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Dynamical notions along filters

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ABSTRACT. We study the localization along a filter of several dynamical notions. This generalizes and extends similar localizations that have been considered in the literature, e.g. near 0 and near an idempotent. Definitions and basic properties of $\mathcal{F}$-syndetic, piecewise $\mathcal{F}$-syndetic, collectionwise $\mathcal{F}$-piecewise syndetic, $\mathcal{F}$-quasi central and $\mathcal{F}$-central sets and their relations with $\mathcal{F}$-uniformly recurrent points and ultrafilters are studied. We provide also the nonstandard characterizations of some of the above notions and we prove the partition regularity of several nonlinear equations along filters under mild general assumptions.

CONTENTS

1. Introduction

H. Furstenberg and B. Weiss first applied dynamical systems (topological dynamics) in Ramsey Theory in [13] and [14], starting the extremely fruitful use of ergodic methods in combinatorics, which has provided many fundamental results over the years. One of the reasons why these methods have been so successful is because, in many cases, dynamical descriptions of Ramsey-theoretical problems are simpler than the algebraic or combinatorial ones; in other cases, the dynamical notions that arise from the study of combinatorial problems are interesting enough to be studied for themselves. Later, starting with the work

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of V. Bergelson and N. Hindman, several ergodic notions found an equivalent characterization in terms of special kinds of ultrafilters: for example, central sets were related to minimal idempotents. In recent years, several papers have faced the problem of localizing known dynamical notions and results: for example, dynamical and combinatorial results near zero have been obtained in [1, 4, 17, 23, 27, 28], and similar studies near an idempotent have been done in [29, 32]. The interplay between algebra and dynamics has been studied near zero in [28] and it has been extended to idempotents in [29]. Motivated by [17] and [31], the most generalized notion of largeness along a filter was introduced in [32].

In this present work, we want to explore a setting, studied also in [10], that unifies and extends all those that have been developed so far, namely the notion of dynamics along a filter. We show that many classical definitions and properties of notions of large sets can be extended to dynamics along a filter: properties of $\mathcal{F}$-syndetic sets and $\mathcal{F}$-uniformly recurrent points are studied in Section 2, $\mathcal{F}$-quasi central sets and their dynamics are studied in Section 4, collectionwise $\mathcal{F}$-piecewise syndetic sets and $\mathcal{F}$-central sets are studied in Section 5 and relations between $\mathcal{F}$-uniformly recurrent points and minimality are studied in Section 6. Finally, in Section 7 we provide some applications of our results to the partition regularity of nonlinear Diophantine equations along filters. Moreover, since nonstandard methods in combinatorics have become ever more used in recent years (see [15], where nonstandard characterizations of all classical notions we will consider here are provided, and [11] for combinatorial applications), we also provide the nonstandard characterizations of the basic dynamical notions along a filter in Section 3; this is the only section where a basic knowledge of nonstandard analysis is required.

2. Basic results

We now present here some basic definitions, results, and theorems related to topological dynamics and the Stone-Čech compactification βS of a discrete semigroup S that will be used frequently in this paper. Throughout this paper, S is considered to be an arbitrary discrete semigroup (unless otherwise stated). A nonempty subset I of S is called a *left ideal* of S if $SI \subseteq I$, a *right ideal* if $IS \subseteq I$, and a *two-sided ideal* (or simply an *ideal*) if it is both a left and a right ideal. A *minimal left ideal* is a left ideal that does not contain any proper left ideal. Similarly, one can define the *minimal right ideal* and the *smallest ideal*. Any compact Hausdorff right topological semigroup S has the unique smallest two-sided ideal

$$\begin{aligned} K(S) &= \bigcup\{L : L \text{ is a minimal left ideal of } S\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } S\}. \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R of S, $L \cap R$ is a group and, in particular, $K(S)$ contains an idempotent(an element $a \in S$ is said to be an idempotent if $a = aa$). An idempotent that belongs to $K(S)$ is called a

minimal idempotent. We consider $E(S)$ to be the set of all idempotents in S throughout this paper. Now we state the well known definition of a dynamical system.

Definition 2.1. A *dynamical system* is a pair $(X, \langle T_s \rangle_{s\in S})$ such that

(1) X is a compact Hausdorff space,
(2) S is a semigroup,
(3) for each $s \in S$, $T_s : X \to X$ and T_s is continuous, and
(4) for all $s, t \in S$, $T_s \circ T_t = T_{st}$.

In topological dynamics several notions of largeness for sets are used. Among these, we recall those of syndeticity and piecewise syndeticity, that were defined as follow in [18, Definition 3.1]. We shall use $\mathcal{P}_f(X)$ to denote the set of all finite nonempty subsets of a set X.

Definition 2.2. Let $(S, \cdot)$ be a semigroup.

(1) A set $A \subseteq S$ is *syndetic* if there exists $G \in \mathcal{P}_f(S)$ with $S \subseteq \bigcup_{t\in G} t^{-1}A$, where $t^{-1}A = \{y \in S : ty \in A\}$.
(2) A set $A \subseteq S$ is *piecewise syndetic* if there exists $G \in \mathcal{P}_f(S)$ such that $\{y^{-1}(\bigcup_{t\in G} t^{-1}A) : y \in S\}$ has the finite intersection property.

We recall the definitions of proximality from [5, Definition 1.2(b)], $U(x)$ from [20, Definition 1.5(b)] and uniform recurrence in a dynamical system from [5, Definition 1.2(c)].

Definition 2.3. Let $(X, \langle T_s \rangle_{s\in S})$ be a dynamical system.

(1) A point $y \in X$ is *uniformly recurrent* if for every neighbourhood U of y, $\{s \in S : T_s(y) \in U\}$ is syndetic.
(2) For $x \in X$, $U(x) = U_X(x) = \{p \in \beta S : T_p(x)$ is[1] uniformly recurrent$\}$.
(3) The points x and y of X are *proximal* if for every neighbourhood U of the diagonal in $X \times X$, there is some $s \in S$ such that $(T_s(x), T_s(y)) \in U$.

To give the dynamical characterization of the members of certain idempotent ultrafilters, we need the following definition [21, Definition 2.1].

Definition 2.4. Let S be a nonempty discrete space, let $K \subseteq S$ and let $\mathcal{K}$ be a filter on S. We set

(a) $\overline{K} = \{p \in \beta S : K \in p\}$;
(b) $\overline{\mathcal{K}} = \{p \in \beta S : \mathcal{K} \subseteq p\}$;
(c) $\mathcal{L}(\mathcal{K}) = \{A \subseteq S : S \setminus A \notin \mathcal{K}\}$.

Clearly, $\overline{\mathcal{K}} \subseteq \beta S$ and $\overline{\mathcal{K}} = \bigcap_{K\in\mathcal{K}} \overline{K}$. It is to be noted that $\overline{\mathcal{K}} \subseteq \beta S$ is closed and contains all the ultrafilters on S that contain $\mathcal{K}$. Conversely, every nonempty closed subset C of βS admits such a representation, as $C = \overline{\bigcap_{p\in C} p}$.

When $\mathcal{K}$ is an idempotent filter, i.e., when $\mathcal{K} \subseteq \mathcal{K} \cdot \mathcal{K}$, $\overline{\mathcal{K}}$ is a semigroup; the converse is not always true. Here we have used the notion of *filter product*

[1] See Definition 2.12 for the definition of T_p.

which is defined as follows: given two filters $\mathcal{F}$ and $\mathcal{G}$ of a discrete semigroup S, $\mathcal{F}\cdot\mathcal{G} := \{A \subseteq S : \{x \in S : x^{-1}A \in \mathcal{G}\} \in \mathcal{F}\}$.

In this paper, we often focus on those filters $\mathcal{F}$ on S such that $\overline{\mathcal{F}}$ is a closed (hence compact) subsemigroup of βS. This condition guarantees the existence of $K(\overline{\mathcal{F}})$ which allows us to use the concept of minimal left ideals and the smallest ideals meaningfully. A characterization of such filters can be found in [8, 26] (see also [7] for a similar study using the notion of $(\mathcal{F},\mathcal{G})$-sets).

Our main goal is to study properties of known dynamical notions when the dynamics is localized along a filter. Let us recall the following definitions from [31].

Definition 2.5. Let $\mathcal{F}$ and $\mathcal{G}$ be two filters on S. A subset A of S is $(\mathcal{F},\mathcal{G})$ -*syndetic* if for every $V \in \mathcal{F}$, there is a finite set $F \subseteq V$ such that $F^{-1}A \in \mathcal{G}$, where $F^{-1}A = \{x \in S : (\exists f \in F) fx \in A\}$.

Definition 2.6. Let T be a closed subsemigroup of βS and $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}} = T$.

(1) A subset A of S is *$\mathcal{F}$-syndetic* if A is $(\mathcal{F},\mathcal{F})$-syndetic.
(2) A subset A of S is *piecewise $\mathcal{F}$-syndetic* if for every $V \in \mathcal{F}$, there is a finite $F_V \subseteq V$ and $W_V \in \mathcal{F}$ such that the family
$$\{(x^{-1}F_V^{-1}A)\cap V : V \in \mathcal{F}, x \in W_V\}$$
has the finite intersection property.

In [10], the analogous notion of uniformly recurrent and proximality along a filter were introduced. These notions will help us to discuss the dynamical characterizations of large sets along a filter.

Definition 2.7. Let $(X,\langle T_s\rangle_{s\in S})$ be a dynamical system. Let T be a closed subsemigroup of βS and $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}} = T$.

(1) A point $x \in X$ is *$\mathcal{F}$-uniformly recurrent* if for each neighbourhood W of x, $\{s \in S : T_s(x) \in W\}$ is $\mathcal{F}$-syndetic.
(2) Points x and y of X are *$\mathcal{F}$-proximal* if for every neighbourhood U of the diagonal in $X\times X$ and for each $V \in \mathcal{F}$ there exists $s \in V$ such that $(T_s(x), T_s(y)) \in U$.

We now recall some useful results from [10], [19], and [31]. The first is [31, Lemma 2.1].

Lemma 2.8. *Let T be a closed subsemigroup of βS, let L be a minimal left ideal of T, let $\mathcal{F}$ and $\mathcal{G}$ be the filters on S such that $\overline{\mathcal{F}} = T$ and $\overline{\mathcal{G}} = L$, and $A \subseteq S$. Then the following statements are equivalent:*

(1) $\overline{A}\cap L \neq \emptyset$;
(2) *A is $\mathcal{G}$-syndetic;*
(3) *A is $(\mathcal{F},\mathcal{G})$ -syndetic.*

As a consequence, we have the following result, that generalizes [31, Theorem 2.2], where only the equivalence between (a) and (b) below was shown.

Theorem 2.9. *Let T be a closed subsemigroup of βS, $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}} = T$, and $p \in T$. Then the following statements are equivalent:*

(a) $p \in K(T)$;
(b) *For all $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is $\mathcal{F}$-syndetic;*
(c) *For all $r \in T$, $p \in T \cdot r \cdot p$.*

Proof. That (a) implies (b) follows from the proof of [31, Theorem 2.2].

To prove that (b) implies (c), let $r \in T$. For each $A \in p$, let $B(A) = \{x \in S : x^{-1}A \in p\}$ and $C(A) = \{t \in S : t^{-1}B(A) \in r\}$. Observe that for any $A_1, A_2 \in p$, one has $B(A_1 \cap A_2) = B(A_1) \cap B(A_2)$ and $C(A_1 \cap A_2) = C(A_1) \cap C(A_2)$.

We claim that for every $A \in p$ and every $V \in \mathcal{F}$, $C(A) \cap V \neq \emptyset$. To see this, let $A \in p$ and $V \in \mathcal{F}$ be given and pick $F \in \mathcal{P}_f(V)$ such that $F^{-1}B(A) \in \mathcal{F}$ so that $F^{-1}B(A) \in r$. Hence there is some $t \in F$ with $t^{-1}B(A) \in r$. Then $t \in C(A) \cap V$. Thus $\{C(A) \cap V : A \in p \text{ and } V \in \mathcal{F}\}$ has the finite intersection property, so pick $q \in \beta S$ with $\{C(A) \cap V : A \in p \text{ and } V \in \mathcal{F}\} \subseteq q$. Then $q \in T$. We claim that $p = qrp$ for which it suffices (since both are ultrafilters) to show that $p \subseteq qrp$.

Let $A \in p$ be given. Then $\{t \in S : t^{-1}B(A) \in r\} = C(A) \in q$ so $B(A) \in qr$ so $A \in qrp$ as required.

Finally, to prove that (c) implies (a) it sufficies to pick $r \in K(T)$. □

The following theorem is based on [4, Theorem 2.5].

