New York Journal of Mathematics

New York J. Math. **28** (2022) 1512–1530.

Representations of the weak Weyl commutation relation

S. Sundar

ABSTRACT. Let G be a locally compact, second countable, Hausdorff abelian group with Pontryagin dual $\widehat{G}$. Suppose P is a closed subsemigroup of G containing the identity element 0. We assume that P has dense interior and P generates G. Let $U := \{U_\chi\}_{\chi \in \widehat{G}}$ be a strongly continuous group of unitaries and let $V := \{V_a\}_{a \in P}$ be a strongly continuous semigroup of isometries. We call (U, V) a weak Weyl pair if $U_\chi V_a = \chi(a) V_a U_\chi$ for every $\chi \in \widehat{G}$ and for every $a \in P$.

We work out the representation theory (the factorial and the irreducible representations) of the above commutation relation under the assumption that $\{V_a V_a^* : a \in P\}$ is a commuting family of projections. Not only does this generalise the results of [4] and [5], our proof brings out the Morita equivalence that lies behind the results. For $P = \mathbb{R}_+^2$, we demonstrate that if we drop the commutativity assumption on the range projections, then the representation theory of the weak Weyl commutation relation becomes very complicated.

CONTENTS

1. Introduction

The classical Stone-von Neumann theorem that asserts the uniqueness of the Weyl commutation relation $U_s V_t = e^{its} V_t U_s$, where $\{U_s\}_{s \geq 0}$ and $\{V_t\}_{t \geq 0}$ are strongly continuous 1-parameter group of unitaries is a fundamental theorem in both quantum mechanics and in operator algebras. In [4] and in [5], a weaker version of the above commutation relation was considered, where $\{V_t\}_{t \geq 0}$ is only assumed to be a semigroup of isometries. The representation theory (the factorial representations and the irreducible representations) of such relations

Received June 13, 2022.
2010 *Mathematics Subject Classification.* Primary 46L05 ; Secondary 81S05.
Key words and phrases. Weak Weyl relations, Semigroups of isometries, Morita equivalence.

ISSN 1076-9803/2022

was worked out by Bracci and Picasso in [4] and in [5]. Bracci and Picasso consider such a weak form of the commutation relation as the quantisation postulate for systems whose configuration space is semibounded like the half-line. This is because, on the half-line, though the position operator generates a group of unitaries, the momentum operator (which is not self-adjoint) generates only a semigroup of isometries. The purpose of this paper is to extend slightly the resuts to systems with d degrees of freedom where $d \geq 2$. We work in the more general setting of subsemigroups of locally compact abelian groups.

Our proof is similar to the operator algebraic proof of the Stone-von Neumann theorem. It is well known ([10], [14]) that the C^*-algebra that encodes the usual Weyl commutation relation is Morita equivalent to $\mathbb{C}$. The Stone-von Neumann theorem is then an immediate consequence of this Morita equivalence. We establish a similar reasoning here. Based on the results obtained in [13], we prove that the C^*-algebra that encodes the weak version of the Weyl commutation relation considered in this paper is Morita equivalent to a commutative C^*-algebra. Thus, it follows at once that every factorial representation is a multiple of an irreducible representation and the irreducible representations are parameterised by the character space of the underlying commutative C^*-algebra. In the one-dimensional case, i.e. when $P = [0, \infty)$ and $G = \mathbb{R}$, the commutative C^*-algebra is $C_0((-\infty, \infty])$. This recovers the results obtained in [4] and in [5].

The results obtained are next explained. Let G be a locally compact, second countable, Hausdorff abelian group. We denote the dual group of G by $\widehat{G}$. We use additive notation for the group operations in G. Let $P \subset G$ be a closed subsemigroup containing 0 such that $P - P = G$. Set $\Omega := Int(P)$. We assume that Ω is dense in P. For $x, y \in G$, we write $x \leq y$ if $y - x \in P$ and $x < y$ if $y - x \in \Omega$

Let $U := \{U_\chi\}_{\chi \in \widehat{G}}$ be a strongly continuous group of unitaries and let $V := \{V_a\}_{a \in P}$ be a strongly continuous semigroup of isometries. We call (U, V) a weak Weyl pair if $U_\chi V_a = \chi(a) V_a U_\chi$ for every $\chi \in \widehat{G}$ and for every $a \in P$.

Let (U, V) be a weak Weyl pair. For $a \in P$, let $E_a := V_a V_a^*$. We say that (U, V) has commuting range projections if $\{E_a : a \in P\}$ is a commuting family of projections. Note that if $P = [0, \infty)$ or, more generally, if the preorder $\leq$ is a total order, every weak Weyl pair has commuting range projections (for a proof see, for example, Example 7.7 of [13]).

Examples of weak Weyl pairs with commuting range projections are given below. Let A be a non-empty closed subset of G which is P-invariant, i.e. $A + P \subset A$. Such subsets will be called P-spaces. Let $\mathcal{K}$ be a Hilbert space whose dimension we denote by k. Consider the Hilbert space $H := L^2(A, \mathcal{K})$. For $\chi \in \widehat{G}$, let U_χ be the unitary on H defined by $U_\chi f(y) = \chi(y) f(y)$. Then, $U := \{U_\chi\}_{\chi \in \widehat{G}}$ is a strongly continuous group of unitaries on H. For $a \in P$, let

V_a be the isometry on H defined by

$$V_a(f)(y) := \begin{cases} f(y-a) & \text{if } y-a \in A, \\ 0 & \text{if } y-a \notin A. \end{cases}$$

Then, $V = \{V_a\}_{a \in P}$ is a strongly continuous semigroup of isometries on H. It is clear that V has commuting range projections. It is routine to verify that (U, V) is a weak Weyl pair. We call (U, V) the weak Weyl pair associated to the P-space A with multiplicity k. If we want to stress the dependence of (U, V) on A and k, we denote (U, V) by $(U^{(A,k)}, V^{(A,k)})$.

The main theorem of this paper is stated below.

Theorem 1.1. *We have the following.*

1. *Let A be a P-space and let $k \in \{1, 2, \cdots\} \cup \{\infty\}$ be given. The weak Weyl pair $(U^{(A,k)}, V^{(A,k)})$ is a factorial representation. Moreover, it is irreducible if and only if $k = 1$.*
2. *Let A, B be P-spaces and let $k, \ell \in \{1, 2, \cdots\} \cup \{\infty\}$. The weak Weyl pair $(U^{(A,k)}, V^{(A,k)})$ is unitarily equivalent to $(U^{(B,\ell)}, V^{(B,\ell)})$ if and only if $A = B$ and $k = \ell$.*
3. *Suppose (U, V) is a weak Weyl pair with commuting range projections. Assume that the von Neumann algebra generated by the set $\{U_\chi, V_a : \chi \in \widehat{G}, a \in P\}$ is a factor. Then, there exists a P-space A and $k \in \{1, 2, \cdots\} \cup \{\infty\}$ such that (U, V) is unitarily equivalent to $(U^{(A,k)}, V^{(A,k)})$.*

   Thus, for weak Weyl pairs with commuting range projections, factorial representations are completely reducible. Moreover, irreducible weak Weyl pairs with commuting range projections are precisely those associated to P-spaces with multiplicity 1.

For $P = [0, \infty)$, as already mentioned, every weak Weyl pair has commuting range projections. Also, every P-space is either $\mathbb{R}$ or of the form $[a, \infty)$ for a unique $a \in \mathbb{R}$. It is now clear that the results obtained in [4] and [5] for the semibounded case follow from Thm. 1.1.

Morever, for irreducible weak Weyl pairs with commuting range projections, we have the following uniqueness result. We need a bit of notation. Let $U := \{U_\chi\}_{\chi \in \widehat{G}}$ be a strongly continuous group of unitaries on a Hilbert space H. Then, U determines a representation π_U of $C_0(G) \cong C^*(\widehat{G})$ on H. We denote the unique closed subset of G that corresponds to the ideal $Ker(\pi_U)$ by $Spec(U)$.

