0, |\alpha^{C_p}| > 1, |\alpha^{C_q}| > 1 \text{ odd} \\ \mathcal{K}_p\langle \mathbb{Z}/p\rangle \oplus \mathcal{K}_q\langle \mathbb{Z}/q\rangle & \text{if } |\alpha| > 0, |\alpha^{C_p}| \le 0, |\alpha^{C_q}| \le 0 \text{ even} \\ \mathcal{K}_p\langle \mathbb{Z}/p\rangle & \text{if } |\alpha| > 0, |\alpha^{C_p}| \le 0, |\alpha^{C_q}| > 0 \text{ even} \\ \mathcal{K}_q\langle \mathbb{Z}/q\rangle & \text{if } |\alpha| > 0, |\alpha^{C_p}| > 0, |\alpha^{C_q}| \le 0 \text{ even} \\ \underline{\mathbb{Z}} & \text{if } |\alpha| = 0, |\alpha^{C_p}| \le 0, |\alpha^{C_q}| \le 0 \\ \underline{\mathbb{Z}}^* & \text{if } |\alpha| = 0, |\alpha^{C_p}| > 0, |\alpha^{C_q}| > 0 \\ \mathcal{K}_p\underline{\mathbb{Z}}^* & \text{if } |\alpha| = 0, |\alpha^{C_p}| > 0, |\alpha^{C_q}| \le 0 \\ \mathcal{K}_q\underline{\mathbb{Z}}^* & \text{if } |\alpha| = 0, |\alpha^{C_p}| \le 0, |\alpha^{C_q}| > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using Theorem 3.4, we have the following equivalences of spectra which simplify computations:

Lemma 3.5. *(a) There are the equivalences*

(1) $S^{\xi} \wedge H\underline{\mathbb{Z}} \simeq S^{\xi^j} \wedge H\underline{\mathbb{Z}}$ *if* $(j, pq) = 1$.
(2) $S^{\xi^p} \wedge H\underline{\mathbb{Z}} \simeq S^{\xi^{jp}} \wedge H\underline{\mathbb{Z}}$ *if* $(j, q) = 1$.
(3) $S^{\xi^q} \wedge H\underline{\mathbb{Z}} \simeq S^{\xi^{jq}} \wedge H\underline{\mathbb{Z}}$ *if* $(j, p) = 1$.
(4) $\Sigma^{\xi^p} H\underline{\mathbb{Z}} \simeq \Sigma^{\xi} H(\mathcal{K}_p \underline{\mathbb{Z}}^*)$.
(5) $H\underline{\mathbb{Z}}^* \simeq \Sigma^{2-\xi} H\underline{\mathbb{Z}}$.

(b) The equivalence (4) induces a map $u_{\xi-\xi^p} : H\underline{\mathbb{Z}} \to \Sigma^{\xi-\xi^p} H\underline{\mathbb{Z}}$ *and equivalence (5) induces* $u_{\xi} : H\underline{\mathbb{Z}} \to \Sigma^{\xi-2} H\underline{\mathbb{Z}}$.

Proof. (a) These follow by computing homotopy groups of $S^{\xi^j-\xi^k} \wedge H\underline{\mathbb{Z}}$ and using Theorem 3.4.

(b) Consider the C_p-Mackey functor map $\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}}$ which is the identity on the orbit C_p/e. By tensoring with the constant C_q-Mackey functor $\underline{\mathbb{Z}}$ and using equivalence (4), we obtain

$$H\underline{\mathbb{Z}} \xrightarrow{\cong} \Sigma^{\xi-\xi^p} H\mathcal{K}_p \underline{\mathbb{Z}}^* \to \Sigma^{\xi-\xi^p} H\underline{\mathbb{Z}}.$$

Next, consider the canonical C_{pq}-Mackey functor map $\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}}$ which is the identity at level C_{pq}/e. Applying (5) yields u_{ξ}. □

Remark 3.6. As a consequence of Lemma 3.5, only the C_{pq}-representations of the form

$$V = a + b\xi + c\xi^p + d\xi^q \text{ for } a, b, c, d \in \mathbb{Z}$$

are useful for our computations.

4. Spherical C_p-slices for the spectrum $S^W \wedge H\underline{\mathbb{Z}}$

In this section, we recall certain general facts about the slice tower of any genuine G-spectra. We also prove a few results for C_p-spectra of the form $S^W \wedge H\underline{\mathbb{Z}}$.

Recall the *localizing subcategories* of Sp^G are those closed under weak equivalences, cofibrations, extensions, coproducts, and well-ordered homotopy colimits ([15, §2]). For each integer n, let $\tau_{\geq n}$ denote the localizing subcategory of Sp^G generated by the G-spectra of the form $\Sigma^{\infty}_G G_+ \wedge_H S^{k\rho_H}$, where H ranges over all subgroups of G, ρ_H is the regular representation of H and $k|H| \geq n$.

Associated to the category $\tau_{\geq n}$ there is a natural localization functor P^{n-1}. As $\tau_{\geq n+1} \subseteq \tau_{\geq n}$, there is a natural transformation $P^n \to P^{n-1}$, which yields the slice tower (1) for each spectrum X. The homotopy fibre of $P^n X \to P^{n-1} X$, denoted by $P^n_n X$, is called the *n-slice* for the spectrum X (see [15, 4]).

A spectrum $X \in \tau_{\geq n}$ is called *n-slice connective* written $X \geq n$. A criterion for this to hold is produced by

Theorem 4.1 (Theorem A, [12]). *A G-spectrum $X \geq n$ if and only if the non equivariant homotopy groups*

$$\pi_k \Phi^H(X) = 0$$

for all $H \leq G$ and $k < \frac{n}{|H|}$.

Here $\Phi^H(X)$ is the H-geometric fixed points of the spectrum X, that is, the geometric fixed point of $i_H^*(X)$, the restriction of the G-spectrum X to $H \leq G$.

Remark 4.2. For a fixed point free G-representation V, $a_V \in \pi_{-V}(S^0)$ is represented by the map $S^0 = \{0, \infty\} \subseteq S^V$. For any $X \in \mathsf{Sp}^{\mathsf{G}}$, $\Phi^G(X)$ is the fixed point spectrum of the localization $X[a_{\bar{\rho}_G}^{-1}]$ by the element $a_{\bar{\rho}_G}$. For $G = C_{pq}$, it is evident that $\xi^p \oplus \xi^q \subset \bar{\rho}_{C_{pq}}$. Therefore, $a_{\bar{\rho}_{C_{pq}}}$ is trivial by the ring structure described in [2]. Hence, the spectrum $\Phi^{C_{pq}}(X)$ has trivial homotopy groups.

