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Strongly continuous composition semigroups on analytic Morrey spaces

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ABSTRACT. For a semigroup $(\varphi_t)_{t\geq 0}$ consisting of analytic self-maps from the unit disk $\mathbb{D}$ to itself, a strongly continuous composition semi-group $(C_t)_{t\geq 0}$ induced by $(\varphi_t)_{t\geq 0}$ on analytic Morrey spaces $H^{2,\lambda}$, $0 < \lambda < 1$, is investigated. By the weak compactness of resolvent operator, we give a complete characterization of $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$ for $0 < \lambda < 1$ in terms of the infinitesimal generator if the Denjoy-Wolff point of $(\varphi_t)_{t\geq 0}$ is in $\mathbb{D}$.

CONTENTS

1. Introduction

Recall that a family $(\varphi_t)_{t\geq 0}$ of analytic self-maps of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ is said to be a semigroup if:

(i) φ_0 is the identity map I, i.e. $\varphi_0(z) = z, z \in \mathbb{D}$;
(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $t, s \geq 0$;
(iii) for each $z \in \mathbb{D}$, $\varphi_t(z) \to z$ as $t \to 0^+$.

A semigroup $(\varphi_t)_{t\geq 0}$ is said to be trivial if each φ_t is the identity of $\mathbb{D}$. By [12], every non-trivial semigroup $(\varphi_t)_{t\geq 0}$ has a unique common fixed point $b \in \overline{\mathbb{D}}$ with $|\varphi_t'(b)| \leq 1$ for all $t \geq 0$, called the Denjoy-Wolff point (DW point) of $(\varphi_t)_{t\geq 0}$. The infinitesimal generator of $(\varphi_t)_{t\geq 0}$ is the function

$$G(z) = \lim_{t\to 0^+} \frac{\varphi_t(z) - z}{t} = \frac{\partial \varphi_t(z)}{\partial t}\Big|_{t=0}, \quad z \in \mathbb{D}.$$

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This convergence holds uniformly on compact subsets of $\mathbb{D}$, so $G \in \mathcal{H}(\mathbb{D})$, the set of all analytic functions on $\mathbb{D}$. Moreover, G has a unique representation

$$G(z) = (\overline{b}z - 1)(z - b)P(z), \qquad z \in \mathbb{D}, \tag{1}$$

where b is the DW point of $(\varphi_t)_{t\geq 0}$ and $P \in \mathcal{H}(\mathbb{D})$ with $\mathrm{Re}(P(z)) \geq 0$ for $z \in \mathbb{D}$. For every non-trivial semigroup $(\varphi_t)_{t\geq 0}$ with the infinitesimal generator G, there exists a unique univalent function h, the Koenigs function of $(\varphi_t)_{t\geq 0}$ on $\mathbb{D}$, corresponding to $(\varphi_t)_{t\geq 0}$. If the DW point $b \in \mathbb{D}$, then $h(b) = 0$, $h'(b) = 1$ and

$$h(\varphi_t(z)) = e^{G'(b)t}h(z), \qquad z \in \mathbb{D},\ t \geq 0.$$

If the DW point $b \in \partial\mathbb{D} = \{z : |z| = 1\}$, then $h(0) = 0$ and

$$h(\varphi_t(z)) = h(z) + it, \qquad z \in \mathbb{D},\ t \geq 0.$$

Without loss of generality, the DW point $b \in \mathbb{D}$ or $b \in \partial\mathbb{D}$ can be written as $b = 0$ or $b = 1$. See [5] and [12] for more results about the composition semigroups.

For a given semigroup $(\varphi_t)_{t\geq 0}$ and a Banach space X consisting of analytic functions on $\mathbb{D}$, we say that $(\varphi_t)_{t\geq 0}$ generates a strongly continuous composition semigroup $(C_t)_{t\geq 0}$ on X if C_t is bounded on X for $t \geq 0$ and