Theorem 2.10. *Let $A \subseteq S$, T be a closed subsemigroup of βS and $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}} = T$. Then the following statements are equivalent:*

(a) *A is piecewise $\mathcal{F}$-syndetic;*
(b) *There exists $e \in E(K(T))$ such that $\{x \in S : x^{-1}A \in e\}$ is $\mathcal{F}$-syndetic;*
(c) *There exists $e \in E(K(T))$ such that for every $V \in \mathcal{F}$ there exists $x \in V$ for which $x^{-1}A \in e$.*

Proof. Let us prove that (a) implies (b). By [31, Theorem 2.3], pick some $p \in K(T)$ with $A \in p$. Let L be a minimal left ideal of T with $p \in L$ and let $e \in L$ be an idempotent. Since $A \in p = pe$, we have $\{x \in S : x^{-1}A \in e\} \in p$. Now $e \in L \subseteq K(T)$ and by Theorem 2.9, we have $\{x \in S : x^{-1}A \in e\}$ is $\mathcal{F}$-syndetic.

That (b) implies (c) is trivial, as any $\mathcal{F}$-syndetic set has a non-empty intersection with sets in $\mathcal{F}$.

Finally, to prove that (c) implies (a) let $e \in E(K(T))$ be such that for every $V \in \mathcal{F}$ there exists $x \in V$ for which $x^{-1}A \in e$. Now choose $p \in T$ such that $E = \{x \in S : x^{-1}A \in e\} \in p$. Then $A \in pe$ and $pe \in K(T)$. So by [31, Theorem 2.3], A is piecewise $\mathcal{F}$-syndetic. □

Let X be a topological space and consider X^X with the product topology. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system; then, $\overline{\{T_s : s \in S\}}$ in X^X is a semigroup under the composition of mappings. The semigroup $\overline{\{T_s : s \in S\}}$ is the enveloping

semigroup of $(X, \langle T_s\rangle_{s\in S})$. [19, Theorem 19.11] shows a connection between a dynamical system and βS via the enveloping semigroup of $(X, \langle T_s\rangle_{s\in S})$.

Theorem 2.11. *Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and define $\theta : S \to X^X$ by $\theta(s) = T_s$. Let $\tilde{\theta}$ be the continuous extension of θ to βS. Then $\tilde{\theta}$ is a continuous homomorphism from βS onto the enveloping semigroup of $(X, \langle T_s\rangle_{s\in S})$.*

We now recall the Definition 19.12 from [19], which makes precise the meaning of T_p for any ultrafilter p.

Definition 2.12. Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and define $\theta : S \to X^X$ by $\theta(s) = T_s$. For each $p \in \beta S$, let $T_p = \tilde{\theta}(p)$.

As an immediate consequence of Theorem 2.11, we have the following remark (see [19, Remark 19.13]).

Remark 2.13. Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and let $p, q \in \beta S$. Then $T_p \circ T_q = T_{pq}$ and for each $x \in X$, $T_p(x) = p\text{-}\lim_{s\in S} T_s(x)$.

The relationships between T_p, for p ultrafilter, and dynamical notions localized to $\mathcal{F}$ will be fundamental in the following. We recall the last three known results that we need for our studies. The first is [10, Lemma 3.1].

Lemma 2.14. *Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and let $x, y \in X$. Then x and y are $\mathcal{F}$-proximal if and only if there is some $p \in \overline{\mathcal{F}}$ such that $T_p(x) = T_p(y)$.*

The second, that shows the relationships between some algebraic and dynamical notions, is [10, Theorem 3.2].

Lemma 2.15. *Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and L be a minimal left ideal of $\overline{\mathcal{F}}$ and $x \in X$. The following statements are equivalent:*

1. *The point x is an $\mathcal{F}$-uniformly recurrent point of $(X, \langle T_s\rangle_{s\in S})$.*
2. *There exists $u \in L$ such that $T_u(x) = x$.*
3. *There exist $y \in X$ and an idempotent $u \in L$ such that $T_u(y) = x$.*
4. *There exists an idempotent $u \in L$ such that $T_u(x) = x$.*

The last is [10, Lemma 3.3].

Lemma 2.16. *Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and let $x \in X$. Then for each $V \in \mathcal{F}$ there is a $\mathcal{F}$-uniformly recurrent point $y \in \overline{\{T_s(x) : s \in V\}}$ such that x and y are $\mathcal{F}$-proximal.*

3. Nonstandard characterizations

It is well known that the notions of syndetic and piecewise syndetic (and many other related ones) have simple characterizations in nonstandard terms. Such characterizations have been fundamental to develop many applications of these notions; we refer to [15] and the references therein. For this reason, we believe that it is relevant to generalize these nonstandard characterizations and proofs to our present setting of dynamics along filters; to this end, we will follow

the notation and, when possible, the proofs of [15]. In the present setting there are some additional technical difficulties with respect to the classical case.

Solely in this section, we assume the reader to be familiar with the basics of nonstandard analysis. The nonstandard take on ultrafilters that we discuss here has been used in many recent papers to produce several results in combinatorial number theory (see e.g. [11, 24, 25]). We recall its basic definition and facts.

We work in a setting that allows for iterated nonstandard extensions, which we assume to be sufficiently saturated. For any $A \subseteq S$, we inductively define $A^{(0)*} := A$ and[2] $A^{(n+1)*} := \left(A^{(n)*}\right)^*$; we set $A^{(\infty)*} := \bigcup_{n\in\mathbb{N}} A^{(n)*}$. We use the same kind of notation when the star map is applied to nonstandard objects: e.g., whenever $\alpha \in S^{(\infty)*}$ we let $\alpha^{(3)*} := \alpha^{***} = \left((\alpha^*)^*\right)^*$.

For $\alpha \in S^{(\infty)*}$, we set

$$\ell(\alpha) := \min\left\{n \in \mathbb{N} \mid \alpha \in S^{(n)*}\right\}.$$

Given $\mathcal{F}$ a filter on S we set

$$\mu(\mathcal{F}) := \{\alpha \in S^* \mid \forall A \in \mathcal{F}\ \alpha \in A^*\}$$

and

$$\mu_\infty(\mathcal{F}) := \left\{\alpha \in S^{(\infty)*} \mid \forall A \in \mathcal{F}\ \alpha \in A^{(\infty)*}\right\}.$$

$\mu_\infty(\mathcal{F})$ will be called the monad of $\mathcal{F}$.

Conversely, given $\alpha \in S^{(\infty)*}$, we let

$$\mathcal{U}_\alpha := \left\{A \subseteq S \mid \alpha \in A^{(\infty)*}\right\} = \left\{A \subseteq S \mid \alpha \in A^{(\ell(\alpha))*}\right\}.$$

For $\alpha, \beta \in S^{(\infty)*}$, we say that α, β are u-equivalent (notation: $\alpha \sim \beta$) if $\mathcal{U}_\alpha = \mathcal{U}_\beta$.

It is rather simple (see e.g. [24] for a detailed study of the properties of these nonstandard characterizations) to prove that, for all $\alpha \in S^{(\infty)*}$, $\mathcal{U}_\alpha \in \beta S$ and, conversely (assuming sufficient saturation), that $\mu(\mathcal{F})$ is nonempty for all $\mathcal{F}$ filters on S; for $\mathcal{U}$ nonprincipal, $\mu(\mathcal{U})$ will be infinite, its cardinality depending on that of S^*. Moreover,

$$\alpha \in \mu(\mathcal{F}) \Leftrightarrow \mathcal{F} \subseteq \mathcal{U}_\alpha,$$

namely $\mu(\mathcal{F}) = \bigcup_{\mathcal{U}\supseteq\mathcal{F}} \mu(\mathcal{U})$.

In general, $\mathcal{U}_\alpha \cdot \mathcal{U}_\beta \neq \mathcal{U}_{\alpha\cdot\beta}$; however, in our nonstandard setting, we have that[3] $\forall \alpha, \beta \in S^{(\infty)*}$

$$\mathcal{U}_\alpha \cdot \mathcal{U}_\beta = \mathcal{U}_{\alpha\cdot\beta^{(\ell(\alpha))*}}.$$

For this reason, we say that $\alpha \in S^{(\infty)*}$ is *idempotent* if $\alpha \sim \alpha \cdot \alpha^{(\ell(\alpha))*}$, i.e. if $\mathcal{U}_\alpha$ is idempotent, and that $\alpha \in S^*$ is $\mathcal{F}$*-minimal* if $\mathcal{U}_\alpha \in \overline{\mathcal{F}} \cap K(\beta S)$. We say that α is an $\mathcal{F}$*-minimal idempotent* if it is $\mathcal{F}$-minimal and idempotent. Notice that if α is $\mathcal{F}$-minimal and $\beta, \gamma \in \mathcal{F}^*$ then also $\alpha\beta^*$, $\gamma\alpha^*$, $\gamma\alpha^*\beta^{**}$ are minimal.

[2] When $n = 1, 2$ we use the more common and simpler notations A^*, A^{**}.

[3] In what follows and in most applications, one uses this formula with $\alpha, \beta \in S^*$; we wrote here the more general formulation as we will need to use it in two proofs.

For $X \subseteq \beta S$, we set

$$\mu(X) := \bigcup_{\mathcal{U} \in X} \mu(\mathcal{U});$$

conversely, for $A \subseteq S^{(\infty)*}$ we let

$$\pi(A) := \{\mathcal{U} \in \beta S \mid \exists \alpha \in A \; \mathcal{U} = \mathcal{U}_\alpha\}.$$

The nonstandard characterizations of $\mathcal{F}$-syndetic and piecewise $\mathcal{F}$-syndetic sets are given in the following proposition.

Proposition 3.1. *Let $A \subseteq S$ and $\mathcal{F}$ filter on S. The following equivalences hold:*

(1) *A is $\mathcal{F}$-syndetic if and only if there exists $\Gamma \subseteq \mu(\mathcal{F})$ hyperfinite with $\Gamma^{-1}A^* \in \mathcal{F}^*$;*
(2) *A is piecewise $\mathcal{F}$-syndetic if and only if there exists $W \in \mathcal{F}^*, \Gamma \subseteq \mu(\mathcal{F})$ hyperfinite and $\beta \in \mu(\mathcal{F})$ such that $\beta^* \in \bigcap_{x \in W} x^{-1}\Gamma^{-1}A^{**}$, i.e. such that $W\beta^* \subseteq \Gamma^{-1}A^{**}$.*

Proof. (1) Assume that A is $\mathcal{F}$-syndetic. For all $V \in \mathcal{F}$ set

$$I_V := \{G \in \wp_{fin}(V) \mid G^{-1}A \in \mathcal{F}\}.$$

The family $\{I_V\}_{V \in \mathcal{F}}$ has the finite intersection property, as $\emptyset \neq I_{V_1 \cap \cdots \cap V_n} \subseteq I_{V_1} \cap \cdots \cap I_{V_n}$ for all $n \in \mathbb{N}$ and $V_1, \ldots, V_n \in \mathcal{F}$. Let $\Gamma \in \bigcap_{V \in \mathcal{F}} V^*$. Then Γ is hyperfinite and $\Gamma \subseteq V^*$ for all $V \in \mathcal{F}$ by construction, hence $\Gamma \subseteq \mu(\mathcal{F})$. And, by transfer, as $\Gamma \in I_V^*$ it follows that $\Gamma^{-1}A^* \in \mathcal{F}^*$.

Conversely, let $\Gamma \subseteq \mu(\mathcal{F})$ hyperfinite be such that $\Gamma^{-1}A^* \in \mathcal{F}^*$. In particular, for all $V \in \mathcal{F}$ $\Gamma \in \wp_{fin}(V)^*$, so (with the same notations used above) $I_V^* \neq \emptyset$. By transfer, $I_V \neq \emptyset$, so A is $\mathcal{F}$-syndetic.

(2) A is piecewise $\mathcal{F}$-syndetic if and only if for all $V \in \mathcal{F}$ there are V_F, W_F as in Definition 2.10 such that the family

$$\mathcal{G} := \{(x^{-1}F_V^{-1}A) \cap V \mid V \in \mathcal{F}, x \in W_V\}$$

has the FIP. This is equivalent to say that there exists an ultrafilter $\mathcal{U}_\beta \in \beta S$ that extends $\mathcal{G}$ which, by definition of $\mathcal{U}_\beta$, is equivalent to say that there exists $\beta \in S^*$ such that $\forall V \in \mathcal{F} \; \forall x \in W_V \; \exists f \in F_V \; fx\beta \in A^*$. Now for all $V \in \mathcal{F}$ let

$$I_V(\beta) = \{(F_V, W_V) \mid \forall x \in W_V \exists f \in F_V fx\beta \in A^*\}.$$

The family $\{I_V\}_{V \in \mathcal{F}}$ has the FIP, as $I_{V_1}(\beta) \cap \cdots \cap I_{V_n}(\beta) \supseteq I_{V_1 \cap \cdots \cap V_n}(\beta) \neq \emptyset$. Pick $(\Gamma, W) \in \bigcap_{V \in \mathcal{F}} I_V(\beta)^*$. Then, by definition, $W \in \mathcal{F}^*, \Gamma \subseteq \mu(\mathcal{F})$ is hyperfinite and $W\beta^* \subseteq \Gamma^{-1}A^{**}$, as required.