The local objects in Sp^{G} with respect to $\tau_{\geq n}$ are those X such that $[Y, X]^G = 0$ for all $Y \in \tau_{\geq n}$. These spectra are called *n-slice coconnective* spectra, and we write $X \leq n$ in this case. Equivalently,

Lemma 4.3. *A spectrum is n-slice coconnective if the restriction $i_H^*(X) \leq n$ and $[S^{k\rho_G}, X]^G = 0$ for all $k \geq 0$ such that $k|G| > n$. We denote such an X by $X \leq n$.*

By [9, Proposition 2.6], if a spectrum X satisfies $k \leq X \leq n$, the same holds for all its restrictions. Moreover, $n \leq X \leq n$ if and only if $X = P_n^n X$ in which case X is called an *n-slice*.

In general, $\Sigma^V P_n^n X$ need not equal $P_{n+\dim(V)}^{n+\dim(V)}(\Sigma^V X)$, but it does commute with suspensions of the following forms.

Proposition 4.4 (Corollary 4.25, [10]). *If X an m-slice, $\Sigma^{k\rho_G} X$ is $(m+k|G|)$-slice for all $k \in \mathbb{Z}$, that is,*

$$P_{m+k|G|}^{m+k|G|}(\Sigma^{k\rho_G} X) \simeq \Sigma^{k\rho_G} P_m^m(X).$$

The homotopy spectral sequence of the slice tower (1) of a G-spectrum X is called the *slice spectral sequence* for X, with

$$E_2^{s,t} = \underline{\pi}_{t-s}(P_t^t X) \Rightarrow \underline{\pi}_{t-s}(X)$$

(more generally, it computes the $RO(G)$-graded homotopy groups $\underline{\pi}_\bigstar(X)$). The associated filtration on the homotopy groups of the spectrum X is

$$F^s \underline{\pi}_t(X) = \mathrm{Ker}(\underline{\pi}_t(X) \to \underline{\pi}_t(P^{t+s-1} X)).$$

In [15], Ullman proved following

Proposition 4.5 (Corollary 8.6, [15]). *If $n > 0$, then for a $(n+1)$-coconnective spectrum X,*

$$F^s \underline{\pi}_n(X) \cong \mathcal{F}^{(s+n-1)/n} \underline{\pi}_n X.$$

The filtration $\mathcal{F}^k\underline{M}$ for a Mackey functor $\underline{M}$ is given by:

$$\mathcal{F}^k\underline{M}(G/H) = \{x \in \underline{M}(G/H) : i^*_{|J|}x = 0, \text{ for all } J \subset H, |J| \leq k\}.$$

Where $i^*_a : \mathsf{Mack_G} \to \mathsf{Mack_G}$ such that

$$i^*_a\underline{M}(G/H) = \begin{cases} 0 & \text{if } |H| > a, \\ \underline{M}(G/H) & \text{otherwise,} \end{cases}$$

with restrictions and transfers induced from $\underline{M}$.

For the rest of this section, $G = C_p$, for p odd prime order. Let W be a (non-virtual) representation C_p of the form $m + n\xi$ with $m, n \geq 0$. To describe the $\dim(W)$-slice of $S^W \wedge H\underline{\mathbb{Z}}$, we begin with the n-coconnective case:

Lemma 4.6. *For a C_p-representation $W = m + n\xi$, the C_p-spectrum $S^W \wedge H\underline{\mathbb{Z}}$ is slice* $\dim(W)$*-coconnective if and only if*

$$m \leq \frac{2n + 3p}{p - 1}.$$

Proof. By Lemma 4.3, we need to find conditions on m, n such that

$$[S^{k\rho}, S^W \wedge H\underline{\mathbb{Z}}]^{C_p} = \tilde{H}^{W-k\rho}_{C_p}(S^0; \underline{\mathbb{Z}}) = 0$$

for all $k \geq 0$ with $kp > m + 2n$. These cohomology groups are computed in [5, Corollary B.10]. For completeness, recall that

$$\underline{H}^{\alpha}_{C_p}(S^0; \underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } |\alpha| = 0, |\alpha^{C_p}| \leq 0 \\ \underline{\mathbb{Z}}^* & \text{if } |\alpha| = 0, |\alpha^{C_p}| > 0 \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| > 0, |\alpha^{C_p}| \leq 0 \text{ even} \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| < 0, |\alpha^{C_p}| \geq 3 \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Let $\alpha = W - k\rho$. Then $(|\alpha|, |\alpha|^{C_p}) = (m + 2n - kp, m - k)$. Therefore, since $m + 2n - kp < 0$, then by (3), the group $\tilde{H}^{W-k\rho}_{C_p}(S^0; \underline{\mathbb{Z}})$ is non-zero if and only if $m - k \geq 3$ odd, equivalently (by transforming the inequalities),

$$\frac{m + 2n}{p} < k \leq m - 3.$$

So there exists k with $0 \leq k$ and $kp > m + 2n$ such that $\tilde{H}^{W-k\rho}_{C_p}(S^0; \underline{\mathbb{Z}})$ is non-trivial if and only if $m - 3 > \frac{m+2n}{p}$. Therefore, $S^W \wedge H\underline{\mathbb{Z}} \leq \dim(W)$ if and only if $m \leq \frac{2n+3p}{p-1}$. The result follows. □

Lemma 4.7. *Let $W = m + n\xi$ be a representation of C_p. Then the spectrum $S^W \wedge H\underline{\mathbb{Z}} \geq \dim(W)$ if and only if*

$$\frac{2n}{p - 1} \leq m.$$

Proof. Write $\dim(W) = d$, the C_p-spectrum $S^W \wedge H\underline{\mathbb{Z}} \geq d$ if and only if $\pi_k \Phi^{C_p}(S^W \wedge H\underline{\mathbb{Z}}) = 0$ for all $k \leq d/p$, as the statement of Theorem 4.1 for $H = e$ is always true for $X = S^W \wedge H\underline{\mathbb{Z}}$. Since $\Phi^{C_p}(\Sigma^W \wedge H\underline{\mathbb{Z}}) \cong S^m \wedge H\mathbb{Z}/p$, $\pi_k \Phi^{C_p}(\Sigma^W \wedge H\underline{\mathbb{Z}}) \neq 0$ if and only if $k = m$. Therefore, the statement is true if and only if $m \geq d/p = (m+2n)/p$. □