$$\lim_{t\to 0^+} \|C_t(f) - f\|_X = 0 \quad \text{for all } f \in X,$$

where $C_t(f) = f\circ\varphi_t$ for $f \in \mathcal{H}(\mathbb{D})$. Here C_0 is the identity operator and $C_{t+s} = C_t \circ C_s$ for $t, s \geq 0$. Denote by $[\varphi_t, X]$ the maximal subspace of X on which $(\varphi_t)_{t\geq 0}$ generates a strongly continuous composition semigroup $(C_t)_{t\geq 0}$. Note that $[\varphi_t, X] \subset X$ is obvious. By [2, 10, 11], we know that every semigroup $(\varphi_t)_{t\geq 0}$ generates a strongly continuous composition semigroup $(C_t)_{t\geq 0}$ on the Hardy space $H^p, 1 \leq p < \infty$, the Bergman space $A^p, 1 \leq p < \infty$, and the Dirichlet space $\mathcal{D}$, respectively. In our notation, $[\varphi_t, H^p] = H^p, [\varphi_t, A^p] = A^p$ for $1 \leq p < \infty$ and $[\varphi_t, \mathcal{D}] = \mathcal{D}$. However, not all analytic function spaces admit the property that the corresponding composition semigroups are strongly continuous on them. For this situation, we choose $X = H^\infty$, the Bloch space $\mathcal{B}$, the spaces $\mathcal{Q}_p$ and $\mathcal{Q}_K$, for examples. See [3, 9, 15] for the details.

The authors of [6] considered the same problems for the analytic Morrey spaces $H^{2,\lambda}$, $0 \leq \lambda \leq 1$. Let H^2 be the Hardy space of all analytic functions f on $\mathbb{D}$ for which

$$\sup_{0\leq r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{\mathrm{d}\theta}{2\pi} < \infty.$$

Note that for $f \in H^2$, the function $f(z)$ converges nontangentially to an L^2 function $f(t)$ almost everywhere on $\partial\mathbb{D}$. For $0 \leq \lambda \leq 1$, the analytic Morrey space $H^{2,\lambda}$ consisting of those functions $f \in H^2$ such that

$$\|f\|_{H^{2,\lambda}} := \sup_{I\subset\partial\mathbb{D}} \left(\frac{1}{|I|^\lambda} \int_I |f(t) - f_I|^2 \frac{|\mathrm{d}t|}{2\pi} \right)^{1/2} < \infty,$$

where f_I denotes the average of f over the arc $I \subset \partial\mathbb{D}$ and $|I|$ denotes the arc length of $I \subset \partial\mathbb{D}$. It is clear that for $\lambda = 0$ or $\lambda = 1$, $H^{2,\lambda}$ reduces to H^2 or $BMOA$, the set of analytic functions in $\mathbb{D}$ with boundary values of bounded mean oscillation. It is known (cf.[14]), that $\|f\|^2_{H^{2,\lambda}}$ is equivalent to

$$\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^\lambda}\int_{S(I)} |f'(z)|^2(1-|z|^2)\mathrm{d}m(z), \tag{2}$$

where $S(I)$ is the Carleson box and $\mathrm{d}m(z)$ is the normalized Lebesgue area measure on $\mathbb{D}$.

It was shown in [6] that for every non-trivial semigroup $(\varphi_t)_{t\geq 0}$,

$$BMOA \subsetneqq H_0^{2,\lambda} \subset [\varphi_t, H^{2,\lambda}] \subsetneqq H^{2,\lambda}, \quad 0<\lambda<1. \tag{3}$$

Here, $H_0^{2,\lambda}$ is the closure of all polynomials in $H^{2,\lambda}$. [6, Theorem 3.1], the analogue of Sarason's characterization of a function in $VMOA$, showed that $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$ for $\varphi_t(z) = e^{-t}z$ with the DW point $b = 0$. However, by choosing

$$\varphi_t(z) = \frac{(e^{-t}((\frac{1+z}{1-z})^{\frac{1-\lambda}{2}}-1)+1)^{\frac{2}{1-\lambda}}-1}{(e^{-t}((\frac{1+z}{1-z})^{\frac{1-\lambda}{2}}-1)+1)^{\frac{2}{1-\lambda}}+1}, \quad 0<\lambda<1,$$

with the DW point $b = 0$, we find that the function

$$f_\lambda(z) = (\frac{1+z}{1-z})^{\frac{1-\lambda}{2}} - 1 \in H^{2,\lambda}\backslash H_0^{2,\lambda}, \quad 0<\lambda<1.$$

Since

$$\|f_\lambda\circ\varphi_t - f_\lambda\|_{H^{2,\lambda}} = (1-e^{-t})\|f_\lambda\|_{H^{2,\lambda}} \to 0$$

as $t\to 0$, $f_\lambda \in [\varphi_t, H^{2,\lambda}]$. It means that $H_0^{2,\lambda} \neq [\varphi_t, H^{2,\lambda}]$ holds for the semigroup $(\varphi_t)_{t\geq 0}$. In addition, we are able to find a semigroup $(\varphi_t)_{t\geq 0} = (e^{-t}z+1-e^{-t})_{t\geq 0}$ with the DW point $b = 1$, for example, such that $H_0^{2,\lambda} \neq [\varphi_t, H^{2,\lambda}]$.

