New York Journal of Mathematics

New York J. Math. **28** (2022) 1637–1655.

Cohomologies and generalized derivations of n-Lie algebras

Begüm Ateşli *, Oğul Esen and Serkan Sütlü

Abstract. A cohomology theory associated to an n-Lie algebra and a representation space of it is introduced. It is shown that this cohomology theory classifies generalized derivations of n-Lie algebras as 1-cocycles, and inner generalized derivations as 1-coboundaries.

Contents

1. Introduction

In an effort to generalize the classical Hamiltonian mechanics to a theory that can accommodate two Hamiltonians, ternary Lie algebras were introduced in [12], wherein it was hinted that the theory may be generalized into one that allows $(n-1)$-many Hamiltonians. This point of view was pursued further in [7] introducing a theory of n-Lie algebras, and then has further been developed in [15].

It was in [8] that, for the first time, a cohomology theory has been associated to n-ary Lie algebras to study their formal deformations. Then, in [2], this cohomology theory has been considered further, in close connection with the Leibniz cohomology. This cohomology theory, associated to n-Lie algebras, upgraded recently to a cohomology theory with arbitrary coefficients in [1] and [16] in the case of $n = 3$, and in [14] to a cohomology theory associated to an n-Hom-Lie algebra along with a representation space of it.

On the other hand, cohomology theories, more precisely the low dimensional cohomology groups, have intimate relations with the extensions/deformations of the algebraic/geometric objects, [9, 10]. In particular, the derivation extensions have been recently considered in [13] for 3-Lie algebras, and it is argued that a

Received August 18, 2022.

2010 *Mathematics Subject Classification.* 17A32; 17A40; 17A42; 17A36; 17B56.

Key words and phrases. n-Lie algebra; Leibniz algebra; generalized derivation.

* Corresponding Author.

ISSN 1076-9803/2022

derivation of a 3-Lie algebra $\mathfrak{G}$ does not fit to define a 3-Lie algebra structure on the 1-dimensional extension $\mathfrak{G} \oplus k$. The authors, therefore, introduced the notion of a generalized derivation for a 3-Lie algebra, which has later been extended to n-Hom-Lie algebras in [14].

It is very well known that the derivations on n-Lie algebras, Leibniz algebras, etc. are in one-to-one correspondence with the 1st cohomology group of this algebraic object in question, with coefficients in itself. However, as is pointed out in [13] for 3-Lie algebras, such a characterization is missing in the case of generalized derivations. It is this gap that the present paper aims to fill in. To this end, we shall introduce a cohomology theory for n-Lie algebras, along which we shall be able to realize a generalized derivation of an n-Lie algebra as a 1-cocycle in this cohomology theory. It is worth to remark that the new cohomology theory we introduce here differs from the known cohomology theory of n-Lie algebras for $n > 3$. In the case of $n = 3$, on the other hand, the two cohomology theories coincide. Furthermore, the coefficient space of the 1st cohomology group that classifies a generalized derivation of an n-Lie algebra does not appear to be the n-Lie algebra itself, as it might be expected. Instead, what we observe is that the coefficients are given by the space of endomorphisms of the n-Lie algebra on which the generalized derivation acts.

The paper may be outlined as follows.

The very first section consists of the preliminary material to be used in the sequel. More precisely, in Subsection 2.1 we recall briefly the representations and the cohomology of Leibniz algebras, which are also carried out in the level of n-Lie algebras in Subsection 2.2. Also in this subsection we prove Proposition 2.1, a generalization of [1, Prop. 2.5(b)], in order to be able to realize the space of endomorphisms on an n-Lie algebra as a representation space. The second section contains the main results of the manuscript. The notion of a generalized derivation on an n-Lie algebra is recalled in Subsection 3.1. Finally, Subsection 3.2 is reserved for the main result of the paper. It is this subsection where we introduce a new cohomology theory for n-Lie algebras via which we realize a generalized derivation on an n-Lie algebra as a 1-cocycle in Proposition 3.7.

Notations and conventions.

Given a linear space V, we shall denote by $g\ell(V)$ the space of endomorphisms on V. On the other hand, given an n-Lie algebra $\mathcal{L}$, we shall employ the notations $\mathcal{L}_n := \wedge^n \mathcal{L}$, $\mathcal{L}_{n-1} := \wedge^{n-1} \mathcal{L}$, and $\mathcal{L}_{n-2} := \wedge^{n-2} \mathcal{L}$.

2. Preliminaries

This introductory section is meant to review the basics of (the cohomology of) Leibniz and n-Lie algebras. To be more precise, we shall take a quick tour towards the cohomology of Leibniz algebras in the first subsection, whereas the second subsection will consist of the representations and the cohomology of n-Lie algebras.

2.1. Cohomology of Leibniz algebras.

Along the way to show that the new cohomological complex we shall introduce is indeed a differential complex, we shall make use of the cohomology of the associated Leibniz algebras. As such, we now recall briefly the cohomology of Leibniz algebras.

Let us begin with the definition of a (left) Leibniz algebra following [13], see also [11, 5].

A *(left) Leibniz algebra* is a vector space $\mathfrak{L}$ equipped with a bilinear map $[\,,]: \mathfrak{L}\otimes\mathfrak{L}\to\mathfrak{L}$ satisfying

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \tag{2.1}$$

for all $x,y,z\in\mathfrak{L}$.

Following [11, Subsect. 1.5], a *representation* of a Leibniz algebra $(\mathfrak{L},[\,,])$ is defined to be a vector space V equipped with the linear maps $\lambda:\mathfrak{L}\to g\ell(V)$ and $\rho:\mathfrak{L}\to g\ell(V)$ such that

$$\lambda([x,y]) = \lambda(x)\circ\lambda(y) - \lambda(y)\circ\lambda(x), \tag{2.2}$$

$$\rho([x,y]) = \lambda(x)\circ\rho(y) - \rho(y)\circ\lambda(x), \tag{2.3}$$

$$\rho(y)\circ\rho(x) = -\rho(y)\circ\lambda(x), \tag{2.4}$$

for any $x,y\in\mathfrak{L}$. Let us note also that, from the point of view of (left and right) *actions*, we may view $\lambda:\mathfrak{L}\to g\ell(V)$ as a left action $\triangleright:\mathfrak{L}\otimes V\to V$ by $x\triangleright v := \lambda(x)(v)$, and $\rho:\mathfrak{L}\to g\ell(V)$ as a right action $\triangleleft: V\otimes\mathfrak{L}\to V$ via $v\triangleleft x := \rho(x)(v)$, for any $x,y\in\mathfrak{L}$, and any $v\in V$. Accordingly, the conditions (2.2)-(2.4) may be re-written as

$$[x,y]\triangleright v = x\triangleright(y\triangleright v) - y\triangleright(x\triangleright v), \tag{2.5}$$

$$v\triangleleft[x,y] = x\triangleright(v\triangleleft y) - (x\triangleright v)\triangleleft y, \tag{2.6}$$

$$(v\triangleleft x)\triangleleft y = -(x\triangleright v)\triangleleft y. \tag{2.7}$$

Let us next recall the semi-direct sum construction for Leibniz algebras. Given a Leibniz algebra $\mathfrak{L}$, and a representation V of $\mathfrak{L}$, it takes a routine verification that $V\rtimes\mathfrak{L} := V\oplus\mathfrak{L}$ is also Leibniz algebra via

$$\Big[(v,x),(w,y)\Big] := \Big(x\triangleright w + v\triangleleft y,\ [x,y]\Big),$$

called the *semi-direct sum* Leibniz algebra.

Given a (left) Leibniz algebra $\mathfrak{L}$, a linear map $D:\mathfrak{L}\to\mathfrak{L}$ is called a *derivation* if

$$D([x,y]) = [D(x),y] + [x,D(y)]$$

for all $x,y\in\mathfrak{L}$. Given a Leibniz algebra $\mathfrak{L}$, the (left) adjoint action $\mathrm{ad}^L_x:\mathfrak{L}\to\mathfrak{L}$ given by $\mathrm{ad}^L(x)(y) := [x,y]$ is a derivation for any $x\in\mathfrak{L}$.