Corollary 1.2. *Let (U, V) and $(\widetilde{U}, \widetilde{V})$ be irreducible weak Weyl pairs with commuting range projections. Assume that (U, V) acts on H and $(\widetilde{U}, \widetilde{V})$ acts on $\widetilde{H}$. Suppose $Spec(U) = Spec(\widetilde{U})$. Then, there exists a unitary $X : H \to \widetilde{H}$ such that for $\chi \in \widehat{G}$ and $a \in P$,*

$$XU_\chi X^* = \widetilde{U}_\chi \, ; \, XV_a X^* = \widetilde{V}_a.$$

What about weak Weyl pairs which do not have commuting range projections? For $P = \mathbb{R}^2_+$, we demonstrate that working out the irreducible weak Weyl pairs is a complicated task. We explain a procedure (preserving factoriality, type and irreducibility) that allows us to build weak Weyl pairs starting from a representation of the free product $c_0(\mathbb{N}) * c_0(\mathbb{N})$. We also prove that Corollary 1.2 no longer stays true if we relax the commutativity assumption on the range projections.

We end this introduction by mentioning that weak Weyl relations, for one degree of freedom, in the unbounded picture were analysed extensively in the literature. Some of the important papers that deal with the unbounded version are [11], [12], [7], [1], [2], and [3]. We do not touch the unbounded version here. The author is of the belief that the C^*-algebra machinery may not be sufficient to handle domain issues.

All the Hilbert spaces considered in this paper are assumed to be separable. Moreover, we use the convention that the inner product is linear in the first variable.

2. The equivalence between $Isom_c(P)$ and $Rep(C_0(Y_u) \rtimes G)$

For the rest of this paper, G stands for an arbitrary but a fixed locally compact, second countable, Hausdorff abelian group. The letter P stands for a closed subsemigroup of G containing the identity element 0. We assume that $\Omega := Int(P)$ is dense in P. We also assume $P - P = G$. For $x, y \in G$, we say $x \leq y$ ($x < y$) if $y - x \in P$ ($y - x \in \Omega$).

Let $Isom_c(P)$ be the collection (up to unitary equivalence) of isometric representations of P with commuting range projections. In this section, we show that there is a bijective correspondence between $Isom_c(P)$ and the collection of non-degenerate representations of a certain crossed product. The proof is based on the results obtained in [13].

Let $\mathcal{C}(G)$ be the set of closed subsets of G endowed with the Fell topology. Let

$$Y_u := \{A \in \mathcal{C}(G) : A \neq \emptyset, -P + A \subset A\}.$$

Endow Y_u with the subspace topology inherited from the Fell topology on $\mathcal{C}(G)$. Then, Y_u is a locally compact, second countable, Hausdorff space. Moreover, the map

$$Y_u \times G \ni (A, x) \to A + x \in Y_u$$

defines an action of G on Y_u. Set

$$X_u := \{A \in Y_u : -P \subset A\} = \{A \in Y_u : 0 \in A\}.$$

Then, X_u is a compact subset of Y_u. Clearly, $X_u + P \subset X_u$.

Let (s_n) be a cofinal sequence in Ω. We claim that

$$Y_u = \bigcup_{n \geq 1} (X_u - s_n). \tag{2.1}$$

Let $A \in Y_u$ be given. Pick a point $x \in A$. Since (s_n) is cofinal, there exists a natural number n such that $-x < s_n$. Hence, $-x - s_n \in -\Omega \subset -P$. Since $-P + A \subset A$, it follows that $-s_n = (-x - s_n) + x \in A$. This implies that $0 \in A + s_n$. Hence, $A + s_n \in X_u$. Equivalently, $A \in X_u - s_n$. This proves the claim.

Lemma 2.1. *The collection $\{X_u + x : x \in G\}$ generates the Borel σ-algebra of Y_u.*

Proof. Let $\mathcal{B}$ be the Borel σ-algebra of Y_u. Denote the σ-algebra generated by the collection $\{X_u + x : x \in G\}$ by $\mathcal{B}_0$. Since X_u is compact, we have $\mathcal{B}_0 \subset \mathcal{B}$. For a compact subset F of G and for an open subset O of G, define

$$\begin{aligned} \mathcal{U}_F &: = \{A \in Y_u : A \cap F = \emptyset\}, \\ \mathcal{U}_O &: = \{A \in Y_u : A \cap O \neq \emptyset\}, \\ \mathcal{U}_O^{'} &: = \{A \in Y_u : A \cap O = \emptyset\}. \end{aligned}$$

The sets $\mathcal{U}_F \cap \mathcal{U}_{O_1} \cap \mathcal{U}_{O_2} \cap \cdots \mathcal{U}_{O_n}$, as F and O_i's vary, form a basis for the Fell topology on Y_u. Moreover, Y_u is second countable. Thus, it suffices to show that for every compact set F and for every open set O, $\mathcal{U}_O, \mathcal{U}_F \in \mathcal{B}_0$.

Fix an open set O of G. Let $D := \{x_1, x_2, \cdots\}$ be a dense subset of O. Let $A \in Y_u$ be given. Observe that $A + \Omega = \bigcup_{a \in A}(a + \Omega)$ is an open set contained in A. Thus, $A + \Omega \subset Int(A)$. Since $0 \in \overline{\Omega}$, $Int(A)$ is dense in A. Thus, for $A \in Y_u$, $A \cap O \neq \emptyset$ if and only if $Int(A) \cap O \neq \emptyset$ if and only if $A \cap D \neq \emptyset$. Therefore,

$$\mathcal{U}_O = \bigcup_{n \geq 1}(X_u + x_n).$$

Hence, $\mathcal{U}_O \in \mathcal{B}_0$ for every open set O of G.

Since $\mathcal{U}_O^{'}$ is the complement of $\mathcal{U}_O$, it follows that $\mathcal{U}_O^{'} \in \mathcal{B}_0$ for every open set O of G. Let F be a compact subset of G. Choose a decreasing sequence of open sets (O_n) such that $\{O_n : n \geq 1\}$ forms a base at F. This means that if O is an open set that contains F, then $O_n \subset O$ eventually. Note that for a closed subset A of G, $A \cap F = \emptyset$ if and only if $A \cap O_n = \emptyset$ eventually. Thus,

$$\mathcal{U}_F = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{U}_{O_n}^{'}.$$

Hence, $\mathcal{U}_F \in \mathcal{B}_0$ for every compact set F of G. This completes the proof. □

Let $Rep(C_0(Y_u) \rtimes G)$ be the collection (up to unitary equivalence) of non-degenerate representations of the crossed product $C_0(Y_u) \rtimes G$. We construct, in this section, maps

$$\Phi : Rep(C_0(Y_u) \rtimes G) \to Isom_c(P)$$

and

$$\Psi : Isom_c(P) \to Rep(C_0(Y_u) \rtimes G)$$

that are inverses of each other.

Let (π, W) be a covariant representation of the dynamical system $(C_0(Y_u), G)$ on a separable Hilbert space K and let R be the projection valued measure on Y_u that corresponds to the homomorphism π. Then, we have the following covariance relation: for a Borel subset $E \subset Y_u$ and $x \in G$,

$$W_x R(E) W_x^* = R(E + x).$$

Set $H := R(X_u)K$. Since $X_u + P \subset X_u$, it follows that the subspace H is invariant under $\{W_a : a \in P\}$. For $a \in P$, let V_a be the operator on H defined by $V_a := W_a|_H$. Then, $V := \{V_a\}_{a \in P}$ is a strongly continuous semigroup of isometries. Moreover, the collection $\{V_a V_a^* : a \in P\}$ is a commuting family of projections on H. If we want to stress the dependence of V on (π, W), we denote V by $V^{(\pi,W)}$.

Define $\Phi : Rep(C_0(Y_u) \rtimes G) \to Isom_c(P)$ by

$$\Phi((\pi, W)) = V^{(\pi,W)}.$$

The construction of the map $\Psi : Isom_c(P) \to Rep(C_0(Y_u) \rtimes G)$ is based on the results obtained in [13]. Let $V := \{V_a\}_{a \in P}$ be a strongly continuous semigroup of isometries on a Hilbert space H. For $a \in P$, set $E_a := V_a V_a^*$. Assume that V has commuting range projections, i.e. $\{E_a : a \in P\}$ is a commuting family of projections. Let $z \in G$ be given. Write $z = a - b$ with $a, b \in P$ and set $T_z := V_b^* V_a$. The following facts were proved in Prop. 3.4 of [13].

(1) The operator T_z is well-defined and is a partial isometry.
(2) For $z \in G$, denote the range projection of T_z by E_z. Then, the family $\{E_z : z \in G\}$ is a commuting family of projections.
(3) The map $G \ni z \to T_z \in B(H)$ is strongly continuous. Consequently, the map $G \ni z \to E_z \in B(H)$ is strongly continuous.