A natural problem is to characterize the semigroup $(\varphi_t)_{t\geq 0}$ such that $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$ holds. The authors of [6] obtained a sufficient condition for $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$ in terms of the infinitesimal generator of $(\varphi_t)_{t\geq 0}$ as follows.

Theorem A ([6]). *Let $(\varphi_t)_{t\geq 0}$ be a semigroup of analytic self-maps of $\mathbb{D}$ with the infinitesimal generator G and $0<\lambda<1$. If*

$$\lim_{|I|\to 0} \frac{1}{|I|}\int_{S(I)} \frac{1-|z|}{|G(z)|^2}\mathrm{d}m(z) = 0, \tag{4}$$

then $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$.

They also gave a neccessary condition on the infinitesimal generator of a semigroup with the DW point $b\in\mathbb{D}$ such that $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$.

Theorem B ([6]). *Let $(\varphi_t)_{t\geq 0}$ be a semigroup of analytic self-maps of $\mathbb{D}$ with the DW point $b \in \mathbb{D}$ and the infinitesimal generator G. If for some $\lambda \in (0,1)$ we have $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$, then*

$$\lim_{|z|\to 1} \frac{(1-|z|)^{\frac{3-\lambda}{2}}}{G(z)} = 0.$$

The following result, Theorem1.1, is our main result in this paper which gives a sufficient and necessary condition for $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$ in terms of the weakly compactness of the resolvent operator when the semigroup $(\varphi_t)_{t\geq 0}$ has a DW point in $\mathbb{D}$. Moreover, this shows that when (5) holds, condition (4) in Theorem A is also necessary for $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$.

Theorem 1.1. *Suppose $0 < \lambda < 1$ and $(\varphi_t)_{t\geq 0}$ is a non-trivial semigroup of analytic self-maps of $\mathbb{D}$ with the DW point $b = 0$ and the infinitesimal generator G. Denote by Γ the infinitesimal generator of the corresponding composition semigroup $(S_t)_{t\geq 0}$ on $H_0^{2,\lambda}$ and denote by $R(\sigma,\Gamma) = (\sigma - \Gamma)^{-1}$ the resolvent operator for $\sigma \in \rho(\Gamma)$, the resolvent set of Γ. Then $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$ if and only if the resolvent operator $R(\sigma,\Gamma)$ is weakly compact on $H_0^{2,\lambda}$. Moreover, if*

$$\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|} \int_{S(I)} \frac{1-|z|}{|G(z)|^2} \mathrm{d}m(z) < \infty, \tag{5}$$

then $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$ if and only if

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_{S(I)} \frac{1-|z|}{|G(z)|^2} \mathrm{d}m(z) = 0. \tag{6}$$

Throughout the paper, the symbol $A \approx B$ means that $A \lesssim B \lesssim A$. We say that $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$.

2. Lemmas

For $g \in \mathcal{H}(\mathbb{D})$, the Volterra type operator V_g on $H^{2,\lambda}$ is defined by

$$V_g(f)(z) = \int_0^z f(\xi)g'(\xi)\mathrm{d}\xi, \quad f \in H^{2,\lambda}.$$

The following Lemma 2.1 and Lemma 2.2 are extensions of the related results in [3].

Lemma 2.1. *Let $0 < \lambda < 1$ and $g \in \mathcal{H}(\mathbb{D})$. Then the following are equivalent:*

(i) *V_g is bounded on $H^{2,\lambda}$.*

(ii) *V_g is bounded on $H_0^{2,\lambda}$.*

Proof. (i) ⇒ (ii). Suppose V_g is bounded on $H^{2,\lambda}$. By [8], $g \in H_0^{2,\lambda}$ since $BMOA \subset H_0^{2,\lambda}$ for $0 < \lambda < 1$. A simple computation shows that

$$V_g(z^n) = \int_0^z \xi^n g'(\xi)\mathrm{d}\xi$$

belong to $H_0^{2,\lambda}$ for all integers $n \geq 1$, and then $V_g(P) \in H_0^{2,\lambda}$ for all polynomials P. Thus, for $f \in H_0^{2,\lambda}$, $V_g(f)$ can be approximated by $H_0^{2,\lambda}$ functions since $H_0^{2,\lambda}$ is the closure of all polynomials in $H^{2,\lambda}$. Bearing in mind that $H_0^{2,\lambda}$ is closed and the assertion follows.