Let us finally recall the Leibniz algebra cohomology from [6, 11]. Let $\mathfrak{L}$ be a Leibniz algebra, and let V be a representation of $\mathfrak{L}$. Then,

$$CL(\mathfrak{L},V) := \bigoplus_{n\geqslant 0} CL^n(\mathfrak{L},V), \qquad CL^n(\mathfrak{L},V) := \mathrm{Hom}\left(\mathfrak{L}^{\otimes n},V\right),$$

together with $d : CL^n(\mathfrak{L}, V) \to CL^{n+1}(\mathfrak{L}, V)$ which is given by

$$\begin{aligned} & df(x_1, \ldots, x_{n+1}) := \\ & \sum_{1 \leqslant i < j \leqslant n+1} (-1)^i f(x_1, \ldots, \widehat{x}_i, \ldots, x_{j-1}, [x_i, x_j], \ldots, x_{n+1}) + \\ & \sum_{k=1}^{n} (-1)^{k+1} x_k \triangleright f(x_1, \ldots, \widehat{x}_k, \ldots, x_{n+1}) + (-1)^{n+1} f(x_1, \ldots, x_n) \triangleleft x_{n+1} \end{aligned} \tag{2.8}$$

for any $f \in CL^n(\mathfrak{L}, V)$, form a differential complex. The homology $H(CL(\mathfrak{L}, V), d)$ is called the *Leibniz algebra cohomology* of $\mathfrak{L}$, with coefficients in V, and is denoted by $HL^*(\mathfrak{L}, V)$.

2.2. Cohomology of *n*-Lie algebras.

We shall next recall the very basics of n-Lie algebras from [2, 12, 15]. A linear space $\mathcal{L}$, equipped with a linear operator $[\,,\ldots,\,] : \mathcal{L}_n \to \mathcal{L}$, is called an n-Lie algebra if it satisfies the *fundamental identity*

$$[x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{2n-1}]] = \sum_{k=0}^{n-1} [x_n, \ldots, [x_1, \ldots, x_{n-1}, x_{n+k}], \ldots, x_{2n-1}], \tag{2.9}$$

for any $x_1, \ldots, x_{2n-1} \in \mathcal{L}$. Along the lines of [2], given an n-Lie algebra $\mathcal{L}$, the space $\mathcal{L}_{n-1}$ has the structure of a Leibniz algebra through

$$[x_1, x_2] := \sum_{k=1}^{n-1} x_2^1 \wedge \cdots \wedge [x_1^1, \ldots, x_1^{n-1}, x_2^k] \wedge \cdots \wedge x_2^{n-1}$$

for any $x_1 := x_1^1 \wedge \cdots \wedge x_1^{n-1}, x_2 := x_2^1 \wedge \cdots \wedge x_2^{n-1} \in \mathcal{L}_{n-1}$. Furthermore, as was shown in [2, Thm. 2], $\mathcal{L}$ happens to be a representation of the Leibniz algebra $\mathcal{L}_{n-1}$.

2.2.1. Representation of an *n*-Lie algebra.

Following [3, 4], given an n-Lie algebra $\mathcal{L}$, a vector space V with a linear map $\mu : \mathcal{L}_{n-1} \to g\ell(V)$ is called a *representation* of $\mathcal{L}$, if

$$\begin{aligned} \mu(x_1, \ldots, x_{n-1})\mu(x_n, \ldots, x_{2n-2}) - \mu(x_n, \ldots, x_{2n-2})\mu(x_1, \ldots, x_{n-1}) \\ = \sum_{k=0}^{n-2} \mu(x_n, \ldots, [x_1, \ldots, x_{n-1}, x_{n+k}], \ldots, x_{2n-2}), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \mu([x_1, \ldots, x_n], x_{n+1}, \ldots, x_{2n-2}) \\ = \sum_{k=1}^{n} (-1)^k \mu(x_1, \ldots, \hat{x}_k, \ldots, x_n)\mu(x_{n+1}, \ldots, x_{2n-2}, x_k), \end{aligned} \tag{2.11}$$

for any $x_1, \ldots, x_{2n-2} \in \mathcal{L}$.

The most immediate example of a representation is $\mathrm{ad} : \mathcal{L}_{n-1} \to g\ell(\mathcal{L})$, that is, the *adjoint representation* of an n-Lie algebra $\mathcal{L}$ on itself, which is given by

$$\mathrm{ad}(x_1, \ldots, x_{n-1})(x_n) := [x_1, \ldots, x_n] \tag{2.12}$$

for any $x_1, \ldots, x_n \in \mathcal{L}$.

In order to present a slightly more serious example, in the spirit of a dual representation, we record the following generalization of [1, Prop. 2.5(b)] regarding the representations of n-Lie algebras.

Proposition 2.1. *Given an n-Lie algebra $\mathcal{L}$, and a representation (V, μ) of $\mathcal{L}$,*

$$\sum_{k=1}^{n} (-1)^k \mu(x_1, \ldots, \widehat{x_k}, \ldots, x_n)\mu(x_{n+1}, \ldots, x_{2n-2}, x_k) + \sum_{k=1}^{n} (-1)^k \mu(x_{n+1}, \ldots, x_{2n-2}, x_k)\mu(x_1, \ldots, \widehat{x_k}, \ldots, x_n) = 0 \tag{2.13}$$

for any $x_1, \ldots, x_{2n-2} \in \mathcal{L}$.

Proof. To begin with, in view of (2.10) we have

$$\begin{aligned}
&\sum_{k=1}^{n} (-1)^k \mu(x_1, \ldots, \widehat{x_k}, \ldots, x_n)\mu(x_{n+1}, \ldots, x_{2n-2}, x_k) + \\
&\qquad \sum_{k=1}^{n} (-1)^k \mu(x_{n+1}, \ldots, x_{2n-2}, x_k)\mu(x_1, \ldots, \widehat{x_k}, \ldots, x_n) = \\
&\sum_{k=1}^{n} (-1)^k \mu(x_{n+1}, \ldots, x_{2n-2}, x_k)\mu(x_1, \ldots, \widehat{x_k}, \ldots, x_n) + \\
&\sum_{p=1}^{n-2} \sum_{k=1}^{n} (-1)^k \mu(x_{n+1}, \ldots, [x_1, \ldots, \widehat{x_k}, \ldots, x_n, x_{n+p}], \ldots, x_{2n-2}, x_k) + \\
&\sum_{k=1}^{n} (-1)^k \mu(x_{n+1}, \ldots, x_{2n-2}, [x_1, \ldots, \widehat{x_k}, \ldots, x_n, x_k]) + \\
&\sum_{k=1}^{n} (-1)^k \mu(x_1, \ldots, \widehat{x_k}, \ldots, x_n)\mu(x_{n+1}, \ldots, x_{2n-2}, x_k) + \\
&\sum_{1 \leqslant r < k \leqslant n} \sum_{k=1}^{n} (-1)^k \mu(x_1, \ldots, [x_{n+1}, \ldots, x_{2n-2}, x_k, x_r], \ldots, \widehat{x_k}, \ldots, x_n) + \\
&\sum_{1 \leqslant k < r \leqslant n} \sum_{k=1}^{n} (-1)^k \mu(x_1, \ldots, \widehat{x_k}, \ldots, [x_{n+1}, \ldots, x_{2n-2}, x_k, x_r], \ldots, x_n).
\end{aligned}$$

That is,

$$n\,\mu([x_1, \ldots, x_n], x_{n+1}, \ldots, x_{2n-2}) +$$

$$\sum_{p=1}^{n-2}\sum_{k=1}^{n}(-1)^{k+p-1}\mu([x_1,\ldots,\widehat{x_k},\ldots,x_n,x_{n+p}],x_{n+1},\ldots,\widehat{x_{n+p}},\ldots,x_{2n-2},x_k)+$$
$$\sum_{1\leqslant r<k\leqslant n}\sum_{k=1}^{n}(-1)^{k+r-1}\mu([x_{n+1},\ldots,x_{2n-2},x_k,x_r],x_1,\ldots,\widehat{x_r},\ldots,\widehat{x_k},\ldots,x_n)+$$
$$\sum_{1\leqslant k<r\leqslant n}\sum_{k=1}^{n}(-1)^{k+r}\mu([x_{n+1},\ldots,x_{2n-2},x_k,x_r],x_1,\ldots,\widehat{x_k},\ldots,\widehat{x_r},\ldots,x_n)=0.$$