For $f \in C_c(G)$, set

$$E_f := \int f(z) E_z dz.$$

Denote the C^*-algebra generated by $\{E_f : f \in C_c(G)\}$ by $\mathcal{D}$. Note that for $f \in C_c(G)$, if $supp(f) \subset -\Omega$ and $\int f(z)dz = 1$, then $E_f = 1$. Thus, $\mathcal{D}$ is a unital commutative C^*-subalgebra of $B(H)$. Let $\widehat{\mathcal{D}}$ be the space of characters of $\mathcal{D}$. By Prop. 4.6 and by Prop. 4.7 of [13], there exists an injective continuous map $\widehat{\mathcal{D}} \ni \psi \to A_\psi \in X_u$, denoted σ, such that

$$\psi\Big(\int f(z) E_z dz\Big) = \int f(z) 1_{A_\psi}(z) dz$$

for every $f \in C_c(G)$.

Let $\tau : C(\widehat{\mathcal{D}}) \to \mathcal{D} \subset B(H)$ be the inverse of the Gelfand transform. Define a unital *-homomorphism $\overline{\pi} : C(X_u) \to \mathcal{D} \subset B(H)$ by $\overline{\pi}(f) = \tau(f \circ \sigma)$. Denote the spectral measure of $\overline{\pi}$ by Q. By Lemma 7.1 of [13], we have the following covariance relations. For a Borel set $E \subset X_u$ and $a \in P$,

$$V_a Q(E) V_a^* = Q(E + a) \tag{2.2}$$

and

$$V_a^* Q(E) V_a = Q((E-a) \cap X_u). \tag{2.3}$$

Let (W, K) be the minimal unitary dilation of V. This means the following.

(1) The Hilbert space K contains H as a closed subspace.
(2) $W = \{W_x\}_{x \in G}$ is a strongly continuous group of unitaries on K.
(3) For $a \in P$ and $\xi \in H$, $W_a \xi = V_a \xi$.
(4) The union $\bigcup_{a \in P} W_a^* H$ is dense in K.

For the existence of the minimal unitary dilation of an isometric representation, we refer the reader to Thm. 3.2 of [8].

Next, we construct a projection valued measure R on Y_u taking values in $B(K)$. We will write operators on K in block matrix form w.r.t the decomposition $K = H \oplus H^\perp$. For $a \in P$, since $W_a|_H = V_a$, the operator W_a is of the form

$$W_a = \begin{bmatrix} V_a & * \\ 0 & * \end{bmatrix}.$$

Let (s_n) be a cofinal sequence in Ω. By replacing s_n with $\sum_{k=1}^n s_k$, we can make the sequence (s_n) increasing, i.e. $s_n < s_{n+1}$ for every n. Since $X_u + P \subset X_u$, we have $X_u - s_n \subset X_u - s_{n+1}$. By Eq. 2.1, $(X_u - s_n)_{n \geq 1} \nearrow Y_u$. Let $n \geq 1$ be given. For a Borel subset $E \subset X_u - s_n$, define

$$R_n(E) := W_{s_n}^* \begin{bmatrix} Q(E + s_n) & 0 \\ 0 & 0 \end{bmatrix} W_{s_n}.$$

Then, R_n is a projection valued measure on $X_u - s_n$.

Let $n \geq 1$ be given. Let $E \subset X_u - s_n \subset X_u - s_{n+1}$ be a Borel subset. We claim that $R_{n+1}(E) = R_n(E)$. Write $s_{n+1} = s_n + t_n$ with $t_n \in \Omega$. Then,

$$\begin{aligned}
R_{n+1}(E) &= W_{s_n+t_n}^* \begin{bmatrix} Q((E+s_n)+t_n) & 0 \\ 0 & 0 \end{bmatrix} W_{s_n+t_n} \\
&= W_{s_n}^* W_{t_n}^* \begin{bmatrix} V_{t_n} Q(E+s_n) V_{t_n}^* & 0 \\ 0 & 0 \end{bmatrix} W_{t_n} W_{s_n} \\
&= W_{s_n}^* W_{t_n}^* \begin{bmatrix} V_{t_n} & * \\ 0 & * \end{bmatrix} \begin{bmatrix} Q(E+s_n) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{t_n} & * \\ 0 & * \end{bmatrix}^* W_{t_n} W_{s_n} \\
&= W_{s_n}^* W_{t_n}^* W_{t_n} \begin{bmatrix} Q(E+s_n) & 0 \\ 0 & 0 \end{bmatrix} W_{t_n}^* W_{t_n} W_{s_n} \\
&= W_{s_n}^* \begin{bmatrix} Q(E+s_n) & 0 \\ 0 & 0 \end{bmatrix} W_{s_n} \\
&= R_n(E).
\end{aligned}$$

This proves the claim.

Since the sequence of projection valued measures $(R_n)_{n \geq 1}$ is consistent and the sequence $(X_u - s_n)_{n \geq 1} \nearrow Y_u$, there exists a unique projection valued measure, denoted R, on Y_u that takes values in $B(K)$ such that for a Borel subset $E \subset X_u - s_n$, $R(E) = R_n(E)$.

Let $b \in P$ be given. Suppose $E \subset X_u - b$ is a Borel subset. We claim that

$$R(E) = W_b^* \begin{bmatrix} Q(E+b) & 0 \\ 0 & 0 \end{bmatrix} W_b. \tag{2.4}$$

Choose $n \geq 1$ such that $s_n > b$. Note that $X_u - b \subset X_u - s_n$ and $R(E) = R_n(E)$. Write $s_n = b + t_n$ for some $t_n \in \Omega$. A computation exactly similar to the one required to prove $R_{n+1}|_{X_u - s_n} = R_n$ shows that

$$R(E) = W_{s_n}^* \begin{bmatrix} Q(E+s_n) & 0 \\ 0 & 0 \end{bmatrix} W_{s_n} = W_b^* \begin{bmatrix} Q(E+b) & 0 \\ 0 & 0 \end{bmatrix} W_b.$$

This proves the claim.

Next, we prove that $R(Y_u) = 1$. Let $b \in P$ be given. It follows from Eq. 2.4 that $R(X_u - b)$ is the orthogonal projection onto $W_b^* H$. Hence, $R(Y_u)K$ contains $W_b^* H$ for every $b \in P$. Consequently, $R(Y_u)K$ contains the union $\bigcup_{b \in P} W_b^* H$ which is dense in K. Therefore, $R(Y_u) = 1$.

Let $a \in P$ be given. Let $n \geq 1$. Suppose $E \subset X_u - s_n$ is a Borel subset. We claim that $W_a^* R(E) W_a = R(E - a)$. Note that $E - a \subset X_u - (s_n + a)$. By Eq. 2.4, we have

$$\begin{aligned} R(E-a) &= W_{s_n+a}^* \begin{bmatrix} Q((E-a)+s_n+a) & 0 \\ 0 & 0 \end{bmatrix} W_{s_n+a} \\ &= W_a^* W_{s_n}^* \begin{bmatrix} Q(E+s_n) & 0 \\ 0 & 0 \end{bmatrix} W_{s_n} W_a \\ &= W_a^* R(E) W_a. \end{aligned}$$

This proves the claim. Since $(X_u - s_n) \nearrow Y_u$, it follows that for every Borel subset $E \subset Y_u$ and for $a \in P$,

$$W_a^* R(E) W_a = R(E - a).$$

Since $P - P = G$, we have the following covariance relation. For $z \in G$ and a Borel subset $E \subset Y_u$,

$$W_z R(E) W_z^* = R(E + z). \tag{2.5}$$

Let π be the non-degenerate representation of $C_0(Y_u)$ associated to the projection valued measure R, i.e. for $f \in C_0(Y_u)$,

$$\pi(f) = \int f dR.$$

Thanks to Eq. 2.5, it follows that (π, W) is a covariant representation of the dynamical system $(C_0(Y_u), G)$. To denote the dependence of (π, W) on V, we denote (π, W) by (π^V, W^V). Define $\Psi : Isom_c(P) \to Rep(C_0(Y_u) \rtimes G)$ by

$$\Psi(V) = (\pi^V, W^V).$$

Theorem 2.2. *The map*

$$\Phi : Rep(C_0(Y_u) \rtimes G) \ni (\pi, W) \to V^{(\pi, W)} \in Isom_c(P)$$

and the map

$$\Psi : Isom_c(P) \ni V \to (\pi^V, W^V) \in Rep(C_0(Y_u) \rtimes G)$$

are inverses of each other.