Conversely, let β, W, Γ as in the hypothesis be given. Then for all $V \in \mathcal{F}$ the following holds true:

$$\exists W \in \mathcal{F}^* \; \exists \Gamma \in \wp_{fin}(V)^* \; \beta^* \in \left(\bigcap_{x \in W} x^{-1}\Gamma^{-1}A^{**} \right) \cap V^{**}.$$

By transfer then, for all $V \in \mathcal{F}$ we have that

$$\exists W_V \in F\ \exists F_V \in \wp_{fin}(V)\ \beta \in \left(\bigcap_{x\in W_V} x^{-1}F_V^{-1}A^*\right) \cap V^*.$$

For all $V \in \mathcal{F}$ take W_V, F_V as above. As $\beta \in \left(\bigcap_{x\in W_V} x^{-1}F_V^{-1}A^*\right) \cap V^*$, in particular it means that $\mathcal{U}_\beta$ extends $\mathcal{G} := \{(x^{-1}F_V^{-1}A) \cap V \mid V \in \mathcal{F}, x \in W_V\}$, therefore $\mathcal{G}$ has the FIP, which proves that A is piecewise $\mathcal{F}$-syndetic. □

Notice that, in the nonstandard characterization of piecewise $\mathcal{F}$-syndeticity, by using Lemma 3.3 below we could have additionally asked that $W \subseteq \mu(\mathcal{F})$.

To generalize the nonstandard characterizations of piecewise syndetic sets in terms of minimal points and central sets to dynamics along a filter $\mathcal{F}$, we will need some useful results about generators of filters $\mathcal{F}$, some of which require $\overline{\mathcal{F}}$ to be a semigroup.

Lemma 3.2. *Let $\mathcal{F}$ be a filter on S. Then for all $W \in \mathcal{F}^*$, for all $\mathcal{U} \in \overline{\mathcal{F}}$ $W \cap \mu(\mathcal{U}) \neq \emptyset$.*

Proof. We just have to observe that $W \in \mathcal{F}^* \subseteq \mathcal{U}^*$, hence $I_A := W \cap A^* \neq \emptyset$ for all $A \in \mathcal{U}$. As the family $\{I_A\}_{A\in\mathcal{U}}$ has the FIP, by saturation we deduce that $\bigcap_{A\in\mathcal{U}} W \cap A^* \neq \emptyset$, and any α in this intersection is, in particular, in the monad of $\mathcal{U}$. □

Lemma 3.3. *Let $\mathcal{F}$ be a filter on S. Then there exists $W \in \mathcal{F}^*$ such that $W \subseteq \mu(\mathcal{F})$.*

Proof. For all $A \in \mathcal{F}$ let $I_A = \{B \in \mathcal{F} \mid B \subseteq A\}$. Clearly, $I_A \neq \emptyset$ and $\{I_A\}_{A\in\mathcal{F}}$ has the FIP, therefore by enlarging $\bigcap_{A\in\mathcal{F}} I_A^* \neq \emptyset$. It remains to notice that any W in this intersection has the desired property. □

Theorem 3.4. *Let $\mathcal{F}$ be a filter on S and assume that $\overline{\mathcal{F}}$ is a semigroup. Let $\Gamma \subseteq \mu(\mathcal{F})$ hyperfinite, $W \in \mathcal{F}^*$ and $\alpha \in \mu(\mathcal{F})$. Let $L \subseteq \overline{\mathcal{F}}$ be the left ideal $L = \mathcal{F}\cdot\mathcal{U}_\alpha$. Then there exists $\tau \in \mu(\mathcal{F}) \cap W$ such that $\tau \sim \alpha$ and $\forall \gamma \in \Gamma\ \gamma\tau \in \mu(L)$.*

Proof. Let $A \subseteq S, \gamma \in \Gamma$. By definition,

$$A \in \mathcal{U}_\gamma \cdot \mathcal{U}_\alpha \Leftrightarrow \{s \in S \mid \{t \in S \mid st \in A\} \in \mathcal{U}_\alpha\} \in \mathcal{U}_\gamma,$$

hence

$$A \in \mathcal{U}_\gamma \cdot \mathcal{U}_\alpha \Leftrightarrow \gamma \in \{s \in S \mid \{t \in S \mid st \in A\} \in \mathcal{U}_\alpha\}^*.$$

As $\{s \in S \mid \{t \in S \mid st \in A\} \in \mathcal{U}_\alpha\}^* = \{\sigma \in S^* \mid \{\tau \in S^* \mid st \in A^*\} \in \mathcal{U}_\alpha^*\}$, this shows that

$$A \in \mathcal{U}_\gamma \cdot \mathcal{U}_\alpha \Leftrightarrow I_A^\gamma := \{\tau \in S^* \mid \gamma\tau \in A^*\} \in \mathcal{U}_\alpha^*.$$

In particular, by Lemma 3.3 pick $T \in \mathcal{U}_\alpha^*$ with $T \subseteq \mu(\mathcal{U}_\alpha)$. Let $\mathcal{G}$ be the filter on S such that $\overline{\mathcal{G}} = L$. If $A \in L$ then, as $\mathcal{U}_\gamma \cdot \mathcal{U}_\alpha \in L$, we have that $A \in \mathcal{U}_\gamma \cdot \mathcal{U}_\alpha$. So for all $\gamma \in \Gamma$ and $A \in \mathcal{G}$, $I_A^\gamma \in \mathcal{U}_\alpha$. Therefore also $I_A^\gamma \cap T \in \mathcal{U}_\alpha$ and, as Γ

is hyperfinite, we have that $I_A = \left(\bigcap_{\gamma\in\Gamma}\{\tau \in S^* \mid \gamma\tau \in A^*\}\right) \cap T \in \mathcal{U}_\alpha$. Notice that I_A is an internal set for all $A \in \mathcal{G}$, and that the family $\{I_A\}_{A\in\mathcal{G}}$ has the FIP. Hence, by saturation, $\bigcap_{A\in\mathcal{G}} I_A \neq \emptyset$.

We claim that any τ in the above nonempty intersection satisfies the conclusions of our Theorem. In fact, $\alpha \sim \tau$ as $\tau \in T \subseteq \mu(\mathcal{U}_\alpha)$ and, for all $\gamma \in \Gamma$, by construction $\gamma\tau \in A^*$ for all $A \in \mathcal{G}$, namely $\gamma\tau \in \mu(L)$. □

In the following, given $\alpha \in S^*$ and $A \subseteq S$, we let

$$A_\alpha := \{s \in S \mid s \cdot \alpha \in A^*\}.$$

First, we provide a nonstandard proof of the nonstandard formulation of Theorem 2.9.

Theorem 3.5. *Let $\alpha \in S^*$, let $\mathcal{F}$ be a filter on S and assume that $\overline{\mathcal{F}}$ is a semigroup. The following facts are equivalent:*

(1) *α is $\mathcal{F}$-minimal;*
(2) *$\forall A \in \mathcal{U}_\alpha$ A_α is $\mathcal{F}$-syndetic;*
(3) *$\forall\beta \in \mu(\mathcal{F})\ \exists\gamma \in \mu(\mathcal{F})$ such that $\alpha \sim \gamma \cdot \beta^* \cdot \alpha^{**}$.*

Proof. (1) ⇒ (2) Fix $A \in \mathcal{U}_\alpha$. As α is minimal, there is L minimal left ideal in $\overline{\mathcal{F}}$ such that $\mathcal{U}_\alpha \in L$. Let $\beta \in \mu(L)$. As L is minimal, there exists $\gamma \in \mu(\mathcal{F})$ such that $\gamma \cdot \beta^* \sim \alpha$. In particular, for all $F \in \mathcal{F}$ we have

$$\exists\gamma \in F^*\ \gamma \cdot \beta^* \in A^{**}.$$

By transfer, it follows that $\forall F \in \mathcal{F}\ \exists f \in F\ f \cdot \beta \in A^*$.

As the above is true for any $\beta \in \mu(L)$, it is in particular true for any object of the form $\delta \cdot \alpha^*$ with $\delta \in \mu(F)$. Thus,

$$\forall\delta \in \mu(\mathcal{F})\ \forall F \in \mathcal{F}\ \delta \cdot \alpha^* \in f^{-1}A^{**}.$$

Therefore,

$$\forall\delta \in \mu(\mathcal{F})\ \forall F \in \mathcal{F}\ \exists f \in F\ \gamma \cdot \alpha^* \in f^{-1}A^{**},$$

which means that

$$\gamma \in \{\eta \in S^* \mid f\eta\alpha^* \in A^{**}\} = \{s \in S \mid fs\alpha \in A^*\}^* = \left(f^{-1}A_\alpha\right)^*.$$

As $\gamma \in \mu(\mathcal{F})$, this shows that $f^{-1}A_\alpha \in \mathcal{F}$, hence that A_α is $\mathcal{F}$-syndetic.

(2) ⇒ (3) Fix $\beta \in \mu(\mathcal{F}), A \in \mathcal{U}_\alpha$. By hypothesis, A_α is $\mathcal{F}$-syndetic, namely $\forall F \in \mathcal{F}\ \exists H \in \wp_{fin}(F)\ H^{-1}A_\alpha \in \mathcal{F}$. As $\beta \in \mu(\mathcal{F})$, we have that $\beta \in \left(H^{-1}A\right)^* = H^{-1}(A^*)$, so $\beta \in f^{-1}A^*_\alpha$ for some $f \in H$. As, by transfer, $A_\alpha = \{s \in S \mid s \cdot \alpha \in A^*\}$, it follows that $\beta \in f^{-1}(A^*_\alpha) \Leftrightarrow f\beta \in A^*_\alpha \Leftrightarrow f\beta\alpha^* \in A^{**}$.

Now, for $F \in \mathcal{F}, A \in \mathcal{U}_\alpha$ let

$$\Gamma^A_F = \{f \in F \mid f\beta\alpha^* \in A^{**}\}.$$

The family $\{\Gamma^A_F\}_{F\in\mathcal{F},A\in\mathcal{U}_\alpha}$ has the FIP as, for all $n \in \mathbb{N}$,

$$\Gamma^{A_1}_{F_1} \cap \cdots \cap \Gamma^{A_1}_{F_1} \supseteq \Gamma^{A_1\cap\cdots\cap A_n}_{F_1\cap\cdots\cap F_n} \neq \emptyset.$$

By enlarging, $\bigcap_{F\in\mathcal{F},A\in\mathcal{U}_\alpha}\left(\Gamma_F^A\right)^* \neq \emptyset$. If γ belongs to this intersection, by construction $\gamma \in \mu(\mathcal{F})$ is such that $\gamma\cdot\beta^*\cdot\alpha^{**}\in\mu(\mathcal{U}_\alpha)$, i.e. $\gamma\cdot\beta^*\cdot\alpha^{**}\sim\alpha$.

(3) $\Rightarrow$ (1) Take $\beta\in\mu(\mathcal{F})$ minimal, take $\gamma\in\mu(\mathcal{F})$ such that $\alpha\sim\gamma\beta^*\alpha^{**}$. We conclude as $\gamma\beta^*\alpha^{**}$ is minimal. □

As a consequence, we have the following:

Theorem 3.6. *Let $A\subseteq S$, let $\mathcal{F}$ be a filter on S and let $\overline{\mathcal{F}}$ be a semigroup. Then A is piecewise $\mathcal{F}$-syndetic if and only if there exists an $\mathcal{F}$-minimal $\alpha\in A^*$.*

Proof. (1) $\Rightarrow$ (2) Let $\beta\in\mu(\mathcal{F}), W\in\mathcal{F}^*, \Gamma\subseteq\mu(\mathcal{F})$ hyperfinite be such that $W\beta^*\subseteq\Gamma^{-1}A^{**}$. By Lemma 3.2 and Lemma 3.3, we can assume that $W\subseteq\mu(\mathcal{F})$ and $\forall\alpha\in\mu(\mathcal{F})\ \exists\tau\in\mu(\mathcal{F})\alpha\sim\tau$. Hence, in particular, there exists $\alpha\in W$ that is $\mathcal{F}$-minimal. Let L be the left $\mathcal{F}$-ideal generated by $\mathcal{U}_\alpha$. By Theorem 3.4, in particular there exists $\tau\in W$ such that $\forall\gamma\in\Gamma\ \gamma\tau\in\mu(L)$. By hypothesis, there exists $\gamma\in\Gamma$ such that $\gamma\tau\beta^*\in A^{**}$. We just have to observe that $\gamma\tau\beta^{**}$ is minimal since $\gamma\tau$ is minimal, $\mathcal{U}_{\gamma\tau}\cdot\mathcal{U}_\beta\in K(\mathcal{F})$ and $\gamma\tau\beta^*\in\mu\left(\mathcal{U}_{\gamma\tau}\cdot\mathcal{U}_\beta\right)$.