We next apply (2.11) to arrive at

$$\begin{aligned}
&n\,\mu([x_1,\ldots,x_n],x_{n+1},\ldots,x_{2n-2})+\\
&\sum_{1\leqslant r<k\leqslant n}\sum_{p=1}^{n-2}\sum_{k=1}^{n}(-1)^{r+k+p-1}\mu(x_1,\ldots,\widehat{x_r},\ldots,\widehat{x_k},\ldots,x_n,x_{n+p})\times\\
&\qquad\qquad\mu(x_{n+1},\ldots,\widehat{x_{n+p}},\ldots,x_{2n-2},x_k,x_r)+\\
&\sum_{1\leqslant k<r\leqslant n}\sum_{p=1}^{n-2}\sum_{k=1}^{n}(-1)^{r+k+p}\mu(x_1,\ldots,\widehat{x_k},\ldots,\widehat{x_r},\ldots,x_n,x_{n+p})\times\\
&\qquad\qquad\mu(x_{n+1},\ldots,\widehat{x_{n+p}},\ldots,x_{2n-2},x_k,x_r)+\\
&\sum_{p=1}^{n-2}\sum_{k=1}^{n}(-1)^{n+k+p-1}\mu(x_1,\ldots,\widehat{x_k},\ldots,x_n)\mu(x_{n+1},\ldots,\widehat{x_{n+p}},\ldots,x_{2n-2},x_k,x_{n+p})+\\
&\sum_{1\leqslant r<k\leqslant n}\sum_{p=1}^{n-2}\sum_{k=1}^{n}(-1)^{p+k+r-1}\mu(x_{n+1},\ldots\widehat{x_{n+p}},\ldots,x_{2n-2},x_k,x_r)\times\\
&\qquad\qquad\mu(x_1,\ldots,\widehat{x_r},\ldots,\widehat{x_k},\ldots,x_n,x_{n+p})+\\
&\sum_{1\leqslant r<k\leqslant n}\sum_{k=1}^{n}(-1)^{n+k+r}\mu(x_{n+1},\ldots,x_{2n-2},x_r)\mu(x_1,\ldots,\widehat{x_r},\ldots,\widehat{x_k},\ldots,x_n,x_k)+\\
&\sum_{1\leqslant r<k\leqslant n}\sum_{k=1}^{n}(-1)^{n+k+r-1}\mu(x_{n+1},\ldots,x_{2n-2},x_k)\mu(x_1,\ldots,\widehat{x_r},\ldots,\widehat{x_k},\ldots,x_n,x_r)+\\
&\sum_{1\leqslant k<r\leqslant n}\sum_{p=1}^{n-2}\sum_{k=1}^{n}(-1)^{k+r+p}\mu(x_{n+1},\ldots,\widehat{x_{n+p}},\ldots,x_{2n-2},x_k,x_r)\times\\
&\qquad\qquad\mu(x_1,\ldots,\widehat{x_k},\ldots,\widehat{x_r},\ldots,x_n,x_{n+p})+\\
&\sum_{1\leqslant k<r\leqslant n}\sum_{k=1}^{n}(-1)^{n+k+r-1}\mu(x_{n+1},\ldots,x_{2n-2},x_r)\mu(x_1,\ldots,\widehat{x_k},\ldots,\widehat{x_r},\ldots,x_n,x_k)+\\
&\sum_{1\leqslant k<r\leqslant n}\sum_{k=1}^{n}(-1)^{n+k+r}\mu(x_{n+1},\ldots,x_{2n-2},x_k)\mu(x_1,\ldots,\widehat{x_k},\ldots,\widehat{x_r},\ldots,x_n,x_r)=0,
\end{aligned}$$

which, calling the left hand side of (2.13) as $F(x_1,\dots,x_{2n-2})$, may be arranged into

$$
\begin{aligned}
&(2n-2)\,F(x_1,\dots,x_{2n-2})+\\
&\sum_{1\leqslant r<k\leqslant n}\sum_{p=1}^{n-2}\sum_{k=1}^{n}(-1)^{r+k+p-1}\,\mu(x_1,\dots,\widehat{x_r},\dots,\widehat{x_k},\dots,x_n,x_{n+p})\times\\
&\qquad\qquad\qquad\qquad\mu(x_{n+1},\dots,\widehat{x_{n+p}},\dots,x_{2n-2},x_k,x_r)+\\
&\sum_{1\leqslant k<r\leqslant n}\sum_{p=1}^{n-2}\sum_{k=1}^{n}(-1)^{r+k+p}\,\mu(x_1,\dots,\widehat{x_k},\dots,\widehat{x_r},\dots,x_n,x_{n+p})\times\\
&\qquad\qquad\qquad\qquad\mu(x_{n+1},\dots,\widehat{x_{n+p}},\dots,x_{2n-2},x_k,x_r)+\\
&\sum_{1\leqslant r<k\leqslant n}\sum_{p=1}^{n-2}\sum_{k=1}^{n}(-1)^{p+k+r-1}\,\mu(x_{n+1},\dots\widehat{x_{n+p}},\dots,x_{2n-2},x_k,x_r)\times\\
&\qquad\qquad\qquad\qquad\mu(x_1,\dots,\widehat{x_r},\dots,\widehat{x_k},\dots,x_n,x_{n+p})+\\
&\sum_{1\leqslant k<r\leqslant n}\sum_{p=1}^{n-2}\sum_{k=1}^{n}(-1)^{k+r+p}\,\mu(x_{n+1},\dots,\widehat{x_{n+p}},\dots,x_{2n-2},x_k,x_r)\times\\
&\qquad\qquad\qquad\qquad\mu(x_1,\dots,\widehat{x_k},\dots,\widehat{x_r},\dots,x_n,x_{n+p})=0.
\end{aligned}
$$

Next, setting for any $a\in\{1,\dots,n\}$ and $b\in\{1,\dots,n-2\}$

$$F_{a,b}:=F(x_1,\dots,x_{a-1},x_{n+b},x_{a+1},\dots,x_n,x_{n+1},\dots,x_{n+b-1},x_a,x_{n+b+1},\dots,x_{2n-2}),$$

the above expression may be presented as

$$\sum_{k=1}^{n}\sum_{p=1}^{n-2}F_{k,p}=nF(x_1,\dots,x_{2n-2}).\tag{2.14}$$

The claim, now, follows from the iteration of (2.14), or equivalently the successive applications of (2.10) and (2.11). □

Remark 2.2. Let us note that in the case of $n=3$, (2.13) is precisely [1, Prop. 2.5(b)] for 3-Lie algebras.

Corollary 2.3. *Let $\mathcal{L}$ be an n-Lie algebra, together with a representation (V,η). Then, the space* $\mathrm{Hom}(V,W)$ *of linear maps from V to a vector space W is also a representation of $\mathcal{L}$ via $\mu:\mathcal{L}_{n-1}\to g\ell(\mathrm{Hom}(V,W))$ with*

$$\Big(\mu(x_1,\dots,x_{n-1})(T)\Big)(v):=-T(\eta(x_1,\dots,x_{n-1})(v)).\tag{2.15}$$

Proof. Indeed, we observe at once that

$$\begin{aligned}
&\Big(\mu(x_1,\ldots,x_{n-1})\mu(x_n,\ldots,x_{2n-2})(T)-\mu(x_n,\ldots,x_{2n-2})\mu(x_1,\ldots,x_{n-1})(T)\Big)(v)=\\
&T(\eta(x_n,\ldots,x_{2n-2})\eta(x_1,\ldots,x_{n-1})(v)-\eta(x_1,\ldots,x_{n-1})\eta(x_n,\ldots,x_{2n-2})(v))=\\
&-\sum_{k=0}^{n-2}T(\eta(x_n,\ldots,[x_1,\ldots,x_{n-1},x_{n+k}],\ldots,x_{2n-2})(v))=\\
&\sum_{k=0}^{n-2}\Big(\mu(x_n,\ldots,[x_1,\ldots,x_{n-1},x_{n+k}],\ldots,x_{2n-2})(T)\Big)(v),
\end{aligned} \tag{2.16}$$

where we employed (2.10) on the second equality. On the other hand,

$$\begin{aligned}
&\Big(\mu([x_1,\ldots,x_n],x_{n+1},\ldots,x_{2n-2})(T)\Big)(v)\\
&\qquad=-T(\eta([x_1,\ldots,x_n],x_{n+1},\ldots,x_{2n-2})(v))\\
&\qquad=-\sum_{k=1}^{n}(-1)^k\,T(\eta(x_1,\ldots,\hat{x}_k,\ldots,x_n)\eta(x_{n+1},\ldots,x_{2n-2},x_k)(v))\\
&\qquad=\sum_{k=1}^{n}(-1)^k\,T(\eta(x_{n+1},\ldots,x_{2n-2},x_k)\eta(x_1,\ldots,\hat{x}_k,\ldots,x_n)(v))\\
&\qquad=\sum_{k=1}^{n}(-1)^k\,\Big(\mu(x_1,\ldots,\hat{x}_k,\ldots,x_n)\mu(x_{n+1},\ldots,x_{2n-2},x_k)(T)\Big)(v),
\end{aligned}$$

where we applied (2.13) on the third equality. □

For a direct proof, one that does not appeal to an analogue of [1, Prop. 2.5(b)], we refer the reader to [14, Thm. 3.9].