Lemma 4.6 and Lemma 4.7 together directly imply the following

Proposition 4.8. *For a C_p-representation W, the spectrum $S^W \wedge H\underline{\mathbb{Z}}$ is a* $\dim(W)$*-slice if and only if*

$$\frac{\dim(W)}{p} \leq \dim(W^{C_p}) \leq \frac{\dim(W)}{p} + 3.$$

Remark 4.9. For $p = 3$, we have an alternative proof: Recall that the regular representation is $\rho_{C_3} = 1 + \xi$. Therefore, we have $S^{m+n\xi} \wedge H\underline{\mathbb{Z}} \cong S^{n\rho_{C_3}} \wedge \Sigma^{m-n} H\underline{\mathbb{Z}}$. So, the problem reduces to showing that $\Sigma^{m-n} H\underline{\mathbb{Z}}$ is an $(m-n)$-slice. Clearly, $\Sigma^{m-n} H\underline{\mathbb{Z}} \geq m-n$. By [5, Corollary B.10], it follows that if $0 \leq m-n \leq 4$, then the spectrum $\Sigma^{m-n} H\underline{\mathbb{Z}} \leq m-n$.

For $n > 0$, recall [20, §3.1] d_n to be the number of the integers of the same parity as n that lie between $\frac{n}{p}$ and $n-2$. It can be expressed using the following formula:

$$d_n := \frac{1}{2}(n - \frac{(n-n_0)}{p} - \delta) \tag{4}$$

where n_0 is the residue of n modulo p and

$$\delta = \begin{cases} 2 & \text{if } n_0 \text{ is even} \\ 1 & \text{if } n_0 \text{ is odd} \\ 0 & \text{if } n_0 = 0. \end{cases} \tag{5}$$

Corollary 4.10. *For a C_p-representation W, if the C_p-spectrum $S^W \wedge H\underline{\mathbb{Z}}$ is slice* $\dim(W)$*-coconnective then the integer*

$$\frac{1}{2}(\dim(W) - \dim(W^{C_p})) - d_{\dim(W)} \geq -1.$$

The equality holds, if and only if $S^W \wedge H\underline{\mathbb{Z}}$ is a $\dim(W)$*-slice.*

Proof. Using (4),

$$\begin{aligned} \frac{1}{2}(\dim(W) - \dim(W^{C_p})) - d_{\dim(W)} &= \frac{1}{2}(\frac{\dim(W)}{p} - \dim(W^{C_p}) - \frac{\dim(W)_0}{p} + \delta) \\ &\geq \frac{1}{2}(-3 - \frac{\dim(W)_0}{p} + \delta) \text{ (by Proposition 4.8)} \\ &\geq -\frac{3}{2} \text{ (by (5))}. \end{aligned}$$

If the equality holds, then

$$\frac{\dim(W)}{p} - \dim(W^{C_p}) = \frac{\dim(W)_0}{p} - 2 - \delta \leq 0 \text{ (by (5))}.$$

Then by Proposition 4.8, the spectrum $S^W \wedge H\underline{\mathbb{Z}}$ is a $\dim(W)$-slice. The result follows. □

For the cyclic group $G = C_{p^k}$, Yarnall ([20, Main Theorem, §4]) computed an explicit formula of the n-slice associated to the C_{p^k}-spectrum $S^n \wedge H\underline{\mathbb{Z}}$ when $n \geq 3$. To summarise her formula for $k = 1$, define two C_p-representations

$$W(n) = (n - 2d_n) + d_n\xi \text{ and } W'(n) = W(n) + (2 - \xi),$$

and then

$$P_n^n(S^n \wedge H\underline{\mathbb{Z}}) = \begin{cases} S^{W(n)} \wedge H\underline{\mathbb{Z}} & \text{if } p \nmid n, n_0 = \text{even} \\ S^{W'(n)} \wedge H\underline{\mathbb{Z}} & \text{if } p \nmid n, n_0 = \text{odd} \\ S^{W'(n)} \wedge H\underline{\mathbb{Z}} & \text{if } p \mid n. \end{cases} \tag{6}$$

Let W be a C_p-representation. The slice formula for the non-trivial $RO(C_p)$-suspensions $S^W \wedge H\underline{\mathbb{Z}}$ of $H\underline{\mathbb{Z}}$ can be derived from the following result.

Theorem 4.11 (Theorem C, [12]). *Let p be odd. For the C_p-spectrum $X = S^W \wedge H\underline{\mathbb{Z}}$, we have equivalences for any $a \in \mathbb{Z}$*

$$P_{ap}^{ap}(X) \simeq S^{a\rho} \wedge H(\underline{H}_{C_p}^{W-a\rho}(S^0; \underline{\mathbb{Z}})).$$

$$P_{ap+2k+1}^{ap+2k+1}(X) \simeq S^{a\rho+k\xi+1} \wedge H(\mathcal{P}^0\underline{H}_{C_p}^{W-a\rho-k\xi-1}(S^0; \underline{\mathbb{Z}})),\ 0 \leq k \leq \frac{p-3}{2}.$$

$$P_{ap+2k+2}^{ap+2k+2}(X) \simeq S^{a\rho+(k+1)\xi} \wedge H(EC_p \otimes \underline{H}_{C_p}^{W-a\rho-(k+1)\xi}(S^0; \underline{\mathbb{Z}})),\ 0 \leq k \leq \frac{p-3}{2}.$$

Here, for a G-Mackey functor $\underline{M}$, $EG \otimes \underline{M}$ denote the subMackey functor generated by $\underline{M}(G/e)$. $\mathcal{P}^0$ is the functor that takes a Mackey functor to the largest quotient in which the restriction maps are injections.

As a consequences of the above discussion one may easily derive the formula for $\dim(W)$-slice for the spectrum $S^W \wedge H\underline{\mathbb{Z}}$ as follows.