Proof. Let $(\pi, W) \in Rep(C_0(Y_u) \rtimes G)$ be given. Let R be the projection valued measure on Y_u associated to π. Suppose that (π, W) acts on K. Set $V := V^{(\pi,W)}$. We need to show that $(\pi^V, W^V) = (\pi, W)$.

First, we claim that W is the minimal unitary dilation of V. Recall that, by definition, V acts on $H = R(X_u)K$ and $V = \{V_a\}_{a\in P}$ is the restriction of $\{W_a\}_{a\in P}$ onto H. Thus, W is a dilation of V. It is enough to show that $\bigcup_{a\in P} W_a^* H$ is dense in K.

Let (s_n) be an increasing cofinal sequence in Ω. Observe that the sequence $(X_u - s_n)_{n\geq 1} \nearrow Y_u$. Thus, $R(X_u - s_n) = W_{s_n}^* R(X_u) W_{s_n} \nearrow 1$ strongly. Clearly, $R(X_u - s_n)$ is the orthogonal projection onto $W_{s_n}^* H$. Thus, $\bigcup_{n\geq 1} W_{s_n}^* H$ is dense in K. This proves that W is the minimal unitary dilation of V. Thus, $W^V = W$.

Next, we show that $\pi = \pi^V$. Denote the projection valued measure associated to π^V by R^V. By definition, for $x \in G$, $R(X_u + x)$ is the orthogonal projection onto $W_x H$ and by Eq. 2.4 and by Eq. 2.5, $R^V(X_u + x)$ is the orthogonal projection onto $W_x H$. Thus,

$$R(X_u + x) = R^V(X_u + x) \tag{2.6}$$

for every $x \in G$. By Lemma 2.1, $R(E) = R^V(E)$ for every Borel set $E \subset G$. Hence, $R = R^V$ and consequently, $\pi = \pi^V$. This completes the proof of the assertion $\Psi\circ\Phi = Id$.

Let $V \in Isom_c(P)$ be given. Suppose that V acts on H. Denote (π^V, W^V) by (π, W) and let K be the Hilbert space on which (π, W) acts. Denote the projection valued measure on Y_u associated to π by R. Let $\widetilde{V} = V^{(\pi,W)}$. For $a \in P$, V_a is the restriction of W_a to H and $\widetilde{V}_a$ is the restriction of W_a to $R(X_u)K$. By Eq. 2.4, $R(X_u)K = H$. Consequently, $\widetilde{V} = V$. Hence $\Phi\circ\Psi = Id$. The proof is now complete. □

3. Proof of the main theorem

With Thm. 2.2 in hand, the conceptual explanation for Thm. 1.1 is quite simple. Having a unitary group, indexed by $\widehat{G}$, implementing the Weyl commutation relation is equivalent to having a unitary group implementing the dual action on $C_0(Y_u) \rtimes G$. Then, Thm. 1.1 is a straightforward consequence of Takai duality. We explain some details below.

Let $\mathcal{W}_c(P, \widehat{G})$ denote the collection (up to unitary equivalence) of weak Weyl pairs with commuting range projections. Consider the dual action of $\widehat{G}$ on $C_0(Y_u) \rtimes G$. We prove below that $\mathcal{W}_c(P, \widehat{G}) \cong Rep((C_0(Y_u) \rtimes G) \rtimes \widehat{G})$.

Theorem 3.1. *There exist maps*

$$\Psi : \mathcal{W}_c(P, \widehat{G}) \to Rep((C_0(Y_u) \rtimes G) \rtimes \widehat{G})$$

and

$$\Phi : Rep((C_0(Y_u) \rtimes G) \rtimes \widehat{G}) \to \mathcal{W}_c(P, \widehat{G})$$

such that Φ *and* Ψ *are inverses of each other.*

Proof. Let $((\pi, W), U) \in Rep((C_0(Y_u) \rtimes G) \rtimes \widehat{G})$ be given. The projection valued measure on Y_u associated to π will be denoted R. Suppose that $((\pi, W), U)$ acts on K. Set $V := V^{(\pi,W)}$. By definition, V acts on $H = R(X_u)K$. Since $U := \{U_\chi\}_{\chi \in \widehat{G}}$ commutes with $\pi(C_0(Y_u))$, it follows that U_χ commutes with $R(X_u)$ and consequently maps H onto H. Clearly, $(U|_H, V^{(\pi,W)})$ is a weak Weyl pair on H with commuting range projections. We define

$$\Phi((\pi, W), U) = (U|_H, V^{(\pi,W)}).$$

Let $(U, V) \in \mathcal{W}_c(P, \widehat{G})$ be given. Suppose that (U, V) acts on H. By Thm. 2.2, there exists $(\pi, W) \in Rep(C_0(Y_u) \rtimes G)$ such that $V = V^{(\pi,W)}$. Suppose that (π, W) acts on K. Then, $H = R(X_u)K$ where R is the projection valued measure associated to π. Recall that W is the minimal unitary dilation of V.

Let $\chi \in \widehat{G}$ be given. We claim that there exists a unique unitary operator $\widetilde{U}_\chi$ on K such that

(C1) for $\xi \in H$, $\widetilde{U}_\chi \xi = U_\chi \xi$, and
(C2) for $x \in G$, $\widetilde{U}_\chi W_x = \chi(x) W_x \widetilde{U}_\chi$.

Conditions $(C1)$ and $(C2)$ together with the fact that $\bigcup_{a \in P} W_a^* H$ is dense in K clearly determine the operator $\widetilde{U}_\chi$ uniquely.

We show below the existence. Define $\widetilde{U}_\chi$ on the dense subspace $\bigcup_{a \in P} W_a^* H$ as follows: for $\xi \in W_a^* H$, set

$$\widetilde{U}_\chi \xi = \overline{\chi(a)} W_a^* U_\chi W_a \xi.$$

Let $a, b \in P$ and let $\xi \in W_a^* H \cap W_b^* H$ be given. Since $W_x H \subset H$ for $x \in P$, it follows that $W_a^* H \cap W_b^* H \subset W_{a+b}^* H$. Calculate as follows to observe that

$$\begin{aligned}
\overline{\chi(a+b)} W_{a+b}^* U_\chi W_{a+b} \xi &= \overline{\chi(a+b)} W_a^* W_b^* U_\chi W_b W_a \xi \\
&= \overline{\chi(a+b)} W_a^* W_b^* U_\chi V_b W_a \xi \text{ (since } W_a \xi \in H) \\
&= \overline{\chi(a+b)} W_a^* W_b^* \chi(b) V_b U_\chi W_a \xi \\
&= \overline{\chi(a)} W_a^* W_b^* W_b U_\chi W_a \xi \text{ (since } U_\chi W_a \xi \in H) \\
&= \overline{\chi(a)} W_a^* U_\chi W_a \xi.
\end{aligned}$$

Similarly, $\overline{\chi(a+b)} W_{a+b}^* U_\chi W_{a+b} \xi = \overline{\chi(b)} W_b^* U_\chi W_b \xi$. This shows that $\widetilde{U}_\chi$ is well defined. It is clear from the definition that $\widetilde{U}_\chi$ is an isometry on $D := \bigcup_{a \in P} W_a^* H$ and maps D onto D. Thus, $\widetilde{U}_\chi$ extends to a unitary operator on K which we again denote by $\widetilde{U}_\chi$.

By definition, $\widetilde{U}_\chi$ restricted to H coincides with U_χ. Next, we check that on the dense subspace $D := \bigcup_{b\in P} W_b^* H$

$$\widetilde{U}_\chi W_a = \chi(a) W_a \widetilde{U}_\chi$$

for $a \in P$.

Let $a \in P$ be given. Suppose $b \in P$ and $\xi \in W_b^* H$. Then, $W_a^*\xi \in W_{a+b}^* H$. Calculate as follows to observe that

$$\begin{aligned} W_a\widetilde{U}_\chi W_a^*\xi &= \overline{\chi(a+b)} W_a W_{a+b}^* U_\chi W_{a+b} W_a^* \xi \text{ (by the defn. of } \widetilde{U}_\chi) \\ &= \overline{\chi(a)}\,\overline{\chi(b)} W_b^* U_\chi W_b \xi \\ &= \overline{\chi(a)} \widetilde{U}_\chi \xi \text{ (by the defn. of } \widetilde{U}_\chi). \end{aligned}$$

Thus, $W_a\widetilde{U}_\chi W_a^* = \overline{\chi(a)}\widetilde{U}_\chi$ on the dense subspace D. Consequently, we have $\widetilde{U}_\chi W_a = \chi(a) W_a \widetilde{U}_\chi$ for every $a \in P$.

