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# Upper bounds on resolvent degree via Sylvester's obliteration algorithm

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ABSTRACT. For each $n$, let $\mathrm{RD}(n)$ denote the minimum $d$ for which there exists a formula for the general polynomial of degree $n$ in algebraic functions of at most $d$ variables. In this paper, we recover an algorithm of Sylvester for determining non-zero solutions of systems of homogeneous polynomials, which we present from a modern algebro-geometric perspective. We then use this geometric algorithm to determine improved thresholds for upper bounds on $\mathrm{RD}(n)$.

CONTENTS



## 1. Introduction

A classical problem in mathematics is to determine the roots of a general degree $n$ polynomial in one variable in terms of its coefficients. Modern work on this problem centers around resolvent degree, an invariant whose ideas permeate classical work, but was not formally defined until the independent definitions of Brauer [Bra1975, p.46] and Arnol'd and Shimura [AS1976, p.46]. Farb and Wolfson greatly expanded the context of resolvent degree in [FW2019, Definition 2.3, Proposition 2.4, Definition 3.1].

Following [Wol2021, Example 4.2], we denote the resolvent degree of the general degree $n$ polynomial by $\mathrm{RD}(n)$. Currently, non-trivial lower bounds on $\mathrm{RD}(n)$ are unknown [FW2019, Section 1.5]; it is possible that $\mathrm{RD}(n) = 1$ for

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all $n$. Nonetheless, Dixmier [Dix1993] noted that "Every reduction of $\mathrm{RD}(n)$ would be serious progress," and Wolfson provided new upper bounds on $\mathrm{RD}(n)$ [Wol2021, Theorems 5.6 and 5.8] by constructing a "bounding function" $F(m)$ such that $\mathrm{RD}(n) \leq n - m$ for $n \geq F(m)$. The current best upper bounds on $\mathrm{RD}(n)$ are given by [Sut2021C, Theorem 3.27], where the second-named author constructs an improved bounding function $G(m)$ and shows that $\lim_{m\to\infty} \frac{F(m)}{G(m)} = \infty$.

In this paper, we recover an algorithm from [Syl1887] (henceforth referred to as the "obliteration algorithm") for solving systems of equations using polynomials of minimal degree. An additional modern description of the Sylvester's work and its relevance to resolvent degree is given in [Heb2021]. Here we present the algorithm primarily from an algebro-geometric viewpoint using the language of "polar cones" introduced in [Sut2021C, Section 2]. We then use the obliteration algorithm to determine the following new upper bounds on resolvent degree:

**Theorem 1.1** (Upper Bounds on Resolvent Degree)**.**

(1) *For* $n \geq 5,250,199$, $\mathrm{RD}(n) \leq n - 13$.
(2) *For each* $14 \leq m \leq 17$ *and* $n > \frac{(m-1)!}{120}$, $\mathrm{RD}(n) \leq n - m$.
(3) *For* $n \geq 381,918,437,071,508,901$, $\mathrm{RD}(n) \leq n - 22$.
(4) *For each* $23 \leq m \leq 25$ *and* $n > \frac{(m-1)!}{720}$, $\mathrm{RD}(n) \leq n - m$.

The above result is found as Theorem 4.6 in Section 4 and leads to the construction of a new bounding function $G'(m)$ such that $\mathrm{RD}(n) \leq n-m$ for $n \geq G'(m)$ and $G'(m) \leq G(m)$ in Corollary 4.9.

*Historical Remarks.* The second-named author uses two distinct methods to construct $G(m)$ [Sut2021C, Theorems 3.7, 3.10, 3.24]. For general $m$ (Theorem 3.24), the second-named author uses a result of Debarre and Manivel [DM1998, Theorem 2.1] to improve on the construction of Wolfson which underlies [Wol2021, Theorem 5.6]. For small $m$ (Theorems 3.7 and 3.10), the second-named author uses iterated polar cone methods which build upon the methods of [Wim1927], [Che1954], and [Seg1945] (note, however, that Wiman and Chebotarev do not use the language of polars at all and Segre refers only to individual polars). An application of Sylvester's obliteration algorithm to certain small $m$ cases is considered in [Heb2021]. By combining Sylvester's obliteration algorithm with the other methods described above, the authors believe they have exhausted the classical methods related to the theory of Tschirnhaus transformations; implications of this are discussed in Subsection 4.4.

*Outline of the Paper.* In Section 2, we recall the relevant background on resolvent degree, polar cones, and Tschirnhaus transformations. In Section 3, we present a modern, geometric version of the obliteration algorithm and related phemonena, as well as a summary of Sylvester's original work. In Section 4, we

apply the geometric obliteration algorithm to obtain upper bounds on resolvent degree. In Section 5, we discuss Python implementations of the geometric obliteration algorithm used for computations relevant for Theorem 4.6.

*Conventions.*

(1) We restrict to fields $K$ which are finitely generated $\mathbb{C}$-algebras. One could instead fix an arbitrary algebraically closed field $F$ of characteristic zero (in lieu of $\mathbb{C}$) and the statements (relative to $F$) would hold.
(2) We follow the conventions of [Har1992] for algebraic varieties. In particular, a projective (respectively, affine) variety is defined to be a closed algebraic set in $\mathbb{P}^r_K$ (respectively, $\mathbb{A}^r_K$). When we say variety without a specific modifier, we mean a quasi-projective variety. Note that we do not assume that varieties are irreducible.
(3) Given $a, b \in \mathbb{Z}_{\geq 0}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.
(4) Given a collection of homogeneous polynomials

$$S = \{f_1, \dots, f_s\} \subseteq K[x_0, \dots, x_r],$$

we write $\mathbb{V}(f_1, \dots, f_s)$ (and occasionally $\mathbb{V}(S)$) for the subvariety of $\mathbb{P}^r_K$ determined by the conditions $f_1 = \cdots = f_s = 0$.
(5) Given a subvariety $V \subseteq \mathbb{P}^r_K$, we write $V(K)$ for the set of $K$-rational points of $V$.
(6) Given points $P_0, \dots, P_\ell \in \mathbb{P}^r(K)$, we write $\Lambda(P_0, \dots, P_\ell)$ for the linear subvariety of $\mathbb{P}^r_K$ that they determine. Additionally, we refer to a linear subvariety $\Lambda \subseteq \mathbb{P}^r_k$ of dimension $k \geq 3$ as a $k$-plane. We refer to linear subvarieties of dimension 1 (respectively, 2) as lines (respectively, planes).
(7) We use the notation $K_n$ to mean $\mathbb{C}(a_1, \dots, a_n)$, a purely transcendental extension of $\mathbb{C}$ with transcendence basis $a_1, \dots, a_n$.

Note that for generic choices of $f_1, \dots, f_s$, $\mathbb{V}(f_1, \dots, f_s)$ is a complete intersection. However, there are examples of such choices which are not complete intersections, such as the twisted cubic curve. Following the convention of [Sut2021C], we refer to a subvariety $\mathbb{V}(f_1, \dots, f_s)$ as an intersection of hypersurfaces.

Consider a system of equations $S$ where each polynomial has degree at most $d$ and where we denote the number of polynomials of degree $j$ by $\ell_j$. In such a case, we say that $S$ is of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$. If $\ell_j = 0$ for any $j \in [1, d-1]$, the corresponding column may be omitted from the presentation. When $d \geq 2$ and each $\ell_j = 1$, we say $S$ is of type $(1, \dots, d)$.

When $V = \mathbb{V}(f_1, \dots, f_s)$, we say that the type of $V$ is the type of the system $\{f_1, \dots, f_s\}$. We note that the type of $V$ explicitly depends on the presentation in terms of $f_1, \dots, f_s$; it is not unique. However, we only consider the type of an

intersection of hypersurfaces when it is defined by an explicit set of polynomials.

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## 2. Resolvent degree, polar cones, and Tschirnhaus transformations

**2.1. Resolvent degree.** We refer the reader to [FW2019] for general definitions of resolvent degree (Definitions 1.3, 2.3), a summary of its history (Section 1), and additional context. We only work over $\mathbb{C}$ and thus provide definitions in this context.

**Definition 2.1** (RD for Fields)**.** Let $K'/K$ be an extension of $\mathbb{C}$-fields. The **resolvent degree** of $K'/K$, denoted $\mathrm{RD}(L/K)$, is the minimal $d$ for which there exists a tower of finite extensions

$$K = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_\ell,$$

such that $K'$ embeds into $E_\ell$ over $K$ and the essential dimension of each $E_{j+1}/E_j$ is at most $d$.

**Definition 2.2** (RD for Maps)**.** Suppose $Y \dashrightarrow X$ is a generically finite, dominant rational map of $\mathbb{C}$-varieties. The **resolvent degree** of $Y \dashrightarrow X$, denoted $RD(Y \dashrightarrow X)$, is the minimal $d$ for which there exists a tower of generically finite, dominant rational maps

$$E_\ell \dashrightarrow \cdots \dashrightarrow E_1 \dashrightarrow E_0 = X,$$

such that $E_\ell \dashrightarrow X$ factors as $E_\ell \dashrightarrow Y \dashrightarrow X$ and the essential dimension of each $E_{j+1} \dashrightarrow E_j$ is at most $d$.

We first note that Definitions 2.1 and 2.2 agree and is induced by sending an irreducible affine variety $X$ to the corresponding field of rational functions $\mathbb{C}(X)$. We refer the reader to [FW2019, Definition 1.3] for a precise definition of essential dimension, but note that we often approximate essential dimension via the bounds $\mathrm{ed}\left(K'/K\right) \leq \mathrm{tr.deg}\,(K)$ and $\mathrm{ed}\,(Y \dashrightarrow X) \leq \dim(X)$.

We write $\mathrm{RD}(n)$ for the resolvent degree of the general degree $n$ polynomial, which is given precisely as

$$\begin{aligned}\mathrm{RD}(n) &= \mathrm{RD}\left(\mathbb{C}^n \dashrightarrow \mathbb{C}^n/S_n\right),\\ &= \mathrm{RD}\left(K_n[z]/\left(z^n + a_1 z^{n-1} + \cdots a_{n-1}z + a_n\right)/K_n\right),\end{aligned}$$

recalling the convention that $K_n = \mathbb{C}\left(a_1, \ldots, a_n\right)$. Additionally, resolvent degree is defined for finite groups [FW2019, Definition 3.1] and

$$\mathrm{RD}(n) = \mathrm{RD}\left(S_n\right) = \mathrm{RD}\left(A_n\right)$$

[FW2019, Theorem 3.3, Corollary 3.17].

While we restrict ourselves to working over $\mathbb{C}$, we do note lose any generality. Theorem 1.2 of [Rei2022] yields that $\mathrm{RD}_{\mathbb{C}}(S_n) = \mathrm{RD}_K(S_n)$ for any field $K$ of characteristic zero and [Rei2022, Theorem 1.3] yields that $\mathrm{RD}_{\mathbb{C}}(S_n) \geq \mathrm{RD}_K(S_n)$ for any field $K$, i.e. resolvent degree can only go down in positive characteristic.

In Lemma 2.3, we summarize several basic results which will be used frequently (and often without explicit reference). Item 1 is the field-theoretic version of [FW2019, Lemma 2.7] and follows immediately from the definition of resolvent degree. Items 2 and 3 are algebraic versions of [FW2019, Lemma 2.9] and can be found explicitly as follows as [Sut2021C, Lemma 2.18, Proposition 2.19]. Note that items 2 and 3 follow directly from the primitive element theorem.

**Lemma 2.3** (Properties of Resolvent Degree)**.**

(1) *Let $E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_\ell$ be a tower of field extensions. Then,*
$$\mathrm{RD}(E_\ell/E_0) = \max\left\{\mathrm{RD}(E_j/E_{j-1}) \mid j \in [1,\ell]\right\}.$$
(2) *Let $K'/K$ be a degree $d$ field extension. Then, $\mathrm{RD}(K'/K) \leq \mathrm{RD}(d)$.*
(3) *Let $V \subseteq \mathbb{P}^r_K$ be a degree $d$ subvariety. Then, there is an extension $K'/K$ with $\mathrm{RD}(K'/K) \leq \mathrm{RD}(d)$ over which we can determine a $K'$-rational point of $V$.*

As a consequence of item 3, we say that we can determine a point of a degree $d$ subvariety $V$ by solving a degree $d$ polynomial.

**2.2. Polar cones and *k*-polar points.** The original theory of polars for hypersurfaces is classical and a classical reference is [Ber1923]; a modern reference on polars is [Dol2012]. We now recall the key definitions and results of [Sut2021C, Section 2]; we use the same notation and begin with the definition of polars.