Proposition 4.12. *Let W be a C_p-representation with dimension ω. Then*

(1) If the spectrum $S^W \wedge H\underline{\mathbb{Z}}$ is slice ω-coconnective, then either $S^W \wedge H\underline{\mathbb{Z}}$ itself a ω-slice or $S^{W(\omega)} \wedge H\underline{\mathbb{Z}}$ is the ω-slice of $S^W \wedge H\underline{\mathbb{Z}}$.

(2) If the spectrum $S^W \wedge H\underline{\mathbb{Z}}$ is slice ω-connective, then either $S^W \wedge H\underline{\mathbb{Z}}$ itself a ω-slice or

$$P_\omega^\omega(S^W \wedge H\underline{\mathbb{Z}}) = \begin{cases} S^{W'(\omega)} \wedge H\underline{\mathbb{Z}} & \text{if } p \mid \omega \\ S^{W'(\omega)} \wedge H\underline{\mathbb{Z}} & \text{if } p \nmid \omega \text{ and } \omega_0 = \text{odd} \\ S^{W(\omega)} \wedge H\underline{\mathbb{Z}} & \text{if } p \nmid \omega \text{ and } \omega_0 = \text{even}. \end{cases}$$

Proof. Here we prove part (1); the proof of part (2) is similar to (1). Set $X := S^W \wedge H\underline{\mathbb{Z}}$. For (1), first consider $p \mid \omega$, that is, $\omega = ap$ for some integer a. One readily computes

$$W(\omega) = a + (\frac{a(p-1)}{2})\xi.$$

Since we consider X as slice ω-coconnective, thus by Lemma 4.10, one derives either $\dim(W^{C_p}) \le a$ or X itself is a ω-slice. Then by Theorem 4.11, the ω-slice of X,

$P^\omega_\omega(X) \simeq S^{a\rho} \wedge H(\underline{H}^{W-a\rho}_{C_p}(S^0;\underline{\mathbb{Z}})) \cong S^{a\rho} \wedge H\underline{\mathbb{Z}} \cong S^{W(\omega)} \wedge H\underline{\mathbb{Z}}$ (by Proposition 2.4).

If ω of the form $ap+2k+1$ where $0 \le k \le \frac{p-3}{2}$, then $W(\omega) = a+1+(\frac{a(p-1)}{2}+k)\xi$. If X is not a ω-slice, then Lemma 4.10 computes $\dim W^{C_p} \le a+1$. Thus by Theorem 4.11,

$$\begin{aligned} P^\omega_\omega(X) &\simeq S^{a\rho+k\xi+1} \wedge H\mathcal{P}^0(\underline{H}^{W-a\rho-k\xi-1}_{C_p}(S^0;\underline{\mathbb{Z}})) \\ &\cong S^{a\rho+k\xi+1} \wedge H\mathcal{P}^0(\underline{\mathbb{Z}}) \\ &= S^{a\rho+k\xi+1} \wedge H\underline{\mathbb{Z}} \\ &\cong S^{W(\omega)} \wedge H\underline{\mathbb{Z}} \text{ (by Proposition 2.4)} \end{aligned}$$

Finally, if ω is of the form $ap+2k+2$ for some $0 \le k \le \frac{p-3}{2}$, then we compute $W(\omega) = a+2+(\frac{a(p-1)}{2}+k)\xi$. The result follows by analogous computations above along with the observation $EC_p \otimes \underline{\mathbb{Z}}^* \cong \underline{\mathbb{Z}}^* \cong \sum^{\xi-2} H\underline{\mathbb{Z}}$. □

Example 4.13. (1) Let us consider the spectrum $X = S^{2+5\xi} \wedge H\underline{\mathbb{Z}}$. Using Proposition 4.8, one observes X is 12-coconnective but not slice 12-connective. Also note that $S^{W(12)} \wedge H\underline{\mathbb{Z}} \cong S^{4+4\xi} \wedge H\underline{\mathbb{Z}}$ is a 12-slice (by Theorem 4.12).
(2) Next, consider $X = S^{8+2\xi} \wedge H\underline{\mathbb{Z}}$, which is slice 12-connective but not slice 12-coconnective. Here we compute $S^{W'(12)} \wedge H\underline{\mathbb{Z}} \cong S^{6+3\xi} \wedge H\underline{\mathbb{Z}}$, and it is the 12-slice of X by Theorem 4.12.

5. C_{pq}-slices for $S^V \wedge H\underline{\mathbb{Z}}$

In this section, we compute the C_{pq}-slices for the spectrum $S^\alpha \wedge H\underline{\mathbb{Z}}$ for each $\alpha \in RO(C_{pq})$. Slices of any C_{pq}-spectrum have a special feature: under a mild condition, the information of the C_p- and C_q-slices of the corresponding restrictions give the slices of C_{pq}-spectrum as follows.

Proposition 5.1. *Let X be a C_{pq}-spectrum such that $\underline{\pi}_\bigstar(X)$ is cohomological. Then X is a k-slice if and only if both $i^*_{C_p}(X)$ and $i^*_{C_q}(X)$ are k-slices.*

Proof. Assume both $i^*_{C_p}(X)$ and $i^*_{C_q}(X)$ are k-slices. Then, by Proposition 3.3 X is a k-slice for the group C_{pq}. The other direction follows from [10, Proposition 4.13]. □

For each $k \geq 0$, the Mackey functor $\underline{\pi}_k S^V \wedge H\underline{\mathbb{Z}}$ is cohomological (see Definition 3.2). Therefore, as a direct consequence of Proposition 5.1, one may extend the detection result (Proposition 4.8) for C_p to C_{pq} case:

Corollary 5.2. *Let $p < q$ be odd primes and V be a C_{pq}-representation of the form $V = a + b\xi + c\xi^p + d\xi^q$. Then the spectrum $S^V \wedge H\underline{\mathbb{Z}}$ is a C_{pq}-$\dim(V)$-slice if and only if*

i) $\frac{2(b+d)}{p-1} \leq a + 2c \leq \frac{2(b+d)+3p}{p-1}$.

ii) $\frac{2(b+c)}{q-1} \leq a + 2d \leq \frac{2(b+c)+3q}{q-1}$.