2.2.2. Cohomology of *n*-Lie algebras.

We shall now recall the cohomology theory for n-Lie algebras from [2], see also [14, Sect. 3].

Let $\mathcal{L}$ be an n-Lie algebra, and let (V,μ) be a representation of $\mathcal{L}$. Then,

$$C(\mathcal{L},V):=\bigoplus_{m\geqslant 0}C^m(\mathcal{L},V),\qquad C^m(\mathcal{L},V):=\operatorname{Hom}\Big(\mathcal{L}_{n-1}^{\wedge\,(m-1)}\wedge\mathcal{L},V\Big),\quad m\geqslant 1, \tag{2.17}$$

with $C^0(\mathcal{L}, V) := \mathcal{L}_{n-2} \otimes V$, is a differential complex via

$$
\begin{aligned}
&\delta : C^0(\mathcal{L}, V) \to C^1(\mathcal{L}, V), \qquad \delta(z_1 \wedge \cdots \wedge z_{n-2} \otimes v)(y) := \mu(z_1, \ldots, z_{n-2}, y)(v),\\
&\delta : C^m(\mathcal{L}, V) \to C^{m+1}(\mathcal{L}, V), \qquad m \geqslant 1\\
&\delta f(x_1, \ldots, x_m, y) :=\\
&\sum_{1 \leqslant i < j \leqslant m} (-1)^i f(x_1, \ldots, \widehat{x_i}, \ldots, [x_i, x_j], \ldots, x_m, y) +\\
&\qquad \sum_{i=1}^{m} (-1)^i f(x_1, \ldots, \widehat{x_i}, \ldots, x_m, [x_i^1, \ldots, x_i^{n-1}, y]) +\\
&\qquad \sum_{i=1}^{m} (-1)^{i+1} \mu(x_i^1, \ldots, x_i^{n-1}) \Big(f(x_1, \ldots, \widehat{x_i}, \ldots, x_m, y) \Big) +\\
&\qquad \sum_{i=1}^{n-1} (-1)^{n-1+m+i} \mu(x_m^1, \ldots, \widehat{x_m^i}, \ldots, x_m^{n-1}, y) \Big(f(x_1, \ldots, x_{m-1}, x_m^i) \Big),
\end{aligned}
\tag{2.18}
$$

for any $f \in C^m(\mathcal{L}, V)$, any $x_i := x_i^1 \wedge \cdots \wedge x_i^{n-1} \in \mathcal{L}_{n-1}$, any $z_1 \wedge \cdots \wedge z_{n-2} \in \mathcal{L}_{n-2}$, and any $y \in \mathcal{L}$.

The homology $H(C(\mathcal{L}, V), \delta)$ of the differential complex (2.17) is called the *n-Lie algebra cohomology* of $\mathcal{L}$, with coefficients in V, and is denoted by $H^*(\mathcal{L}, V)$.

We shall now point out, in view of [2], the relation between the cohomology of an n-Lie algebra $\mathcal{L}$, and the cohomology of the associated Leibniz algebra $\mathcal{L}_{n-1}$.

The following auxiliary result on the representations may be obtained by a routine verification of (2.5), (2.6), and (2.7), and hence the proof is omitted.

Proposition 2.4. *Given an n-Lie algebra $\mathcal{L}$, together with a representation (V, μ), the linear space $\mathcal{L}_{n-2} \otimes V$ is a representation of the Leibniz algebra $\mathcal{L}_{n-1}$ through*

$$
\begin{aligned}
&x \triangleright (z_1 \wedge \cdots \wedge z_{n-2} \otimes v) := \qquad (2.19)\\
&\quad \sum_{k=1}^{n-2} z_1 \wedge \cdots \wedge [x^1, \ldots, x^{n-1}, z_k] \wedge \cdots \wedge z_{n-2} \otimes v +\\
&\qquad\qquad z_1 \wedge \cdots \wedge z_{n-2} \otimes \mu(x^1, \ldots, x^{n-1})(v),\\
&(z_1 \wedge \cdots \wedge z_{n-2} \otimes v) \triangleleft x := \qquad (2.20)\\
&\quad \sum_{k=1}^{n-1} (-1)^{n+k} x^1 \wedge \cdots \wedge \widehat{x^k} \wedge \cdots \wedge x^{n-1} \otimes \mu(z_1, \ldots, z_{n-2}, x^k)(v).
\end{aligned}
$$

The relation between the cohomology of an n-Lie algebra, and its associated Leibniz algebra of fundamental objects, is now given below.

Proposition 2.5. *Given an n-Lie algebra $\mathcal{L}$, together with a representation (V,μ), the diagram*

$$\begin{tikzcd} C^m(\mathcal{L},V) \arrow[r, "\Delta^m"] \arrow[d, "\delta"'] & CL^m(\mathcal{L}_{n-1},\mathcal{L}_{n-2}\otimes V) \arrow[d, "d"] \\ C^{m+1}(\mathcal{L},V) \arrow[r, "\Delta^{m+1}"] & CL^{m+1}(\mathcal{L}_{n-1},\mathcal{L}_{n-2}\otimes V) \end{tikzcd}$$

is commutative for any $m \geqslant 0$, where

$$\Delta^m : C^m(\mathcal{L},V) \to CL^m(\mathcal{L}_{n-1},\mathcal{L}_{n-2}\otimes V), \qquad f \mapsto \Delta^m(f),$$

$$\Delta^m(f)(x_1,\dots,x_m) := \sum_{k=1}^{n-1} (-1)^k\, x_m^1 \wedge \cdots \wedge \widehat{x_m^k} \wedge \cdots \wedge x_m^{n-1} \otimes f(x_1,\dots,x_{m-1},x_m^k), \qquad m \geqslant 1,$$

$$\Delta^0 := -\operatorname{Id} : \mathcal{L}_{n-2}\otimes V \to \mathcal{L}_{n-2}\otimes V.$$

Proof. On the one hand we have

$$\Delta^{m+1}(\delta f)(x_1,\dots,x_{m+1}) =$$
$$\sum_{k=1}^{n-1} (-1)^k\, x_{m+1}^1 \wedge \cdots \wedge \widehat{x_{m+1}^k} \wedge \cdots \wedge x_{m+1}^{n-1} \otimes \delta f(x_1,\dots,x_m,x_{m+1}^k) =$$
$$\sum_{1\leqslant i<j\leqslant m}\sum_{k=1}^{n-1} (-1)^{i+k}\, x_{m+1}^1 \wedge \cdots \wedge \widehat{x_{m+1}^k} \wedge \cdots \wedge x_{m+1}^{n-1}\otimes$$
$$f(x_1,\dots,\widehat{x_i},\dots,[x_i,x_j],\dots,x_m,x_{m+1}^k)+$$
$$\sum_{i=1}^{m}\sum_{k=1}^{n-1} (-1)^{i+k}\, x_{m+1}^1 \wedge \cdots \wedge \widehat{x_{m+1}^k} \wedge \cdots \wedge x_{m+1}^{n-1}\otimes$$
$$f(x_1,\dots,\widehat{x_i},\dots,x_m,[x_i^1,\dots,x_i^{n-1},x_{m+1}^k])+$$
$$\sum_{i=1}^{m}\sum_{k=1}^{n-1} (-1)^{i+1+k}\, x_{m+1}^1 \wedge \cdots \wedge \widehat{x_{m+1}^k} \wedge \cdots \wedge x_{m+1}^{n-1}\otimes$$
$$\mu(x_i^1,\dots,x_i^{n-1})\Big(f(x_1,\dots,\widehat{x_i},\dots,x_m,x_{m+1}^k)\Big)+$$
$$\sum_{i=1}^{n-1}\sum_{k=1}^{n-1} (-1)^{n-1+m+i+k}\, x_{m+1}^1 \wedge \cdots \wedge \widehat{x_{m+1}^k} \wedge \cdots$$
$$\cdots \wedge x_{m+1}^{n-1} \otimes \mu(x_m^1 \wedge \cdots \wedge \widehat{x_m^i} \wedge \cdots \wedge x_m^{n-1},x_{m+1}^k)\Big(f(x_1,\dots,x_{m-1},x_m^i)\Big),$$