**Definition 2.4** (Polars and Polar Cones)**.** Let $f \in K[x_0,\ldots,x_r]$ be a homogeneous polynomial of degree $d$ and $P \in \mathbb{P}^r(K)$. Observe that the set
$$I_j^* := \mathrm{Hom}_{\mathbf{Set}}([1,j],[0,r])$$
indexes the (ordered) $j^{th}$ partial derivatives of $f$ for each $j \in [0,d]$. We also use the shorthand
$$\partial_0^{j_0}\cdots\partial_\ell^{j_\ell} = \frac{\partial^{j_0+\cdots+j_\ell}}{\partial x_0^{j_0}\cdots\partial x_\ell^{j_\ell}}.$$
For each $j \in [0,d]$, the $j^{th}$ **polar of** $f$ **at** $P$ is the homogeneous polynomial
$$t(j,f,P) := \sum_{\iota\in I^*_{d-j}} \left(\partial_0^{|\iota^{-1}(0)|}\cdots\partial_r^{|\iota^{-1}(r)|} f\right)\Big|_P x_0^{|\iota^{-1}(0)|}\cdots x_r^{|\iota^{-1}(r)|}, \tag{1}$$

which is of degree $d-j$. Next, consider the hypersurface $H = \mathbb{V}(f)$. The $j^{th}$ **polar of** $H$ **at** $P$ is

$$T(j,f,P) := \mathbb{V}(t(j,f,P)) \subseteq \mathbb{P}^r_K.$$

Finally, the **(first) polar cone of** $H$ **at** $P$ is

$$\mathcal{C}(H;P) := \bigcap_{j=0}^{d-1} T(j,f,P).$$

Note that $T(0,f,P) = H$ for all $P$ and $T(d,f,P) = \mathbb{P}^r_K$ if $P \in H(K)$. If $H$ is smooth at $P$, then $T(d-1,f,P)$ is the tangent hyperplane of $H$ at $P$. Our interest in polars stems from our interest in polar cones, which are themselves motivated by the following classical result (which is stated as a fact in [Seg1945, I.5, p.292]; Segre refers readers to [Ber1923, p.203]).

**Lemma 2.5** (Bertini's Lemma for Hypersurfaces)**.** *Let $H \subseteq \mathbb{P}^r_K$ be a hypersurface and $P \in H(K)$. Then, $\mathcal{C}(H;P) \subseteq H$ is a cone with vertex $P$.*

In particular, for any point $Q \in \mathcal{C}(H;P) \setminus \{P\}$, the line $\Lambda(P,Q)$ lies in $H$.

Observe that for an intersection of hypersurfaces $\mathbb{V}(f_1,\dots,f_s)$, a line $\Lambda$ lies on $\mathbb{V}(f_1,\dots,f_s)$ exactly when $\Lambda$ lies on each hypersurface $\mathbb{V}(f_j)$. This observation motivates the following definition and lemma, which are originally given as [Sut2021C, Definition 2.10, Lemma 2.11].

**Definition 2.6** (Polar Cones for Intersections)**.** Let $V = \mathbb{V}(f_1,\dots,f_s) \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces and $P \in V(K)$. The **(first) polar cone of** $V$ **at** $P$ is

$$\mathcal{C}(V;P) := \bigcap_{j=1}^{s} \mathcal{C}(\mathbb{V}(f_j);P).$$

**Lemma 2.7** (Bertini's Lemma for Intersections)**.** *Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces and $P \in V(K)$. Then, $\mathcal{C}(V;P) \subseteq V$ is a cone with vertex $P$.*

Iterating the polar cone construction yields a method for determining $k$-planes on intersections of hypersurfaces. We now recall the associated definitions, first given as [Sut2021C, Definition 2.22].

**Definition 2.8** (Iterated Polar Cones and k-Polar Points)**.** Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces and $P_0 \in V(K)$. First, set $\mathcal{C}^1(V;P_0) := \mathcal{C}(V;P_0)$. Given additional points $P_1,\dots,P_{k-1} \in V(K)$ such that

$$P_\ell \in \mathcal{C}^\ell(V;P_0,\dots,P_{\ell-1}) \setminus \Lambda(P_0,\dots,P_{\ell-1})$$

for $\ell \in [1,k-1]$, the $k^{th}$ **polar cone of** $V$ **at** $P_0,\dots,P_{k-1}$ is

$$\mathcal{C}^k(V;P_0,\dots,P_{k-1}) := \mathcal{C}\left(\mathcal{C}^{k-1}(V;P_0,\dots,P_{k-2});P_{k-1}\right).$$

We refer to an ordered collection of such points $(P_0,\dots,P_k)$ as a $k$-**polar point** of $V$.

If the points $P_0, \dots, P_{k-1}$ have already been chosen, we refer to $\mathcal{C}^k(V; P_0, \dots, P_{k-1})$ as *the* $k^{th}$ polar cone of $V$. In the event that such points exist, but have not been explicitly chosen, we refer to *a* $k^{th}$ polar cone of $V$. Additionally, it is sometimes useful to refer to $V$ itself as a zeroth polar cone of $V$ (at any of its $K$-points).

By noting that iterated polar cones are nested, i.e.

$$\mathcal{C}^k(V; P_0, \dots, P_{k-1}) \subseteq \mathcal{C}^{k-1}(V; P_0, \dots, P_{k-2}) \subseteq \cdots \subseteq \mathcal{C}(V; P_0) \subseteq V,$$

and that the points $P_0, \dots, P_k$ defining a $k$-polar point $(P_0, \dots, P_k)$ are in general position, we arrive at the following $k$-plane analogue of Lemma 2.7, which is originally [Sut2021C, Lemma 2.24]:

**Lemma 2.9** (Polar Point Lemma)**.** *Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces and let $(P_0, \dots, P_k)$ be a $k$-polar point of $V$. Then,*

$$\Lambda(P_0, \dots, P_k) \subseteq \mathcal{C}^k(V; P_0, \dots, P_{k-1}) \subseteq V$$

*is a $k$-plane.*

**2.3. Tschirnhaus transformations.** We use the notation and conventions of [Sut2021C, Subsection 3.1] for Tschirnhaus transformations and refer the reader there for details. Note also that Wolfson provides a more complete history of Tschirnhaus transformations in [Wol2021, Section 2 and Appendix B]. Let $K_n = \mathbb{C}(a_1, \dots, a_n)$ be a purely transcendental extension of $\mathbb{C}$ with transcendence basis $a_1, \dots, a_n$.

**Definition 2.10** (General Polynomials)**.** The **general polynomial of degree** $n$ is the polynomial

$$\phi_n(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \in K_n[z].$$

**Definition 2.11** (Tschirnhaus Transformations)**.** A **Tschirnhaus transformation** of the general degree $n$ polynomial is an isomorphism of $K_n$-fields

$$\Upsilon : K_n[z]/(\phi_n(z)) \to K_n[z]/(\psi(z)),$$

where $\psi(z) = z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n$. We say that $\Upsilon$ has type $(j_1, \dots, j_k)$ if $b_{j_1} = \cdots = b_{j_k} = 0$.

As per Remark 3.3 of [Sut2021C], the space of all Tschirnhaus transformations of the general degree $n$ polynomial (up to re-scaling) is

$$\mathcal{T}^n_{K_n} := \mathbb{P}^{n-1}_{K_n} \setminus [1 : 0 : \cdots : 0] \subseteq \mathbb{P}^{n-1}_{K_n}.$$

Note that each $b_j$ in Definition 2.11 is a homogeneous polynomial of degree $j$ in $a_1, \dots, a_n$.

**Definition 2.12** (Tschirnhaus Complete Intersections)**.** Fix $n \in \mathbb{Z}_{\geq 1}$. For any $m \in [1, n-1]$, the $m^{th}$ **extended Tschirnhaus hypersurface is**

$$\tau_m := \mathbb{V}(b_m) \subseteq \mathbb{P}^{n-1}_{K_n},$$

and the $m^{th}$ **extended Tschirnhaus complete intersection** is

$$\tau_{1,\dots,m} := \bigcap_{j=1}^{m} \tau_j \subseteq \mathbb{P}^{n-1}_{K_n}.$$

Additionally, the $m^{th}$ **Tschirnhaus hypersurface** is

$$\tau^\circ_m := \tau_m \cap \mathcal{T}^n_{K_n} = \tau_m \setminus \{[1:0:\cdots:0]\},$$

and the $m^{th}$ **Tscihrnhaus complete intersection is**

$$\tau^\circ_{1,\dots,m} := \tau_{1,\dots,m} \cap \mathcal{T}^n_{K_n} = \tau_{1,\dots,m} \setminus \{[1:0:\cdots:0]\}.$$

**Remark 2.13** (Strategy for Upper Bounds on RD($n$))**.** If we can determine a $K'$-rational point of $\tau^\circ_{1,\dots,m-1}$ over an extension $K'/K_n$ of sufficiently small resolvent degree, then we can conclude that $\mathrm{RD}(n) \leq n - m$. Notice that if we can determine an $(m-d-1)$-plane $\Lambda \subseteq \tau^\circ_{1,\dots,d}$ over an extension $L/K_n$ of low resolvent degree, then we need only further pass to an extension $K'/L$ with $\mathrm{RD}(K'/L) \leq \mathrm{RD}\left(\frac{(m-1)!}{d!}\right)$, by Lemma 2.3.

Lemma 2.9 yields that every $k$-polar point determines a $k$-plane, hence Remark 2.13 yields that it will suffice to determine $k$-polar points on the Tschirnhaus complete intersections $\tau^\circ_{1,\dots,d}$.

## 3. The obliteration algorithms

In [Syl1887], Sylvester gives an algorithm to determine an upper bound on the number of variables required to determine a non-trivial solution for a system of homogeneous polynomials of given degrees by solving polynomials of the same, or lower, degrees. The algorithm centers on Sylvester's "formula of obliteration" [Syl1887, p.475], which will be covered in detail in Corollary 3.12 and Proposition 3.15. Consequently, we refer to Sylvester's method as the "obliteration algorithm." In Subsection 3.1, we give a modern description of the obliteration algorithm via geometry (in terms of varieties, rational points, and polar cones). In Subsection 3.2, we describe the obliteration algorithm in terms of systems of homogeneous polynomials and explain Sylvester's classical language.

### 3.1. The Geometric obliteration algorithm.

We now give a geometric construction of Sylvester's obliteration algorithm. More specifically, given an intersection of hypersurfaces $V \subseteq \mathbb{P}^r_K$, we give a bound on the ambient dimension required to be able to determine a point of $V$ over an extension $K'/K$ of bounded resolvent degree. Note that this bound depends only on the type of $V$.

**Definition 3.1** (Minimal dimension bound)**.** The **minimal dimension bound of type** $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$, which we will denote by $r(d; \ell_d, \dots, \ell_1)$, is the minimal

$r' \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ such that whenever $r \geq r'$, we can determine a point of any intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ in $\mathbb{P}^r_K$ over an extension $K'/K$ with $\mathrm{RD}\left(K'/K\right) \leq \mathrm{RD}(d)$. Given an intersection of hypersurfaces $V$ of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$, we set $r(V) := r(d; \ell_d, \ldots, \ell_1)$.

**Remark 3.2** (Finiteness of the Minimal Dimension Bound)**.** The main goal of this section is to establish an upper bound on $r(d; \ell_d, \ldots, \ell_1)$. More specifically, we introduce a recursive, combinatorial bound $g(d; \ell_d, \ldots, \ell_1)$ in Definition 3.3 which we will show satisfies

$$r(d; \ell_d, \ldots, \ell_1) \leq g(d; \ell_d, \ldots, \ell_1). \tag{2}$$

The proof of inequality (2) is exactly the geometric version of the obliteration algorithm.

We now give Definition 3.3 and note that the underlying geometric intuition is explained in Lemma 3.5 and Remark 3.6.