Proof. If $S^V \wedge H\underline{\mathbb{Z}}$ is a slice, it must be a $\dim(V)$-slice. Using Proposition 5.1, it is enough to show that both the spectra $i^*_{C_p}(S^V \wedge H\underline{\mathbb{Z}})$ and $i^*_{C_q}(S^V \wedge H\underline{\mathbb{Z}})$ are $\dim(V)$-slices. Hence, the result follows from Proposition 4.8. □

Remark 5.3. If the spectrum $S^V \wedge H\underline{\mathbb{Z}}$ is not an n-slice, then we will construct certain (co)fiber sequences to study the slices of this spectrum. For $\ell \geq 1$, repeated applications of the map $u_{(\xi-\xi^p)}$ in Lemma 3.5 yields a map $H\underline{\mathbb{Z}} \to S^{\ell(\xi-\xi^p)} \wedge H\underline{\mathbb{Z}}$. Then smashing with S^V yields a map $S^V \wedge H\underline{\mathbb{Z}} \to S^{V+\ell(\xi-\xi^p)} \wedge H\underline{\mathbb{Z}}$. We denote it by $u^V_{\ell(\xi-\xi^p)}$ and its cofiber by $C(V,\ell,p)$. For the prime q, one has the similar construction. This cofiber spectrum plays an important role in analyzing slices of the spectrum $S^V \wedge H\underline{\mathbb{Z}}$. To understand the slices of $C(V,\ell,p)$, we begin with the following result.

Lemma 5.4. *The spectrum $\Sigma^n H\mathcal{K}_p\langle \mathbb{Z}/p\rangle$ is a pn-slice and $\Sigma^n H(\mathcal{K}_p\langle \mathbb{Z}/p\rangle \oplus \mathcal{K}_q\langle \mathbb{Z}/q\rangle)$ is a npq-slice.*

Proof. As $\Sigma^n H\mathcal{K}_p\langle \mathbb{Z}/p\rangle$ has homotopy groups concentrated only in degree n, it is $(n+1)$-coconnective. So, by Proposition 4.5 $F^{s+1}\underline{\pi}_n \Sigma^n H\mathcal{K}_p\langle \mathbb{Z}/p\rangle \neq F^s\underline{\pi}_n \Sigma^n H\mathcal{K}_p\langle \mathbb{Z}/p\rangle$ if and only if $\mathcal{F}^{(s+n)/n}\mathcal{K}_p\langle \mathbb{Z}/p\rangle \neq \mathcal{F}^{(s+n-1)/n}\mathcal{K}_p\langle \mathbb{Z}/p\rangle$. This can only happen when

$$(s+n-1)/n < p \text{ and } (s+n)/n \geq p.$$

This gives $s = n(p-1)$ and so,

$$F^{s+1}\underline{\pi}_n \Sigma^n H\mathcal{K}_p\langle \mathbb{Z}/p\rangle = 0 \text{ and } F^s\underline{\pi}_n \Sigma^n H\mathcal{K}_p\langle \mathbb{Z}/p\rangle \cong \mathcal{K}_p\langle \mathbb{Z}/p\rangle.$$

The quotient $F^s\underline{\pi}_{t-s}(\Sigma^n H\mathcal{K}_p\langle \mathbb{Z}/p\rangle)/F^{s+1}\underline{\pi}_{t-s}(\Sigma^n H\mathcal{K}_p\langle \mathbb{Z}/p\rangle)$ can only be non-zero when $t-s = n$ and hence, $t = np$. Therefore, $\Sigma^n H\mathcal{K}_p\langle \mathbb{Z}/p\rangle$ is an np-slice.

The result for $\mathcal{K}_p\langle \mathbb{Z}/p\rangle \oplus \mathcal{K}_q\langle \mathbb{Z}/q\rangle$ can be proved analogously. □

Proposition 5.5. *For $\ell \geq 1$, the cofiber spectrum $C(V,\ell,p)$ has only kp-slices for each $k \in \{\dim(V^{C_p}) - 2\ell, \cdots, \dim(V^{C_p}) - 2\}$.*

Proof. First, we compute the Mackey functor valued homotopy groups of the cofiber $C(V,\ell,p)$. Note that the restriction of the cofiber to the subgroup C_q is trivial and the homotopy groups of $C(V,\ell,p)$ are cohomological. Using Proposition 3.3, we conclude that $\underline{\pi}_k C(V,\ell,p)$ is non-zero if and only if $\underline{\pi}_k i^*_{C_p} C(V,\ell,p)$ is non-zero. Set $i^*_{C_p}(V) = m + n\xi$ for $m, n \geq 0$. The cofiber sequence

$$S^{m+n\xi} \wedge H\underline{\mathbb{Z}} \to S^{m+n\xi+\ell(\xi-2)} \wedge H\underline{\mathbb{Z}} \to i^*_{C_p} C(V,\ell,p)$$

yields the long exact sequence:

$$\cdots \underline{H}^{m-k+n\xi}_{C_p}(S^0) \to \underline{H}^{m-k-2\ell+(n+\ell)\xi}_{C_p}(S^0) \to \underline{\pi}_k i^*_{C_p} C(V,\ell,p) \to \underline{H}^{m+1-k+n\xi}_{C_p}(S^0) \cdots$$

Incorporating the computation (3) in the above long exact sequence, one yields

$$\underline{\pi}_k(C(V,\ell,p)) \cong \begin{cases} \mathcal{K}_p\langle \mathbb{Z}/p\rangle & \text{if } k \in \{m-2\ell, \cdots, m-2\} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the map $u^V_{\xi-\xi^p}$ induces a cofiber sequence as follows

$$C(V,\ell-1,p) \xrightarrow{u^V_{\xi-\xi^p}} C(V,\ell,p) \to \Sigma^{m-2l} H\mathcal{K}_p\langle \mathbb{Z}/p\rangle \tag{7}$$

with $C(V,1,p) \cong \Sigma^{m-2} H\mathcal{K}_p\langle \mathbb{Z}/p\rangle$.