while on the other hand

$$d(\Delta^m f)(x_1,\dots,x_{m+1}) = \sum_{1\leqslant i<j\leqslant m+1} (-1)^i\, (\Delta^m f)(x_1,\dots,\widehat{x_i},\dots,[x_i,x_j],\dots,x_{m+1})+$$
$$\sum_{k=1}^{m} (-1)^{k+1}\, x_k \triangleright (\Delta^m f)(x_1,\dots,\widehat{x_k},\dots,x_{m+1}) + (-1)^{m+1}\,(\Delta^m f)(x_1,\dots,x_m) \triangleleft x_{m+1} =$$

$$\sum_{1\leqslant i<j\leqslant m} (-1)^i\,(\Delta^m f)(x_1,\dots,\widehat{x_i},\dots,[x_i,x_j],\dots,x_{m+1})+$$
$$\sum_{1\leqslant i\leqslant m} (-1)^i\,(\Delta^m f)(x_1,\dots,\widehat{x_i},\dots,x_m,[x_i,x_{m+1}])+$$
$$\sum_{k=1}^{m} (-1)^{k+1}\,x_k \triangleright (\Delta^m f)(x_1,\dots,\widehat{x}_k,\dots,x_{m+1}) + (-1)^{m+1}\,(\Delta^m f)(x_1,\dots,x_m) \triangleleft x_{m+1}.$$

Accordingly,

$$d(\Delta^m f)(x_1,\dots,x_{m+1}) =$$
$$\sum_{1\leqslant i<j\leqslant m}\sum_{k=1}^{n-1} (-1)^{i+k}\,x^1_{m+1}\wedge\cdots\wedge\widehat{x^k_{m+1}}\wedge\cdots\wedge x^{n-1}_{m+1}\otimes f(x_1,\dots,\widehat{x_i},\dots,[x_i,x_j],\dots,x^k_{m+1})+$$
$$\sum_{p=1}^{n-1}\sum_{\substack{k=1\\k\neq p}}^{n-1}\sum_{i=1}^{m} (-1)^{k+i}\,x^1_{m+1}\wedge\cdots\wedge\widehat{x^k_{m+1}}\wedge\cdots\wedge[x^1_i\dots,x^{n-1}_i,x^p_{m+1}]\wedge\cdots$$
$$\cdots\wedge x^{n-1}_{m+1}\otimes f(x_1,\dots,\widehat{x_i},\dots,x_m,x^k_{m+1})+$$
$$\sum_{p=1}^{n-1}\sum_{i=1}^{m} (-1)^{p+i}\,x^1_{m+1}\wedge\cdots\wedge\widehat{x^p_{m+1}}\wedge\cdots\wedge x^{n-1}_{m+1}\otimes f(x_1,\dots,\widehat{x_i},\dots,x_m,[x^1_i\dots,x^{n-1}_i,x^p_{m+1}])+$$
$$\sum_{i=1}^{m} (-1)^{i+1}\,x_i \triangleright (\Delta^m f)(x_1,\dots,\widehat{x_i},\dots,x_{m+1}) + (-1)^{m+1}\,(\Delta^m f)(x_1,\dots,x_m) \triangleleft x_{m+1}.$$

Next, we note that

$$\sum_{i=1}^{m} (-1)^{i+1}\,x_i \triangleright (\Delta^m f)(x_1,\dots,\widehat{x_i},\dots,x_{m+1}) =$$
$$\sum_{k=1}^{n-1}\sum_{i=1}^{m} (-1)^{i+1+k}\,x_i \triangleright \Big(x^1_{m+1}\wedge\cdots\wedge\widehat{x^k_{m+1}}\wedge\cdots\wedge x^{n-1}_{m+1}\otimes f(x_1,\dots,\widehat{x_i},\dots,x^k_{m+1})\Big) =$$
$$\sum_{\substack{p=1\\p\neq k}}^{n-1}\sum_{k=1}^{n-1}\sum_{i=1}^{m} (-1)^{i+1+k}\,x^1_{m+1}\wedge\cdots\wedge\widehat{x^k_{m+1}}\wedge\cdots\wedge[x^1_i\dots,x^{n-1}_i,x^p_{m+1}]\wedge\cdots$$
$$\cdots\wedge x^{n-1}_{m+1}\otimes f(x_1,\dots,\widehat{x_i},\dots,x_m,x^k_{m+1})+$$
$$\sum_{k=1}^{n-1}\sum_{i=1}^{m} (-1)^{i+1+k}\,x^1_{m+1}\wedge\cdots\wedge\widehat{x^k_{m+1}}\wedge\cdots\wedge x^{n-1}_{m+1}\otimes$$
$$\mu(x^1_i,\dots,x^{n-1}_i)\Big(f(x_1,\dots,\widehat{x_i},\dots,x^k_{m+1})\Big),$$

and that

$$(-1)^{m+1}\,(\Delta^m f)(x_1,\dots,x_m) \triangleleft x_{m+1} =$$

$$\sum_{k=1}^{n-1}(-1)^{m+1+k}\left(x_m^1\wedge\cdots\wedge\widehat{x_m^k}\wedge\cdots\wedge x_m^{n-1}\otimes f(x_1,\ldots,x_{m-1},x_m^k)\right)\triangleleft x_{m+1}=$$

$$\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}(-1)^{n+i+m+1+k}\,x_{m+1}^1\wedge\cdots\wedge\widehat{x_{m+1}^i}\wedge\cdots$$

$$\cdots\wedge x_{m+1}^{n-1}\otimes\mu(x_m^1\wedge\cdots\wedge\widehat{x_m^k}\wedge\cdots\wedge x_m^{n-1},x_{m+1}^i)\Big(f(x_1,\ldots,x_{m-1},x_m^k)\Big).$$

The result, then, follows. ◻

3. Generalized derivation extensions of n-Lie algebras

In the present section we shall present the cohomological classification of generalized derivations of n-Lie algebras. In the case of $n=3$ the result answers the question raised in [13, Rk. 3.7]. More precisely, we shall realize a generalized derivation of an n-Lie algebra as a 1-cocycle in a cohomology theory associated to n-Lie algebras, and an inner generalized derivation as a 1-coboundary.

3.1. Generalized derivations.

Given an n-Lie algebra $\mathcal{L}$, a linear operator $D:\mathcal{L}\to\mathcal{L}$ is called a *derivation* if

$$D([x_1,\ldots,x_n])=\sum_{k=1}^{n}[x_1,\ldots,D(x_k),\ldots,x_n]$$

for any $x_1,\ldots,x_n\in\mathcal{L}$. Accordingly, it follows at once from the fundamental identity (2.9) that

$$\mathrm{ad}(x_1,\ldots,x_{n-1}):\mathcal{L}\to\mathcal{L}$$

given by (2.12) is a derivation for any $x_1,\ldots,x_{n-1}\in\mathcal{L}$. Such derivations are called *inner derivations*.

A *generalized derivation* of an n-Lie algebra $\mathcal{L}$, on the other hand, is defined to be a linear map $D:\mathcal{L}_{n-1}\to\mathcal{L}$ satisfying

$$\begin{aligned}D(x_1,\ldots,x_{n-2},[x_{n-1},\ldots,x_{2n-2}])\\ =\sum_{k=0}^{n-1}[x_{n-1},\ldots,D(x_1,\ldots,x_{n-2},x_{n-1+k}),\ldots,x_{2n-2}],\end{aligned}\tag{3.1}$$

$$\begin{aligned}[x_1,\ldots,x_{n-1},D(x_n,\ldots,x_{2n-2})]+(-1)^n[D(x_1,\ldots,x_{n-1}),x_n,\ldots,x_{2n-2}]\\ =\sum_{k=0}^{n-2}D(x_n,\ldots,[x_1,\ldots,x_{n-1},x_{n+k}],\ldots,x_{2n-2}),\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}D(x_1,\ldots,x_{n-2},D(x_{n-1},\ldots,x_{2n-3}))\\ =\sum_{k=0}^{n-2}D(x_{n-1},\ldots,D(x_1,\ldots,x_{n-2},x_{n-1+k}),\ldots,x_{2n-3}),\end{aligned}\tag{3.3}$$

for any $x_1, \dots, x_{n-1} \in \mathcal{L}$, see [14, Def. 5.1].