(2) $\Rightarrow$ (1) Let $\alpha\in A^*$ minimal. By Theorem 3.5 it follows that A_α is $\mathcal{F}$-syndetic. Let $\Gamma\subseteq\mu(\mathcal{F})$ hyperfinite be such that $W=\Gamma^{-1}A_\alpha^*\in\mathcal{F}^*$. We claim that

$$\alpha^*\in\bigcap_{w\in W}w^{-1}\Gamma^{-1}A^{**},$$

which would conclude by Proposition 3.1. By definition, $w\in W$ if and only if there exists $\gamma\in\Gamma, a\in A_\alpha^*$ such that $\gamma w=a\in A_\alpha^*$. As $A_\alpha^*=\{\eta\in S^*\mid\eta\cdot\alpha^*\in A^{**}\}$, this shows that for all $w\in W$ there exists $\gamma\in\Gamma$ such that $\gamma w\alpha^*\in A^{**}$, which proves our claim. □

We can now prove a nonstandard version of Theorem 2.9.

Nonstandard Proof of Theorem 2.9. (1) $\Rightarrow$ (2) By Theorem 3.6, there is $\alpha\in A^*$ $\mathcal{F}$-minimal. Let L be the minimal left $\mathcal{F}$-ideal such that $\alpha\in L$. Let $\beta\in\mu(L)$ be a minimal idempotent. Then $A_\beta=\{s\in S\mid s\beta\in A^*\}\subseteq B$. If we prove that A_β is $\mathcal{F}$-syndetic, B is then $\mathcal{F}$-syndetic as well. As $\alpha,\beta\in\mu(L)$, there is $\gamma\in\mu(\mathcal{F})$ such that $\alpha\sim\gamma\beta^*$. Then $\alpha\beta^*\sim\gamma\beta^*\beta^{**}\sim\gamma\beta^*\sim\alpha$. As $\alpha\in A^*$, it follows that also $\alpha\beta^*\in A^*$, and this holds if and only if

$$\alpha\in\{s\in S\mid s\beta\in A^*\}^*.$$

This shows that $A_\beta\in\mathcal{U}_\alpha$ and, as α is minimal, we conclude that A_β is $\mathcal{F}$-syndetic by Theorem 3.5.

(2) $\Rightarrow$ (3) As B is $\mathcal{F}$-syndetic, there exists $\Gamma\subseteq\mu(\mathcal{F})$ hyperfinite with $\Gamma^{-1}B^*\in\mathcal{F}^*$. Given $\alpha\in\mu(\mathcal{F})$, let $\beta\in\mu(\mathcal{F})\cap\Gamma^{-1}B^*$ be given by Theorem 3.4, namely $\beta\sim\alpha$ and $\gamma\cdot\beta\in\mu(L)$ for all $\gamma\in\Gamma$, where L is the left $\mathcal{F}$-ideal generated by $\mathcal{U}_\alpha$. As $\beta\in\Gamma^{-1}B^*$, there exists $\gamma\in\Gamma$ such that $\gamma\beta\in B^*$. As $\gamma\cdot\beta\in L$ and $\overline{\mathcal{F}}$ is a semigroup, $\gamma\beta\in\mu(\mathcal{F})$; in particular, $\gamma\beta\in V^*\cap B^*$. By transfer, we have that $B\cap V\neq\emptyset$.

(3) $\Rightarrow$ (1) For all $V \in \mathcal{F}$ let

$$I_V = \{\mathcal{U} \in \beta S \mid \mathcal{U} \in \overline{K(\overline{F})} \text{ and } \exists x \in V\ x^{-1}A \in \mathcal{U}\}.$$

I_V is closed, $\overline{K(\overline{\mathcal{F}})}$ is compact and $\{I_V\}_{V\in\mathcal{F}}$ has the FIP, hence $\bigcap_{V\in\mathcal{F}} I_V \neq \emptyset$. Pick $\mathcal{U}$ in this intersection. Let $\beta \in \mu(\mathcal{U})$. Then for all $V \in \mathcal{F}$, by construction, the set $J_V = \{x \in V \mid \beta \in x^{-1}A^*\} \neq \emptyset$. As the family $\{J_V\}_{V\in\mathcal{F}}$ has the FIP, we find $\gamma \in \bigcap_{V\in\mathcal{F}} J_V^*$. In particular, $\gamma \in \mu(\mathcal{F})$ and $\gamma\beta^* \in A^{**}$. As β is minimal and $\gamma \in \mu(\mathcal{F})$, so is $\gamma\beta^*$, and we conclude by Theorem 3.6. □

4. $\mathcal{F}$-quasi-central sets and their dynamics

Quasi-central sets were first introduced in [18]. In [5] they were dynamically characterized. The second author of this paper studied quasi-central near zero in [28]. In [29] quasi-central sets near an idempotent of a semitopological semigroup were discussed extensively.

In this present section we study quasi-central sets along a filter which will generalize all the above settings. To move forward, we need to state the following definition and theorem from [21].

Definition 4.1. Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system, x and y be two points in X, and $\mathcal{K}$ be a filter on S. The pair (x, y) is called *jointly $\mathcal{K}$-recurrent* if and only if for every neighbourhood U of y we have that $\{s \in S : T_s(x) \in U \text{ and } T_s(y) \in U\} \in \mathcal{L}(\mathcal{K})$.

Theorem 4.2. *Let S be a semigroup, $\mathcal{K}$ be a filter on S such that $\overline{\mathcal{K}}$ is a compact subsemigroup of βS, and let $A \subseteq S$. Then A is a member of an idempotent in $\overline{\mathcal{K}}$ if there exists a dynamical system $(X, \langle T_s\rangle_{s\in S})$ with points x and y in X and there exists a neighbourhood U of y such that the pair (x, y) is jointly $\mathcal{K}$-recurrent and $A = \{s \in S : T_s(x) \in U\}$.*

Proof. See [21, Theorem 3.3]. □

In the above theorem (Theorem 4.2) the hypothesis that $\overline{\mathcal{K}}$ is a compact subsemigroup of βS guarantees the existence of $K(\overline{\mathcal{K}})$. Theorem 4.2 shows a beautiful relation between jointly $\mathcal{K}$-recurrent pairs and idempotent ultrafilters. Now we define quasi-central sets along a filter.

Definition 4.3. Let $\mathcal{F}$ be a filter on a semigroup S such that $\overline{\mathcal{F}}$ is a compact subsemigroup of βS and let $C \subseteq S$. Then C is said to be *$\mathcal{F}$-quasi-central* if and only if there is an idempotent p in $clK(\overline{\mathcal{F}})$ such that $C \in p$.

It is well known that piecewise syndetic sets in S can be characterized in terms of the closure of the smallest bilateral ideal of βS. [31, Theorem 2.3] generalizes this fact to piecewise $\mathcal{F}$-syndeticity.

Theorem 4.4. *Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a compact subsemigroup of βS and $A \subseteq S$. Then $K(\overline{\mathcal{F}}) \cap cl_{\beta S_d}(A) \neq \emptyset$ if A is piecewise $\mathcal{F}$-syndetic.*

As an immediate consequence, we get the following characterization.

Lemma 4.5. *Let $\mathcal{F}$ be a filter on a semigroup S such that $\overline{\mathcal{F}}$ is a compact subsemigroup of βS and let*

$$\mathcal{K} = \{A \subseteq S : S \setminus A \text{ is not piecewise } \mathcal{F}\text{-syndetic }\}.$$

Then $\mathcal{K}$ is a filter on S with $clK(\overline{\mathcal{F}}) = \overline{\mathcal{K}}$, which is a compact subsemigroup of βS_d.

Proof. By the construction of $\mathcal{K}$ and Theorem 4.4, we have $\mathcal{K} = \bigcap K(\overline{\mathcal{F}})$. Using Theorem 3.20(b) of [19], we have $\mathcal{K}$ is a filter and $\overline{\mathcal{K}} = clK(\overline{\mathcal{F}})$. By [19, Theorem 2.15], $clK(\overline{\mathcal{F}})$ is a right ideal of $\overline{\mathcal{F}}$, so in particular, $\overline{\mathcal{K}}$ is a compact subsemigroup of $\overline{\mathcal{F}}$. Therefore $clK(\overline{\mathcal{F}}) = \overline{\mathcal{K}}$ is a compact subsemigroup of βS. □

Let us now define jointly intermittently $\mathcal{F}$-uniform recurrence which will be helpful to give a dynamical characterization of quasi-central sets along a filter.

Definition 4.6. Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and let $x, y \in X$. The pair (x, y) is *jointly intermittently $\mathcal{F}$-uniformly recurrent* (abbreviated as $JI\mathcal{F}UR$) if for every neighbourhood U of y, the set $\{s \in S : T_s(x) \in U \text{ and } T_s(y) \in U\}$ is piecewise $\mathcal{F}$-syndetic.

Now we are in the position to characterize quasi-central sets dynamically along a filter in terms of $JI\mathcal{F}UR$ pairs.

Theorem 4.7. *Let $\mathcal{F}$ be a filter on a semigroup S such that $\overline{\mathcal{F}}$ is a compact subsemigroup of βS and let $A \subseteq S$. The set A is $\mathcal{F}$-quasi-central if and only if there exists a dynamical system $(X, \langle T_s\rangle_{s\in S})$, points x and y in X, and a neighbourhood U of y such that the pair (x, y) is $JI\mathcal{F}UR$ and $A = \{s \in S : T_s(x) \in U\}$.*

Proof. We shall prove this theorem using Theorem 4.2. Let

$$\mathcal{K} = \{B \subseteq S : S \setminus B \text{ is not a piecewise } \mathcal{F}\text{-syndetic set}\}.$$

Clearly, $\mathcal{L}(\mathcal{K}) = \{A \subseteq S : A \text{ is piecewise } \mathcal{F}\text{-syndetic }\}$. By Lemma 4.5, we have that $\mathcal{K}$ is a filter and $\overline{\mathcal{K}} = clK(\overline{\mathcal{F}})$ which is a compact subsemigroup of βS. Now we can apply Theorem 4.2 to prove our required statement. □

5. Combinatorial characterization of large sets along a filter

In [18] Hindman, Maleki, and Strauss gave combinatorial characterizations of central sets and quasi-central sets. To characterize these large sets, syndetic sets, piecewise syndetic sets, and collectionwise piecewise syndetic sets played significant roles. Motivated by this, in this section we want to study the combinatorial characterizations of large sets along a filter using the notions of $\mathcal{F}$-syndetic sets, piecewise $\mathcal{F}$-syndetic, and collectionwise piecewise $\mathcal{F}$-syndetic. So, at first, we need to define the notion of collectionwise piecewise $\mathcal{F}$-syndeticity for further discussions.

Definition 5.1. Let T be a closed subsemigroup of βS, $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}} = T$. A family $\mathcal{A} \subseteq \mathcal{P}(S)$ is *collectionwise piecewise $\mathcal{F}$-syndetic* if for every $V \in \mathcal{F}$ there exist functions $G_V : \mathcal{P}_f(\mathcal{A}) \longrightarrow \mathcal{P}_f(V)$ and $\delta_V : \mathcal{P}_f(\mathcal{A}) \longrightarrow \mathcal{P}(V) \cap \mathcal{F}$ such that for every $U \in \mathcal{F}$, every $F \in \mathcal{P}_f(S)$, every $\mathcal{H} \in \mathcal{P}_f(\mathcal{F})$, and every $\mathcal{C} \in \mathcal{P}_f(\mathcal{A})$, there is some $t \in U$ such that for every $V \in H$ and every $\mathcal{B} \in \mathcal{P}_f(\mathcal{C})$, $(F \cap \delta_V(\mathcal{B}))t \subseteq (G_V(\mathcal{B}))^{-1}(\cap\mathcal{B})$.

Note that if $\mathcal{F} = \{S\}$, then collectionwise piecewise $\mathcal{F}$-syndetic sets reduce to collectionwise piecewise syndetic collections.

In the following theorem we provide an algebraic characterization of collectionwise piecewise $\mathcal{F}$-syndetic sets.

Theorem 5.2. *Let T be a closed subsemigroup of βS, $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}} = T$ and $\mathcal{A} \subseteq \mathcal{P}(S)$. Then there exists $p \in K(T)$ such that $\mathcal{A} \subseteq p$ if and only if $\mathcal{A}$ is collectionwise piecewise $\mathcal{F}$-syndetic.*

Proof. To prove the necessity, we set $M(\mathcal{B}) = \{x \in S : x^{-1}(\cap\mathcal{B}) \in p\}$, for each $\mathcal{B} \in \mathcal{P}_f(\mathcal{A})$. Then by Theorem 2.9, $M(\mathcal{B})$ is a $\mathcal{F}$-syndetic in S. Thus for each $V \in \mathcal{F}$, pick $G_V(\mathcal{B}) \in \mathcal{P}_f(V)$ such that $(G_V(\mathcal{B}))^{-1}M(\mathcal{B}) \in \mathcal{F}$.