To compute the slices of $C(V,\ell,p)$, we use induction on ℓ. For $\ell = 1$, it is clear from Lemma 5.4 that $C(V,1,p)$ has only one $(m-2)p$-slice. Using (7) it is clear that $C(V,\ell,p)$ has an $(m-2)p$-slice as $\Sigma^{m-2} H\mathcal{K}_p\langle \mathbb{Z}/p\rangle$. The other slices of $C(V,\ell,p)$ are obtained from the induction hypothesis and [19, Proposition 2.32]. Hence, the result follows. □

Remark 5.6. Using Lemma 3.5 (b), construct a map

$$u^V_{\ell\xi} : S^V \wedge H\underline{\mathbb{Z}} \to S^{V+\ell(\xi-2)} \wedge H\underline{\mathbb{Z}}$$

analogous to $u^V_{\ell(\xi-\xi^p)}$. We let $\mathcal{S}(p) = \{\dim(V^{C_p}) - 2\ell, \cdots, \dim(V^{C_p}) - 2\}$. Then direct computation yields

$$\underline{\pi}_k(\text{cofiber}(u_{l\xi})) = \begin{cases} \mathcal{K}_p\langle \mathbb{Z}/p\rangle \oplus \mathcal{K}_q\langle \mathbb{Z}/q\rangle & \text{if } k \in \mathcal{S}(p) \cap \mathcal{S}(q) \\ \mathcal{K}_p\langle \mathbb{Z}/p\rangle & \text{if } k \in \mathcal{S}(p) \cap \mathcal{S}(q)^c \\ \mathcal{K}_q\langle \mathbb{Z}/q\rangle & \text{if } k \in \mathcal{S}(p)^c \cap \mathcal{S}(q) \\ 0, & \text{otherwise.} \end{cases}$$

This allows us to identifies the slices for the cofiber of $u_{l\xi}$ using Lemma 5.4.

Theorem 5.7. *For a real C_{pq}-representation V with dimension ν, the ν-slice of the C_{pq}-spectrum $S^V \wedge H\underline{\mathbb{Z}}$ is spherical. In particular, if $i^*_{C_p}(S^V \wedge H\underline{\mathbb{Z}}) \leq \nu$ and $i^*_{C_q}(S^V \wedge H\underline{\mathbb{Z}}) \leq \nu$, then for $0 \leq k < \nu$, the k-slice of $S^V \wedge H\underline{\mathbb{Z}}$ is either $\Sigma^{k/p} H\mathcal{K}_p\langle \mathbb{Z}/p\rangle$ or $\Sigma^{k/q} H\mathcal{K}_q\langle \mathbb{Z}/q\rangle$ or the wedge of the two.*

Proof. Using Remark 3.6, assume $V = a + b\xi + c\xi^p + d\xi^q$ for some a, b, c and d are non-negative integers. Note that for any C_p-representation W, by Lemma 4.6 and Lemma 4.7, the C_p-spectrum $S^W \wedge H\underline{\mathbb{Z}}$ is either slice d-connective or slice d-coconnective, for $d = \dim(W)$. Therefore, to identify the slices of the C_{pq}-spectrum $S^V \wedge H\underline{\mathbb{Z}}$ it is enough to prove the following four cases:

Case (i). $i^*_{C_p}(S^V \wedge H\underline{\mathbb{Z}}) \leq \nu$ and $i^*_{C_q}(S^V \wedge H\underline{\mathbb{Z}}) \leq \nu$.

We want to construct a spherical ν-slice mapping to $S^V \wedge H\underline{\mathbb{Z}}$ with cofiber $\leq \nu - 1$. By Lemma 4.6, the hypothesis of (i) yields

$$a + 2c \leq \frac{2(b+d) + 3p}{p-1} \text{ and } a + 2d \leq \frac{2(b+c) + 3q}{q-1}.$$

By Proposition 4.12, there exist C_p- and C_q-representations $m + n\xi$ and $m' + n'\xi$, respectively, such that $S^{m+n\xi} \wedge H\underline{\mathbb{Z}}$ is the ν-slice of $i^*_{C_p}(S^V \wedge H\underline{\mathbb{Z}})$ and $S^{m'+n'\xi} \wedge H\underline{\mathbb{Z}}$ is the ν-slice of $i^*_{C_q}(S^V \wedge H\underline{\mathbb{Z}})$. By Lemma 4.10, it follows that $b + d > n$ and $b + c > n'$.

For $V' = V - (b + d - n)(\xi - \xi^p)$, we have

$$i^*_{C_p}(V') = m + n\xi \text{ and } i^*_{C_q}(V') = i^*_{C_q}(V).$$

By Proposition 5.5, there exists a map

$$u^{V'}_{(b+d-n)(\xi-\xi^p)} : S^{V'} \wedge H\underline{\mathbb{Z}} \to S^V \wedge H\underline{\mathbb{Z}},$$

and the cofiber $C(V', b + d - n, p)$ has slices only in filtrations $p(m - 2(b + d - n)), \cdots, p(m - 2)$.

By Proposition 4.12,

$$\begin{aligned} m = \dim(V'^{C_p}) = \dim(V) - 2d_{\dim(V)} &= \frac{\dim(V) - \dim(V)_0}{p} + \delta \\ &\leq \frac{\dim(V) - 1}{p} + 1 \text{ (by (5))}. \end{aligned}$$

It readily follows $p(m-2) < \nu$. Therefore, the cofiber $C(V', b+d-n, p)$ is slice $(\nu - 1)$-connective.

Since $i^*_{C_q}(S^V \wedge H\underline{\mathbb{Z}}) \leq \nu$, we may define a representation

$$V'' := V - (b + c - n')(\xi - \xi^q),$$

and deduce $i^*_{C_q}(V'') = m' + n'\xi$ and $i^*_{C_p}(V'') = i^*_{C_p}(V)$. By Proposition 5.5, there is a map $u_{(b+c-n')(\xi-\xi^q)} : S^{V''} \wedge H\underline{\mathbb{Z}} \to S^V \wedge H\underline{\mathbb{Z}}$ and as above the associated cofiber $C(V'', b + c - n', q)$ has slices in filtration $< \nu$, so $C(V'', b + c - n', q) \leq \nu - 1$.

Finally, set

$$\widehat{V} := V - (b + d - n)(\xi - \xi^p) - (b + c - n')(\xi - \xi^q).$$

Then, $i^*_{C_p}(S^{\widehat{V}} \wedge H\underline{\mathbb{Z}})$ (respectively, $i^*_{C_q}(S^{\widehat{V}} \wedge H\underline{\mathbb{Z}})$) is the ν-slice for $i^*_{C_p}(S^V \wedge H\underline{\mathbb{Z}})$ (respectively, $i^*_{C_q}(S^V \wedge H\underline{\mathbb{Z}})$). By Proposition 5.1, the C_{pq}-spectrum $S^{\widehat{V}} \wedge H\underline{\mathbb{Z}}$ is thus the ν-slice of $S^V \wedge H\underline{\mathbb{Z}}$.