It follows at once, as was noted in [14, Lemma 5.3], that the map $\mathrm{ad}_y : \mathcal{L}_{n-1} \to \mathcal{L}$ which is given by $\mathrm{ad}_y(z_1, \dots, z_{n-1}) := [y, z_1, \dots, z_{n-1}]$ is a generalized derivation for any $y \in \mathcal{L}$. Such a generalized derivation is called an *inner generalized derivation*.

Let us note also that $D : \mathcal{L}_{n-1} \to \mathcal{L}$ being a generalized derivation, $\mathcal{L} \oplus_D k := \mathcal{L} \oplus k$ is an n-Lie algebra through

$$[x_1 + \alpha_1 D, \dots, x_n + \alpha_n D] := [x_1, \dots, x_n] + \sum_{k=1}^{n} (-1)^{k+1} \alpha_k D(x_1, \dots, \widehat{x_k}, \dots, x_n) \tag{3.4}$$

for any $x_1 + \alpha_1 D, \dots, x_n + \alpha_n D \in \mathcal{L} \oplus_D k$.

Remarks on the conditions (3.1) and (3.3) are in order now.

Remark 3.1. Setting

$$D^\sharp : \mathcal{L}_{n-2} \to g\ell(\mathcal{L}), \qquad D^\sharp(x_1, \dots, x_{n-2})(y) := D(x_1, \dots, x_{n-2}, y), \tag{3.5}$$

we see at once that (3.1) is equivalent to $D^\sharp(x_1, \dots, x_{n-2}) \in g\ell(\mathcal{L})$ being a derivation of the n-Lie algebra $\mathcal{L}$, for any $x_1, \dots, x_{n-2}, y \in \mathcal{L}$.

Remark 3.2. The condition (3.3) is equivalent for the n-Lie algebra $\mathcal{L}$ to have the structure of an $(n-1)$-Lie algebra through

$$D : \mathcal{L}_{n-1} \to \mathcal{L}, \qquad x_1 \wedge \cdots \wedge x_{n-1} \mapsto D(x_1, \dots, x_{n-1}).$$

As for the condition (3.2), we have the following analogue of [13, Lemma 3.5]. Let us, however, record first the semi-direct sum Leibniz algebra $\mathcal{L} \rtimes \mathcal{L}_{n-1} := \mathcal{L} \oplus \mathcal{L}_{n-1}$ associated to an n-Lie algebra $\mathcal{L}$.

To begin with, $\mathcal{L}$ being an adjoint representation over itself via (2.12), it becomes a representation of the Leibniz algebra $\mathcal{L}_{n-1}$ through

$$x \triangleright y := [x^1, \dots, x^{n-1}, y], \qquad y \triangleleft x := -[x^1, \dots, x^{n-1}, y],$$

for any $y \in \mathcal{L}$ and any $x := x^1 \wedge \cdots \wedge x^{n-1} \in \mathcal{L}_{n-1}$. Then, $\mathcal{L} \oplus \mathcal{L}_{n-1}$ is a Leibniz algebra through

$$[y_1 + x_1, y_2 + x_2]_s := [x_1^1, \dots, x_1^{n-1}, y_2] - [x_2^1, \dots, x_2^{n-1}, y_1] + [x_1, x_2],$$

for any $y_1, y_2 \in \mathcal{L}$, and any $x_1 := x_1^1 \wedge \cdots \wedge x_1^{n-1}, x_2 := x_2^1 \wedge \cdots \wedge x_2^{n-1} \in \mathcal{L}_{n-1}$, where

$$[x_1, x_2] := \sum_{k=1}^{n-1} x_2^1 \wedge \cdots \wedge [x_1^1, \dots, x_1^{n-1}, x_2^k] \wedge \cdots \wedge x_2^{n-1} \in \mathcal{L}_{n-1}.$$

Accordingly, [13, Lemma 3.5] extends verbatim to the following.

Proposition 3.3. *Given an n-Lie algebra $\mathcal{L}$, let $D : \mathcal{L}_{n-1} \to \mathcal{L}$ be a generalized derivation. Then,*

$$\hat{D} : \mathcal{L} \rtimes \mathcal{L}_{n-1} \to \mathcal{L} \rtimes \mathcal{L}_{n-1}, \qquad \hat{D}(y + x) := D(x),$$

for any $y \in \mathcal{L}$ and any $x := x^1 \wedge \cdots \wedge x^{n-1} \in \mathcal{L}_{n-1}$, is a derivation of Leibniz algebras.

3.2. The cohomological classification of generalized derivations.

The present subsection accommodates the main result of the paper. Namely, we shall now introduce a new cohomology theory for n-Lie algebras that captures generalized derivations of n-Lie algebras as 1-cocycles. Accordingly, we shall begin with the presentation of this differential complex for n-Lie algebras.

Let $\mathcal{L}$ be an n-Lie algebra, and let (V, μ) be a representation of $\mathcal{L}$. Let also

$$C(\mathcal{L}, V) := \bigoplus_{m \geqslant 0} C^m(\mathcal{L}, V), \qquad C^m(\mathcal{L}, V) := \operatorname{Hom}\Big(\mathcal{L}_{n-1}^{\wedge (m-1)} \wedge \mathcal{L}_{n-2}, V\Big), \quad m \geqslant 1, \tag{3.6}$$

with $C^0(\mathcal{L}, V) := \mathcal{L} \otimes V$, and

$$\begin{aligned}
&\delta : C^0(\mathcal{L}, V) \to C^1(\mathcal{L}, V), \qquad \delta(z \otimes v)(y) := \mu(z, y^1, \ldots, y^{n-2})(v),\\
&\delta : C^m(\mathcal{L}, V) \to C^{m+1}(\mathcal{L}, V), \qquad m \geqslant 1\\
&\delta f(x_1, \ldots, x_m, y) :=\\
&\sum_{1 \leqslant i < j \leqslant m} (-1)^i f(x_1, \ldots, \widehat{x_i}, \ldots, [x_i, x_j], \ldots, x_m, y) +\\
&\qquad \sum_{i=1}^{m} (-1)^i f(x_1, \ldots, \widehat{x_i}, \ldots, x_m, [x_i, y]) +\\
&\qquad \sum_{i=1}^{m} (-1)^{i+1} \mu(x_i^1, \ldots, x_i^{n-1})\Big(f(x_1, \ldots, \widehat{x_i}, \ldots, x_m, y)\Big) +\\
&\qquad \sum_{i=1}^{n-1} (-1)^{m+i+1} \mu(x_m^i, y^1, \ldots, y^{n-2})\Big(f(x_1, \ldots, x_{m-1}, X_m^i)\Big),
\end{aligned} \tag{3.7}$$

for any $f \in C^m(\mathcal{L}, V)$, any $x_i := x_i^1 \wedge \cdots \wedge x_i^{n-1} \in \mathcal{L}_{n-1}$, and any $y := y^1 \wedge \cdots \wedge y^{n-2} \in \mathcal{L}_{n-2}$, where $X_m^i := x_m^1 \wedge \cdots \wedge \widehat{x_m^i} \wedge \cdots \wedge x_m^{n-1} \in \mathcal{L}_{n-2}$,

$$[x_i, x_j] := \sum_{k=1}^{n-1} x_j^1 \wedge \cdots \wedge [x_i^1, \ldots, x_i^{n-1}, x_j^k] \wedge \cdots \wedge x_j^{n-1} \in \mathcal{L}_{n-1},$$

and

$$[x_i, y] := \sum_{k=1}^{n-2} y^1 \wedge \cdots \wedge [x_i^1, \ldots, x_i^{n-1}, y^k] \wedge \cdots \wedge y^{n-2} \in \mathcal{L}_{n-2}.$$

We shall now show that (3.7) is indeed a differential. To this end, we first note from Proposition 2.4 that $\mathcal{L} \otimes V$ is a representation through

$$x \triangleright (z \otimes v) := [x^1, \ldots, x^{n-1}, z] \otimes v + z \otimes \mu(x^1, \ldots, x^{n-1})(v), \tag{3.8}$$

$$(z \otimes v) \triangleleft x := \sum_{k=1}^{n-1} (-1)^k x^k \wedge \mu(z, X^k)(v), \tag{3.9}$$

where $x := x^1 \wedge \cdots \wedge x^{n-1} \in \mathcal{L}_{n-1}$, and $X^k := x^1 \wedge \cdots \wedge \widehat{x^k} \wedge \cdots \wedge x^{n-1} \in \mathcal{L}_{n-2}$. Then we observe the following analogue of Proposition 2.5.