Let $\delta_V(\mathcal{B}) = (G_V(\mathcal{B}))^{-1}M(\mathcal{B}) \cap V$, then $\delta_V(\mathcal{B}) \in \mathcal{P}(V) \cap \mathcal{F}$. Hence for each $V \in \mathcal{F}$, we can define $G_V : \mathcal{P}_f(\mathcal{A}) \longrightarrow \mathcal{P}_f(V)$ and $\delta_V : \mathcal{P}_f(\mathcal{A}) \longrightarrow \mathcal{P}(V) \cap \mathcal{F}$. To see that these functions are as required, let $U \in \mathcal{F}$, $F \in \mathcal{P}_f(S)$, $\mathcal{H} \in \mathcal{P}_f(\mathcal{F})$, and $\mathcal{C} \in \mathcal{P}_f(\mathcal{A})$ be given. For each $(y, V, \mathcal{B})$ such that $V \in \mathcal{H}$, $\mathcal{B} \in \mathcal{P}_f(\mathcal{C})$, and $y \in \delta_V(\mathcal{B}) \cap F$, pick $x(y, V, \mathcal{B}) \in G_V(\mathcal{B})$ such that $x(y, V, \mathcal{B})y \in M(\mathcal{B})$, that is $(x(y, V, \mathcal{B})y)^{-1}(\cap\mathcal{B}) \in p$.

Let

$$\mathcal{D} = \{(x(y, V, \mathcal{B})y)^{-1}(\cap\mathcal{B}) : V \in \mathcal{H}, \mathcal{B} \in \mathcal{P}_f(\mathcal{C}) \text{ and } y \in \delta_V(\mathcal{B}) \cap F\}.$$

If $\mathcal{D} = \emptyset$, the conclusion is trivial, so we may assume $\mathcal{D} \neq \emptyset$ and hence $\mathcal{D} \in \mathcal{P}_f(p)$. Pick $t \in (\cap\mathcal{D}) \cap U$. Let $V \in \mathcal{H}$ and $\mathcal{B} \in \mathcal{P}_f(\mathcal{C})$ be given. If $F \cap \delta_V(\mathcal{B}) = \emptyset$, the conclusion holds, so assume $F \cap \delta_V(\mathcal{B}) \neq \emptyset$ and let $y \in F \cap \delta_V(\mathcal{B})$. Thus $yt \in (x(y, V, \mathcal{B}))^{-1}(\cap\mathcal{B}) \subseteq (G_V(\mathcal{B}))^{-1}(\cap\mathcal{B})$.

To prove the sufficiency, pick functions G_V and δ_V for each $V \in \mathcal{F}$ as guaranteed by the assumption that $\mathcal{A}$ is collectionwise piecewise $\mathcal{F}$-syndetic. Given $U \in \mathcal{F}$, $F \in \mathcal{P}_f(S)$, $\mathcal{H} \in \mathcal{P}_f(\mathcal{F})$, and $\mathcal{C} \in \mathcal{P}_f(\mathcal{A})$; pick $t(\mathcal{C}, \mathcal{H}, F, U) \in U$ such that for every $V \in \mathcal{H}$ and $\mathcal{B} \in \mathcal{P}_f(\mathcal{C})$, we have

$$(\delta_V(\mathcal{B}) \cap F)(t(\mathcal{C}, \mathcal{H}, F, U)) \subseteq (G_V(\mathcal{B}))^{-1}(\cap\mathcal{B}).$$

Now for each $\mathcal{B} \in \mathcal{P}_f(\mathcal{A})$, each $\mathcal{H} \in \mathcal{P}_f(\mathcal{F})$, and every $y \in S$, let $D(\mathcal{B}, \mathcal{H}, y) = \{t(\mathcal{C}, \mathcal{H}, F, U) : \mathcal{C} \in \mathcal{P}_f(\mathcal{A}) \text{ with } \mathcal{B} \subseteq \mathcal{C}, F \in \mathcal{P}_f(S) \text{ with } y \in F, \text{ and } U \in \mathcal{F}\}$. Then

$$\mathcal{D} = \{D(\mathcal{B}, \mathcal{H}, y) : \mathcal{B} \in \mathcal{P}_f(\mathcal{A}), \mathcal{H} \in \mathcal{P}_f(\mathcal{F}), y \in S\} \cup \mathcal{F}$$

has the finite intersection property.

Indeed, given sets

$$\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n \in \mathcal{P}_f(\mathcal{A}), \quad \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n \in \mathcal{P}_f(\mathcal{F}), \quad U_1, U_2, \ldots, U_n \in \mathcal{F},$$

and points $y_1, y_2, \dots, y_n \in S$, let $\mathcal{B} = \bigcup_{i=1}^n \mathcal{B}_i, \mathcal{H} = \bigcup_{i=1}^n \mathcal{H}_i, F = \{y_1, y_2, \cdots, y_n\}$, and $U = \bigcap_{i=1}^n U_i$. Then $t(\mathcal{B}, \mathcal{H}, F, U) \in \bigcap_{i=1}^n (D(\mathcal{B}_i, \mathcal{H}_i, y_i) \cap U_i)$. So pick $u \in T$ such that $\{D(\mathcal{B}, \mathcal{H}, y) : \mathcal{B} \in \mathcal{P}_f(\mathcal{A}), \mathcal{H} \in \mathcal{P}_f(\mathcal{F}), y \in S\} \subseteq u$.

Now we assert that for each $\mathcal{B} \in \mathcal{P}_f(\mathcal{A})$, each $\mathcal{H} \in \mathcal{P}_f(\mathcal{F})$, and each $V \in \mathcal{H}$, $Tu \subseteq \overline{(G_V(\mathcal{B}))^{-1}(\cap\mathcal{B})}$. To this end, pick $q \in T$ and let $A = (G_V(\mathcal{B}))^{-1}(\cap\mathcal{B})$. Now for all $V \in \mathcal{H}$, we claim that $\delta_V(\mathcal{B}) \subseteq \{y \in S : y^{-1}A \in u\}$ so that since $\delta_V(\mathcal{B}) \in q$, we have $A \in qu$. To prove our claim it suffices to show that $D(\mathcal{B}, \mathcal{H}, y) \subseteq y^{-1}A$ for all $y \in \delta_V(\mathcal{B})$. So, let $\mathcal{C} \in \mathcal{P}_f(\mathcal{A})$ with $\mathcal{B} \subseteq \mathcal{C}$, let $F \in \mathcal{P}_f(S)$ with $y \in F$, and let $V \in \mathcal{H}$ be given. Then for each $V \in \mathcal{H}$, $y \in F \cap \delta_V(\mathcal{B})$, so $yt(\mathcal{C}, \mathcal{H}, F, U) \in A$ as required. Let $L := Tu$, then

$$L \subseteq \bigcap_{\mathcal{B} \in \mathcal{P}_f(\mathcal{A})} \bigcap_{\mathcal{H} \in \mathcal{P}_f(\mathcal{F})} \bigcap_{V \in \mathcal{H}} \overline{(G_V(\mathcal{B}))^{-1}(\cap\mathcal{B})}.$$

We may assume L is a minimal left ideal of T. Pick $r \in L$. For each $\mathcal{B} \in \mathcal{P}_f(\mathcal{A})$, $\mathcal{H} \in \mathcal{P}_f(\mathcal{F})$, and each $V \in \mathcal{H}$, pick $x(\mathcal{B}, \mathcal{H}, V) \in G_V(\mathcal{B})$ such that $(x(\mathcal{B}, \mathcal{H}, V))^{-1}(\cap\mathcal{B}) \in r$. For each $\mathcal{B} \in \mathcal{P}_f(\mathcal{A})$ and $\mathcal{H} \in \mathcal{P}_f(\mathcal{F})$, let $\varepsilon(\mathcal{B}, \mathcal{H}) = \{x(\mathcal{C}, \mathcal{H}, V) : \mathcal{C} \in \mathcal{P}_f(\mathcal{A}), \mathcal{B} \subseteq \mathcal{C}, \text{ and } V \in \mathcal{H}\}$. We establish that

$$\{\varepsilon(\mathcal{B}, \mathcal{H}) : \mathcal{B} \in \mathcal{P}_f(\mathcal{A}), \mathcal{H} \in \mathcal{P}_f(\mathcal{F})\} \cup \mathcal{F}$$

has the finite intersection property.

Indeed, given

$$\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_m \in \mathcal{P}_f(\mathcal{A}); \quad \mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m \in \mathcal{P}_f(\mathcal{F}); \quad U_1, U_2, \dots, U_m \in \mathcal{F},$$

let $V = \bigcap_{i=1}^m U_i$, $\mathcal{H} = (\bigcup_{i=1}^m \mathcal{H}_i) \cup \{V\}$, and $\mathcal{C} = \bigcup_{i=1}^m \mathcal{B}_i$. Then $x(\mathcal{C}, \mathcal{H}, V) \in \bigcap_{i=1}^m \varepsilon(\mathcal{B}_i, \mathcal{H}_i) \cap \bigcap_{i=1}^m U_i$. So, pick $w \in T$ such that $\{\varepsilon(\mathcal{B}, \mathcal{H}) : \mathcal{B} \in \mathcal{P}_f(\mathcal{A}), \mathcal{H} \in \mathcal{P}_f(\mathcal{F})\} \subseteq w$. Let $p = wr$. Then $p \in L \subseteq K(T)$. To see that $\mathcal{A} \subseteq p$, let $A \in \mathcal{A}$. We show that $\varepsilon(\{A\}, \mathcal{H}) \subseteq \{x \in S : x^{-1}A \in r\}$ for all $\mathcal{H} \in \mathcal{P}_f(\mathcal{F})$. Let $\mathcal{B} \in \mathcal{P}_f(\mathcal{A})$ with $A \in \mathcal{B}$ and let $V \in \mathcal{H}$. Then $(x(\mathcal{B}, \mathcal{H}, V))^{-1}(\cap\mathcal{B}) \in r$. So, $(x(\mathcal{B}, \mathcal{H}, V))^{-1}A \in r$ because $\cap\mathcal{B} \subseteq A$. Hence $\varepsilon(\{A\}, \mathcal{H}) \subseteq \{x \in S : x^{-1}A \in r\}$ for each $\mathcal{H} \in \mathcal{P}_f(\mathcal{F})$. Therefore, $\{x \in S : x^{-1}A \in r\} \in w$ and so $A \in wr = p$, where $p \in L \subseteq K(T)$, as required. □

We recall the notion of tree below. We let $\omega = \{0, 1, 2, ...\}$ be the first transfinite ordinal number; we recall that in Von Neumann representation each ordinal can be identified with the set of its predecessors.

Definition 5.3. $\mathcal{T}$ is a *tree* in A if $\mathcal{T}$ is a set of functions and for each $f \in \mathcal{T}$, $\text{domain}(f) \in \omega$ and $\text{range}(f) \subseteq A$ and if $\text{domain}(f) = n > 0$, then $f|_{n-1} \in \mathcal{T}$. $\mathcal{T}$ is a tree if for some A, $\mathcal{T}$ is a tree in A.

Definition 5.4. We fix the following notations.

(a) Let f be a function with $\text{domain}(f) = n \in \omega$ and let x be given. Then $f \frown x = f \cup \{(n, x)\}$.
(b) Given a tree $\mathcal{T}$ and $f \in \mathcal{T}$, $B_f = B_f(\mathcal{T}) = \{x : f \frown x \in \mathcal{T}\}$

(c) Let S be semigroup and let $A \subseteq S$. Then $\mathcal{T}$ is a *$*$-tree* in A if $\mathcal{T}$ is a tree in A and for all $f \in \mathcal{T}$ and all $x \in B_f$, $B_{f^\frown x} \subseteq x^{-1}B_f$.

(d) Let S be semigroup and let $A \subseteq S$. Then $\mathcal{T}$ is a *FS-tree* in A if $\mathcal{T}$ is a tree in A and for all $f \in \mathcal{T}$,

$$B_f = \left\{\prod_{t\in F} g(t) : g \in \mathcal{T}, f \subseteq g, \text{ and } \emptyset \neq F \subseteq \text{domain}(g) \setminus \text{domain}(f)\right\}.$$

First, let us recall two results about FS-trees. The first is [18, Lemma 3.6].

Lemma 5.5. *Let S be semigroup and let p be an idempotent in βS. If $A \in p$, then there is a FS-tree $\mathcal{T}$ in A such that for each $f \in \mathcal{T}$, $B_f \in p$.*

The second is [17, Lemma 4.6].

Lemma 5.6. *Any FS-tree is a $*$-tree.*

We can now prove the equivalences of several tree properties when localizing along a filter using the notion of *$\mathcal{F}$-central set*. This notion has been studied deeply in [16].

Definition 5.7. Let S be a discrete semigroup and let $\mathcal{F}$ be a filter on S where $\overline{\mathcal{F}}$ is a subsemigroup of βS. Then a set $C \subseteq S$ is said to be *$\mathcal{F}$-central* if there exists an idempotent $p \in K(\overline{\mathcal{F}})$ such that $C \in p$.

Theorem 5.8. *Let T be a closed subsemigroup of βS, $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}} = T$ and let $A \subseteq S$.*
Statements (a), (b), (c), and (d) are equivalent and implied by statement (e). If S is countable, all five statements are equivalent.