Since P spans G and $\{W_x\}_{x\in G}$ is a group of unitaries, it follows that

$$\widetilde{U}_\chi W_x = \chi(x) W_x \widetilde{U}_\chi$$

for $x \in G$. Thus, we have established the existence of the unitary operator $\widetilde{U}_\chi$ on K for which $(C1)$ and $(C2)$ are satisfied.

Note that for $x \in G$, $R(X_u + x)$ is the orthogonal projection onto $W_x H$. Let $\chi \in \widehat{G}$ be given. By the definition of $\widetilde{U}_\chi$, $\widetilde{U}_\chi$ maps $W_a^* H$ onto $W_a^* H$ for every $a \in P$. Thus, $\widetilde{U}_\chi$ commutes with $\{R(X_u - a) : a \in P\}$. Let $x \in G$ be given. Write $x = a - b$ with $a, b \in P$. Then, $R(X_u + x) = W_a R(X_u - b) W_a^*$. Thanks to the Weyl commutation relation and the fact that $\widetilde{U}_\chi$ commutes with $R(X_u - b)$, it follows that $\widetilde{U}_\chi$ commutes with $R(X_u + x)$ for every $x \in G$. By Lemma 2.1, $\widetilde{U}_\chi$ commutes with $R(E)$ for every Borel set $E \subset Y_u$. Thus, $\widetilde{U}_\chi \in \pi(C_0(Y_u))'$.

We leave it to the reader to verify that $\widetilde{U} := \{\widetilde{U}_\chi\}_{\chi\in\widehat{G}}$ is a strongly continuous group of unitaries on K. We have now proved that $((\pi, W), \widetilde{U})$ is a representation of the dynamical system $(C_0(Y_u) \rtimes G, \widehat{G})$. Set

$$\Psi(U, V) = ((\pi, W), \widetilde{U}).$$

Then, Ψ and Φ are inverses of each other. We omit this routine verification. □

Next, we show that the maps Φ and Ψ of Thm. 3.1 take factorial representations to factorial representations and take irreducible representations to irreducible representations.

Let $((\pi, W), U)$ be a representation of $(C_0(Y_u) \rtimes G, \widehat{G})$ and let K be the Hilbert space on which it acts. The projection valued measure on Y_u associated to π will be denoted by R. Let $\Phi((\pi, W), U) = (U|_H, V)$ where $H = R(X_u)K$. Denote the von Neumann algebra generated by $\{\pi(\phi), W_x, U_\chi : \phi \in C_0(Y_u), x \in G, \chi \in \widehat{G}\}$ by M and the von Neumann algebra on H generated by $\{V_a, U_\chi|_H : a \in P, \chi \in \widehat{G}\}$ by N.

Proposition 3.2. *Keep the foregoing notation. The map*

$$M' \ni T \to T|_H \in N'$$

is an isomorphism of von Neumann algebras. Consequently, the maps Φ *and* Ψ *map factorial representations to factorial representations and irreducible representations to irreducible representations.*

Proof. Let $T \in M'$ be given. Since T commutes with $\{\pi(\phi) : \phi \in C_0(Y_u)\}$, it follows that T commutes with $R(E)$ for every Borel subset $E \subset Y_u$. In particular, T commutes with $R(X_u)$ which is the projection onto H. We decompose K as $K = H \oplus H^{\perp}$ and write operators on K in block matrix form. Since T commutes with $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, T is of the form

$$T = \begin{bmatrix} T_0 & 0 \\ 0 & T_1 \end{bmatrix}.$$

Let $\chi \in \widehat{G}$ and $a \in P$ be given. Then, U_χ and W_a are of the form

$$U_\chi = \begin{bmatrix} U_\chi|_H & 0 \\ 0 & U_\chi|_{H^\perp} \end{bmatrix}$$

and

$$W_a = \begin{bmatrix} V_a & * \\ 0 & * \end{bmatrix}.$$

Since T commutes with U_χ, it follows that T_0 commutes with $U_\chi|_H$. Since T commutes with W_a and W_a^*, it follows that T commutes with V_a and V_a^*. Hence, $T_0 \in N'$.

Thus, the map $M' \ni T \to T_0 = T|_H \in N'$ is well defined. Writing operators in M' in the block diagonal form as above and performing routine computations, we see that the map

$$M' \ni T \to T_0 = T|_H \in N'$$

is a *-algebra homomorphism.

Let $T \in M'$. Suppose $T|_H = 0$. Let $a \in P$ and $\xi \in H$ be given. Since T commutes with W_a^*, it follows that

$$TW_a^*\xi = W_a^*T\xi = 0.$$

Therefore, T vanishes on $\bigcup_{a \in P} W_a^* H$ which is dense in K. Hence, $T = 0$. This shows that the map $M' \ni T \to T_0 = T|_H \in N'$ is injective.

Let $U_0 \in N'$ be a unitary operator. We claim that there exists a unitary operator U on K such that

(D1) for $\xi \in H$, $U\xi = U_0\xi$, and
(D2) for $x \in G$, $UW_x = W_xU$.

The construction of U is similar to the construction of $\widetilde{U}_\chi$ done in Thm. 3.1. We define U on the dense subspace $D := \bigcup_{a\in P} W_a^*H$ as follows: for $\xi \in W_a^*H$, set

$$U\xi = W_a^* U_0 W_a \xi.$$

Calculations similar to the one carried out in Thm. 3.1 show that U is a well defined unitary operator that satisfy $(D1)$ and $(D2)$.

Since U agrees with U_0 on H, it follows that U maps H onto H. Moreover, U commutes with W_x for every $x \in G$. Thus, U maps W_xH onto W_xH for every $x \in G$. Consequently, for $x \in G$, U commutes with the orthogonal projection onto W_xH which is $R(X_u + x)$. By Lemma 2.1, it follows that U commutes with $R(E)$ for every Borel subset $E \subset Y_u$. Thus, $U \in \pi(C_0(Y_u))^{'}$.

Let $\chi \in \widehat{G}$ be given. Suppose $a \in P$ and $\xi \in W_a^*H$. Observe that $U_\chi\xi \in W_a^*H$. Calculate as follows to observe that

$$\begin{aligned} UU_\chi\xi &= W_a^*U_0W_aU_\chi\xi \\ &= \overline{\chi(a)}W_a^*U_0U_\chi W_a\xi \\ &= \overline{\chi(a)}W_a^*U_\chi U_0W_a\xi \ \big(\text{since } W_a\xi \in H \text{ and } U_\chi U_0\eta = U_0U_\chi\eta \text{ if } \eta \in H\big) \\ &= \overline{\chi(a)}\chi(a)U_\chi W_a^*U_0W_a\xi \\ &= U_\chi U\xi. \end{aligned}$$

Thus, UU_χ and $U_\chi U$ agree on the dense subspace $\bigcup_{a\in P} W_a^*H$. Therefore, $UU_\chi = U_\chi U$ for every $\chi \in \widehat{G}$.

We have now shown that $U \in M^{'}$ and by $(D1)$, $U|_H = U_0$. This shows that every unitary operator in $N^{'}$ lies in the image of the homomorphism $M^{'} \ni T \to T|_H \in N^{'}$. Consequently, the map

$$M^{'} \ni T \to T|_H \in N^{'}$$

is surjective. This completes the proof. □

Thm. 1.1 is an immediate consequence of Thm. 3.1, Prop. 3.2 and Takai duality. Takai duality asserts that $(C_0(Y_u) \rtimes G) \rtimes \widehat{G} \cong C_0(Y_u) \otimes \mathcal{K}(L^2(G))$. As a consequence, $Rep((C_0(Y_u) \rtimes G) \rtimes \widehat{G}) \cong Rep(C_0(Y_u))$. The proof of Thm. 1.1 is essentially transporting the representation theory of $C_0(Y_u)$ to $\mathcal{W}_c(P, \widehat{G})$ using Thm. 3.1 and by making using of the explicit isomorphism between the C^*-algebras $(C_0(Y_u) \rtimes G) \rtimes \widehat{G}$ and $C_0(Y_u) \otimes \mathcal{K}(L^2(G))$. For the explicit isomorphism involved in Takai duality, we refer the reader to either [9] or [14].