Proposition 3.4. *Given an n-Lie algebra $\mathcal{L}$, together with a representation (V, μ), the diagram*

$$\begin{array}{ccc} C^m(\mathcal{L}, V) & \xrightarrow{\Theta^m} & CL^m(\mathcal{L}_{n-1}, \mathcal{L} \otimes V) \\ \delta \downarrow & & \downarrow d \\ C^{m+1}(\mathcal{L}, V) & \xrightarrow{\Theta^{m+1}} & CL^{m+1}(\mathcal{L}_{n-1}, \mathcal{L} \otimes V) \end{array}$$

is commutative for any $m \geqslant 0$, where

$$\begin{aligned} &\Theta^m : C^m(\mathcal{L}, V) \to CL^m(\mathcal{L}_{n-1}, \mathcal{L} \otimes V), \qquad f \mapsto \Theta^m(f), \\ &\Theta^m(f)(x_1, \ldots, x_m) := \sum_{k=1}^{n-1} (-1)^{k+1}\, x_m^k \otimes f(x_1, \ldots, x_{m-1}, X_m^k), \qquad m \geqslant 1, \\ &\Theta^0 := -\operatorname{Id} : \mathcal{L} \otimes V \to \mathcal{L} \otimes V, \end{aligned} \tag{3.10}$$

and $X_m^k := x_m^1 \wedge \cdots \wedge \widehat{x_m^k} \wedge \cdots \wedge x_m^{n-1} \in \mathcal{L}_{n-2}$.

Proof. On the one hand we have

$$\begin{aligned} &\Theta^{m+1}(\delta f)(x_1, \ldots, x_{m+1}) = \sum_{k=1}^{n-1} (-1)^{k+1}\, x_{m+1}^k \otimes (\delta f)(x_1, \ldots, x_m, X_{m+1}^k) = \\ &\sum_{k=1}^{n-1} \sum_{1 \leqslant i < j \leqslant m} (-1)^{i+k+1}\, x_{m+1}^k \otimes f(x_1, \ldots, \widehat{x_i}, \ldots, [x_i, x_j], \ldots, x_m, X_{m+1}^k) + \\ &\sum_{k=1}^{n-1} \sum_{i=1}^{m} (-1)^{i+k+1}\, x_{m+1}^k \otimes f(x_1, \ldots, \widehat{x_i}, \ldots, x_m, [x_i, X_{m+1}^k]) + \\ &\sum_{k=1}^{n-1} \sum_{i=1}^{m} (-1)^{i+k}\, x_{m+1}^k \otimes \mu(x_i^1, \ldots, x_i^{n-1}) \Big(f(x_1, \ldots, \widehat{x_i}, \ldots, x_m, X_{m+1}^k) \Big) + \\ &\sum_{k=1}^{n-1} \sum_{i=1}^{n-1} (-1)^{i+k+m}\, x_{m+1}^k \otimes \mu(x_m^i, X_{m+1}^k) \Big(f(x_1, \ldots, x_m, X_m^i) \Big), \end{aligned}$$

while on the other hand,

$$\begin{aligned} &d(\Theta^m(f))(x_1, \ldots, x_{m+1}) = \\ &\quad \sum_{1 \leqslant i < j \leqslant m+1} (-1)^i\, (\Theta^m(f))(x_1, \ldots, \widehat{x_i}, \ldots, [x_i, x_j], \ldots, x_{m+1}) \\ &\quad + \sum_{i=1}^{m} (-1)^{i+1}\, x_i \triangleright \Big((\Theta^m(f))(x_1, \ldots, \widehat{x_i}, \ldots, x_{m+1}) \Big) \\ &\quad + (-1)^{m+1} \Big((\Theta^m(f))(x_1, \ldots, x_m) \Big) \triangleleft x_{m+1}, \end{aligned}$$

where on the latter, we note that

$$\sum_{1\leqslant i<j\leqslant m+1}(-1)^i\,(\Theta^m(f))(x_1,\ldots,\widehat{x_i},\ldots,[x_i,x_j],\ldots,x_{m+1})=$$
$$\sum_{1\leqslant i<j\leqslant m}(-1)^i\,(\Theta^m(f))(x_1,\ldots,\widehat{x_i},\ldots,[x_i,x_j],\ldots,x_{m+1})+$$
$$\sum_{i=1}^{m}(-1)^i\,(\Theta^m(f))(x_1,\ldots,\widehat{x_i},\ldots,[x_i,x_{m+1}])=$$
$$\sum_{k=1}^{n-1}\sum_{1\leqslant i<j\leqslant m}(-1)^{i+k+1}\,x^k_{m+1}\otimes f(x_1,\ldots,\widehat{x_i},\ldots,[x_i,x_j],\ldots,x_m,X^k_{m+1})+$$
$$\sum_{k=1}^{n-1}\sum_{i=1}^{m}(-1)^{i+k+1}\,x^k_{m+1}\otimes f(x_1,\ldots,\widehat{x_i},\ldots,[x_i,X^k_{m+1}])+$$
$$\sum_{k=1}^{n-1}\sum_{i=1}^{m}(-1)^{i+k+1}\,[x^1_i,\ldots,x^{n-1}_i,x^k_{m+1}]\otimes f(x_1,\ldots,\widehat{x_i},\ldots,X^k_{m+1}).$$

The result, now, follows from (3.8) and (3.9). More precisely,

$$\sum_{i=1}^{m}(-1)^{i+1}\,x_i\triangleright\Big((\Theta^m(f))(x_1,\ldots,\widehat{x_i},\ldots,x_{m+1})\Big)=$$
$$\sum_{k=1}^{n-1}\sum_{i=1}^{m}(-1)^{i+k}\,[x^1_i,\ldots,x^{n-1}_i,x^k_{m+1}]\otimes f(x_1,\ldots,\widehat{x_i},\ldots,X^k_{m+1})+$$
$$\sum_{k=1}^{n-1}\sum_{i=1}^{m}(-1)^{i+k}\,x^k_{m+1}\otimes\mu(x^1_i,\ldots,x^{n-1}_i)\Big(f(x_1,\ldots,\widehat{x_i},\ldots,x_m,X^k_{m+1})\Big),$$

and

$$(-1)^{m+1}\Big((\Theta^m(f))(x_1,\ldots,x_m)\Big)\triangleleft x_{m+1}=$$
$$(-1)^{m+1}\left(\sum_{k=1}^{n-1}(-1)^{k+1}\,x^k_m\otimes f(x_1,\ldots,x_{m-1},X^k_m)\right)\triangleleft x_{m+1}$$
$$\sum_{k=1}^{n-1}\sum_{i=1}^{n-1}(-1)^{i+k+m}\,x^i_{m+1}\otimes\mu(x^k_m,X^i_{m+1})\Big(f(x_1,\ldots,x_m,X^k_m)\Big).$$

□

Corollary 3.5. *Given an n-Lie algebra $\mathcal{L}$, together with a representation (V,μ), the mapping $\delta : C^n(\mathcal{L},V)\to C^{n+1}(\mathcal{L},V)$ for $n\geqslant 0$, given by (3.7) is a differential map, that is, it satisfies $\delta^2=0$.*

Proof. The claim follows at once from the injectivity of (3.10). □

We shall denote the homology $H(C(\mathcal{L},V),\delta)$ of the differential complex (3.6) by $\mathcal{H}^*(\mathcal{L},V)$. As is customary, we shall denote by $\mathcal{Z}^m(\mathcal{L},V)$ the space of m-cocycles, and by $\mathcal{B}^m(\mathcal{L},V)$ the space of m-coboundaries.

Remark 3.6. It is worth to note that the differentials (2.18) and (3.7), and also the differential complexes (2.17) and (3.6), coincide (only) for $n = 3$.

We are now ready to prove our main result.