(a) *A is $\mathcal{F}$-central.*

(b) *There is a FS-tree $\mathcal{T}$ in A such that $\{B_f : f \in \mathcal{T}\}$ is collectionwise piecewise $\mathcal{F}$-syndetic.*

(c) *There is a $*$-tree $\mathcal{T}$ in A such that $\{B_f : f \in \mathcal{T}\}$ is collectionwise piecewise $\mathcal{F}$-syndetic.*

(d) *There is a downward directed family $\langle C_F\rangle_{F\in I}$ of subsets of A such that*
  (i) *for all $F \in I$ and all $x \in C_F$, there is some $G \in I$ with $C_G \subseteq x^{-1}C_F$*
  (ii) *$\{C_F : F \in I\}$ is collectionwise piecewise $\mathcal{F}$-syndetic.*

(e) *There is a decreasing sequence $\langle C_n\rangle_{n=1}^{\infty}$ of subsets of A such that*
  (i) *for all $n \in \mathbb{N}$ and all $x \in C_n$, there is some $m \in \mathbb{N}$ with $C_m \subseteq x^{-1}C_n$ and*
  (ii) *$\{C_n : n \in \mathbb{N}\}$ is collectionwise piecewise $\mathcal{F}$-syndetic.*

Proof. (a) implies (b). By Lemma 5.5, pick a FS-tree $\mathcal{T}$ in A such that for each $f \in \mathcal{T}$, $B_f \in p$. By Theorem 5.2, $\{B_f : f \in \mathcal{T}\}$ is collectionwise piecewise $\mathcal{F}$-syndetic.

That (b) implies (c) follows from Lemma 5.6.

(c) implies (d). Let $\mathcal{T}$ be given as guaranteed by (c). Let $I = \mathcal{P}_f(\mathcal{T})$ and for $F \in I$, let $C_F = \bigcap_{f\in F} B_f$. Since $\{B_f : f \in T\}$ is collectionwise piecewise $\mathcal{F}$-syndetic, so is $\{C_F : F \in I\}$. Given $F \in I$ and $x \in C_F$, let $G = \{f^\frown x :$

$f \in F\}$. For each $f \in F$ we have $B_{f\frown x} \subseteq x^{-1}B_f$ by the definition of $*$-tree so $C_G = \bigcap_{f\in F} B_{f\frown x} \subseteq \bigcap_{f\in F} x^{-1}B_f = x^{-1}C_F$.

(d) implies (a). Let $M = \bigcap_{F\in I} \overline{C_F}$. We claim that M is a subsemigroup of βS. To this end, let $p, q \in M$ and let $F \in I$. To see that $C_F \in pq$, we show that $C_F \subseteq \{x \in S : x^{-1}C_F \in q\}$. Let $x \in C_F$ and pick $G \in I$ such that $C_G \subseteq x^{-1}C_F$. Then $C_G \in q$ so $x^{-1}C_F \in q$.

By Theorem 5.2 we have $M \cap K(T) \neq \emptyset$. Since K is the union of all minimal left ideal of T (see [3, Theorem 1.3.11]), pick a minimal left ideal L of $K(T)$ with $M \cap L \neq \emptyset$. Then $M \cap L$ is a compact semigroup so by [12, Corollary 2.10], there is some $p = p \cdot p$ in $M \cap L$. Since each $C_F \subseteq A$, we have $p \in K(T) \cap \overline{A}$.

That (e) implies (d) is trivial.

Now assume that S is countable. We show that (c) implies (e), so let $\mathcal{T}$ be as guaranteed by (c). Since $\mathcal{T}$ is countable, enumerate $\mathcal{T}$ as $\langle f_n\rangle_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, let $C_n = \bigcap_{k=1}^{n} B_{f_k}$. Then $\{C_n : n \in \mathbb{N}\}$ is collectionwise piecewise $\mathcal{F}$-syndetic. Let $n \in \mathbb{N}$ be given and let $x \in C_n$. Then for each $k \in \mathbb{N}$, $B_{f_k\frown x} \subseteq x^{-1}B_{f_k}$. Pick $m \in \mathbb{N}$ such that $\{f_k \frown x : k \in \{1, 2, \dots, n\}\} \subseteq \{f_k : k \in \{1, 2, \dots, m\}\}$. Then $C_m \subseteq x^{-1}C_n$. □

Theorem 5.9. *Let T be a closed subsemigroup of βS, $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}} = T$ and let $A \subseteq S$.*
Statements (a), (b), (c), and (d) are equivalent and implied by statement (e). If S is countable, all five statements are equivalent.

(a) *A is $\mathcal{F}$-quasi-central.*
(b) *There is a FS-tree $\mathcal{T}$ in A such that for each $F \in \mathcal{P}_f(\mathcal{T})$, $\bigcap_{f\in F} B_f$ is piecewise $\mathcal{F}$-syndetic.*
(c) *There is a $*$-tree $\mathcal{T}$ in A such that for each $F \in \mathcal{P}_f(\mathcal{T})$, $\bigcap_{f\in F} B_f$ is piecewise $\mathcal{F}$-syndetic.*
(d) *There is a downward directed family $\langle C_F\rangle_{F\in I}$ of subsets of A such that*
  (i) *for each $F \in I$ and each $x \in C_F$, there exists $G \in I$ with $C_G \subseteq x^{-1}C_F$ and*
  (ii) *for each $F \in I$, C_F is piecewise $\mathcal{F}$-syndetic.*
(e) *There is a decreasing sequence $\langle C_n\rangle_{n=1}^{\infty}$ of subsets of A such that*
  (i) *for each $n \in \mathbb{N}$ and each $x \in C_n$, there exists $m \in \mathbb{N}$ with $C_m \subseteq x^{-1}C_n$ and*
  (ii) *For each $n \in \mathbb{N}$, C_n is piecewise $\mathcal{F}$-syndetic.*

Proof. The proof is the same as Theorem 5.8. □

6. Minimal systems along filters

In Section 2 of [20] Hindman, Strauss, and Zamboni presented some well known results about $U(x)$ (see Definition 2.3(2)) that are true in an arbitrary dynamical system as well as the few simple results in $(\beta S, \langle\lambda_s\rangle_{s\in S})$ such as:

(i) $U(x) = \beta S$ if x is uniformly recurrent;

(ii) for every $x \in X$, $U(x)$ is a left ideal of βS containing $K(\beta S)$;
(iii) $\bigcap_{x\in X} U(x)$ is a left as well as a right ideal of βS;
(iv) $K(\beta S)$ is not prime, under some weak cancellation assumptions.

These results were studied near an idempotent of a semitopological semigroup in [29]. In this present section, we shall establish these results along a filter, i.e., in a more general setting.

Definition 6.1. Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and $x \in X$. Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a closed subsemigroup of βS, then

(1) $U_{\mathcal{F}}(x) = U_{\mathcal{F}_X}(x) = \{p \in \beta S : T_p(x)$ is $\mathcal{F}$-uniformly recurrent$\}$.
(2) A subspace Z of X is called *$\mathcal{F}$-invariant* if $T_p(Z) \subseteq Z$ for every $p \in \overline{\mathcal{F}}$.

Lemma 6.2. *Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a closed subsemigroup of βS. Then the following are equivalent:-*

(1) *x is $\mathcal{F}$-uniformly recurrent.*
(2) *There exists $q \in K(\overline{\mathcal{F}})$ such that $T_q(x) = x$.*
(3) *There exists $y \in X$ and $q \in K(\overline{\mathcal{F}})$ such that $T_q(y) = x$.*

Proof. Since the smallest ideal $K(\overline{\mathcal{F}})$ is the union of all minimal left ideals of $\overline{\mathcal{F}}$, the equivalences all follow from Lemma 2.15. □

Corollary 6.3. *Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and $x \in X$. Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a closed subsemigroup of βS, then*

(1) *if x is $\mathcal{F}$-uniformly recurrent then $\overline{\mathcal{F}} \subseteq U_{\mathcal{F}}(x)$,*
(2) *for each $x \in X$, $K(\overline{\mathcal{F}}) \subseteq U_{\mathcal{F}}(x)$,*
(3) *for each $x \in X$, $U_{\mathcal{F}}(x) \cap \overline{\mathcal{F}}$ is a left ideal of $\overline{\mathcal{F}}$,*
(4) *$(\bigcap_{x\in X} U_{\mathcal{F}}(x)) \bigcap \overline{\mathcal{F}}$ is a two-sided ideal of $\overline{\mathcal{F}}$.*

Proof. (1) Suppose that x is $\mathcal{F}$-uniformly recurrent. Then by Lemma 6.2, $T_u(x) = x$ for some $u \in K(\overline{\mathcal{F}})$. Thus for every $v \in \overline{\mathcal{F}}$, $T_v(x) = T_v(T_u(x)) = T_{vu}(x)$. Now since $vu \in K(\overline{\mathcal{F}})$, by Lemma 2.15 $T_v(x)$ is $\mathcal{F}$-uniformly recurrent and thus, $v \in U_{\mathcal{F}}(x)$. Therefore $\overline{\mathcal{F}} \subseteq U_{\mathcal{F}}(x)$.
(2) This is immediate from Lemma 2.15.
(3) Let $x \in X$, $p \in U_{\mathcal{F}}(x) \cap \overline{\mathcal{F}}$ and $r \in \overline{\mathcal{F}}$. By Lemma 2.15 pick $q \in K(\overline{\mathcal{F}})$ such that $T_q(T_p(x)) = T_p(x)$. Then $T_{rp}(x) = T_r(T_q(T_p(x))) = T_{rqp}(x)$. Now $rqp \in K(\overline{\mathcal{F}})$. So by Lemma 2.15, $T_{rp}(x)$ is $\mathcal{F}$-uniformly recurrent and hence $rp \in U_{\mathcal{F}}(x) \cap \overline{\mathcal{F}}$. Therefore $U_{\mathcal{F}}(x) \cap \overline{\mathcal{F}}$ is a left ideal of $\overline{\mathcal{F}}$.
(4) By (2) $(\bigcap_{x\in X} U_{\mathcal{F}}(x)) \bigcap \overline{\mathcal{F}}$ is nonempty. So by (3), $(\bigcap_{x\in X} U_{\mathcal{F}}(x)) \bigcap \overline{\mathcal{F}}$ is a left ideal of $\overline{\mathcal{F}}$. So it is enough to show that $(\bigcap_{x\in X} U_{\mathcal{F}}(x)) \bigcap \overline{\mathcal{F}}$ is a right ideal of $\overline{\mathcal{F}}$. To this end, let $p \in (\bigcap_{x\in X} U_{\mathcal{F}}(x)) \bigcap \overline{\mathcal{F}}$ and $q \in \overline{\mathcal{F}}$. Suppose $y \in X$ then $p \in U_{\mathcal{F}}(T_q(y))$. Thus $T_{pq}(y)$ is $\mathcal{F}$-uniformly recurrent and so $pq \in U_{\mathcal{F}}(y)$. □

The proofs of the following Lemma and the next Theorem follow closely the arguments of Hindman, Strauss, and Zamboni in [20].

Lemma 6.4. *Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system and let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a closed subsemigroup of βS. Let L be a minimal left ideal of $\overline{\mathcal{F}}$, then*

(1) *a subspace Y of X is minimal among all closed and $\mathcal{F}$-invariant subspaces of X if and only if there is some $x \in X$ such that $Y = \{T_p(x) : p \in L\}$,*
(2) *let Y be a subspace of X which is minimal among all closed and $\mathcal{F}$-invariant subspaces of X. Then every element of Y is $\mathcal{F}$-uniformly recurrent,*
(3) *if $x \in X$ is $\mathcal{F}$-uniformly recurrent and $Y = \{T_p(x) : p \in \overline{\mathcal{F}}\}$, then Y is minimal among all closed and $\mathcal{F}$-invariant subspaces of X,*
(4) *if $x \in X$ is $\mathcal{F}$-uniformly recurrent then $T_p(x)$ is $\mathcal{F}$-uniformly recurrent for every $p \in \overline{\mathcal{F}}$.*

Proof. (1) Suppose that Y is minimal among all closed and $\mathcal{F}$-invariant subspaces of X. Pick $x \in Y$ and let $Z = \{T_p(x) : p \in L\}$. We show that Z is a closed and $\mathcal{F}$-invariant subspace of Y and this is equal to Y. If $p \in L$ and $q \in \overline{\mathcal{F}}$, then $T_q(T_p(x)) = T_{qp}(x)$ and $qp \in L$. So Z is $\mathcal{F}$-invariant and obviously $Z \subseteq Y$. To prove that Z is closed, it is enough to show that any net in Z has a cluster point in Z.

To this end, let $\langle p_\alpha\rangle_{\alpha\in D}$ be a net in L and pick a cluster point p in L of $\langle p_\alpha\rangle_{\alpha\in D}$. Then $T_p(x)$ is a cluster point of $\langle T_{p_\alpha}(x)\rangle_{\alpha\in D}$.

Conversely, let $x \in X$ and $Y = \{T_p(x) : p \in L\}$. Then Y is $\mathcal{F}$-invariant and is closed as above. We now show that Y is minimal among all closed $\mathcal{F}$-invariant subspaces of X. Suppose that Z is a subspace of Y which is closed and $\mathcal{F}$-invariant. We shall show that $Y \subseteq Z$. So let $y \in Y$ and pick $z \in Z$. Then $y = T_p(x)$ and $z = T_q(x)$ for some p and q in L. Since $Lq = L$, there exists $r \in L$ such that $rq = p$. It follows that $T_r(z) = T_r(T_q(x)) = T_{rq}(x) = T_p(x) = y$ and thus $y \in Z$ as required.