Case (ii). $i^*_{C_p}(S^V \wedge H\underline{\mathbb{Z}}) \geq \nu$ and $i^*_{C_q}(S^V \wedge H\underline{\mathbb{Z}}) \leq \nu$.

By Lemma 4.6,

$$\ell_p = \lceil \frac{p\dim(V^{C_p}) - \dim(V) - 3p}{2p} \rceil \qquad (8)(p)$$

is the least positive integer such that the C_{pq}-spectrum $S^{V+\ell_p(\xi-\xi^p)} \wedge H\underline{\mathbb{Z}} \leq \nu$. Then the spectrum $i^*_{C_p}(S^{V+\ell_p(\xi-\xi^p)} \wedge H\underline{\mathbb{Z}}) \leq \nu$. For such l, by Proposition 5.5, there exists a map

$$u^V_{\ell_p(\xi-\xi^p)} : S^V \wedge H\underline{\mathbb{Z}} \to S^{V+\ell_p(\xi-\xi^p)} \wedge H\underline{\mathbb{Z}}$$

and the fiber of this map (equivalently, $\Sigma^{-1}C(V, \ell_p, p)$) has slice filtration $\geq \nu+1$. Now, as $i^*_{C_p}(S^{V+\ell_p(\xi-\xi^p)} \wedge H\underline{\mathbb{Z}}) \leq \nu$ and $i^*_{C_q}(S^{V+\ell_p(\xi-\xi^p)} \wedge H\underline{\mathbb{Z}}) \leq \nu$, by case (i), we have the spherical ν-slice and also all the lower slices can be determined. Hence, the result follows.

Case (iii). $i^*_{C_p}(S^V \wedge H\underline{\mathbb{Z}}) \leq \nu$ and $i^*_{C_q}(S^V \wedge H\underline{\mathbb{Z}}) \geq \nu$.

It is analogous to case (ii).

Case (iv). $i^*_{C_p}(S^V \wedge H\underline{\mathbb{Z}}) \geq \nu$ and $i^*_{C_q}(S^V \wedge H\underline{\mathbb{Z}}) \geq \nu$.

Then $\ell_0 := \max\{\ell_p, \ell_q\}$ is the positive integer such that $i^*_{C_p}S^{V+\ell_0(\xi-2)} \wedge H\underline{\mathbb{Z}} \leq \nu$ and $i^*_{C_p}S^{V+\ell_0(\xi-2)} \wedge H\underline{\mathbb{Z}} \leq \nu$. Now we are in case (i). Therefore, by Proposition 4.12, there exists a C_p-representation $m+n\xi$ and a C_q-representation $m'+n'\xi$ such that $S^{m+n\xi} \wedge H\underline{\mathbb{Z}}$ (resp., $S^{m'+n'\xi} \wedge H\underline{\mathbb{Z}}$) is the spherical ν-slice of $i^*_{C_p}(S^{V+\ell_0(\xi-2)} \wedge H\underline{\mathbb{Z}})$ (resp., $i^*_{C_q}(S^{V+\ell_0(\xi-2)} \wedge H\underline{\mathbb{Z}})$).

By hypothesis (iv), Proposition 4.1 yields

$$a + 2c \geq \frac{\nu}{p} \text{ and } a + 2d \geq \frac{\nu}{q},$$

which implies that $b + d + \ell_0 \leq n$ and $b + c + l_0 \leq n'$. So, we set

$$\widehat{V} = V + \ell_0(\xi - 2) + (n - b - d - \ell_0)(\xi - \xi^p) + (n' - b - c - \ell_0)(\xi - \xi^q),$$

as in case (i) we see that the C_{pq}-spectrum $S^{\widehat{V}} \wedge H\underline{\mathbb{Z}}$ is the ν-slice for $S^V \wedge H\underline{\mathbb{Z}}$, which is spherical. □

Corollary 5.8. *Any $\alpha \in RO(C_{pq})$ has $\beta \in RO(C_{pq})$ such that $S^\alpha \wedge H\underline{\mathbb{Z}}$ has a* $\dim(\alpha)$*-slice $S^\beta \wedge H\underline{\mathbb{Z}}$. The other slices of $S^\alpha \wedge H\underline{\mathbb{Z}}$ are suspensions of $H\mathcal{K}_p\langle\mathbb{Z}/p\rangle$ or $H\mathcal{K}_q\langle\mathbb{Z}/q\rangle$ or wedges of the two.*

Proof. We can always find some $k \in \mathbb{Z}$ such that $\alpha + k\rho_{C_{pq}}$ is a non-virtual representation of C_{pq}. Therefore, using Proposition 4.4 it is enough to consider $\alpha = V = a + b\xi + c\xi^p + d\xi^q$ for a, b, c, and d non-negative integers. Proposition 5.5 and Theorem 5.7 together then imply the result. □

Example 5.9. For the cyclic group C_{15} with $p = 3$ and $q = 5$ we shall write a slice tower of the C_{15}-spectrum $S^6 \wedge H\underline{\mathbb{Z}}$. Note that this type of spectrum was studied by Yarnall in [20] for $G = C_{p^k}$, in particular for C_p. In our case, the restrictions satisfy

$$i^*_{C_3}(S^6 \wedge H\underline{\mathbb{Z}}) \geq 6 \text{ and } i^*_{C_5}(S^6 \wedge H\underline{\mathbb{Z}}) \geq 6,$$

so, we are in case (iv) of the Theorem 5.7. Here $\ell_0 = \max\{\ell_3, \ell_5\} = 2$ and $n = 2 = n'$. Therefore, by construction of $\widehat{V}$ in case (iv) of the theorem:

$$\widehat{V} = 6 + 2(\xi - 2) + (2-2)(\xi - \xi^3) + (2-2)(\xi - \xi^5) = 2 + 2\xi.$$

Hence, using Remark 5.6, the slice tower is:

$$\begin{array}{lcr} 45 - slice: & \Sigma^3 H(\mathcal{K}_q\langle \mathbb{Z}/q\rangle \oplus \mathcal{K}_p\langle \mathbb{Z}/p\rangle) \longrightarrow & S^6 \wedge H\underline{\mathbb{Z}} \\ & & \big\downarrow u_\xi \\ 6 - slice: & & S^{4+\xi} \wedge H\underline{\mathbb{Z}}. \end{array}$$

Example 5.10. Consider the C_{15}-spectrum $X = S^{11\xi^5} \wedge H\underline{\mathbb{Z}}$. Note that

$$i^*_{C_3}(X) \cong S^{11\xi} \wedge H\underline{\mathbb{Z}} \text{ and } i^*_{C_5}(X) \cong S^{22} \wedge H\underline{\mathbb{Z}}.$$

Therefore, we are in case (iv) of Theorem 5.7 as $i^*_{C_3}(X) \leq 22$ and $i^*_{C_5}(X) \geq 22$.