(ii) ⇒ (i). Suppose V_g is bounded on $H_0^{2,\lambda}$. From [13], we know that the second dual of $H_0^{2,\lambda}$ is isomorphic to $H^{2,\lambda}$ under the pairing:

$$\langle f, h\rangle = \frac{1}{2\pi}\int_{\partial\mathbb{D}} f(\zeta)\overline{h(\zeta)}|\mathrm{d}\zeta| \tag{7}$$

for $f \in H_0^{2,\lambda}$ and $h \in (H_0^{2,\lambda})^*$. Let V_g^* be the adjoint of V_g acting on the dual space $(H_0^{2,\lambda})^*$ under (2.1), and let V_g^{**} be the adjoint of V_g^* acting on $H^{2,\lambda}$. Thus, by the definition of the adjoint operator,

$$\langle V_g(f), h\rangle = \langle f, V_g^*(h)\rangle = \overline{\langle V_g^*(h), f\rangle} = \overline{\langle h, V_g^{**}(f)\rangle} = \langle V_g^{**}(f), h\rangle$$

hold for all $f \in H_0^{2,\lambda}$ and $h \in (H_0^{2,\lambda})^*$. Owing to $H_0^{2,\lambda}$ is weak* dense in $H^{2,\lambda}$, we say that $V_g^{**} = V_g$ as operators on $H^{2,\lambda}$. Hence, V_g is bounded on $H^{2,\lambda}$. □

Lemma 2.2. *Suppose $0 < \lambda < 1$ and $g \in \mathcal{H}(\mathbb{D})$. If V_g is bounded on $H^{2,\lambda}$, then the following statements are equivalent.*

(i) *V_g is weakly compact on $H^{2,\lambda}$.*
(ii) *V_g is weakly compact on $H_0^{2,\lambda}$.*
(iii) *V_g is compact on $H^{2,\lambda}$.*
(iv) *V_g is compact on $H_0^{2,\lambda}$.*
(v) *$V_g(H^{2,\lambda}) \subset H_0^{2,\lambda}$.*

Proof. By the proof of Lemma 2.1, we conclude that $V_g^{**} = V_g$. According to [4], the equivalence of (i), (ii) and (v) can be easily obtained.

Next, we show that (iii) and (iv) are equivalent. Because $H_0^{2,\lambda}$ is a subspace of $H^{2,\lambda}$ and they share the same norm, (iii) implies (iv). Conversely, let V_g be compact on $H_0^{2,\lambda}$. Using $V_g^{**} = V_g$ again, and together with [4, Theorem VI.5.2], we get that (iv) and (iii) are equivalent.

Finally, we verify that (i) and (iii) are equivalent. (iii) ⇒ (i) is obvious. To finish the proof, for a given subarc $I \subset \partial\mathbb{D}$, we consider the functions

$$f_w(z) = \frac{1}{(1-\overline{w}z)^{\frac{1-\lambda}{2}}}, \quad z \in \mathbb{D},$$

where $w = (1 - |I|)\zeta$ and ζ is the center of I. Note that $f_w \in H^{2,\lambda}$ and

$$\sup_{w \in \mathbb{D}} \|f_w\|_{H^{2,\lambda}} < \infty.$$

If (i) is true, then the equivalence of (i) and (v) gives that $V_g(H^{2,\lambda}) \subset H_0^{2,\lambda}$. It follows that

$$V_g(f_w)(z) = \int_0^z f_w(\xi) g'(\xi) \mathrm{d}\xi, \qquad w \in \mathbb{D},$$

belong to $H_0^{2,\lambda}$. Similar to (2), we have

$$\lim_{|I| \to 0} \frac{1}{|I|^\lambda} \int_{S(I)} |f_w(z)|^2 |g'(z)|^2 (1 - |z|^2) \mathrm{d}m(z) = 0.$$