**Definition 3.3** (Geometric Dimension Bound)**.** The **geometric dimension bound of type** $\begin{bmatrix} 1 \\ \ell_1 \end{bmatrix}$ is $g(1; \ell_1) := \ell_1$. Similarly, the geometric dimension bound **of type** $\begin{bmatrix} 2 & 1 \\ 1 & \ell_1 \end{bmatrix}$ is $g(2; 1, \ell_1) := 1 + \ell_1$ and the geometric dimension bound **of type** $\begin{bmatrix} 2 & 1 \\ \ell_2 & \ell_1 \end{bmatrix}$ with $\ell_2 \geq 2$ is

$$g(2; \ell_2, \ell_1) := g(2; \ell_2 - 1, \ell_2 + \ell_1 + 1).$$

For $d \geq 3$, the geometric dimension bound **of type** $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ 1 & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ is

$$\begin{aligned} &g(d; 1, \ell_{d-1}, \ldots, \ell_2, \ell_1) \\ &:= g\left(d-1; \ell_{d-1}, (\ell_{d-1} + \ell_{d-2}), \ldots, \sum_{j=2}^{d-1} \ell_j, \left(\sum_{j=1}^{d-1} \ell_j\right) + 1\right). \end{aligned}$$

For $d \geq 3$ and $\ell_d \geq 2$, the geometric dimension bound **of type** $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ 1 & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ is

$$\begin{aligned} &g(d; \ell_d, \ell_{d-1}, \ldots, \ell_2, \ell_1) \\ &:= g\left(d; \ell_d - 1, (\ell_d + \ell_{d-1}) - 1, \ldots, \left(\sum_{j=2}^{d} \ell_j\right) - 1, \sum_{j=1}^{d-1} \ell_j\right). \end{aligned}$$

Finally, given an intersection of hypersurfaces $V$ of type $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$, we set

$$g(V) := g(d; \ell_d, \ldots, \ell_1).$$

**Remark 3.4** (Hyperplane Identities)**.** The definitions of both the minimal and geometric dimension bounds admit a "hyperplane identity," which we use without explicit reference:

$$1 + r(d; \ell_d, \ldots, \ell_2, \ell_1) = r(d; \ell_d, \ldots, \ell_2, \ell_1 + 1),$$
$$1 + g(d; \ell_d, \ldots, \ell_2, \ell_1) = g(d; \ell_d, \ldots, \ell_2, \ell_1 + 1).$$

We next state Lemma 3.5, which is the technical underpinning of the geometric obliteration algorithm and which specializes to give the geometric version of Sylvester's formula of reduction.

**Lemma 3.5** (The Reduction Lemma)**.** *Let $V$ be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $d \geq 2$ and which is not a hypersurface. Take $V_d$ to be a degree $d$ hypersurface and $V^{red}$ to be an intersection of hypersurfaces of type*

$$\begin{bmatrix} d & \cdots & 1 \\ \ell_d - 1 & \cdots & \ell_1 \end{bmatrix}$$

*if $\ell_d \geq 2$ and of type*

$$\begin{bmatrix} d-1 & \cdots & 1 \\ \ell_{d-1} & \cdots & \ell_1 \end{bmatrix}$$

*if $\ell_d = 1$, such that $V = V^{red} \cap V_d$. Let $P \in V^{red}(K)$ and take $H$ to be a hyperplane which does not contain $P$. Then,*

$$g(V) = g(H \cap \mathcal{C}(V^{red}; P)) = g(\mathcal{C}(V^{red}; P)) + 1.$$

**Proof.** First, consider when $\ell_d \geq 2$. From Definition 2.6, observe that $\mathcal{C}(V^{\mathrm{red}}; P)$ has type

$$\begin{bmatrix} d & \cdots & 1 \\ \ell_d - 1 & \cdots & \left(\sum_{j=1}^{d} \ell_j\right) - 1 \end{bmatrix}.$$

.

From Definition 3.3, it follows that

$$\begin{aligned} g(V) &= g(d; \ell_d, \ldots, \ell_1) \\ &= g\left(d; \ell_d - 1, (\ell_d + \ell_{d-1}) - 1, \ldots, \left(\sum_{j=2}^{d} \ell_j\right) - 1, \sum_{j=1}^{d} \ell_j\right) \\ &= g\left(\mathcal{C}(V^{\mathrm{red}}; P)\right) + 1 \\ &= g\left(H \cap \mathcal{C}(V^{\mathrm{red}}; P)\right). \end{aligned}$$

Similarly, when $\ell_d = 1$, we have

$$\begin{aligned} g(V) &= g(d; \ell_d, \dots, \ell_1) \\ &= g\left(d-1; \ell_{d-1}, (\ell_d + \ell_{d-1}), \dots, \sum_{j=2}^{d} \ell_j, \left(\sum_{j=1}^{d} \ell_j\right) + 1\right) \\ &= g\left(\mathcal{C}(V^{\mathrm{red}}; P)\right) + 1 \\ &= g\left(H \cap \mathcal{C}(V^{\mathrm{red}}; P)\right). \end{aligned}$$

□

**Remark 3.6** (Geometric insight for the reduction lemma)**.** The proof of Lemma 3.5 follows immediately from Definition 3.3, but we wish to address the geometric reasoning underlying the lemma. Suppose our goal is to determine a point $Q$ of $V$ over an extension of bounded resolvent degree. Observe that if we can determine a line $\Lambda \subseteq V^{\mathrm{red}}$, then we need only solve a degree $d$ polynomial to determine a point of $V$. As $V^{\mathrm{red}}$ is $V$ with $V_d$ removed, it is already "less difficult" to determine the point $P \in V^{\mathrm{red}}(K)$ given by assumption (i.e. $g(V) \geq g(V^{\mathrm{red}})$). Additionally, we can determine a line $\Lambda \subseteq V^{\mathrm{red}}$ by determining a point $P' \neq P$ of $\mathcal{C}(V^{\mathrm{red}}; P)$. As $H$ is taken to be a hyperplane which does not contain $P$, it suffices to determine any point of $\mathcal{C}(V^{\mathrm{red}}; P) \cap H$, which is also "less difficult" as $\mathcal{C}(V^{\mathrm{red}}; P)$ is defined by fewer top degree hypersurfaces.

As in Lemma 3.5, we will frequently want to split an intersection of hypersurfaces $V$ into parts analogous to $V^{\mathrm{red}}$ and $V_d$, and so we introduce the following terminology and notation.

**Definition 3.7** (Reduction and Complement)**.** Given an intersection of hypersurfaces $V$ of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $\ell_d \geq 2$, a **reduction** of $V$ is an intersection of hypersurfaces $V^{\mathrm{red}}$ of type $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d - 1 & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ such that $V = V^{\mathrm{red}} \cap V_d$ for some degree $d$ hypersurface $V_d$, we which refer to as a **complement of $V^{\mathrm{red}}$ for $V$**.

If $V$ is an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $\ell_d = 1$, a reduction of $V$ is an intersection of hypersurfaces $V^{\mathrm{red}}$ of type $\begin{bmatrix} d-1 & \cdots & 1 \\ \ell_{d-1} & \cdots & \ell_1 \end{bmatrix}$ such that $V = V^{\mathrm{red}} \cap V_d$ for some degree $d$ hypersurface $V_d$, we which refer to as a complement of $V^{\mathrm{red}}$ for $V$.

With Lemma 3.5 and Definition 3.7 in place, we now state the geometric version of Sylvester's "formula of reduction" [Syl1887, p.475].

**Corollary 3.8** (Geometric Formula of Reduction)**.** *Let $W$ be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$. Then, for any $P_0 \in W(K)$, any reduction*

*$\mathcal{C}(W;P_0)^{red}$, and any $P_1 \in \mathcal{C}(W;P_0)^{red}(K)$, we have*

$$g\left(\mathcal{C}(W;P_0)\right) = g\left(\mathcal{C}\left(\mathcal{C}(W;P_0)^{red};P_1\right)\right) + 1.$$

**Proof.** This follows immediately as a special case of Lemma 3.5 applied to $V = \mathcal{C}(W;P_0)$. □

We will soon want to successively iterate Lemma 3.5 so that we can eliminate the hypersurfaces of largest degree from any intersection of hypersurfaces by introducing many hypersurfaces of strictly lower degree. This is achieved in Proposition 3.10. However, we first introduce additional language and notation to refer to the varieties which arise in this process of reduction.

**Definition 3.9** (Sylvester Reductions)**.** Let $V$ be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $d \geq 2$ and which is not a hypersurface. A **first partial Sylvester reduction of** $V$ is

$$V^{\text{Syl}}(d;1) := \mathcal{C}(V^{\text{red}};P_0),$$

where $V^{\text{red}}$ is any reduction of $V$ and $P_0 \in V^{\text{red}}(K)$. Proceeding inductively, for any $j \in [2,\ell_d]$, a $j^{th}$ **partial Sylvester reduction of** $V$ is

$$V^{\text{Syl}}(d;j) := \mathcal{C}(H_{j-1} \cap V^{\text{Syl}}_{j-1};P_k) = H_{j-1} \cap \mathcal{C}(V^{\text{Syl}}_{j-1};P_k),$$

where $H_{j-1}$ is a hyperplane which does not contain $P_{j-1}$ and $P_j \in \left(H_{k-1} \cap V^{\text{Syl}}_{k-1}(d;j-1)\right)(K)$.

When $d \geq 3$, a **first Sylvester reduction of** $V$ is

$$V^{\text{Syl}}_1 := V^{\text{Syl}}(d;\ell_d).$$

For each $j \in [2,d-1]$, let $\lambda_{d-j+1}$ be the number of degree $d-j+1$ hypersurfaces defining a $(j-1)^{st}$ Sylvester reduction $V^{\text{Syl}}_{j-1}$. Then, a $j^{th}$ **Sylvester reduction of** $V$ is

$$V^{\text{Syl}}_j := \left(V^{\text{Syl}}_{j-1}\right)^{\text{Syl}}(d-j+1;\lambda_{d-j+1}).$$

Continuing with the notation of Definition 3.9, note that $V^{\text{Syl}}_j$ is a variety obtained by repeatedly applying Lemma 3.5 to $V$ to remove all hypersurfaces of degree $> d-j$.

**Proposition 3.10** (The Obliteration Proposition)**.** *Let $V$ be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $d \geq 2$ which is not a hypersurface. Then,*

$$g(V) = g\left(V^{Syl}_1\right),$$

*for any first Sylvester reduction $V^{Syl}_1$ of $V$.*

**Proof.** From Lemma 3.5 and Definition 3.9, it follows immediately that

$$g\left(V^{\mathrm{Syl}}(d;j)\right) = g\left(V^{\mathrm{Syl}}(d;j+1)\right),$$

for each $j \in [1, \ell_d - 1]$. Consequently, applying Lemma 3.5 to $V$ and its partial Sylvester reductions yields

$$g(V) = g\left(V^{\mathrm{Syl}}(d;1)\right) = \cdots = g\left(V^{\mathrm{Syl}}(d;\ell_d)\right) = g\left(V_1^{\mathrm{Syl}}\right). \qquad \square$$

**Remark 3.11** (Geometric Dimension Bound via Obliteration)**.** From the definition of the $j^{th}$ Sylvester reductions, we can iteratively apply Proposition 3.10 to observe that

$$g(V) = g\left(V_1^{\mathrm{Syl}}\right) = \cdots = g\left(V_{d-2}^{\mathrm{Syl}}\right) = g\left(V_{d-1}^{\mathrm{Syl}}\right),$$

which provides the most succinct description of the central argument of the geometric obliteration algorithm.

We now arrive at the geometric version of Sylvester's "formula of obliteration" as a specialization of Proposition 3.10.

**Corollary 3.12** (Geometric Formula of Obliteration)**.** *Let $W$ be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $d \geq 2$. For any $P_0 \in W(K)$ and any Sylvester reduction $\mathcal{C}(W;P_0)_1^{Syl}$, we have*

$$g(\mathcal{C}(W;P_0)) = g\left(\mathcal{C}(W;P_0)_1^{Syl}\right). \tag{3}$$

**Proof.** This follows immediately as a special case of Proposition 3.10 with $V = \mathcal{C}(W;P_0)$. $\square$

**Remark 3.13** (Explicit numerics of the formula of obliteration)**.** Sylvester's formula of obliteration [Syl1887, p.475], which we address in Proposition 3.15, is given numerically and, for notational reasons, he chooses to write the statement in terms of "linear solutions" of $\mathcal{C}(W;P_0)^{\mathrm{Syl}}(d;\ell_d - 1)$ instead of $g\left(\mathcal{C}(W;P_0)_1^{\mathrm{Syl}}\right)$. For this reason, we delay the discussion of numerics of the formula of obliteration to Subsection 3.2.

As we have established the reduction lemma and the obliteration proposition, which we used to recover Sylvester's formula of reduction and formula of obliteration, we proceed to prove inequality (2).