(2) Let Y be a subspace of X, which is minimal among all closed and $\mathcal{F}$-invariant subspaces of X and $x \in Y$. Pick $y \in X$ such that $Y = \{T_p(y) : p \in L\}$. Pick $p \in L$ such that $x = T_p(y)$. By Lemma 2.15, x is $\mathcal{F}$-uniformly recurrent.

(3) Let $x \in X$ be $\mathcal{F}$-uniformly recurrent and $Y = \{T_p(x) : p \in \overline{\mathcal{F}}\}$. By Lemma 2.15, pick $q \in L$ such that $T_q(x) = x$. By (1), it suffices to show that $Y = \{T_p(x) : p \in L\}$. To prove this, let $y \in Y$ and pick $p \in \overline{\mathcal{F}}$ such that $y = T_p(x)$. Then $y = T_p(T_q(x)) = T_{pq}(x)$ and $pq \in L$ as required.

(4) Let $x \in X$ be $\mathcal{F}$-uniformly recurrent and $Y = \{T_p(x) : p \in \overline{\mathcal{F}}\}$. By (3) Y is minimal among all closed and $\mathcal{F}$-invariant subspaces of X so (2) applies. □

Theorem 6.5. *Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a closed subsemigroup of βS and let $x \in \overline{\mathcal{F}}$. Statements (a) and (b) are equivalent and imply (c). If $\overline{\mathcal{F}}$ has a left cancelable element, all three are equivalent.*

(a) $x \in K(\overline{\mathcal{F}})$.
(b) *$x \in X$ is $\mathcal{F}$-uniformly recurrent in the dynamical system $(\beta S, \langle\lambda_s\rangle_{s\in S})$.*
(c) *$\overline{\mathcal{F}}x$ is a minimal left ideal of $\overline{\mathcal{F}}$.*

Proof. (a) implies (b). Let $x \in K(\overline{\mathcal{F}})$ and let u be the identity of the group in $K(\overline{\mathcal{F}})$ to which x belongs. Then $x = \lambda_u(x)$ so by Lemma 6.2, x is $\mathcal{F}$-uniformly recurrent in the dynamical system $(\beta S, \langle\lambda_s\rangle_{s\in S})$.

(b) implies (a). Let x be $\mathcal{F}$-uniformly recurrent in the dynamical system $(\beta S, \langle\lambda_s\rangle_{s\in S})$. By Lemma 6.2, there exists $q \in K(\overline{\mathcal{F}})$ such that $\lambda_q(x) = x$. Then $x = qx \in K(\overline{\mathcal{F}})$.

(a) implies (c). Assume that $x \in K(\overline{\mathcal{F}})$ and pick the minimal left ideal L of $\overline{\mathcal{F}}$ such that $x \in L$. Then $\overline{\mathcal{F}}x$ is a left ideal of $\overline{\mathcal{F}}$ contained in L. So $L = \overline{\mathcal{F}}x$. Now assume that $\overline{\mathcal{F}}$ has a left cancelable element z and $\overline{\mathcal{F}}x$ is a minimal left ideal of $\overline{\mathcal{F}}$. Pick an idempotent $u \in \overline{\mathcal{F}}x$. Then $zx \in \overline{\mathcal{F}}x$. So by [19, Lemma 1.30], $zx = zxu$ and therefore $x = xu \in \overline{\mathcal{F}}x \subseteq K(\overline{\mathcal{F}})$. □

Corollary 6.6. *Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a closed subsemigroup of βS and let $x \in K(\overline{\mathcal{F}})$. Then $\overline{\mathcal{F}} \subseteq U_{\mathcal{F}}(x)$ with respect to the dynamical system $(\beta S, \langle\lambda_s\rangle_{s\in S})$.*

Proof. By Theorem 6.5, x is $\mathcal{F}$-uniformly recurrent, so by Lemma 6.4, $\overline{\mathcal{F}} \subseteq U_{\mathcal{F}}(x)$. □

Corollary 6.7. *Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a closed subsemigroup of βS and let $p, q \in \overline{\mathcal{F}}$. Statements (a) and (b) are equivalent and imply (c). If $\overline{\mathcal{F}}$ has a left cancellable element, all three are equivalent.*

(a) $qp \in K(\overline{\mathcal{F}})$.
(b) $q \in U_{\mathcal{F}}(p)$ *with respect to the dynamical system* $(\beta S, \langle\lambda_s\rangle_{s\in S})$.
(c) $\overline{\mathcal{F}}qp$ *is a minimal left ideal of* $\overline{\mathcal{F}}$.

Proof. We have that $q \in U_{\mathcal{F}}(p)$ if and only if $\lambda_q(p)$ is $\mathcal{F}$-uniformly recurrent and $\lambda_q(p) = qp$, so Theorem 6.5 applies. □

Corollary 6.8. *Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a closed subsemigroup of βS. Then the following statements are equivalent.*

(a) *There exists $p \in \overline{\mathcal{F}} \setminus K(\overline{\mathcal{F}})$ such that $K(\overline{\mathcal{F}}) \subsetneq U_{\mathcal{F}}(p)$ with respect to the dynamical system $(\beta S, \langle\lambda_s\rangle_{s\in S})$.*
(b) *$K(\overline{\mathcal{F}})$ is not prime.*

Proof. The proof is an immediate consequence of Corollary 6.7. □

7. Partition regularity along filters

As an example of application of the notions developed above, we discuss here some results about the partition regularity of equations along filters. One of the major problems in Ramsey theory regards the so-called partition regularity of equations (see [11] for a general introduction to the topic). In what follows, we let $T \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ and $S \subseteq T$. We will use the following notation: for $\mathcal{F}$ a filter on S, we let

$$\overline{\mathcal{F}}_T := \{\mathcal{U} \in \beta T \mid \mathcal{F} \subseteq \mathcal{U}\}.$$

Notice that, if we let $\mathcal{F}(T) := \{A \in \mathcal{P}(T) \mid \exists B \in \mathcal{F}\ B \subseteq A\}$, then $\overline{\mathcal{F}}_T = \overline{\mathcal{F}(T)}$.

Definition 7.1. Let $S \subseteq T$, let $\mathbb{K} = \mathbb{R}$ if $T = \mathbb{R}$, $\mathbb{K} = \mathbb{Q}$ otherwise. Let $P_1(x_1, \dots, x_n), \dots, P_m(x_1, \dots, x_n) \in \mathbb{K}[x_1, \dots, x_n]$. Let

$$\sigma(x_1, \dots, x_n) = \begin{cases} P_1(x_1, \dots, x_n), \\ \vdots \\ P_m(x_1, \dots, x_n). \end{cases}$$

We say that the system of equations $\sigma(x_1, \dots, x_n) = (0, \dots, 0)$ is[4] *partition regular* on S if it has a monochromatic solution in every finite coloring of $S \setminus \{0\}$, namely if for every natural number r, for every partition $S \setminus \{0\} = \bigcup_{i=1}^{r} A_i$, there is an index $i \leq r$ and numbers $a_1, \dots, a_n \in A_i$ such that for all $j \in \{1, \dots, m\}$ we have $P_j(a_1, \dots, a_n) = 0$.

The notion of partition regularity near a filter can be introduced as follows:

Definition 7.2. Let $S \subseteq T$ and let $\mathcal{F}$ be a filter on S. Let $P_1(x_1, \dots, x_n)$, ..., $P_m(x_1, \dots, x_n) \in T[x_1, \dots, x_n]$. Let

$$\sigma(x_1, \dots, x_n) = \begin{cases} P_1(x_1, \dots, x_n), \\ \vdots \\ P_m(x_1, \dots, x_n). \end{cases}$$

We say that the system of equations $\sigma(x_1, \dots, x_n) = 0$ is *$\mathcal{F}$-partition regular* on S if for all $V \in \mathcal{F}$, for all finite partitions $S = A_1 \cup \dots \cup A_k$ there exists $j \leq k$ and $a_1, \dots, a_n \in A_j \cap V$ such that $\sigma(a_1, \dots, a_n) = 0$.

In [27], the second and third authors of this paper started the study of the partition regularity of equations in the case where $T = \mathbb{R}$, S is an HL-semigroup and $\mathcal{F} = \{(0, \varepsilon) \cap S \mid \varepsilon \in \mathbb{R}^+\}$ (see [27]). These results where then extended by the first author in [23]; the methods used in [23] actually use two generic properties of $\mathcal{F}$ and can, as such, be generalized, which is what we aim to do in this Section.

It is well known that partition regularity problems can be rephrased in terms of ultrafilters. As for $\mathcal{F}$-partition regularity, the following characterization (whose proof we omit) holds:

Proposition 7.3. *Let $S \subseteq T$ and let $\mathcal{F}$ be a filter on S. Let $P_1(x_1, \dots, x_n)$, ..., $P_m(x_1, \dots, x_n) \in T[x_1, \dots, x_n]$. Let*

$$\sigma(x_1, \dots, x_n) = \begin{cases} P_1(x_1, \dots, x_n), \\ \vdots \\ P_m(x_1, \dots, x_n). \end{cases}$$

The system of equations $\sigma(x_1, \dots, x_n) = 0$ is $\mathcal{F}$-partition regular if and only there exists an ultrafilter $\mathcal{U} \in \overline{\mathcal{F}}$ such that $\forall A \in \mathcal{U}\ \exists a_1, \dots, a_n \in A\ \sigma(a_1, \dots, a_n) = 0$.

[4] From now on, we will simply write $\sigma(x_1, \dots, x_n) = 0$ to simplify the notation.

Definition 7.4. Under the conditions of Proposition 7.3, we say that $\mathcal{U}$ *witnesses* the $\mathcal{F}$-partition regularity of the system $\sigma(x_1,\dots,x_n)=0$, and we call it a ι_σ*-ultrafilter*.

We now recall two results (see e.g. [11] for the proofs[5]) that we will be used in the following.

Theorem 7.5. *Let $P_1(x_1,\dots,x_n),\dots,P_m(x_1,\dots,x_n)\in T[x_1,\dots,x_n]$ be homogeneous. Let*

$$\sigma(x_1,\dots,x_n)=\begin{cases}P_1(x_1,\dots,x_n),\\ \vdots\\ P_m(x_1,\dots,x_n).\end{cases}$$

Assume that the system of equations $\sigma(x_1,\dots,x_n)=0$ is partition regular on S. Then the set

$$I_\sigma=\{\mathcal{U}\in\beta S\mid \mathcal{U}\text{ is a }\iota_\sigma\text{-ultrafilter}\}$$

is a closed bilateral ideal in $(\beta S,\odot)$. In particular, every ultrafilter in $\overline{K(\beta S,\odot)}$ witnesses the partition regularity of all homogeneous partition regular systems on S.

Two immediate consequences of Theorem 7.5 are the following:

Corollary 7.6. *Let $\mathcal{F}$ be a filter on $S\subseteq T$. If every set in $\mathcal{F}$ is piecewise syndetic in $(S,\cdot)$, then all homogeneous partition regular systems on S are also $\mathcal{F}$-partition regular.*

Proof. By our hypothesis, there exists $\mathcal{U}\in\overline{K(\beta S,\odot)}$ that extends $\mathcal{F}$, and we conclude by Theorem 7.5. □

For example, let $\mathcal{F}=\{A\subseteq\mathbb{N}\mid\exists n\in\mathbb{N}\{m\in\mathbb{N}\mid n|m\}\subseteq A\}$. Then every set in $\mathcal{F}$ is piecewise syndetic in $(\mathbb{N},\cdot)$, so every partition regular system is also $\mathcal{F}$-partition regular.

Corollary 7.7. *Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a left or a right ideal in βS. Then an homogeneous system is partition regular on S if and only if it is $\mathcal{F}$-partition regular.*

Proof. Any $\mathcal{F}$-partition regular system is trivially partition regular. Conversely, assume that $\overline{\mathcal{F}}$ is a left ideal (the proof is similar when $\overline{\mathcal{F}}$ is a right ideal). Let σ be an homogeneous partition regular system. If $\mathcal{U}$ is a witness of the partition regularity of σ and $\mathcal{V}\in\overline{\mathcal{F}}$ then $\mathcal{U}\cdot\mathcal{V}\in\overline{\mathcal{F}}$ is a witness of the $\mathcal{F}$-partition regularity of σ by Theorem 7.5. □

For example, from Corollary 7.7 it follows that all homogeneous partition regular systems on $\mathbb{R}$ are also $\mathcal{F}$-partition regular for $\mathcal{F}=\{(0,\varepsilon)\mid\varepsilon>0\}$, as well as for $\mathcal{F}=\{(r,+\infty)\mid r>0\}$, as $\overline{\mathcal{F}}$ is a left ideal in $\beta\mathbb{R}_d$ in both these cases.