Proof of Thm. 1.1. We denote the set $\{1, 2, \cdots\} \cup \{\infty\}$ by $\mathbb{N}_\infty$. Suppose $\mathcal{K}$ is a Hilbert space of dimension $k \in \mathbb{N}_\infty$. Let $K := L^2(G, \mathcal{K})$. Fix an element $B \in Y_u$. Define a representation $\pi^{(B,k)}$ of $C_0(Y_u)$ on K by

$$\pi^{(B,k)}(f)\xi(x) = f(B + x)\xi(x).$$

For $x \in G$, let W_x be the unitary on K defined by

$$W_x\xi(y) = \xi(y - x).$$

For $\chi \in \widehat{G}$, let U_χ be the unitary operator on K defined by

$$U_\chi\xi(y) = \chi(y)\xi(y).$$

Let $V := V^{(\pi^{(B,k)},W)}$ and let $R := R^{(B,k)}$ be the projection valued measure associated to $\pi^{(B,k)}$. Observe that for $\xi \in K$,

$$R(X_u)\xi(x) = 1_{X_u}(B + x)\xi(x) = 1_B(-x)\xi(x) = 1_{-B}(x)\xi(x).$$

Thus, $R(X_u)K = L^2(-B, \mathcal{K})$. By definition, V is the compression of the left regular representation onto $L^2(-B, \mathcal{K})$. Thus, V coincides with the isometric representation $V^{(-B,k)}$ defined in the introduction. Clearly, the restriction of $\{U_\chi\}_{\chi \in \widehat{G}}$ to the subspace $R(X_u)K = L^2(-B, \mathcal{K})$ coincides with $U^{(-B,k)}$ defined in the introduction. Thus,

$$\Phi((\pi^{(B,k)}, W), U) = (U^{(-B,k)}, V^{(-B,k)}).$$

By Takai duality, it follows that $\{((\pi^{(B,k)}, W), U) : (B, k) \in Y_u \times \mathbb{N}_\infty\}$ form a mutually inequivalent exhaustive list of factorial representations of $(C_0(Y_u) \rtimes G) \rtimes \widehat{G}$. By Thm. 3.1 and by Prop. 3.2, it follows that

$$\{\Phi(((\pi^{(B,k)}, W), U)) : (B, k) \in Y_u \times \mathbb{N}_\infty\},$$

which coincides with the collection

$$\{(U^{(-B,k)}, V^{(-B,k)}) : (B, k) \in Y_u \times \mathbb{N}_\infty\},$$

form a mutually inequivalent exhaustive list of weak Weyl pairs with commuting range projections that are factorial.

Again by Takai duality, it follows that $\{((\pi^{(B,1)}, W), U) : B \in Y_u\}$ form a mutually inequivalent exhaustive list of irreducible representations of the C^*-algebra $(C_0(Y_u) \rtimes G) \rtimes \widehat{G}$. Thanks to Thm. 3.1 and Prop. 3.2, it follows that

$$\{\Phi(((\pi^{(B,1)}, W), U)) : B \in Y_u\},$$

which coincides with the collection

$$\{(U^{(-B,1)}, V^{(-B,1)}) : B \in Y_u\},$$

form a mutually inequivalent exhaustive list of irreducible weak Weyl pairs with commuting range projections. This completes the proof of Thm. 1.1. □

What about weak Weyl pairs which do not have commuting range projections? If we drop the assumption that the range projections commute, then we show that, for $P = \mathbb{R}_+^2$, we can construct weak Weyl pairs that generate a factor of both type II and type III. Moreover, we also illustrate that classifying all the irreducible weak Weyl pairs is a complicated task, a task that is at least as hard as classifying the irreducible representations of the free product $c_0(\mathbb{N}) * c_0(\mathbb{N})$. More precisely, we explain a procedure (preserving factoriality, type and irreducibility) that allows us to build weak Weyl pairs starting from a non-degenerate representation of the free product $c_0(\mathbb{N}) * c_0(\mathbb{N})$.

For the rest of this paper, we assume that $P = \mathbb{R}^2_+ = [0,\infty) \times [0,\infty)$ and $G = \mathbb{R}^2$. We identify $\widehat{G}$ with $\mathbb{R}^2$ in the usual way. Let $\{P_m\}_{m\geq 1}$ and $\{Q_n\}_{n\geq 1}$ be two sequences of projections on a Hilbert space K such that $P_iP_j = \delta_{ij}P_i$ and $Q_kQ_\ell = \delta_{k\ell}Q_k$. Writing down two such sequences of projections on a Hilbert space K is clearly equivalent to defining a representation of the free product $c_0(\mathbb{N}) * c_0(\mathbb{N})$ on K.

Denote the set of projections on K by $P(K)$. Define $F : \mathbb{Z}^2 \to P(K)$ by

$$F_{(m,n)} := \begin{cases} \sum_{k=1}^m P_k & \text{if } m \geq 1 \text{ and } n = 0, \\ \sum_{k=1}^n Q_k & \text{if } m = 0 \text{ and } n \geq 1, \\ 1 & \text{if } m \geq 1, n \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $(m,n) \in \mathbb{Z}^2$ and $(p,q) \in \mathbb{N}^2$, $F_{(m+p,n+q)} \geq F_{(m,n)}$.

Let $R := [0,1] \times [0,1]$ be the unit square and suppose that λ is the Lebesgue measure on R. Consider $L^\infty(R, d\lambda)$ as a C^*-algebra and let X be the character space of $L^\infty(R, d\lambda)$. Fix $a, b, c, d \in (0,1)$ such that $a < b$ and $c < d$. Fix a point $z_0 \in X$ such that $1_{[a,b]\times[c,d]}(z_0) \neq 0$.

Define a map $E : \mathbb{R}^2_+ \to P(K)$ as follows. Let $(s,t) \in \mathbb{R}^2_+$ be given. Let m be the integral part of s and let n be the integral part of t. Set

$$\begin{aligned} R_0(s,t) &: = [0, m+1-s] \times [0, n+1-t] \\ R_1(s,t) &: = [0, m+1-s] \times [n+1-t, 1] \\ R_2(s,t) &: = [m+1-s, 1] \times [0, n+1-t] \\ R_3(s,t) &: = [m+1-s, 1] \times [n+1-t, 1]. \end{aligned}$$

Define $E_{(s,t)}$ by the following formula.

$$\begin{aligned} E_{(s,t)} :=& 1_{R_0(s,t)}(z_0)F_{(m,n)} + 1_{R_1(s,t)}(z_0)F_{(m,n+1)} \\ &+ 1_{R_2(s,t)}(z_0)F_{(m+1,n)} + 1_{R_3(s,t)}(z_0)F_{(m+1,n+1)}. \end{aligned}$$

Since $\{1_{R_i}\}_{i=0}^3$ is an orthogonal family in $L^\infty(R, d\lambda)$ that sum up to 1, exactly one term survives in the above expression. Consequently, $E_{(s,t)}$ is a projection.

Lemma 3.3. *With the foregoing notation, we have the following.*

(1) *The map $E : \mathbb{R}^2_+ \to P(K)$ is increasing, i.e $E_{(s,t)} \leq E_{(s+s_0,t+t_0)}$ for $(s,t) \in \mathbb{R}^2_+$ and for every $(s_0,t_0) \in \mathbb{R}^2_+$.*
(2) *For $\xi, \eta \in K$, the map*

$$\mathbb{R}^2_+ \ni (s,t) \to \langle E_{(s,t)}\xi|\eta\rangle \in \mathbb{C}$$

is Lebesgue measurable.
(3) *Let $(m,n) \in \mathbb{N}^2$ be given. The set $\{(s,t) \in \mathbb{R}^2_+ : E_{(s,t)} = F_{(m,n)}\}$ contains a Lebesgue measurable set of positive measure.*

Proof. The proof of (1) is a case by case verification. Let $(s,t) \in \mathbb{R}_+^2$ be given. Suppose $s_1 > s$. Let m be the integral part of s, p the integral part of s_1 and n the integral part of t.

Case 1: $m < p$.

Let $r := m+1-s$ and $r_1 = p+1-s_1$.