**Proposition 3.14** (Minimal vs. Geometric Dimension Bound)**.** *For every type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ of an intersection of hypersurfaces,*

$$r(d;\ell_d,\ldots,\ell_1) \leq g(d;\ell_d,\ldots,\ell_1) < \infty.$$

**Proof: The Geometric Obliteration Algorithm.** We proceed by induction on $d$. First, observe that when $d = 1$, it is immediate that

$$r(1;\ell_1) = \ell_1 = g(1;\ell_1).$$

We additionally consider the case $d = 2$ before considering the general case. For the $d = 2$ case, we proceed via induction on $\ell_2$. When $\ell_2 = 1$, $\deg(V) = 2$ and thus we can determine a point of $V$ by solving a quadratic polynomial when

$$\dim(V) \geq r - (\ell_1 + 1) = 0.$$

It follows that

$$r(2; 1, \ell_1) = \ell_1 + 1 = g(2; 1, \ell_1).$$

Now, consider the case where $\ell_2 \geq 2$ is arbitrary. Our inductive hypothesis yields

$$r(2; \ell_2 - 1, \lambda_1) \leq g(2; \ell_2 - 1, \lambda_1),$$

for any $\lambda_1 \geq 0$. Let $V^{\mathrm{red}}$ be a reduction of $V$ with complement $V_2$. As $V^{\mathrm{red}}$ is of type $\begin{bmatrix} 2 & 1 \\ \ell_2 - 1 & \ell_1 \end{bmatrix}$, we can determine a point $P_0$ of $V^{\mathrm{red}}$ over an iterated quadratic extension whenever $r \geq g(V^{\mathrm{red}})$. Let $H$ be a hypersurface which does not contain $P_0$. Note that $H \cap \mathcal{C}(V^{\mathrm{red}}; P_0)$ is of type $\begin{bmatrix} 2 & 1 \\ \ell_2 - 1 & \ell_2 + \ell_1 \end{bmatrix}$ and so we can similarly determine a point $P_1$ of $H \cap \mathcal{C}(V^{\mathrm{red}}; P_0)$ over an iterated quadratic extension whenever $r \geq g(V^{\mathrm{red}}) + 1$. From Lemma 2.7, we have that

$$\Lambda(P_0, P_1) \subseteq \mathcal{C}(V^{\mathrm{red}}; P_0) \subseteq V^{\mathrm{red}}.$$

Thus, we can determine a point of $\Lambda(P_0, P_1) \cap V_2 \subseteq V$ over an additional quadratic extension. From Lemma 3.5, it follows that

$$\begin{aligned} r(2; \ell_2, \ell_1) &\leq \max\{g(2; \ell_2 - 1, \ell_1), g(2; \ell_2 - 1, \ell_1 + \ell_2)\}, \\ &= g(2; \ell_2 - 1, \ell_1 + \ell_2), \\ &= g(2; \ell_2, \ell_1). \end{aligned}$$

Now, let us return to our induction on $d$ and consider the case of general $d \geq 2$. Our inductive hypothesis for $d$ yields that $r(d-1; \lambda_{d-1}, \ldots, \lambda_1) \leq g(d-1; \lambda_{d-1}, \ldots, \lambda_1)$ for any $\lambda_{d-1} \geq 1$ and $\lambda_j \geq 0$ for all $j \in [1, d-2]$. We proceed by induction on $\ell_d$. Let $V^{\mathrm{red}}$ be a reduction of $V$ with complement $V_d$. When $\ell_d = 1$, the inductive hypothesis on $d$ yields that we can determine a point $P_0$ of $V^{\mathrm{red}}$ by solving polynomials of degree at most $d-1$ when $r \geq g(V^{\mathrm{red}})$. Letting $H$ denote a hyperplane which does not contain $P_0$, we can similarly determine a point $P_1$ of $H \cap \mathcal{C}(V^{\mathrm{red}}; P_0)$ over by solving polynomials of degree at most $d-1$ when $r \geq g\left(\mathcal{C}(V^{\mathrm{red}}; P_0)\right) + 1$. It follows that

$$\Lambda(P_0, P_1) \subseteq \mathcal{C}(V^{\mathrm{red}}; P_0) \subseteq V^{\mathrm{red}},$$

and so we can determine a point of $\Lambda(P_0, P_1) \cap V_d \subseteq V$ by solving a degree $d$ polynomial. As a result,

$$\begin{aligned} r(d; 1, \ell_{d-1}, \ldots, \ell_1) &\leq \max\left\{g(V^{\mathrm{red}}), g\left(\mathcal{C}(V^{\mathrm{red}}; P_0)\right) + 1\right\}, \\ &= g\left(\mathcal{C}(V^{\mathrm{red}}; P_0)\right) + 1, \\ &= g(V), \end{aligned}$$

$$= g(d; 1, \ell_{d-1}, \dots, \ell_1).$$

Next, we consider the case of arbitrary $\ell_d \geq 2$. Our inductive hypothesis for $\ell_d$ yields that

$$r(d; \ell_d - 1, \lambda_{d-1}, \dots, \lambda_1) \leq g(d; \ell_d - 1, \lambda_{d-1}, \dots, \lambda_1),$$

for all $\lambda_j \geq 0$, $j \in [1, d-1]$. As a result, we can determine a point $P_0$ of $V^{\text{red}}$ by solving polynomials of degree at most $d$ when $r \geq g(V^{\text{red}})$. Taking $H$ to be a hyperplane which does not contain $P_0$, we can determine a point $P_1$ of $H \cap \mathcal{C}(V^{\text{red}}; P_0)$ by solving polynomials of degree at most $d$ when $r \geq g\left(\mathcal{C}(V^{\text{red}}; P_0)\right) + 1$. Therefore,

$$\Lambda(P_0, P_1) \subseteq \mathcal{C}(V^{\text{red}}; P_0) \subseteq V^{\text{red}},$$

and we can determine a point a point of $\Lambda(P_0, P_1) \cap V_d \subseteq V$ by solving an additional degree $d$ polynomial. Consequently,

$$\begin{aligned} r(d; \ell_d, \dots, \ell_1) &\leq \max\left\{g(V^{\text{red}}), g\left(\mathcal{C}(V^{\text{red}}; P_0)\right) + 1\right\}, \\ &= g\left(\mathcal{C}(V^{\text{red}}; P_0)\right) + 1, \\ &= g(V), \\ &= g(d; \ell_d, \dots, \ell_1). \end{aligned}$$

Finally, we note that the polar cone construction introduces only finitely many hypersurfaces, all of which are strictly smaller degree. Consequently, iterating Lemma 3.5 yields that $g(d; \ell_d, \dots, \ell_1)$ is finite for every type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$. □

**3.2. Sylvester's obliteration algorithm.** In [Syl1887], Sylvester writes

> "In the following memoir I propose to present *Hamilton's* process under what appears to me to be a clearer and more easily intelligible form, to extend his numerical results and to establish the principles of a more general method than that to which he has confined himself."

We now propose to serve the analogous role for Sylvester that Sylvester served for Hamilton. Note that [Syl1887] begins with a "a somewhat more extended statement of the Law of Inertia (Trägheitsgesetz) for quadratic forms" and provides a brief history of the theory of Tschirnhaus transformations, both of which we omit here. Sylvester's law of inertia is well-known (see [Ost1959, Section 1]) and not necessary for our purposes. We refer the reader to [Wol2021, Section 2 and Appendix B] for a more complete history of Tschirnhaus transformations.

Throughout this subsection, we consider a system $S = \{f_1, \dots, f_s\}$ of homogeneous polynomials. Given a solution $P_0$ of $S$, the "first emanant" [Syl1887, p.471] of $S$ at $P_0$ is

$$S(1; P_0) := \left\{t(\ell, f_j, P_0) \mid j \in [1, s], \ell \in [0, \deg(f_j) - 1]\right\},$$

where $t(\ell, f_j, P_0)$ is as in equation (1) of Definition 2.4. Given a solution $P_1$ of $S(1; P_0)$, Sylvester's sub-lemma [Syl1887, p.472] states that any linear combination $\lambda_0 P_0 + \lambda_1 P_1$ (what he calls an "alliance" of $P_0$ and $P_1$) is a solution of $S(1; P_0)$, where $[\lambda_0 : \lambda_1] \in \mathbb{P}^1(K)$. Consequently, Sylvester says that $P_0$ and $P_1$ define a "linear solution" of $S(1; P_0)$ (and thus also of $S$, since $S \subseteq S(1; P_0)$).

Note that the geometric version of Sylvester's sub-lemma [Syl1887, p.472] is Lemma 2.7. The core algebraic computation reduces to the case of hypersurfaces; see Lemma 2.8 of [Sut2021C]. Additionally, just as the second-named author constructs iterated polar cones in [Sut2021C], Sylvester analogously introduces "$r^{th}$ emanants" [Syl1887, p.472] and "the Lemma"
[Syl1887, p.472] is the analogue of the polar point lemma (Lemma 2.9). His proof follows from iterating the sub-lemma.

Sylvester now focuses on linear solutions [Syl1887, p.475] of systems of equations. First, he introduces "completed emanants" [Syl1887, p.475] to ensure that $P_1$ is distinct from $P_0$ (and thus $P_0$ and $P_1$ determine a genuine linear solution). More specifically, a completed emanant is a system of equations $T = S(1; P_0) \cup \{g\}$, where $g$ is a homogeneous linear polynomials such that $g(P_0) \neq 0$. Next, let $S$ be of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$. Sylvester introduces notation [Syl1887, p.475] to denote the number of variables necessary to determine a linear solution of $S$. We modify his notation slightly for clarity and write $[d; \ell_d, \dots, \ell_1]$ instead of $[p, q, r, \dots, \eta, \theta]$. Note that

$$[d; \ell_d, \dots, \ell_1] = r\left(\mathcal{C}(\mathbb{V}(S); P_0)\right) + 1,$$

for any $P_0 \in \mathbb{V}(S)(K)$. It follows that Sylvester's formula of reduction [Syl1887, p.475] is

$$[d; \ell_d, \dots, \ell_1] \leq \left[ d; \ell_d - 1, \ell_d + \ell_{d-1}, \dots, \sum_{j=2}^{d} \ell_j, \sum_{j=1}^{d} \ell_j \right] + 1,$$

when $\ell_d \geq 2$. When $\ell_d = 1$, let $d'$ be the largest $j \leq d-1$ such that $\ell_j$ is non-zero. Then, Sylvester's formula of reduction is

$$[d; \ell_d, \dots, \ell_1] \leq \left[ d'; \ell_{d'}, \ell_{d'} + \ell_{d'-1}, \dots, \sum_{j=2}^{d'} \ell_j, \sum_{j=1}^{d'} \ell_j \right] + 1.$$

Sylvester then claims the his formula of obliteration [Syl1887, p.475] without proof. We state his formula of obliteration and provide a proof, for the sake of completeness.

**Proposition 3.15** (Sylvester's Formula of Obliteration)**.** *Let $S$ be a system of homogeneous polynomials of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $d \geq 2$ and $\ell_d \geq 2$. Then,*

$$\begin{aligned} [d; \ell_d, \dots, \ell_1] &\leq [d-1; \lambda_{d-1}, \lambda_{d-2}, \dots, \lambda_2, \lambda_1] + \ell_d, \\ &= [d-1; \lambda_{d-1}, \lambda_{d-2}, \dots, \lambda_2, \lambda_1 + \ell_d], \end{aligned}$$

*where*

$$\lambda_{d-j} = \binom{\ell_d + j - 1}{j} \frac{j\ell_d + 1}{j+1} + \sum_{\nu=0}^{j-1} \binom{\ell_d + \nu - 1}{\nu} \ell_{d-j+\nu}.$$

**Proof.** It is straightforward to see that iteratively applying Sylvester's formula of reduction allows us to reduce to a system of equations of degree at most $d-1$. For the explicit numerics, we give a proof via induction on $\ell_d$. Note that to determine a linear solution of $S$, it suffices to determine a point solution of a completed emanant $T_0$ of $S$ at some point solution $P_0$. Additionally, we note that the type of $T_0$ is

$$\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d & \ell_d + \ell_{d-1} & \cdots & \sum_{j=2}^{d} \ell_j & \left(\sum_{j=1}^{d} \ell_j\right) + 1 \end{bmatrix}.$$

Now, suppose that $\ell_d = 1$. We can determine a point solution $P_1$ of $T_0$ by determining a linear solution of the subsystem $T_0'$, which is of type

$$\begin{bmatrix} d-1 & \cdots & 2 & 1 \\ 1 + \ell_{d-1} & \cdots & 1 + \sum_{j=2}^{d-1} \ell_j & \left(1 + \sum_{j=1}^{d-1} \ell_j\right) + 1 \end{bmatrix}.$$

Futhermore, we see that

$$\begin{aligned} \lambda_{d-j} &= \binom{1 + j - 1}{j} \frac{j(1) + 1}{j+1} + \sum_{\nu=0}^{j-1} \binom{1 + \nu - 1}{\nu} \ell_{d-j+\nu}, \\ &= 1 + \sum_{\nu=0}^{j-1} \ell_{d-j+\nu}, \\ &= 1 + \sum_{\mu=d-j}^{d-1} \ell_\mu, \end{aligned}$$

so the claim holds when $\ell_d = 1$. Now, consider the case where $\ell_d \geq 2$ is arbitrary. To determine a point solution of $T_0$, it suffices to determine a linear solution of a subsystem $T_0'$, which is of type

$$\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d - 1 & \ell_d + \ell_{d-1} & \cdots & \sum_{j=2}^{d} \ell_j & \left(\sum_{j=1}^{d} \ell_j\right) + 1 \end{bmatrix}.$$