Proposition 3.7. *Given an n-Lie algebra $\mathcal{L}$, let $D : \mathcal{L}_{n-1} \to \mathcal{L}$ be a linear mapping, with $D^\sharp : \mathcal{L}_{n-2} \to g\ell(\mathcal{L})$ being the associated mapping given by* (3.5). *Then, $D^\sharp \in \mathcal{Z}^1(\mathcal{L}, g\ell(\mathcal{L}))$ if* (3.2) *is satisfied.*

Proof. We have, for any $x := x^1\wedge\cdots\wedge x^{n-1} \in \mathcal{L}_{n-1}$, any $y := y^1\wedge\cdots\wedge y^{n-2} \in \mathcal{L}_{n-2}$, and any $z \in \mathcal{L}$,

$$(\delta D^\sharp)(x,y)(z) = -D^\sharp([x,y])(z) + \Big(\mu(x^1,\ldots,x^{n-1})(D^\sharp(y))\Big)(z)+$$
$$\sum_{i=1}^{n-1}(-1)^i\Big(\mu(x^i,y)(D^\sharp(X^i))\Big)(z),$$

where $X^i := x^1\wedge\cdots\wedge\widehat{x^i}\wedge\cdots\wedge x^{n-1} \in \mathcal{L}_{n-2}$, and

$$[x,y] := \sum_{k=1}^{n-2} y^1\wedge\cdots\wedge[x^1,\ldots,x^{n-1},y^k]\wedge\cdots\wedge y^{n-2} \in \mathcal{L}_{n-2}.$$

Accordingly,

$$(\delta D^\sharp)(x,y)(z) = -D([x,y],z) - D^\sharp(y)([x,z]) + \sum_{i=1}^{n-1}(-1)^{i+1}D^\sharp(X^i)([x^i,y,z]) =$$
$$-D([x,y],z) - D(y,[x,z]) - \sum_{i=1}^{n-1} D(x^1,\ldots,[y,z,x^i],\ldots,x^{n-1}).$$

Now, if (3.2) is satisfied, then

$$(\delta D^\sharp)(x,y)(z) =$$
$$-\Big([x,D(y,z)] + (-1)^n[D(x),y,z]\Big) - \Big([y,z,D(x)] + (-1)^n[D(y,z),x]\Big) = 0.$$

□

Remark 3.8. It is worth to note for $n = 3$ that assuming $D : \mathcal{L}_2 \to \mathcal{L}$ to satisfy (3.1), we may further conclude $D^\sharp \in \mathcal{Z}^1(\mathcal{L}, g\ell(\mathcal{L}))$ if and only if (3.2) holds.

Remark 3.9. Given $D : \mathcal{L}_{n-1} \to \mathcal{L}$, it follows at once from the definition of $\delta : C^0(\mathcal{L},V) \to C^1(\mathcal{L},V)$ that $D^\sharp \in \mathcal{B}^1(\mathcal{L}, g\ell(\mathcal{L}))$ if and only if there is $\xi\otimes f \in \mathcal{L}\otimes g\ell(\mathcal{L})$ so that $D^\sharp = \delta(\xi\otimes f)$, or equivalently

$$D(y,z) = D^\sharp(y)(z) = \Big(\mu(\xi,y)(f)\Big)(z) = -f([\xi,y,z]) = \mathrm{ad}_\xi(f)(y,z).$$

Accordingly, we may say that $D^\sharp \in \mathcal{B}^1(\mathcal{L}, g\ell(\mathcal{L}))$ if and only if $D = \mathrm{ad}_\xi(f)$ for some $\xi \in \mathcal{L}$ and $f \in g\ell(\mathcal{L})$. Compared to the definition of an inner generalized derivation in [13, Lemma 3.6] and [14, Lemma 5.3], inner generalized derivations $D : \mathcal{L}_{n-1} \to \mathcal{L}$ correspond to those that satisfy $D^\sharp = \delta(\xi \otimes \mathrm{Id})$ for some $\xi \in \mathcal{L}$, where $\mathrm{Id} \in g\ell(\mathcal{L})$ denotes the identity transformation.

Acknowledgement

The authors acknowledge the support by TÜBİTAK (the Scientific and Technological Research Council of Turkey) through the project "Matched pairs of Lagrangian and Hamiltonian Systems" with the project number 117F426.

References

[1] Bai, C.; Guo, L.; Sheng, Y. Bialgebras, the classical Yang-Baxter equation and Manin triples for 3-Lie algebras. *Adv. Theor. Math. Phys.* **23** (2019), no. 1, 27–74. MR4020309 Zbl 1479.17008 arXiv:1604.05996 doi: 10.4310/ATMP.2019.v23.n1.a2 1637, 1638, 1641, 1643, 1644

[2] Daletskii, Y.L.; Takhtajan, L.A. Leibniz and Lie algebra structures for Nambu algebra. *Lett. Math. Phys.* **39** (1997), no. 2, 127–141. MR1437747 Zbl 0869.58024 doi: 10.1023/A:1007316732705 1637, 1640, 1644, 1645

[3] Dzhumadil'daev, A.S. Representations of vector product n-Lie algebras. *Comm. Algebra* **32** (2004), no. 9, 3315–3326. MR2097463 Zbl 1121.17003 doi: 10.1081/AGB-120038644 1640

[4] Dzhumadil'daev, A.S. Representations of n-Lie algebras. arXiv:math/0202041 doi: 10.48550/arXiv.math/0202041 1640

[5] Feldvoss, J. Leibniz algebras as non-associative algebras. *Nonassociative mathematics and its applications*, 115–149. Contemp. Math., 721. *Amer. Math. Soc., [Providence], RI*, 2019. MR3898506 Zbl 1436.17004 arXiv:1802.07219 doi: 10.1090/conm/721/14502 1639

[6] Feldvoss, J.; Wagemann, F. On Leibniz cohomology. *J. Algebra* **569** (2021), 276–317. MR4187237 Zbl 1465.17006 arXiv:1902.06128 doi: 10.1016/j.jalgebra.2020.11.003 1639

[7] Filippov, V.T. n-Lie algebras. *Sibirsk. Mat. Zh.* **26** (1985), no. 6, 126–140, 191. 1637

[8] Gautheron, P. Some remarks concerning Nambu mechanics. *Lett. Math. Phys.* **37** (1996), no. 1, 103–116. MR1392151 Zbl 0849.70014 doi: 10.1007/BF00400143 1637

[9] Hochschild, G.; Serre, J.P. Cohomology of group extensions. *Trans. Amer. Math. Soc.* **74** (1953), 110–134. MR52438 Zbl 0050.02104 doi: 10.2307/1990851 1637

[10] Hochschild, G.; Serre, J.P. Cohomology of Lie algebras. *Ann. of Math. (2)* **57** (1953), 591–603. MR54581 Zbl 0053.01402 doi: 10.2307/1969740 1637

[11] Loday, J.L.; Pirashvili, T. Universal enveloping algebras of Leibniz algebras and (co)homology. *Math. Ann.* **296** (1993), no. 1, 139–158. MR1213376 Zbl 0821.17022 doi: 10.1007/BF01445099 1639

[12] Nambu, Y. Generalized Hamiltonian dynamics. *Phys. Rev. D (3)* **7** (1973), 2405–2412. MR455611 Zbl 1027.70503 doi: 10.1103/PhysRevD.7.2405 1637, 1640

[13] Song, L.; Jiang, J. Generalized derivation extensions of 3-Lie algebras and corresponding Nambu-Poisson structures. *J. Geom. Phys.* **124** (2018), 74–85. MR3754499 Zbl 1430.17010 doi: 10.1016/j.geomphys.2017.10.011 1637, 1638, 1639, 1648, 1649, 1654

[14] Song, L.; Tang, R. Cohomologies, deformations and extensions of n-Hom-Lie algebras. *J. Geom. Phys.* **141** (2019), 65–78. MR3927936 Zbl 1473.17013 arXiv:1707.07954 doi: 10.1016/j.geomphys.2019.03.003 1637, 1638, 1644, 1649, 1654

[15] Takhtajan, L. On foundation of the generalized Nambu mechanics. *Comm. Math. Phys.* **160** (1994), no. 2, 295–315. MR1262199 Zbl 0808.70015 arXiv:hep-th/9301111 doi: 10.1007/BF02103278 1637, 1640

[16] Zhang, T. Extending structures for 3-Lie algebras. *Comm. Algebra* **50** (2022), no. 4, 1469–1497. MR4391501 Zbl 7488676 arXiv:1401.4656 doi: 10.1080/00927872.2021.1984493 1637

(B. Ateşli) Gebze Technical University, Kocaeli, Turkey
b.atesli@gtu.edu.tr

(O. Esen) Gebze Technical University, Kocaeli, Turkey
oesen@gtu.edu.tr

(S. Sütlü) Işik University, Istanbul, Turkey
serkan.sutlu@isikun.edu.tr

This paper is available via http://nyjm.albany.edu/j/2022/28-72.html.