Case (a): $r \leq r_1$.

Case (i)**:** $1_{R_0(s,t)}(z_0) = 1$. In this case, $E_{(s,t)} = F_{(m,n)}$. Note that $R_0(s_1,t)$ contains $R_0(s,t)$. Thus, $1_{R_0(s,t)} \leq 1_{R_0(s_1,t)}$ in $L^\infty(R, d\lambda)$. Consequently, $1_{R_0(s_1,t)}(z_0) = 1$. Therefore, $E_{(s_1,t)} = F_{(p,n)}$. Since $F_{(p,n)} \geq F_{(m,n)}$, we have $E_{(s_1,t)} \geq E_{(s,t)}$.

Case (ii)**:** $1_{R_1(s,t)}(z_0) = 1$. We can argue as in Case (i) and deduce $E_{(s_1,t)} \geq E_{(s,t)}$.

Case (iii)**:** $1_{R_2(s,t)}(z_0) = 1$. In this case, $E_{(s,t)} = F_{(m+1,n)}$. Note that the union $R_2(s_1,t) \cup R_0(s_1,t)$ contains $R_2(s,t)$. Therefore, either $1_{R_0(s_1,t)}(z_0) = 1$ or $1_{R_2(s_1,t)}(z_0) = 1$. This means that $E_{(s_1,t)}$ is either $F_{(p,n)}$ or $F_{(p+1,n)}$. Both $F_{(p,n)}$ and $F_{(p+1,n)}$ are greater than $F_{(m+1,n)}$ as F is increasing and as $p \geq m+1$. Thus, $E_{(s_1,t)} \geq E_{(s,t)}$.

Case (iv)**:** $1_{R_3(s,t)}(z_0) = 1$. In this case, $E_{(s,t)} = F_{(m+1,n+1)}$. Note that the union $R_3(s_1,t) \cup R_1(s_1,t)$ contains $R_3(s,t)$. Therefore, either $1_{R_3(s_1,t)}(z_0) = 1$ or $1_{R_1(s_1,t)}(z_0) = 1$. This means that $E_{(s_1,t)}$ is either $F_{(p+1,n+1)}$ or $F_{(p,n+1)}$. In either case, $E_{(s_1,t)} \geq E_{(s,t)}$.

Case (b): $r > r_1$. The analysis here is similar and we can conclude $E_{(s,t)} \leq E_{(s_1,t)}$.

Case 2: $m = p$. The analysis here is similar to Case 1 (in this case, Case (a) does not arise) and we can conclude that $E_{(s,t)} \leq E_{(s_1,t)}$.

Thus, we have proved that $E_{(s,t)} \leq E_{(s+s_0,t)}$ for every $(s,t) \in \mathbb{R}_+^2$ and for every $s_0 \geq 0$. An exactly similar argument shows $E_{(s,t)} \leq E_{(s,t+t_0)}$ for every $(s,t) \in \mathbb{R}_+^2$ and $t_0 \geq 0$. Hence, the function E is increasing. This proves (1).

To prove (2), thanks to the polarisation identity, it suffices to show, that for every $\xi \in K$, the map $\mathbb{R}_+^2 \ni (s,t) \to \langle E_{(s,t)}\xi|\xi\rangle \in \mathbb{R}$ is Lebesgue measurable. To that effect, let $\xi \in K$ be given and define $\phi : \mathbb{R}^2 \to [0,\infty)$ by

$$\phi(s,t) := \begin{cases} \langle E_{(s,t)}\xi|\xi\rangle & \text{if } (s,t) \in \mathbb{R}_+^2, \\ \\ 0 & \text{if } (s,t) \notin \mathbb{R}_+^2. \end{cases}$$

Then, ϕ is increasing, i.e. if $s_1 \leq s_2$ and $t_1 \leq t_2$, then $\phi(s_1,t_1) \leq \phi(s_2,t_2)$. It follows from Thm. 4 of [6] that ϕ is Lebesgue measurable. This proves (2).

Let $(m,n) \in \mathbb{N}^2$. Let

$$A := \{(s,t) \in [m,m+1) \times [n,n+1) : (m+1-s, n+1-t) \in (b,1) \times (d,1)\}.$$

Then, A is a Borel set of positive measure. Let $(s,t) \in A$ be given. Note that $R_0(s,t)$ contains $[a,b] \times [c,d]$. Since $1_{[a,b]\times[c,d]}(z_0) = 1$, $1_{R_0(s,t)}(z_0) = 1$. Consequently, for $(s,t) \in A$, $E_{(s,t)} = F_{(m,n)}$. This proves that A is contained in $\{(s,t) \in \mathbb{R}_+^2 : E_{(s,t)} = F_{(m,n)}\}$. The proof of (3) is complete. ☐

Extend E to the whole of $\mathbb{R}^2$ by setting $E_{(s,t)} = 0$ if $(s,t) \notin \mathbb{R}^2_+$. Then, the extended map $E : \mathbb{R}^2 \to P(K)$ is still increasing and Lebesgue measurable.

Let $L := L^2(\mathbb{R}^2, K)$ be the space of square integrable Lebesgue measurable functions taking values in K. For $(x,y) \in \mathbb{R}^2$, let $U_{(x,y)}$ be the unitary on L defined by

$$U_{(x,y)}f(u,v) := e^{i(ux+vy)}f(u,v).$$

For $(s,t) \in \mathbb{R}^2$, let $W_{(s,t)}$ be the unitary on L defined by

$$W_{(s,t)}f(u,v) = f(u-s, v-t).$$

Define a projection $\widetilde{E} : L \to L$ by

$$\widetilde{E}f(u,v) = E_{(u,v)}f(u,v).$$

Set $H := Ran(\widetilde{E})$. Note that $U_{(x,y)}$ commutes with $\widetilde{E}$. Thus, $U_{(x,y)}$ maps H onto H. We denote the restriction of $U_{(x,y)}$ to H again by $U_{(x,y)}$.

Using the fact that E is increasing, it is routine to verify that if $(s,t) \in \mathbb{R}^2_+$, then $\widetilde{E}W_{(s,t)}\widetilde{E} = W_{(s,t)}\widetilde{E}$. In other words, the subspace H is invariant under $\{W_{(s,t)} : (s,t) \in \mathbb{R}^2_+\}$. For $(s,t) \in \mathbb{R}^2_+$, let $V_{(s,t)}$ be the isometry on H defined by

$$V_{(s,t)} = W_{(s,t)}|_H.$$

Then, $V := \{V_{(s,t)}\}_{(s,t)\in\mathbb{R}^2_+}$ is a strongly continuous semigroup of isometries on H. Similarly, $U := \{U_{(x,y)}|_H\}_{(x,y)\in\mathbb{R}^2}$ is a strongly continuous group of unitaries. Clearly, (U,V) is a weak Weyl pair.

Let us fix notation. Define

$$\begin{aligned} M_0 &: = W^*\{U_{(x,y)}|_H, V_{(s,t)} : (x,y) \in \mathbb{R}^2, (s,t) \in \mathbb{R}^2_+\}, \\ M_1 &: = W^*\{U_{(x,y)}, W_{(s,t)}, \widetilde{E} : (x,y) \in \mathbb{R}^2, (s,t) \in \mathbb{R}^2\}, \\ N &: = W^*\{F_{(m,n)} : (m,n) \in \mathbb{N}^2\} = W^*\{1, P_m, Q_n : m \in \mathbb{N}, n \in \mathbb{N}\}. \end{aligned}$$

Note that M_0 acts on H, M_1 acts on L and N acts on K. For a bounded operator T on K, let $\widetilde{T}$ be the operator on L defined by

$$\widetilde{T}f(y) = Tf(y).$$

Proposition 3.4. *With the foregoing notation, we have the following.*

1. *The map $N' \ni T \to \widetilde{T} \in M_1'$ is an isomorphism.*
2. *(W,L) is the minimal unitary dilation of V.*
3. *Let $t \in \{I, II, III\}$. The von Neumann algebra M_0 is a factor of type t if and only if N is a factor of type t.*
4. *The weak Weyl pair (U,V) is irreducible if and only if $N' = \mathbb{C}$.*

Proof. From a routine computation, we see that if $T \in N'$, then $\widetilde{T} \in M_1'$. Let $S \in M_1'$ be given. Note that

$$W^*\{U_{(x,y)}, W_{(s,t)} : (x,y) \in \mathbb{R}^2, (s,t) \in \mathbb{R}^2\} = B(L^2(\mathbb{R}^2)) \otimes 1 \subset B(L^2(\mathbb{R}^2) \otimes K).$$

Therefore, there exists $T \in B(K)$ such that $S = \widetilde{T}$. The fact that S commutes with $\widetilde{E}$ translates to the equation

$$TE_{(s,t)} = E_{(s,t)}T$$

for almost all $(s,t) \in \mathbb{R}_+^2$. By (3) of Lemma 3.3, T commutes with $F_{(m,n)}$ for every $(m,n) \in \mathbb{N}^2$. Thus, $T \in N'$. This completes the proof of (1).