Thus,

$$[d; \ell_d, \ldots, \ell_1] \leq \left[ d; \ell_d - 1, (\ell_d + \ell_{d-1}), \ldots, \left(\sum_{j=2}^{d} \ell_d\right), \left(\sum_{j=1}^{d} \ell_j\right) + 1 \right].$$

By induction, however, we have that

$$\left[d; \ell_d - 1, (\ell_d + \ell_{d-1}), \dots, \left(\sum_{j=2}^{d} \ell_d\right), \left(\sum_{j=1}^{d} \ell_j\right) + 1\right] \leq [d-1; \theta_{d-1}, \dots, \theta_1 + \ell_d],$$

where

$$\begin{aligned}
&\theta_{d-j} \\
&= \binom{(\ell_d - 1) + j - 1}{j} \frac{j(\ell_d - 1) + 1}{j+1} \\
&\quad + \sum_{\nu=0}^{j-1} \binom{(\ell_d - 1) + \nu - 1}{\nu} \left(\sum_{\mu=0}^{j} \ell_{d-j+\mu}\right), \\
&= \binom{\ell_d + j - 2}{j} \frac{j\ell_d - j + 1}{j+1} + \sum_{\nu=0}^{j-1} \binom{\ell_d + \nu - 2}{\nu} \left(\sum_{\mu=0}^{j} \ell_{d-j+\mu}\right).
\end{aligned}$$

Note that for each $\mu' \in [0, j-1]$, there are exactly $\mu' + 1$ summands containing $\ell_{d-j+\mu'}$, namely

$$\binom{\ell_d - 2}{0}\ell_{\mu'}, \binom{\ell_d - 1}{1}\ell_{\mu'}, \dots, \binom{\ell_d + \mu' - 2}{\mu'}\ell_{\mu'}.$$

Additionally, there are exactly $j$ summands containing $\ell_d$, namely

$$\binom{\ell_d - 2}{0}\ell_d, \binom{\ell_d - 1}{1}\ell_d, \dots, \binom{\ell_d + j - 3}{j-1}\ell_d.$$

As a result,

$$\begin{aligned}
&\theta_{d-j} \\
&= \binom{\ell_d + j - 2}{j} \frac{j\ell_d - j + 1}{j+1} + \sum_{\nu'=0}^{j-1} \binom{\ell_d + \nu' - 2}{\nu'}\ell_d \\
&\quad + \sum_{\mu_1=0}^{j-1} \left(\sum_{\mu_2=0}^{\mu_1} \binom{\ell_d + \mu_2 - 2}{\mu_2}\right) \ell_{d-j+\mu_1}, \\
&= \binom{\ell_d + j - 2}{j} \frac{j\ell_d - j + 1}{j+1} + \binom{\ell_d + j - 2}{j-1}\ell_d + \sum_{\mu_1=0}^{j-1} \binom{\ell_d + \mu_1 - 1}{\mu_1}\ell_{d-j+\mu_1}.
\end{aligned}$$

Next, we see that

$$\binom{\ell_d + j - 2}{j} \frac{j\ell_d - j + 1}{j+1} = \binom{\ell_d + j - 2}{j} \frac{j\ell_d + 1}{j+1} - \binom{\ell_d + j - 2}{j} \frac{j}{j+1},$$

and

$$\binom{\ell_d + j - 2}{j-1}\ell_d = \binom{\ell_d + j - 2}{j-1} \frac{j\ell_d + 1}{j+1} + \binom{\ell_d + j - 2}{j-1} \frac{\ell_d - 1}{j+1}.$$

Noting that $\binom{\ell_d+j-2}{j} + \binom{\ell_d+j-2}{j-1} = \binom{\ell_d+j-1}{j}$, it follows that

$$\theta_{d-j} = \binom{\ell_d+j-1}{j}\frac{j\ell_d+1}{j+1} + \binom{\ell_d+j-2}{j-1}\frac{\ell_d-1}{j+1} - \binom{\ell_d+j-2}{j}\frac{j}{j+1} + \sum_{\mu_1=0}^{j-1}\binom{\ell_d+\mu_1-1}{\mu_1}\ell_{d-j+\mu_1}.$$

However,

$$\begin{aligned}
&\binom{\ell_d+j-2}{j-1}\frac{\ell_d-1}{j+1} - \binom{\ell_d+j-2}{j}\frac{j}{j+1}\\
&= \frac{(\ell_d+j-2)!(\ell_d-1)}{(j-1)!(\ell_d-1)!(j+1)} - \frac{(\ell_d+j-2)!j}{j!(\ell_d-2)!(j+1)},\\
&= \frac{(\ell_d+j-2)!}{(j-1)!(\ell_d-2)!(j+1)} - \frac{(\ell_d+j-2)!}{(j-1)!(\ell_d-2)!(j+1)},\\
&= 0,
\end{aligned}$$

and thus

$$\theta_{d-j} = \binom{\ell_d+j-1}{j}\frac{j\ell_d+1}{j+1} + \sum_{\mu_1=0}^{j-1}\binom{\ell_d+\mu_1-1}{\mu_1}\ell_{d-j+\mu_1} = \lambda_{d-j},$$

which proves the claim. □

Sylvester then applies his formula of obliteration to the question of determining non-zero solutions of equations which define the Tschirnhaus complete intersections $\tau_{1,\dots,m-1}$, including his Triangle of Obliteration. We omit his discussion here as the bounds he obtains are succeeded by the bounds of [Bra1975], [Wol2021], [Sut2021C], and the next section.

## 4. Upper bounds on resolvent degree

**4.1. Previous bounds.** The current upper bounds on $\mathrm{RD}(n)$ were determined by the second-named author in [Sut2021C, Theorem 3.27], which improved upon those of Wolfson [Wol2021, Theorem 5.6]. The general framework used by both the second-named author (with polar cones) and Wolfson (without polar cones) for constructing their respective bounding functions $G(m)$ and $F(m)$ was outlined in Remark 2.13. We define $G(m)$ below, but first we highlight the function's key properties (and recall that property 1, which both $F(m)$ and $G(m)$ share, is why we refer to $F(m)$ and $G(m)$ as bounding functions).

**Theorem 4.1** (Theorem 1.3 of [Sut2021C]). *The function $G(m)$ of* [Sut2021C, Definition 3.26] *has the following properties:*

(1) *For each $m \geq 1$ and $n \geq G(m)$, $\mathrm{RD}(n) \leq n - m$.*

(2) *For each* $d \geq 4$, $G(2d^2+7d+6) \leq \frac{(2d^2+7d+5)!}{d!}$. *In particular, for* $d \geq 4$ *and* $n \geq \frac{(2d^2+7d+5)!}{d!}$,

$$\mathrm{RD}(n) \leq n - 2d^2 - 7d - 6.$$

(3) *For each* $m \geq 1$, $G(m) \leq F(m)$ *with equality only when* $m$ *belongs to* $\{1,2,3,4,5,15,16\}$ *and*

$$\lim_{m\to\infty} \frac{F(m)}{G(m)} = \infty.$$

We will now numerically define $G(m)$ (which will require two additional functions) and then a summary of the construction of $G(m)$. We refer the reader to [Sut2021C, Section 3] for the full construction of $G(m)$ and proofs of the statements in Theorem 4.1.

**Definition 4.2** (The Function $G(m)$)**.** We first define $\vartheta : \mathbb{Z}_{\geq 3} \times \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ so that $\vartheta(d,k)$ is the minimal $r \in \mathbb{Z}_{\geq 1}$ such that

$$(k+1)(r-k) - \sum_{j=2}^{d} \binom{k+i}{i} \geq 0.$$

Explicitly, we have

$$\vartheta(d,k) = k + \left\lceil \frac{1}{k+1}\left(\binom{k+d+1}{d} - (k+2)\right)\right\rceil.$$

Next, we define $\varphi : \mathbb{Z}_{\geq 15} \times \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ by setting $\varphi(d,k)$ equal to

$$\max\left\{\frac{(d+k)!}{d!}, \binom{\vartheta(d,k)+d+1}{d} - (\vartheta(d,k)+1)^2 - (\vartheta(d,k)+d)\right\}.$$

Finally, we define $G : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$. For $m \in [1,14]$, we define $G(m)$ by

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $G(m)$ | 2 | 3 | 4 | 5 | 9 | 21 | 109 | 325 | 1681 | 15121 |

| $m$ | 11 | 12 | 13 | 14 |
|---|---|---|---|---|
| $G(m)$ | 151,201 | 1,663,201 | 19,958,401 | 259,459,201 |

and for $m \geq 15$ by

$$G(m) = 1 + \min\{\varphi(d, m-d-1) \mid 4 \leq d \leq m-1\}.$$

The values of $G(m)$ for $m \in [1,5]$ are classical and described in [Wol2021, Appendix B]. In [Che1954], Chebotarev gave an argument that $\mathrm{RD}(n) \leq n-6$ for $n \geq 21$, however his argument had a gap which was fixed by [Sut2021C, Theorem 3.7]. More specifically, Chebotarev (like Wiman before him in [Wim1927]) assumed certain intersections of hypersurfaces were generic without proof.

For $m \in [6,14]$, the second-named author determined $k$-polar points on extended Tschirnhaus complete intersections $\tau^{\circ}_{1,\ldots,d}$ [Sut2021C, Theorems 3.7, 3.10]. However, the degrees of iterated polar cones grow exponentially and this method could not be further extended [Sut2021C, Remark 3.19]. For general

$m$, the second-named author was able to improve on the bounds of Wolfson by using [DM1998, Theorem 2.1] to minimize the ambient dimension required for Wolfson's algorithm [Sut2021C, Theorem 3.24].

**4.2. New bounds.** We will now improve on $G(m)$ for $m \in [13,17] \cup [22,25]$. For $m \in [7,16]$, $G(m)$ is obtained by determining an $(m-5)$-plane on $\tau^{\circ}_{1,2,3,4}$. Additionally, for $m \in [17,24]$, $G(m)$ is obtained by determining an $(m-6)$-plane on $\tau^{\circ}_{1,2,3,4,5}$. Finally, for $m \in [25,33]$, $G(m)$ is obtained by determining an $(m-7)$-plane on $\tau^{\circ}_{1,2,3,4,5,6}$.

Our improvements will come from determining an $(m-6)$-plane on $\tau^{\circ}_{1,2,3,4,5}$ for $m \in [13,17]$ and from determining an $(m-7)$-plane on $\tau^{\circ}_{1,2,3,4,5,6}$ for $m \in [22,25]$. Note that in each of these cases, one can apply the geometric obliteration algorithm to obtain improved bounds. However, we will use a slight modification which allows for a minor optimization.

**Remark 4.3** (A Modification of the Geometric Obliteration Algorithm)**.** Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$. Recall that successive uses of Proposition 3.10 yield that

$$g(V) = g\left(V_1^{\mathrm{Syl}}\right) = \cdots = g\left(V_{d-3}^{\mathrm{Syl}}\right) = g\left(V_{d-2}^{\mathrm{Syl}}\right),$$

and that $V_{d-2}^{\mathrm{Syl}}$ is an intersection of type $\begin{bmatrix} 2 & 1 \\ \lambda_2 & \lambda_1 \end{bmatrix}$. In the spirit of the obliteration algorithm, we could indeed continue to apply Lemma 3.5 until there is a single quadric left, at which point we need only solve a final quadratic polynomial.

However, we also note that $\deg\left(V_{d-2}^{\mathrm{Syl}}\right)$ is $2^{\lambda_2}$ and thus we can determine a point of $W_V$ by solving a polynomial of degree $2^{\lambda_2}$ whenever $r \geq \lambda_2 + \lambda_1$. Consequently, we obtain a slight improvement in the forthcoming bounds on $\mathrm{RD}(n)$ by reducing only to a $j^{th}$ partial Sylvester reduction of $V_{d-2}^{\mathrm{Syl}}$ for some $j < \lambda_2$ instead of $V_{d-1}^{\mathrm{Syl}}$.