[5] In [11], the proofs are done for $T=\mathbb{N}$, but the same exact proof would work for any $T\in\{\mathbb{N},\mathbb{Z},\mathbb{Q},\mathbb{R}\}$.

The second result, which is just a reformulation of [11, Lemma 2.1], allows us to mix different partition regular systems to produce new ones.

Lemma 7.8. *Let $S \subseteq T$. Let $P_1(x_1,\dots,x_n),\dots,P_m(x_1,\dots,x_n) \in T[x_1,\dots,x_n]$, $Q_1(y_1,\dots,y_l)$, …, $Q_t(y_1,\dots,y_l) \in T[y_1,\dots,y_l]$. Let $\mathcal{U} \in \beta S$ be a witness of the partition regularity of the systems of equations $\sigma_1(x_1,\dots,x_n)=0$, $\sigma_2(y_1,\dots,y_l)=0$, where*

$$\sigma_1(x_1,\dots,x_n)=\begin{cases}P_1(x_1,\dots,x_n),\\ \vdots\\ P_m(x_1,\dots,x_n)\end{cases}$$

and

$$\sigma_2(y_1,\dots,y_l)=\begin{cases}Q_1(y_1,\dots,y_l),\\ \vdots\\ Q_t(y_1,\dots,y_l).\end{cases}$$

Then $\mathcal{U}$ witnesses also the partition regularity of $\sigma_3(x_1,\dots,x_n,y_1,\dots,y_l)=0$, where

$$\sigma_3(x_1,\dots,x_n,y_1,\dots,y_l)=\begin{cases}P_1(x_1,\dots,x_n),\\ \vdots\\ P_m(x_1,\dots,x_n),\\ Q_1(y_1,\dots,y_l),\\ \vdots\\ Q_t(y_1,\dots,y_l),\\ x_1-y_1.\end{cases}$$

Lemma 7.8 is useful to work with ultrafilters with multiple structures. For example, as first shown in [25], if $\mathcal{U}$ is a multiplicatively idempotent ultrafilter in $\overline{K(\beta T,\odot)}$ then $\mathcal{U}$ witnesses the partition regularity of all equations of the following form

$$\sum_{i=1}^{n} c_i x_i Q_{F_i}(y_1,\dots,y_m)=0,$$

where $n \geq 2$ is a natural number, $R(x_1,\dots,x_n)=\sum_{i=1}^{n} c_i x_i \in T[x_1,\dots,x_n]$ is partition regular on T, m is a positive natural number and, for every $i \leq n$, $F_i \subseteq \{1,\dots,m\}$ and $Q_{F_i} := \prod_{j\in F_i} y_j$ (if $F_i=\emptyset$, we let $Q_\emptyset = 1$).

In analogy with what was done in [23] for the partition regularity near 0, we can prove the following general result about $\mathcal{F}$-partition regularity.

Theorem 7.9. *Let $S \subseteq T$ and $\mathcal{F}$ be a filter on S. Assume that $\overline{\mathcal{F}}_T \cap \overline{K(\beta T,\odot)}$ contains a multiplicative idempotent. Let $\mathcal{C}_\mathcal{F}$ be the set of polynomial systems that are $\mathcal{F}$-partition regular. Then $\mathcal{C}_\mathcal{F}$ includes:*

(1) *all partition regular homogeneous systems on T;*

(2) *all equations of the form*

$$P(x_1, \dots, x_n, y_1, \dots, y_m) = \sum_{i=1}^{n} a_i x_i Q_{F_i}(y_1, \dots, y_m)$$

where $\sum_{i=1}^{n} a_i x_i \in T[x_1, \dots, x_n]$ *is partition regular on* T *and* $F_1, \dots, F_n \subseteq \{1, \dots, m\}$.

Moreover, if

$$\sigma_1(x_1, \dots, x_n) = \begin{cases} P_1(x_1, \dots, x_n), \\ \vdots \\ P_m(x_1, \dots, x_n) \end{cases}$$

and

$$\sigma_2(y_1, \dots, y_l) = \begin{cases} Q_1(y_1, \dots, y_l), \\ \vdots \\ Q_t(y_1, \dots, y_l) \end{cases}$$

belong to $\mathcal{C}_{\mathcal{F}}$*, then also*

$$\sigma_3(x_1, \dots, x_n, y_1, \dots, y_l) = \begin{cases} P_1(x_1, \dots, x_n), \\ \vdots \\ P_m(x_1, \dots, x_n), \\ Q_1(y_1, \dots, y_l), \\ \vdots \\ Q_t(y_1, \dots, y_l), \\ x_1 - y_1 \end{cases}$$

belongs to $\mathcal{C}_{\mathcal{F}}$.

Proof. Let $\mathcal{U}$ be a multiplicative idempotent in $\overline{\mathcal{F}}_T \cap \overline{K(\beta T, \odot)}$. We just have to observe that, by Corollary 7.7, $\mathcal{U}$ is a witness of all $\mathcal{F}$-partition regular homogeneous systems, which are precisely all partition regular homogeneous systems, whilst the partition regularity of equations of the form (2) has been discussed before the Theorem. The closure with respect to composition follows by Lemma 7.8. As $\mathcal{F} \subseteq \mathcal{U}$, we conclude by Proposition 7.3. □

Notice that the request that $\overline{\mathcal{F}}_T \cap \overline{K(\beta T, \odot)}$ contains an idempotent ultrafilter is always true in the examples considered in this Section and, more in general, whenever $\overline{\mathcal{F}}_T$ is a closed subsemigroup of βT with $\overline{\mathcal{F}}_T \cap \overline{K(\beta T, \odot)} \neq \emptyset$.

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References

[1] E. Bayatmanesh and M. A. Tootkaboni, Central sets theorem near zero. *Top. App.* **210** (2016), 70-80. MR3539726, Zbl 1391.22003, doi: 10.1016/j.topol.2018.03.002. 793

[2] V. Bergelson and N. Hindman, Nonmetrizable topological dynamics and Ramsey theory. *Trans. Amer. Math. Soc.* **320** (1990), 293-320. MR0982232, Zbl 0725.22001, doi: 10.1090/s0002-9947-1990-0982232-5.

[3] J. Berglund, H. Junghenn, and P. Milnes, Analysis on semigroups. *Wiley, New York,* 1989. MR0690846. 808

[4] T. Bhattacharya, S. Chakraborty, S. K. Patra, Combined algebraic properties of C^*-sets near zero. *Semigroup Forum* 2019. MR4101031, doi: 10.1007/s00233-019-10005-4. 793, 796

[5] S. Burns, N. Hindman, Quasi-central sets and their dynamical characterization. *Topology Proc.* **31** (2007), 445-455. MR2476623, Zbl 1420.54057. 794, 803

[6] C. Christopherson, Closed ideal in the Stone-Čech compactification of a countable semigroup and some applications to ergodic theory and topological dynamics. *Ph. D. Dissertation* (2014) Ohio State University. MR3321912 .

[7] C. Christopherson and J. H. Johnson Jr., Algebraic characterizations of some relative notions of size. *Semigroup Forum* **104.1** (2022), pp. 28–44. issn: 0037-1912. MR4368920, Zbl 7463823, doi: 10.1007/s00233-021-10215-9. 795

[8] D. Davenport, The minimal ideal of compact subsemigroups of βS. *Semigroup Forum* **41.2** (1990), pp. 201–213. MR1057591, Zbl 0842.22001, doi: 10.1007/BF02573391. 795

[9] D. De, N. Hindman, and D. Strauss, A new and stronger Central Sets Theorem. *Fundamenta Mathematicae* **199** (2008), 155–175. MR2410923, Zbl 1148.05052, doi: 10.4064/fm199-2-5.

[10] P. Debnath and S. Goswami, Dynamical characterization of central sets along filter. *Top. App.* **300** (2021) 107777. MR4293082, Zbl 1478.37013, doi: 10.1016/j.topol.2021.107777. 793, 795, 797

[11] M. Di Nasso and L. Luperi Baglini, Ramsey properties of nonlinear Diophantine equations. *Adv. Math.,* Vol. 324 (2018), 84–117. MR3733882, Zbl 1376.05159, doi: 10.1016/j.aim.2017.11.003. 793, 798, 811, 813, 814

[12] R. Ellis, Lectures on Topological Dynamics. Benjamin, New York, 1969. MR0267561, Zbl 0193.51502. 808

[13] H. Furstenberg, Recurrence in ergodic theory and combinatorial number theory. Princeton University Press, Princeton, 1981. MR0603625, Zbl 0459.28023. 792

[14] H. Furstenberg and B. Weiss, Topological dynamics and combinatorial number theory. *J. d'Analyse Math.* **34** (1978), 61-85. MR0531271, Zbl 0425.54023, doi: 10.1007/BF02790008. 792

[15] I. Goldbring, A nonstandard take on central sets. *To appear in the volume Research Trends in Contemporary Logic.* 793, 797, 798

[16] S. Goswami and J. Poddar, Central Sets Theorem along filters and some combinatorial consequences. *Indagationes Mathematicae* Volume **33**, Issue 6, November 2022, Pages 1312-1325. MR4498235, doi: 10.1016/j.indag.2022.08.006. 807

[17] N. Hindman and I. Leader, The semigroup of ultrafilters near 0. *Semigroup Forum* **59** (1999), 33-55. MR1847941, Zbl 0942.22003, doi: 10.1007/s002339900031. 793, 807

[18] 794, 803, 804, 807
N. Hindman, A. Maleki, and D. Strauss, Central sets and their combinatorial characterization. *J. Comb. Theory (Series A)* **74** (1996), 188-208. MR1385369, Zbl 0858.22003, doi: 10.1006/jcta.1996.0048.

[19] N. Hindman and D. Strauss, Algebra in the Stone-Čech compactification - theory and application. W. de Gruyter and Co., Berlin, 2012. MRMR2893605, Zbl 1241.22001. 795, 797, 804, 811

[20] N. HINDMAN, D. STRAUSS AND L. Q. ZAMBONI, Recurrence in the dynamical system $(X, \langle T_s \rangle_{s\in S})$ and ideals of βS. *Indagationes Mathematicae* **29** (2018), 293-312. MR3739619, Zbl 1385.37016, doi: 10.1016/j.indag.2016.12.004. 794, 808, 809

[21] J. H. JOHNSON, A dynamical characterization of C-sets. https: //arxiv.org/abs/ 1112.0715. 794, 803

[22] J. LI, Dynamical characterization of C-sets and its application. *Fund. Math.* **216** (2012), 259-286. MR2890544, Zbl 1278.37022, doi: 10.4064/fm216-3-4.

[23] L. LUPERI BAGLINI, Partition regularity of polynomial systems near 0. *Semigroup Forum* **103** (2021), 191–208. MR4276610, doi: 10.1007/s00233-021-10179-w. 793, 812, 814

[24] L. LUPERI BAGLINI, Nonstandard characterisations of tensor products and monads in the theory of ultrafilters. *Math. Log. Quart.* **65** (2019), 347–369. MR4030959, Zbl 7197328, doi: 10.1002/malq.201800089. 798

[25] L. LUPERI BAGLINI, Partition regularity of nonlinear polynomials: a nonstandard approach. *Integers* Vol. 14 (2014), A-30. MR3239578, Zbl 1297.05234. 798, 814

[26] T. PAPAZYAN, Filters and semigroup properties. *Semigroup Forum* **41.3** (1990), pp. 329–338. MR1065960, Zbl 0702.22005, doi: 10.1007/BF02573399. 795

[27] S. K. PATRA AND MD. M. SHAIKH, Monochromatic sums equal to products near zero. *Integers* Vol. 20 (2020), A-66. MR4142503, Zbl 1459.05331. 793, 812

[28] S. K. PATRA, Dynamical characterizations of combinatorially rich sets near zero. *Top. App.* **240** (2018), 173-182. MR3784404, Zbl 1392.37008, doi: 10.1016/j.topol.2018.03.014. 793, 803

[29] MD. M. SHAIKH, S. K. PATRA, AND M. K. RAM, Dynamics near an idempotent. *Top. App.* **282** (2020), 107328. MR4128671, Zbl 1456.37023, doi: 10.1016/j.topol.2020.107328. 793, 803, 809

[30] H. SHI AND H. YANG, Nonmetrizable topological dynamical characterization of central sets. *Fund. Math.* **150** (1996), 1-9. MR1387952, Zbl 0887.54036, doi: 10.4064/fm-150-1-1-9.

[31] O. SHUUNGULA, Y. ZELENYUK, AND Y. ZELENYUK, The closure of the smallest ideal of an ultrafilter semigroup. *Semigroup Forum* **79** (2009) : 531–539. MR2564062 , Zbl 1184.22001, doi: 10.1007/s00233-009-9173-x. 793, 795, 796, 803

[32] M. A. TOOTKABONI, T. VAHED, The semigroup of ultrafilters near an idempotent of semitopological semigroup. *Top. App.* **159** (2012) 3494-3503. MR2964862, Zbl 1285.54017, doi: 10.1016/j.topol.2012.08.009. 793

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