Hence,

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} |g'(z)|^2 (1 - |z|^2) \mathrm{d}m(z) = 0,$$

which means that $g \in VMOA$ by [7]. Combining this with [8] implies that V_g is compact on $H^{2,\lambda}$. □

Suppose now that $(\varphi_t)_{t \geq 0}$ is a semigroup of self-maps of $\mathbb{D}$ and $(C_t)_{t \geq 0}$ is the corresponding composition semigroup on $H^{2,\lambda}$. Since each φ_t is univalent, we know that C_t is bounded on $H^{2,\lambda}$ ([16, Corollary 1]), and $\sup_{t \in [0,1]} \|C_t\| < \infty$. If $f \in H_0^{2,\lambda}$ and $\epsilon > 0$, then there exists a polynomial P such that $\|f - P\|_{H^{2,\lambda}} < \epsilon$ ([13, Lemma 2.8]). Hence,

$$\|C_t(f) - C_t(P)\|_{H^{2,\lambda}} < \epsilon \Big(\frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \Big)^{\frac{1-\lambda}{2}}.$$

Since $C_t(P) \in H_0^{2,\lambda}$, it follows that $C_t(f) \in H_0^{2,\lambda}$. Therefore $C_t : H_0^{2,\lambda} \to H_0^{2,\lambda}$ exists as a bounded operator with $\|C_t\| \leq \big(\frac{1+|\varphi_t(0)|}{1-|\varphi_t(0)|} \big)^{\frac{1-\lambda}{2}}$. Thus, we can define the composition operator $S_t = C_t|_{H_0^{2,\lambda}}$ on $H_0^{2,\lambda}$. It is clear that $(S_t)_{t \geq 0}$ is strongly continuous on $H_0^{2,\lambda}$, $0 < \lambda < 1$, by [1, Corollary 1.3].

Lemma 2.3. *Let $(\varphi_t)_{t \geq 0}$ be a semigroup of self-maps of $\mathbb{D}$, $(C_t)_{t \geq 0}$ be the corresponding composition semigroup on $H^{2,\lambda}$, and $S_t = C_t|_{H_0^{2,\lambda}}$ for $0 < \lambda < 1$. Then $S_t^{**} = C_t$ for all $t \geq 0$, where S_t^{**} means the second adjoint operator of S_t under the pairing (7).*

Proof. For $f \in H_0^{2,\lambda}$ and $h \in (H_0^{2,\lambda})^*$, we have

$$\langle S_t(f), h \rangle = \langle f, S_t^*(h) \rangle = \overline{\langle S_t^*(h), f \rangle} = \overline{\langle h, S_t^{**}(f) \rangle} = \langle S_t^{**}(f), h \rangle,$$

which gives

$$S_t^{**}(f) = S_t(f) \quad \text{for all } f \in H_0^{2,\lambda}.$$

Therefore,

$$C_t|_{H_0^{2,\lambda}} = S_t = S_t^{**}|_{H_0^{2,\lambda}}.$$

Since $H_0^{2,\lambda}$ is weak* dense in $H^{2,\lambda}$, the conclusion follows. □

Lemma C ([5]). *Let $(T_t)_{t\geq 0}$ be a strongly continuous composition semigroup on a Banach space X with the infinitesimal generator A and let ω_0 be the growth bound of $(T_t)_{t\geq 0}$, i.e.*

$$\omega_0 = \lim_{t\to\infty} \frac{\log \|T_t\|}{t}.$$

(i) *If $\delta > \omega_0$, then there is a constant M_δ such that $\|T_t\| \leq M_\delta e^{\delta t},\ t \geq 0$;*
(ii) *If $Re(\sigma) > \omega_0$, then $\sigma \in \rho(A)$ and*

$$R(\sigma, A)(f) = \int_0^\infty e^{-\sigma t} T_t(f)\mathrm{d}t, \qquad f \in X.$$