**Definition 4.4** (Optimal Reduction of Tschirnhaus Complete Intersections)**.** For each $d \geq 3$ and $m \geq d+2$, consider

$$W = \left(\mathcal{C}^{m-d-1}(\tau_{1,\dots,d}; P_0, \dots, P_{m-d-2})\right)_{d-2}^{\mathrm{Syl}},$$

a $(d-2)^{nd}$ Sylvester reduction of an $(m-d-1)^{st}$ polar cone of $\tau_{1,\dots,d}$, which is of type $\begin{bmatrix} 2 & 1 \\ \lambda_2 & \lambda_1 \end{bmatrix}$. For each $j \in [1, \lambda_2 - 1]$, note that a $j^{th}$ partial Sylvester reduction $W^{\mathrm{Syl}}(2; j)$ of $W$ has type $\begin{bmatrix} 2 & 1 \\ \lambda_2 - j & \lambda_1 + \sum_{\nu=\lambda_2-j}^{\lambda_2-1} \nu \end{bmatrix}$. Further,

$$\deg\left(W^{\mathrm{Syl}}(2; j)\right) = 2^{\lambda_2 - j}.$$

For each such $j$, set $\xi(m,d;j)$ equal to

$$\max\left\{(m-d+1)+(\lambda_2-j)+\left(\lambda_1+\sum_{\nu=\lambda_2-j}^{\lambda_2-1}\nu\right), 2^{\lambda_2-j}+1\right\}.$$

The **optimal reduction bound of $\tau_{1,\dots,d}$ for $m$**, is

$$\Xi(m,d) := \min\{\xi(m,d;j) \mid j \in [0,\lambda_2-1]\}.$$

In particular, $\Xi(m,d)$ is defined exactly so that for $n \geq \Xi(m,d)$, we can determine an $(m-d-1)^{th}$ polar point of $\tau^{\circ}_{1,\dots,d}$ in $\mathbb{P}^{n-1}_{K_n}$ over an extension $K'/K_n$ with $\mathrm{RD}(K'/K_n) \leq \mathrm{RD}(\Xi(m;d))$.

**Remark 4.5** ($\Xi(m,d)$ is Non-Decreasing in $m$)**.** We note that $\Xi(m,d)$ is non-decreasing in $m$ for fixed $d$. This can be seen geometrically from the fact if $(P_0,\dots,P_{m-d-1})$ is an $(m-d-1)^{st}$ polar point of $\tau_{1,\dots,d}$, then $(P_0,\dots,P_{m-d-2})$ must be an $(m-d-2)^{nd}$ polar point of $d$, so $\Xi(m,d) \geq \Xi(m-1,d)$.

We are now ready to state and prove the main theorem.

**Theorem 4.6** (Bounds from the Geometric Obliteration Algorithm)**.**

(1) *For $n \geq 5,250,198$, $\mathrm{RD}(n) \leq n-13$.*
(2) *For each $m \in [14,17]$ and $n > \frac{(m-1)!}{120}$, $\mathrm{RD}(n) \leq n-m$.*
(3) *For $n \geq 381,918,437,071,508,900$, $\mathrm{RD}(n) \leq n-22$.*
(4) *For each $m \in [23,25]$ and $n > \frac{(m-1)!}{720}$, $\mathrm{RD}(n) \leq n-m$.*

**Proof.** We continue to use the notation established in Definition 4.4. For each $m \in [13,17]$, we set

$$G'(m) = \max\left\{\Xi(m,5), \frac{(m-1)!}{120}+1\right\},$$

and for each $m \in [22,25]$, we set

$$G'(m) = \max\left\{\Xi(m,6), \frac{(m-1)!}{720}+1\right\}.$$

In each case, it suffices to show the claim when $n = G'(m)$. Further, note that $G'(m) = \Xi(m,5)$ exactly when $m = 13$ and $G'(m) = \Xi(m,6)$ exactly when $m = 22$; this claim is justified by explicit computation and is given in the tables at the end of the proof. Recall that the space of Tschirnhaus transformations up to re-scaling is $\mathbb{P}^{G'(m)-1}_{K_{G'(m)}}$.

Let us first consider the case of $m \in [13,17]$ and let $H \subseteq \mathbb{P}^{G'(m)-1}_{K_n}$ be a hyperplane which does not contain $[1:0:\cdots:0]$. Note that $H \cong \mathbb{P}^{G'(m)-2}_{K_n}$ and $H \cap \tau_{1,\dots,5} = H \cap \tau^{\circ}_{1,\dots,5}$. Since $\Xi(m,5) \geq \Xi(m-1,5)$, we can assume that we

have an $(m-7)$-polar point $(P_0,\dots,P_{m-7})$ of $H\cap\tau^{\circ}_{1,\dots,5}$. Consider the minimal $j$ such that $\Xi(m,5)=\xi(m,5;j)$. By definition of $\xi(m,5;j)$, we have that

$$\dim\left(\left(\left(\mathcal{C}^{m-6}(H\cap\tau^{\circ}_{1,\dots,5};P_0,\dots,P_{m-7})\right)^{\mathrm{Syl}}_{3}\right)^{\mathrm{Syl}}(2;j)\right)\geq m-6.$$

Since $\dim(\Lambda(P_0,\dots,P_{m-7}))=m-7$, we can determine a point of

$$\mathcal{C}^{m-6}\left(H\cap\tau^{\circ}_{1,\dots,5};P_0,\dots,P_{m-7}\right)\setminus\Lambda(P_0,\dots,P_{m-7}),$$

by solving a polynomial of degree at most $\Xi(m;5)$. By construction $(P_0,\dots,P_{m-6})$ is an $(m-6)$-polar point and Lemma 2.9 yields that $\Lambda=\Lambda(P_0,\dots,P_{m-6})\subseteq\tau^{\circ}_{1,\dots,5}$ is an $(m-6)$-plane. We can then determine a point of $\Lambda\cap\tau^{\circ}_{1,\dots,m-1}$ by solving a polynomial of degree $\frac{(m-1)!}{120}$.

We now consider the similar case of $m\in[22,25]$. Let $H\subseteq\mathbb{P}^{G'(m)-1}_{K_n}$ be a hyperplane which does not contain $[1:0:\cdots:0]$. Note that $H\cong\mathbb{P}^{G'(m)-2}_{K_n}$ and $H\cap\tau_{1,\dots,5}=H\cap\tau^{\circ}_{1,\dots,5}$. Since $\Xi(m,6)\geq\Xi(m-1,6)$, we can assume that we have an $(m-8)$ polar point $(P_0,\dots,P_{m-8})$ of $H\cap\tau^{\circ}_{1,\dots,6}$. Consider the minimal $j$ such that $\Xi(m,6)=\xi(m,6;j)$. Observe that

$$\dim\left(\left(\left(\mathcal{C}^{m-7}(H\cap\tau^{\circ}_{1,\dots,6};P_0,\dots,P_{m-8})\right)^{\mathrm{Syl}}_{4}\right)^{\mathrm{Syl}}(2;j)\right)\geq m-7,$$

and so we can determine a point $P_{m-6}$ of

$$\mathcal{C}^{m-7}\left(H\cap\tau^{\circ}_{1,\dots,6};P_0,\dots,P_{m-8}\right)\setminus\Lambda(P_0,\dots,P_{m-8}),$$

by solving a polynomial of degree at most $\Xi(m;6)$. it follows that $(P_0,\dots,P_{m-7})$ is an $(m-7)$-polar point of $\tau^{\circ}_{1,\dots,6}$ and so $\Lambda=\Lambda(P_0,\dots,P_{m-7})\subseteq\tau^{\circ}_{1,\dots,6}$ is an $(m-7)$-plane. Consequently, we can determine a point of $\Lambda\cap\tau^{\circ}_{1,\dots,m-1}$ by solving a polynomial of degree $\frac{(m-1)!}{720}$.

We now show that $G'(m)=\Xi(m,5)$ exactly when $m=13$ and $G'(m)=\Xi(m,6)$ exactly when $m=22$. In the following tables, we note the values of $\Xi(m,5)$ and $\frac{(m-1)!}{120}+1$ for $m\in[13,17]$ and the approximate values of $\Xi(m,6)$ and $\frac{(m-1)!}{720}+1$ for $m\in[22,25]$. The exact values of $\Xi(m,5)$ for $m\in[13,17]$ and of $\Xi(m,6)$ for $m\in[22,25]$ were computed using Algorithm A.6, which can be found in Subsection A.4.

| $m$ | $\Xi(m,5)$ | $\frac{(m-1)!}{120}+1$ |
|---|---|---|
| 13 | 5,250,198 | 3,991,681 |
| 14 | 12,253,482 | 51,891,841 |
| 15 | 26,357,165 | 726,485,761 |
| 16 | 53,008,668 | 10,897,286,401 |
| 17 | 100,769,994 | 174,356,582,401 |

| $m$ | $\Xi(m,6)$ | $\frac{(m-1)!}{720}+1$ |
|---|---|---|
| 22 | $\sim 3.819\times 10^{17}$ | $\sim 7.096\times 10^{16}$ |
| 23 | $\sim 9.526\times 10^{17}$ | $\sim 1.561\times 10^{18}$ |
| 24 | $\sim 2.262\times 10^{18}$ | $\sim 3.591\times 10^{19}$ |
| 25 | $\sim 5.137\times 10^{18}$ | $\sim 8.617\times 10^{20}$ |

□

**4.3. Obstruction to further bounds via the geometric obliteration algorithm.** Unfortunately, the proof strategy of Theorem 4.6 does not yield further bounds on $\mathrm{RD}(n)$. Recall that for $m \geq 15$, $G(m)$ is defined by

$$G(m) = 1 + \min\{\varphi(d, m-d-1) \mid d \in [4, m-1]\},$$

where $\varphi(d,k)$ equals

$$\max\left\{\frac{(d+k)!}{d!}, \binom{\vartheta(d,k)+d+1}{d} - (\vartheta(d,k)+1)^2 - (\vartheta(d,k)+d)\right\}.$$

For each $d$, the values of $m$ for which $G(m) = 1+\varphi(d,m-d-1)$ is a set of consecutive integers. Equivalently, there are positive integers $m_d$ and $m'_d$ such that $G(m) = 1+\varphi(d,m-d-1)$ if and only if $m \in [m_d, m'_d]$; see [Sut2021C, Lemma 3.33] for details.

Similarly, we briefly introduce the notation

$$\varrho(d,k) = \max\left\{\Xi(d+k+1,d), \frac{(d+k)!}{d!}+1\right\}$$

for $d \geq 4$ and $k \geq 1$, as well as

$$H(m) = \min\{\varrho(d, m-d-1) \mid d \in [4, m-1]\}$$

for $m \geq 13$. For fixed $d$, note that $\Xi(m,d)$ is a polynomial in $m$, whereas $\frac{(d+k)!}{d!} = \frac{(m-1)!}{d!}$ grows factorially. It follows that for each $d$, there are positive integers $M_d$ and $M'_d$ such that $H(m) = \varrho(d, m-d-1)$ if and only if $m \in [M_d, M'_d]$.

In the following table, we compare the values $m_d$ and $M_d$ for $d = 5,6,7,8$.

| $d$ | $m_d$ | $M_d$ |
|---|---|---|
| 5 | 17 | 13 |
| 6 | 25 | 22 |
| 7 | 34 | 41 |
| 8 | 44 | 78 |

This provides further evidence, along with [Sut2021C, Remark 3.19], that iterated polar cone methods are most effective for intersections of hypersurfaces of small types. Next, we determine an explicit lower bound on $\Xi(m,d)$.