By definition, (W, L) is a dilation of V. Let Q be the projection onto the closure of the subspace $\bigcup_{(s,t)\in\mathbb{R}_+^2} W_{(s,t)}^* H$. Note that $Ran(Q)$ is invariant under $U_{(x,y)}$ and $W_{(s,t)}$ for every $(x,y) \in \mathbb{R}^2$ and for every $(s,t) \in \mathbb{R}^2$. Thus, $Q \in \{U_{(x,y)}, W_{(s,t)} : (x,y),(s,t) \in \mathbb{R}^2\}'$. Consequently, $Q = \widetilde{R}$ for some projection R on K.

The condition $Q \geq \widetilde{E}$ translates to the fact that $R \geq E_{(s,t)}$ for almost all $(s,t) \in \mathbb{R}_+^2$. Thanks to (3) of Lemma 3.3, $R \geq F_{(1,1)} = 1$. Thus, $R = 1$ and hence $Q = 1$. This proves (2).

As in Prop. 3.2, it is not difficult to prove using the fact that (W, L) is the minimal unitary dilation of V that $M_1' \ni T \to T|_H \in M_0'$ is an isomorphism of von Neumann algebras. Now, (3) and (4) follow from (1). □

Remark 3.5. *We conclude this paper with the following remarks.*

(1) *Thanks to Prop. 3.4, we can construct an irreducible weak Weyl pair starting from an irreducible representation of the free product $c_0(\mathbb{N}) * c_0(\mathbb{N})$. Moreover, inequivalent irreducible representations of $c_0(\mathbb{N}) * c_0(\mathbb{N})$ lead to inequivalent weak Weyl pairs. Thus, listing out all the irreducible weak Weyl pairs is at least as hard as describing the dual of $c_0(\mathbb{N}) * c_0(\mathbb{N})$. Up to the author's knowledge, a "good description" of the dual of $c_0(\mathbb{N}) * c_0(\mathbb{N})$ (or even the dual of some of its natural quotients like $C^*(\mathbb{Z}_n * \mathbb{Z}_m)$) is not available in the literature.*

(2) *Observe that for the weak Weyl pair (U, V) constructed in Prop. 3.4, $Spec(U)$ is independent of the underlying representation of the free product $c_0(\mathbb{N}) * c_0(\mathbb{N})$ as long as $F_{(m,n)} \neq 0$ for $(m,n) \in \mathbb{N}^2 \backslash \{(0,0)\}$. Thus, Corollary 1.2 is not true without the commutativity assumption on the range projections.*

(3) *Prop. 3.4 allows us to construct weak Weyl pairs that generate a factor of both type II and type III. This is because $c_0(\mathbb{N}) * c_0(\mathbb{N})$ admit factorial representations of type II and type III as the C^*-algebra $c_0(\mathbb{N}) * c_0(\mathbb{N})$ is not of type I.*

(4) *Let P be a closed convex cone in $\mathbb{R}^d$ which we assume is spanning, i.e. $P - P = \mathbb{R}^d$ and pointed, i.e. $P \cap -P = \{0\}$. Assume $d \geq 2$. Building on the two dimensional case, it is not difficult to construct, in this case, weak Weyl pairs (U, V) that generate a factor of both type II and type III. Also, it is possible to construct a continuum of irreducible weak Weyl pairs which do not have commuting range projections.*

References

[1] ARAI, ASAO. Generalized weak Weyl relations and decay of quantum dynamics. *Rev. Math. Phys.* **17** (2005), no. 9, 1071–1109. MR2178243, Zbl 1111.81053, doi: 10.1142/S0129055X05002479. 1515

[2] ARAI, ASAO. Spectrum of time operators. *Lett. Math. Phys.* **80** (2007), no. 3, 211–221. MR2333509, Zbl 1117.81054, doi: 10.1007/S11005-007-0158-y. 1515

[3] ARAI, ASAO. On the uniqueness of weak Weyl representations of the canonical commutation relation. *Lett. Math.Phys.* **85** (2008), no. 1, 15–25. MR2425658, Zbl 1162.81017, doi: 10.1007/S11005-008-0252-9. 1515

[4] BRACCI, LUCIANO; PICASSO, LUIGI E. On the Weyl algebra for systems with semibounded and bounded configuration space. *J. Math. Phys.* **47** (2006), no.11, 112102–112118. MR2277602, Zbl 1112.81063, doi: 10.1063/1.2363256. 1512, 1513, 1514

[5] BRACCI, LUCIANO; PICASSO, LUIGI E. On the reducibility of the Weyl algebra for a semibounded space. *Lett. Math. Phys.* **89** (2009), no.3, 277–285. MR2551184, Zbl 1190.47041, doi: 10.1007/S11005-009-0334-3. 1512, 1513, 1514

[6] CHABRILLAC, YVES; CROUZEIX, J.-P. Continuity and differentiability properties of monotone real functions of several real variables. *Nonlinear analysis and optimization (Louvain-la-Neuve, 1983)*, no.30, 1987, 1–16. MR874128, Zbl 0611.26008, doi: 10.1007/bfb0121151. 1527

[7] JORGENSEN, PALLE T.; MUHLY, PAUL S.. Selfadjoint extensions satisfying the Weyl operator commutation relations. *J. Analyse Math.* **37** (1980), 46–99. MR583632, Zbl 0448.47012, doi: 10.1007/BF02797680. 1515

[8] LACA, MARCELO; RAEBURN, IAIN. Extending multipliers from semigroups. *Proc. Amer. Math. Soc.* **123** (1995), no.2, 355–362. MR1227519, Zbl 0841.20058, doi: 10.2307/2160888. 1518

[9] RAEBURN, IAIN. On crossed products and Takai duality. *Proc. Edinburgh Math. Soc. (2)* **31** (1988), no.2, 321–330. MR989764, Zbl 0674.46038, doi: 10.1017/S0013091500003436. 1524

[10] ROSENBERG, JONATHAN A selective history of the Stone-von Neumann theorem. Operator algebras, quantization, and noncommutative geometry, *Contemp. Math., Amer. Math. Soc., Providence, RI.* **365** (2004), 331–353. MR2106827, Zbl 1073.01020, doi: 10.1090/conm/365/06710. 1513

[11] SCHMÜDGEN, KONRAD. On the Heisenberg commutation relation. I. *J. Funct. Anal.* **50** (1983), no.1, 8–49. MR689997, Zbl 0507.47010, doi: 10.1016/0022-1236(83)90058-7. 1515

[12] SCHMÜDGEN, KONRAD. On the Heisenberg commutation relation. II. *Publ. Res. Inst. Math. Sci.* **19** (1983), no.2, 601–671. MR716970, Zbl 0557.47023, doi: 10.2977/prims/1195182446. 1515

[13] SUNDAR, S.. C^*-algebras associated to topological Ore semigroups. *Münster J. Math.* **9** (2016), no.1, 155–185. MR3549547, Zbl 1365.46050, doi: 10.17879/35209716203. 1513, 1515, 1517

[14] WILLIAMS, DANA P. Crossed products of C^*-algebras, Mathematical Surveys and Monographs. *Amer. Math. Soc., Providence, RI*, **134**, 2007. xvi+528 pp. ISBN: 978-0-8218-4242-3. MR2288954, Zbl 1119.46002, doi: 10.1090/surv/134. 1513, 1524

(S. Sundar) INSTITUTE OF MATHEMATICAL SCIENCES, A CI OF HOMI BHABHA NATIONAL INSTITUTE, 4TH CROSS STREET, CIT CAMPUS, TARAMANI, CHENNAI 600113, INDIA
sundarsobers@gmail.com

This paper is available via http://nyjm.albany.edu/j/2022/28-66.html.