Lemma 2.4. *Let $(\varphi_t)_{t\geq 0}$ be a non-trivial semigroup of self-maps of $\mathbb{D}$ with the DW point $b = 0$, the infinitesimal generator G and Koenigs function h. Suppose S_t is the corresponding composition semigroup on $H_0^{2,\lambda}$, $0 < \lambda < 1$, with the infinitesimal generator Γ. Then for $\sigma \in \rho(\Gamma)$, the resolvent operator of Γ has the following representation:*

$$R(\sigma, \Gamma)f(z) = -\frac{1}{G'(0)} \frac{1}{(h(z))^{-\frac{\sigma}{G'(0)}}} \int_0^z f(\zeta)(h(\zeta))^{-\frac{\sigma}{G'(0)}-1} h'(\zeta)\mathrm{d}\zeta. \tag{8}$$

In particular, $-G'(0)$ belongs to $\rho(\Gamma)$ and hence

$$R(-G'(0), \Gamma)f(z) = -\frac{1}{G'(0)h(z)} \int_0^z f(\zeta)h'(\zeta)\mathrm{d}\zeta. \tag{9}$$

Proof. Write

$$R := -\frac{1}{G'(0)} \frac{1}{(h(z))^{-\frac{\sigma}{G'(0)}}} \int_0^z f(\zeta)(h(\zeta))^{-\frac{\sigma}{G'(0)}-1} h'(\zeta)\mathrm{d}\zeta.$$

It is easy to check that

$$(\sigma I - \Gamma)R = R(\sigma I - \Gamma) = I,$$

which shows that R is the resolvent operator of Γ and (8) holds. Since each φ_t is univalent, we immediately get that each S_t maps $H_0^{2,\lambda}$ into itself and so

$$\omega_0 := \lim_{t\to\infty} \frac{\log \|S_t\|}{t} = 0.$$

By (1), we have

$$G(z) = -zP(z), \qquad \mathrm{Re}(P(z)) \geq 0,\ z \in \mathbb{D},$$

and

$$\mathrm{Re}(-G'(0)) = \mathrm{Re}(P(0)) \geq 0.$$

If $\mathrm{Re}(-G'(0)) > 0$, by (ii) of Lemma C, $-G'(0) \in \rho(\Gamma)$. If $\mathrm{Re}(-G'(0)) = 0$, write $G(z) = -i\alpha z$, where $\alpha \in \mathbb{R} \setminus \{0\}$. By [3, Theorem 2],

$$\Gamma(f)(z) = G(z)f'(z) = -i\alpha z f'(z).$$

Thus, $(i\alpha I - \Gamma)(f) = g$ has the unique analytic solution

$$f(z) = \frac{1}{i\alpha z}\int_0^z g(\zeta)\mathrm{d}\zeta.$$

It is not difficult to see that the operator

$$g \to \frac{1}{i\alpha z}\int_0^z g(\zeta)\mathrm{d}\zeta$$

is bounded on $H^{2,\lambda}$. Hence, it is bounded on $H_0^{2,\lambda}$. Therefore, $-G'(0) \in \rho(\Gamma)$. Choosing $\sigma = -G'(0)$ in (8), we obtain (9). □

3. The proof of Theorem 1.1

Now we are going to prove Theorem 1.1. Suppose $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$. By (i) of Lemma C, there are two positive constants δ and M_δ such that $\|S_u\| \le M_\delta e^{\delta u}$ for $u \ge 0$. By (ii) of Lemma C, we choose a large enough real number $\sigma > \delta$ such that $\sigma \in \rho(\Gamma)$ and we have

$$R(\sigma, \Gamma)(f) = \int_0^\infty e^{-\sigma u} S_u(f)\mathrm{d}u, \qquad f \in H_0^{2,\lambda}.$$

Thus,

$$S_t \circ R(\sigma, \Gamma)(f) = \int_0^\infty e^{-\sigma u} S_{t+u}(f)\mathrm{d}u = e^{\sigma t}\int_t^\infty e^{-\sigma u} S_u(f)\mathrm{d}u.$$

Accordingly,

$$S_t \circ R(\sigma, \Gamma)(f) - R(\sigma, \Gamma)(f) = (e^{\sigma t} - 1)\int_t^\infty e^{-\sigma u} S_u(f)\mathrm{d}u - \int_0^t e^{-\sigma u} S_u(f)\mathrm{d}u.$$

Therefore,

$$\|S_t \circ R(\sigma, \Gamma)(f) - R(\sigma, \Gamma)(f)\|_{H^{2,\lambda}} \le \Big(|e^{\sigma t} - 1|\int_t^\infty e^{-\sigma u}\|S_u\|\mathrm{d}u + \int_0^t e^{-\sigma u}\|S_u\|\mathrm{d}u\Big)\|f\|_{H^{2,\lambda}}.$$