**Lemma 4.7** (Lower Approximation). *Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces of type $\begin{bmatrix} d \\ \ell_d \end{bmatrix}$ with $d \geq 3$ and $\ell_d \geq 2$. Denote the type of a $(d-2)^{nd}$ Sylvester*

*reduction* $V_{d-2}^{Syl}$ *by* $\begin{bmatrix} 2 & 1 \\ \lambda_2 & \lambda_1 \end{bmatrix}$. *Then,*

$$\lambda_1 \geq \lambda_2 \geq \left\lceil 2^{5-2d} \left(\ell_d - 1\right)^{2d-4} \right\rceil.$$

**Proof.** Note that the number of degree $d-1$ hypersurfaces of $V_1^{\mathrm{Syl}}$ is

$$\theta_{d-1} = \sum_{j=1}^{\ell_d - 1} \ell_d - j = \frac{1}{2}(\ell_d - 1)\ell_d \geq \left\lceil \frac{1}{2}(\ell_d - 1)^2 \right\rceil.$$

The same argument yields that the number of degree $d-2$ hypersurfaces of $V_2^{\mathrm{Syl}}$ is

$$\theta_{d-2} \geq \left\lceil \frac{1}{2} \left\lceil \frac{1}{2}(\ell_d - 1)^2 \right\rceil^2 \right\rceil \geq \left\lceil 2^{-3}(\ell_d - 1)^4 \right\rceil.$$

Proceeding similarly, we see that

$$\lambda_2 = \theta_2 \geq \left\lceil 2^{5-2d} \left(\ell_d - 1\right)^{2d-4} \right\rceil.$$

Finally, note that $\lambda_1 \geq \lambda_2$ follows immediately from the polar cone construction. □

**Corollary 4.8** (Lower Bound for $\Xi(m,d)$)**.** *Let* $d \geq 4$ *and* $m \geq d+2$. *Then,*

$$\Xi(m,d) \geq \left\lceil 4 \left( \frac{m-d-1}{2} \right)^{2d-4} \right\rceil.$$

**Proof.** First, [Sut2021C, Proposition 2.26] yields that an $(m-d-1)^{th}$ polar cone of $\tau_{1,\ldots,d}$ is of type

$$\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ 1 & \binom{m-d}{1} & \cdots & \binom{m-3}{d-2} & \binom{m-2}{d-1} \end{bmatrix}.$$

Thus, the number of degree $d-1$ hypersurfaces of $V = \left(\tau_{1,\ldots,d}\right)_1^{\mathrm{Syl}}$ is $m-d$. Let $\sigma(m,d)$ be as in Definition 4.4. It follows from Lemma 4.7 that

$$\lambda_1 \geq \lambda_2 \geq \left\lceil 2^{5-2d}(m-d-1)^{2d-4} \right\rceil.$$

Moreover, for each $j$,

$$\begin{aligned} \xi(m,d;j) \geq \lambda_1 + \lambda_2 &\geq \left\lceil 2^{5-2d}(m-d-1)^{2d-4} \right\rceil + \left\lceil 2^{5-2d}(m-d-1)^{2d-4} \right\rceil, \\ &\geq \left\lceil 4 \left( \frac{m-d-1}{2} \right)^{2d-4} \right\rceil, \end{aligned}$$

and thus it follows that

$$\Xi(m,d) = \min\left\{\xi(m,d;j) \mid 0 \leq j \leq \lambda_2 - 1\right\} \geq \left\lceil 4 \left( \frac{m-d-1}{2} \right)^{2d-4} \right\rceil. \qquad \square$$

While we do not provide a full comparison here, we note that the key obstruction to obtaining further bounds on $\mathrm{RD}(n)$ using the methods of Theorem 4.6 is that $\Xi(m,d)$ has a lower bound which grows exponentially in $d$ and that $m-d-1$ grows much more quickly than $d$ (for example, $m-d-1 \geq 19$ for $m \geq 26$).

Having indicated the obstruction to obtaining further upper bounds on $\mathrm{RD}(n)$ using these methods, we now combine Theorems 4.1 and 4.6 to immediately construct a new bounding function with the same key properties of $G(m)$.

**Corollary 4.9** (The New Bounding Function)**.** *Let $G' : \mathbb{Z}_{\geq 2} \to \mathbb{Z}_{\geq 1}$ be the function with*

$$G'(m) = \max\left\{\Xi(m,5), \frac{(m-1)!}{120} + 1\right\},$$

*for $m \in [13,17]$, with*

$$G'(m) = \max\left\{\Xi(m,6), \frac{(m-1)!}{720} + 1\right\},$$

*for $m \in [22,25]$, and with $G'(m) = G(m)$ for $m \notin [13,17] \cup [22,25]$. Then, $G'(m)$ has the following properties:*

(1) *For each $m \geq 1$ and $n \geq G'(m)$, $\mathrm{RD}(n) \leq n - m$.*
(2) *For each $d \geq 4$, $G'(2d^2+7d+6) \leq \frac{(2d^2+7d+5)!}{d!}$. In particular, for $d \geq 4$ and $n \geq \frac{(2d^2+7d+5)!}{d!}$,*

$$\mathrm{RD}(n) \leq n - 2d^2 - 7d - 6.$$

**4.4. Remaining questions.** To the best of the authors' knowledge, the bounding function $G'(m)$ of Corollary 4.9 exhausts the techniques and methods for determining upper bounds on resolvent degree from the classical literature (including [Bri1786, Che1954, Ham1836, Hil1927, Seg1945, Syl1887, SH1887, SH1888, Tsc1683, Wim1927]), as well as the modern insights from [Bra1975, Sut2021C, Wol2021].

The bounding functions of Brauer, Hamilton, Sylvester, Wolfson, and the second-named author are constructed by determining points on the Tschirnhaus complete intersections $\tau^{\circ}_{1,\ldots,m-1}$ over extensions of bounded resolvent degree. However, there are solutions of the quintic and the sextic which use alternative constructions of Tschirnhaus transformations (see [Kle1884, Kle1905] for the respective original works or [Mor1956, Sut2019] for the respective English translations). We believe it would be insightful to understand whether one can reduce the general question of determining $\mathrm{RD}(n)$ to the more specific question of determining points on the Tschirnhaus complete intersections $\tau^{\circ}_{1,\ldots,m-1}$.

**Question 4.10** (Optimal Formulas via Tschirnhaus Complete Intersections)**.** *For every $n$, let $m_n$ be such that $\mathrm{RD}(n) \leq n - m_n$. Is there a formula in $n - m_n$ variables for the general degree $n$ polynomial obtained by determining a point of $\tau^{\circ}_{1,\ldots,m_n-1}$ over an extension $K'/K_n$ of bounded resolvent degree?*

For general $m$, the definition of $G'(m) = G(m)$ uses the combinatorial condition of [DM1998, Theorem 2.1] to guarantee the existence of $k$-planes on the $\tau^{\circ}_{1,\dots,d}$ and then uses the dimension of the relevant moduli space [Sut2021C, Subsection 3.3]. Notably, this combinatorial condition is non-constructive and relies only on the type of $\tau^{\circ}_{1,\dots,d}$. One might hope that such formulas could be determined using constructive methods and one approach may be to leverage the specific geometry of the $\tau^{\circ}_{1,\dots,d}$ (e.g., using more information than its type).

**Question 4.11** (RD Bounds via Explicit Constructions of $k$-Planes)**.** *Is there a bounding function $\mathfrak{G}(m)$ with $\mathfrak{G}(m) \leq G'(m)$ which arises from an explicit construction of $k$-planes on the $\tau^{\circ}_{1,\dots,d}$? If so, is it possible to determine the bounding function $\mathfrak{G}(m)$ such that*

$$\lim_{m\to\infty} \frac{G(m)}{\mathfrak{G}(m)} = \lim_{m\to\infty} \frac{G'(m)}{\mathfrak{G}(m)} = \infty?$$

Theorem 4.6 was proved using a consequence of the geometric obliteration algorithm, namely that $r(V) \leq g(V)$ for any intersection of hypersurfaces $V$. Further examination of the relationship between $r(V)$ and $g(V)$ is of interest.

**Question 4.12** (Minimal vs. Geometric Dimension Bound)**.** *For which intersections of hypersurfaces $V$ is the inequality $r(V) \leq g(V)$ strict? Are there classical examples of types of intersections of hypersurfaces where the inequality is not strict?*

Let us now briefly consider a cubic hypersurface $H = \mathbb{V}(f) \subseteq \mathbb{P}^r_K$. When $r = 3$ and $H$ is smooth, the Cayley-Salmon theorem yields that $H$ contains exactly 27 lines. The resolvent degree of determining a line on $H$ is at most 3, as was established by Farb and Wolfson [FW2019, Theorem 8.2]. Additionally, that $H$ has exactly 27 lines is consistent with [DM1998, Theorem 2.1], which states that the Fano variety of lines of a cubic surface in $\mathbb{P}^3_K$ is non-empty and has dimension 0. In particular, when $r = 3$, most points $P \in H(K)$ do not lie on a line of $H$ over an algebraic closure $\overline{K}$. When $r = 4$, however, any polar cone $\mathcal{C}(V;P)$ has dimension at least one and thus every point $P \in V(H)$ lies on at least one line $\Lambda = \Lambda(P,Q) \subseteq H$ over an algebraic closure $\overline{K}$. To determine such a point $Q$ directly, we must solve a polynomial of degree $6 = 3! = \deg(\mathcal{C}(V;P))$. Hence, we can determine a line through any point $P$ over an extension with $K'/K$ with $\mathrm{RD}(K'/K) \leq \mathrm{RD}(6) \leq 2$.

Additionally, observe that

$$g(\mathcal{C}(V;P)) = g(3;1,1,1) = g(2;1,3) = 5.$$

Thus, when $r \geq 5$, we can determine a point $Q \in \mathcal{C}(V;P) \setminus \{P\}$ over an extension determined by solving at most cubic polynomials (i.e., over a solvable extension).

Now, let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$. For each $k \geq 1$, take $s_k(V)$ to be the minimal $s$ such that

$$(k+1)(s-k) - \sum_{j=1}^{d} \ell_j \binom{k+j}{j} \geq 0.$$

One implication of Theorem 2.1 of [DM1998] is that $V$ contains a $k$-plane for all $r \geq s_k(V)$. We expect $s_k(V)$ to be the minimal ambient dimension required for $V$ to contain a $k$-plane; however, we expect the resolvent degree of determining such a $k$-plane to be large. Conversely, we expect $r\left(\mathcal{C}^k(V; P_0, \ldots, P_{k-1})\right) + k$, the ambient dimension required to determine a $k$-polar point over an extension $K'/K$ of small resolvent degree ($\mathrm{RD}(K'/K) \leq \mathrm{RD}(d)$), to be large.

**Question 4.13** (Minimizing Ambient Dimension vs. Minimizing RD of Extensions)**.** *Let $V$ be an intersection of hypersurfaces. How do $g\left(\mathcal{C}^k(V; P_0, \ldots, P_{k-1})\right) + k$, $r\left(\mathcal{C}^k(V; P_0, \ldots, P_{k-1})\right) + k$, and $s_k(V)$ compare?*

Finally, we recall that we have worked entirely in characteristic zero (more specifically, over $\mathbb{C}$). As we discussed in 2.1, we do not lose any generality from the perspective of resolvent degree, as

$$\mathrm{RD}(n) = \mathrm{RD}_{\mathbb{C}}(S_n) \geq \mathrm{RD}_K(S_n)$$

by [Rei2022, Theorem 1.3], with equality when $K$ has characteristic zero by [Rei2022, Theorem 1.2]. The foundational result for the polar cone framework we use is the technical lemma [Sut2021C, Lemma 2.8]. For those who wish to work in characteristic $p$, one would need to be careful of how the relevant combinatorics, such as [Sut2021C, Proposition 2.26], change. Additionally, the modern reference for Tschirnhaus transformations [Wol2021] works over $\mathbb{Z}$. To consider Tschirnhaus transformations in characteristic $p$, one would need to give extra consideration to the Tschirnhaus hypersurfaces of degree $p^k$.

## Appendix A. Python implementations of the obliteration algorithm and related phenomena

In Appendix A.1, we provide an implementation of the geometric obliteration algorithm in Python (Algorithm A.1). In Appendix A.2, we prove several lemmata which make the computations for the proof of Theorem 4.6 feasible. Algorithm A.5 in Appendix A.3 takes the same input and provides the same output as Algorithm A.1, but uses the lemmata of Appendix A.2 to decrease computation time. Finally, Algorithm A.6 in Appendix A.4 computes the information necessary for Theorem 4.6.

**A.1. The geometric obliteration algorithm.** We begin with an implementation of the geometric obliteration algorithm in Python (Algorithm A.1)

---

**Algorithm A.1.** ***(The Geometric Obliteration Algorithm)***

---

- Input: An intersection of hypersurfaces $V$ of type $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ with $d \geq 2$, encoded as the list DegreeList $= [\ell_d, \ell_{d-1}, \ldots, \ell_2, \ell_1]$.
- Output: The geometric dimension bound $g(d; \ell_d, \ldots, \ell_1)$.

---

The function COMPUTEPOLARCONE inputs a list which contains the type of an intersection of hypersurfaces $W$. It then returns a list which contains the type of a polar cone $\mathcal{C}(W; P)$. In particular, recall that for each $d' < d$, each hypersurface $H$ with $\deg(H) > d'$ defining $W$ contributes exactly one new degree $d'$ hypersurface defining $\mathcal{C}(W; P)$ and each hypersurface defining $\mathcal{C}(W; P)$ arises in this manner.

```
1: function ComputePolarCone(List):
2:     counter = List[0]
3:     ReturnList = [counter]
4:     for index in range(1,len(List)):
5:         counter += List[index]
6:         ReturnList.append(counter)
7:     end for
8:     return ReturnList
9: end function
```

The function OBLITERATELARGESTDEGREEHYPERSURFACES inputs a list which contains the type of an intersection of hypersurfaces $W$ whose largest degree hypersurface has degree $d \geq 3$. It identifies the number of hypersurfaces of largest degree and proceeds to iteratively remove a hypersurface $H$ of largest degree and compute a polar cone of the remaining intersection of hypersurfaces $W'$ (with an additional hyperplane included).

Note that an additional hyperplane is added each time to avoid repeated polar cone points, i.e. if $P$ was the cone point of the previous polar cone point, we pass to a hyperplane which does not contain $P$ to ensure that the cone point $Q$ of the next polar cone satisfies $Q \neq P$. Also, the polar cone of a hyperplane plane at any point is just the hyperplane itself, so to compute the combinatorics, it suffices to add one after computing the polar cone instead of doing it beforehand.