By (8)(q), $l_5 = 8$, so $i^*_{C_5} S^{8(\xi - \xi^5)} \wedge X \leq 22$, and $i^*_{C_5}(S^{8\xi + 3\xi^5} \wedge H\underline{\mathbb{Z}}) \leq 22$. Above the 22-slice, all higher dimension slices are obtained by the computations of the fiber of the map $u_{\xi - \xi^q}$. By repeated use of Theorem 5.7 (iv), we obtain the slice tower of $S^{11\xi^5} \wedge H\underline{\mathbb{Z}}$ in filtrations > 22.

Since, the spectrum $S^{8\xi+3\xi^5} \wedge H\underline{\mathbb{Z}}$ satisfies both

$$i^*_{C_3}(S^{8\xi+3\xi^5} \wedge H\underline{\mathbb{Z}}) \leq 22 \text{ and } i^*_{C_5}(S^{8\xi+3\xi^5} \wedge H\underline{\mathbb{Z}}) \leq 22,$$

so we are in case (i) of Theorem 5.7. Now we compute $d_{22} = 7$ for $p = 3$ and $d_{22} = 8$ for $p = 5$. (see Remark 4.9) By Theorem 5.7 we have

$$\widehat{V} := 8\xi + 3\xi^5 - (11 - 7)(\xi - \xi^3) = 4\xi + 4\xi^3 + 3\xi^5$$

with $S^{\widehat{V}} \wedge H\underline{\mathbb{Z}}$ the 22-slice of $S^{11\xi^5}$.

Thus, the slice tower for $S^{11\xi^q} \wedge H\underline{\mathbb{Z}}$ is

$$
\begin{array}{lccc}
\text{95-slice:} & \Sigma^{19} H\mathcal{K}_q\langle \mathbb{Z}/q\rangle & \longrightarrow & S^{11\xi^q} \wedge H\underline{\mathbb{Z}} \\
 & & & \big\downarrow u_{\xi-\xi^q} \\
\text{85-slice:} & \Sigma^{17} H\mathcal{K}_q\langle \mathbb{Z}/q\rangle & \longrightarrow & S^{\xi+10\xi^q} \wedge H\underline{\mathbb{Z}} \\
 & & & \big\downarrow u_{\xi-\xi^q} \\
\text{75-slice:} & \Sigma^{15} H\mathcal{K}_q\langle \mathbb{Z}/q\rangle & \longrightarrow & S^{2\xi+9\xi^q} \wedge H\underline{\mathbb{Z}} \\
 & & & \big\downarrow u_{\xi-\xi^q} \\
\text{65-slice:} & \Sigma^{13} H\mathcal{K}_q\langle \mathbb{Z}/q\rangle & \longrightarrow & S^{3\xi+8\xi^q} \wedge H\underline{\mathbb{Z}} \\
 & & & \big\downarrow u_{\xi-\xi^q} \\
\text{55-slice:} & \Sigma^{11} H\mathcal{K}_q\langle \mathbb{Z}/q\rangle & \longrightarrow & S^{4\xi+7\xi^q} \wedge H\underline{\mathbb{Z}} \\
 & & & \big\downarrow u_{\xi-\xi^q} \\
\text{45-slice:} & \Sigma^{9} H\mathcal{K}_q\langle \mathbb{Z}/q\rangle & \longrightarrow & S^{5\xi+6\xi^q} \wedge H\underline{\mathbb{Z}} \\
 & & & \big\downarrow u_{\xi-\xi^q} \\
\text{35-slice:} & \Sigma^{7} H\mathcal{K}_q\langle \mathbb{Z}/q\rangle & \longrightarrow & S^{6\xi+5\xi^q} \wedge H\underline{\mathbb{Z}} \\
 & & & \big\downarrow u_{\xi-\xi^q} \\
\text{25-slice:} & \Sigma^{5} H\mathcal{K}_q\langle \mathbb{Z}/q\rangle & \longrightarrow & S^{7\xi+4\xi^q} \wedge H\underline{\mathbb{Z}} \\
 & & & \big\downarrow u_{\xi-\xi^q} \\
\text{22-slice:} & S^{4\xi+4\xi^3+3\xi^5} \wedge H\underline{\mathbb{Z}} & \longrightarrow & S^{8\xi+3\xi^q} \wedge H\underline{\mathbb{Z}} \\
 & & & \big\downarrow u_{\xi-\xi^p} \\
\text{18-slice:} & \Sigma^{6} H\mathcal{K}_p\langle \mathbb{Z}/p\rangle & \longrightarrow & \bigvee_{i\in\{0,1,2,3\}} \sum^{2i} HK_p < Z/p > \\
 & & & \big\downarrow u_{\xi-\xi^p} \\
\text{12-slice:} & \Sigma^{4} H\mathcal{K}_p\langle \mathbb{Z}/p\rangle & \longrightarrow & \bigvee_{i\in\{0,1,2\}} \sum^{2i} HK_p < Z/p > \\
 & & & \big\downarrow u_{\xi-\xi^p} \\
\text{6-slice:} & \Sigma^{2} H\mathcal{K}_p\langle \mathbb{Z}/p\rangle & \longrightarrow & H\mathcal{K}_p\langle \mathbb{Z}/p\rangle \vee \Sigma^2 H\mathcal{K}_p\langle \mathbb{Z}/p\rangle \\
 & & & \big\downarrow u_{\xi-\xi^p} \\
\text{0-slice:} & & & H\mathcal{K}_p\langle \mathbb{Z}/p\rangle
\end{array}
$$

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