Thus,

$$\|S_t \circ R(\sigma, \Gamma) - R(\sigma, \Gamma)\| \le M_\delta\Big(|e^{\sigma t} - 1|\int_t^\infty e^{-(\sigma-\delta)u}\mathrm{d}u + \int_0^t e^{-(\sigma-\delta)u}\mathrm{d}u\Big),$$

and so

$$\lim_{t\to 0}\|S_t \circ R(\sigma, \Gamma) - R(\sigma, \Gamma)\| = 0.$$

By Lemma 2.3, $S_t^{**} = C_t$. Recalling that S_t commutes with $R(\sigma, \Gamma)$, we have

$$\lim_{t\to 0}\|C_t \circ R(\sigma, \Gamma)^{**} - R(\sigma, \Gamma)^{**}\| = 0.$$

This implies

$$\lim_{t\to 0}\|C_t\circ R(\sigma,\Gamma)^{**}(f)-R(\sigma,\Gamma)^{**}(f)\|_{H^{2,\lambda}}=0,\qquad f\in H^{2,\lambda},$$

which yeilds that $R(\sigma,\Gamma)^{**}(H^{2,\lambda})\subset[\varphi_t,H^{2,\lambda}]=H_0^{2,\lambda}$. According to [4, Theorem VI.4.2], we know that $R(\sigma,\Gamma)$ is weakly compact on $H_0^{2,\lambda}$ for a large enough real number σ. For a general $\sigma\in\rho(\Gamma)$, using the resolvent equation

$$R(\sigma,\Gamma)-R(\mu,\Gamma)=(\mu-\sigma)R(\sigma,\Gamma)R(\mu,\Gamma),\quad \sigma,\mu\in\rho(\Gamma),$$

we obtain that $R(\sigma,\Gamma)$ is weakly compact for some $\sigma\in\rho(\Gamma)$ if and only if it is weakly compact for every $\sigma\in\rho(\Gamma)$.

Conversely, write $Y=[\varphi_t,H^{2,\lambda}]$ and then $H_0^{2,\lambda}\subset Y\subsetneqq H^{2,\lambda}$ by [6]. By [3, Theorem 2], the restriction of $(C_t)_{t\geq 0}$ on Y is a strongly continuous semigroup with the infinitesimal generator $\Delta(f)=Gf'$. It is clear that the domain of Γ

$$D(\Gamma)=\{f\in H_0^{2,\lambda}\,:\,Gf'\in H_0^{2,\lambda}\}\subset D(\Delta)=\{f\in Y\,:\,Gf'\in Y\},$$

and that Δ is an extension of Γ. Let σ be a large enough real number such that $\sigma\in\rho(\Gamma)\cap\rho(\Delta)$. An argument similar to that in the proof of Lemma 2.3 shows that

$$R(\sigma,\Gamma)^{**}|_{H_0^{2,\lambda}}=R(\sigma,\Gamma),\qquad R(\sigma,\Gamma)^{**}|_Y=R(\sigma,\Delta).$$

On the other hand,

$$\begin{aligned}D(\Delta)&=\{f\in Y\,:\,Gf'\in Y\}\\&=\{f\in Y\,:\,g=Gf'-\sigma f\in Y\}\\&=\{f\in Y\,:\,f=R(\sigma,\Delta)(g),g\in Y\}\\&=R(\sigma,\Delta)(Y).\end{aligned}$$

Thus,

$$D(\Delta)=R(\sigma,\Gamma)^{**}|_Y(Y)\subset R(\sigma,\Gamma)^{**}(H^{2,\lambda})\subset H_0^{2,\lambda}.$$

By [3, Theorem 1], we have

$$Y=[\varphi_t,H^{2,\lambda}]=\overline{D(\Delta)}\subset H_0^{2,\lambda},$$

which means that

$$H_0^{2,\lambda}=[\varphi_t,H^{2,\lambda}].$$

Next, we are going to prove the second part of Theorem 1.1. By Lemma 2.4, we know that $-G'(0)\in\rho(\Gamma)$ and

$$R_h(f):=R(-G'(0),\Gamma)f(z)=-\frac{1}{G'(0)h(z)}\int_0^z f(\zeta)h'(\zeta)\mathrm{d}\zeta.$$

By using the techniques mentioned in [12], the operator R_h and the multiplier operator

$$M_I(f)(z)=I(z)f(z)=zf(z)$$

satisfy the following identities:

$$M_IP_h=-G'(0)R_hM_I,\qquad Q_h=P_h+Q_hP_h,\tag{10}$$

where

$$P_h f(z) = \frac{1}{zh(z)} \int_0^z f(\zeta)\zeta h'(\zeta)\mathrm{d}\zeta$$

and

$$Q_h f(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{\zeta h'(\zeta)}{h(\zeta)} \mathrm{d}\zeta.$$

To finish our proof, by the first part of the theorem, it suffices to show that R_h is weakly compact on $H_0^{2,\lambda}$ if and only if (6) holds. A simple computation shows that

$$Q_h(f)(z) = J(f)(z) + L_h M_I(f)(z),$$

where

$$J(f)(z) = \frac{1}{z} \int_0^z f(\zeta)\mathrm{d}\zeta$$

and

$$L_h f(z) = \frac{1}{z} \int_0^z f(\zeta) \Big(\log \frac{h(\zeta)}{\zeta} \Big)' \mathrm{d}\zeta.$$

Since the DW point $b = 0$, we have

$$h'(z)G(z) = G'(0)h(z), \qquad z \in \mathbb{D}.$$

Thus, (5) gives

$$\sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|} \int_{S(I)} \left| \frac{zh'(z)}{h(z)} \right|^2 (1 - |z|)\mathrm{d}m(z) < \infty,$$

which shows that $\log \frac{h(z)}{z} \in BMOA$. By [8] and Lemma 2.1, L_h is bounded on $H_0^{2,\lambda}$, and so Q_h is bounded on $H_0^{2,\lambda}$. By (10), R_h is bounded on $H_0^{2,\lambda}$ and therefore, P_h is bounded on $H_0^{2,\lambda}$. Meanwhile, (6) is equivalent to

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} \left| \frac{zh'(z)}{h(z)} \right|^2 (1 - |z|^2)\mathrm{d}m(z) = 0,$$

which shows that $\log \frac{h(z)}{z} \in VMOA$. Similarly, we obtain that (6) is equivalent to that R_h is weakly compact on $H_0^{2,\lambda}$ see [4, Theorem VI.4.5]. The proof is complete.

The following corollary is closely related to Theorem B.

Corollary 3.1. *Suppose $0 < \lambda < 1$ and $(\varphi_t)_{t \geq 0}$ is a non-trivial semigroup of analytic self-maps of $\mathbb{D}$ with the DW point in $\mathbb{D}$ and infinitesimal generator G. If condition (5) holds, then $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$ implies that*

$$\lim_{|z| \to 1} \frac{1 - |z|}{G(z)} = 0.$$

Proof. Suppose $H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$. By Theorem 1.1, we have that (6) holds. A standard argument (cf. [7]) gives

$$\lim_{|a|\to 1} \int_{\mathbb{D}} \frac{1}{|G(z)|^2}(1-|\sigma_a(z)|^2)\mathrm{d}m(z) = 0, \tag{11}$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $a \in \mathbb{D}$, is the Möbius transformation of $\mathbb{D}$. For $0 < r < 1$, let $\mathbb{D}(a,r) = \{a \in \mathbb{D} : |\sigma_a(z)| < r\}$ be the pseudohyperbolic disk with center $a \in \mathbb{D}$ and radius r. By [17], we see that

$$|1-\bar{a}z|^2 \approx (1-|z|^2)^2 \approx (1-|a|^2)^2 \approx m(\mathbb{D}(a,r)), \qquad z \in \mathbb{D}(a,r).$$

Choose an $r_0 \in (0,1)$. By the subharmonicity, we obtain

$$\int_{\mathbb{D}} \frac{1}{|G(z)|^2}(1-|\sigma_a(z)|^2)\mathrm{d}m(z) \\ \geq (1-r_0^2)\int_{\mathbb{D}(a,r_0)} \frac{1}{|G(z)|^2}\mathrm{d}m(z) \geq (1-r_0^2)\frac{(1-|a|^2)^2}{|G(a)|^2}.$$

Letting $|a| \to 1$, by (11) we obtain

$$\lim_{|a|\to 1} \frac{1-|a|}{G(a)} = 0.$$

Thus, Corollary 3.1 is proved. □

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