---

As taking the polar cone of a hypersurface $H$ introduces only hypersurfaces of strictly smaller degree, this process terminates and OBLITERATELARGESTDEGREEHYPERSURFACES returns a list whose data is the multi-degree of an intersection of hypersurfaces $V'$ whose largest degree hypersurface has degree $d-1$.

```
10: function ObliterateLargestDegreeHypersurfaces(List):
11:     while List[0] > 0:
12:         List[0] -= 1
13:         TempList = ComputePolarCone(List)
14:         List = TempList
15:         List[len(List)-1] += 1
16:     end while
17:     ReturnList = []
18:     for index in range(1,len(List):
19:         ReturnList.append(List[index])
20:     end for
21:     return ReturnList
22: end function
```

The function OBLITERATEQUADRICSVIALOOPS works similarly to OBLITERATELARGESTDEGREEHYPERSURFACES, but the input is the multi-degree of an intersection of hypersurfaces of type $\begin{bmatrix} 2 & 1 \\ \ell_2 & \ell_1 \end{bmatrix}$ and the loop ends with a single quadric remaining instead of zero quadrics remaining.

```
23: function ObliterateQuadricsViaLoops(List):
24:     while List[0] > 1:
25:         List[0] -= 1
26:         TempList = ComputePolarCone(List)
27:         List = TempList
28:         List[len(List)-1] += 1
29:     end while
30:     return [List[0],List[1]]
31: end function
```

The procedure MAIN inputs the multi-degree of an intersection of hypersurfaces $V$ as the list DegreeList and proceeds to successively "obliterate" the hypersurfaces of largest degree. The final step of the procedure is to return a list of the form $[1, \alpha]$, which is the requisite intersection of a single quadric and $\alpha$ hyperplanes.

```
32: procedure MAIN(DegreeList):
33:     for index in range(1,len(DegreeList)-1):
34:         TempDegreeList = OBLITERATELARGESTDEGREEHYPERSUR
            FACES(DegreeList)
35:         DegreeList = TempDegreeList
36:     end for
37:     FinalList = OBLITERATEQUADRICSVIALOOPS(DegreeList)
38:     Sum = FinalList[0] + FinalList[1]
39:     return Sum
40: end procedure
```

**A.2. Lemmata for computational improvements.** In this appendix, we give explicit numerics for Proposition 3.10 when $d = 2, 3, 4$.

**Lemma A.2** (Obliterating Quadrics)**.** *Consider an intersection of hypersurfaces $V$ of type $\begin{bmatrix} 2 & 1 \\ \ell_2 & \ell_1 \end{bmatrix}$. Then,*

$$g(V) = 1 + \ell_1 + \frac{1}{2}(\ell_2 - 1)(\ell_2 + 2).$$

**Proof.** First, observe that $V^{\mathrm{Syl}}(2;1)$ has type

$$\begin{bmatrix} 2 & 1 \\ \ell_2 - 1 & \ell_1 + \ell_2 \end{bmatrix},$$

by Definition 3.9. Similarly, $V^{\mathrm{Syl}}(2;2)$ has type

$$\begin{bmatrix} 2 & 1 \\ \ell_2 - 2 & \ell_1 + \ell_2 + \ell_2 - 1 \end{bmatrix}.$$

Proceeding in this manner yields that $V^{\mathrm{Syl}}(2;\lambda_2 - 1)$ has type

$$\begin{bmatrix} 2 & 1 \\ 1 & \ell_1 + \sum_{j=1}^{\ell_2 - 1} (\ell_2 - j + 1) \end{bmatrix},$$

and we note that

$$\sum_{j=1}^{\ell_2 - 1} (\ell_2 - j + 1) = \frac{1}{2}(\ell_2 - 1)(\ell_2 + 2).$$

From Lemma 3.5 and Definition 3.9, we see that

$$g(V) = g\left(V^{\mathrm{Syl}}(2;\lambda_2 - 1)\right) = 1 + \ell_1 + \frac{1}{2}(\ell_2 - 1)(\ell_2 + 2). \qquad \square$$

**Lemma A.3** (Obliterating Cubics)**.** *Consider an intersection of hypersurfaces $V$ of type $\begin{bmatrix} 3 & 2 & 1 \\ \ell_3 & \ell_2 & \ell_1 \end{bmatrix}$. Then, $V_1^{Syl}$ is of type $\begin{bmatrix} 2 & 1 \\ \beta_3 & \alpha_3 \end{bmatrix}$, where*

$$\beta_3 = \ell_2 + \frac{1}{2}(\ell_3 - 1)\ell_3,$$

$$\alpha_3 = \ell_1 + \ell_2\ell_3 + \frac{1}{2}\ell_3(\ell_3+1) + \frac{1}{6}\ell_3\left(2\ell_3^2 - 3\ell_3 + 1\right).$$

**Proof.** An argument analogous to the proof of Lemma A.2 yields that

$$\beta_3 = \ell_2 + \sum_{j=1}^{\ell_3}(\ell_3 - j) = \ell_2 + \frac{1}{2}(\ell_3 - 1)\ell_3.$$

Next, observe that $V^{\mathrm{Syl}}(3; j)$ has type

$$\begin{bmatrix} 3 & 2 & 1 \\ \ell_3 - j & \ell_2 + \sum_{k=1}^{j}(\ell_3 - k) & \lambda_j \end{bmatrix}.$$

Consequently,

$$\lambda_{j+1} = \lambda_j + (\ell_3 - j - 1) + \left(\ell_2 + \sum_{k=1}^{j}(\ell_3 - k)\right) + 1.$$

Combined with the initial condition $\lambda_0 = \ell_1$, we obtain that

$$\begin{aligned}
\alpha_3 &= \ell_1 + \left(\sum_{j_1=1}^{\ell_3}(\ell_3 - j_1 + 1)\right) + \left(\sum_{j_2=1}^{\ell_3}\ell_2 + \sum_{j_3=2}^{\ell_3}\sum_{j_4=1}^{j_4-1}\ell_3 - j_4\right), \\
&= \ell_1 + \frac{1}{2}\ell_3(\ell_3+1) + \left(\ell_2\ell_3 + \sum_{j_3=2}^{\ell_3}\sum_{j_4=1}^{j_3-1}\ell_3 - j_4\right), \\
&= \ell_1 + \ell_2\ell_3 + \frac{1}{2}\ell_3(\ell_3+1) + \sum_{j_3=2}^{\ell_3}\sum_{j_4=1}^{j_3-1}(\ell_3 - j_2), \\
&= \ell_1 + \ell_2\ell_3 + \frac{1}{2}\ell_3(\ell_3+1) + \frac{1}{6}\ell_3\left(2\ell_3^2 - 3\ell_3 + 1\right).
\end{aligned}$$

□

**Lemma A.4** (Obliterating Quartics)**.** *Consider an intersection of hypersurfaces $V \subseteq \mathbb{P}^r_K$ of type $\begin{bmatrix} 4 & 3 & 2 & 1 \\ \ell_4 & \ell_3 & \ell_2 & \ell_1 \end{bmatrix}$. Then, $V_1^{Syl}$ is of type $\begin{bmatrix} 3 & 2 & 1 \\ \gamma_4 & \beta_4 & \alpha_4 \end{bmatrix}$, where*

$$\begin{aligned}
\gamma_4 &= \ell_3 + \frac{1}{2}(\ell_4 - 1)\ell_4, \\
\beta_4 &= \ell_2 + \ell_3\ell_4 + \frac{1}{2}(\ell_4 - 1)\ell_4 + \frac{1}{6}\ell_4\left(2\ell_4^2 - 3\ell_4 + 1\right), \\
\alpha_4 &= \ell_1 + \ell_4\left(\ell_2 + \ell_3 + \frac{1}{2}(\ell_4+1)\right) + \ell_4\left(\frac{1}{2}\ell_3(\ell_4+1) + \frac{1}{3}\left(2\ell_4^2 - 3\ell_4 + 1\right)\right) \\
&\quad + \frac{1}{24}(\ell_4 - 2)(\ell_4 - 1)\ell_4(3\ell_4 - 1).
\end{aligned}$$

**Proof.** The proofs of Lemmata A.2 and A.3 generalize to determine $\gamma_4$ and $\beta_4$ in a straightforward manner. It remains to determine $\alpha_4$. Note that $V^{\mathrm{Syl}}(4; j)$

has type

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ \ell_4 - j & \ell_3 + \sum_{k_1=1}^{j} (\ell_4 - k_1) & \ell_2 + \left( \sum_{k_2=1}^{j} \ell_4 - k_2 \right) + \sum_{k_3=1}^{j} \left( \ell_3 + \sum_{k_4=1}^{j-1} (\ell_4 - k_4) \right) & \lambda_j \end{bmatrix}.$$

As a result,

$$\lambda_{j+1} = \lambda_j + (\ell_4 - j - 1) + \left( \ell_3 + \sum_{k=1}^{j} (\ell_4 - k) \right) + \left( \ell_2 + \left( \sum_{k_1=1}^{j} \ell_4 - k_1 \right) + \sum_{k_2=1}^{j} \left( \ell_3 + \sum_{k_3=1}^{j-1} (\ell_4 - k_3) \right) \right) + 1.$$

Given the initial condition $\lambda_0 = \ell_1$, it follows that

$$\begin{aligned} \alpha_4 &= \ell_1 + \left( \sum_{j_1=1}^{\ell_4} \ell_4 - j_1 + 1 \right) + \left( \sum_{j_2=1}^{\ell_4} \ell_3 + \sum_{j_3=2}^{\ell_4} \sum_{j_4=1}^{j_3-1} (\ell_4 - j_4) \right) \\ &\quad + \left( \sum_{j_5=1}^{\ell_4} \ell_2 + \sum_{j_6=2}^{\ell_4} \sum_{j_7=1}^{j_6-1} (\ell_4 - j_7) + \sum_{j_8=2}^{\ell_4} \sum_{j_9=1}^{j_8-1} \ell_3 \right. \\ &\quad \left. + \sum_{j_{10}=3}^{\ell_4} \sum_{j_{11}=2}^{j_{10}-1} \sum_{j_{12}=1}^{j_{11}-1} (\ell_4 - j_{12}) \right), \\ &= \ell_1 + \left( \frac{1}{2} \ell_4 (\ell_4 + 1) \right) + \left( \ell_3 \ell_4 + \frac{1}{6} \ell_4 \left( 2\ell_4^2 - 3\ell_4 + 1 \right) \right) + \left( \ell_2 \ell_4 \right. \\ &\quad \left. + \frac{1}{6} \ell_4 \left( 2\ell_4^2 - 3\ell_4 + 1 \right) + \frac{1}{2} (\ell_4 - 1) \ell_4 \ell_3 + \sum_{j_{10}=3}^{\ell_4} \sum_{j_{11}=2}^{j_{10}-1} \sum_{j_{12}=1}^{j_{11}-1} (\ell_4 - j_{12}) \right), \\ &= \ell_1 + \ell_4 \left( \ell_2 + \ell_3 + \frac{1}{2} (\ell_4 + 1) \right) + \ell_4 \left( \frac{1}{2} \ell_3 (\ell_4 - 1) + \frac{1}{3} \left( 2\ell_4^2 - 3\ell_4 + 1 \right) \right) \\ &\quad + \sum_{j_{10}=3}^{\ell_4} \sum_{j_{11}=2}^{j_{10}-1} \sum_{j_{12}=1}^{j_{11}-1} (\ell_4 - j_{12}), \\ &= \ell_1 + \ell_4 \left( \ell_2 + \ell_3 + \frac{1}{2} (\ell_4 + 1) \right) + \ell_4 \left( \frac{1}{2} \ell_3 (\ell_4 + 1) + \frac{1}{3} \left( 2\ell_4^2 - 3\ell_4 + 1 \right) \right) \\ &\quad + \frac{1}{24} (\ell_4 - 2)(\ell_4 - 1) \ell_4 (3\ell_4 - 1). \end{aligned}$$

□

### A.3. The geometric obliteration algorithm with computational improvements.

**Algorithm A.5.** ***(The Geometric Obliteration Algorithm with Computational Improvements)***

- Input: An intersection of hypersurfaces $V$ of type $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ with $d \geq 2$, encoded as the list DegreeList $= [\ell_d, \ell_{d-1}, \ldots, \ell_2, \ell_1]$.
- Output: The geometric dimension bound $g(d; \ell_d, \ldots, \ell_1)$.

We will use the same functions ComputePolarCone and ObliterateLargestDegreeHypersurfaces which were originally defined in Algorithm A.1.

We now implement Lemmata A.4, A.3, and A.2 via the following three functions, respectively.

```
function ObliterateQuartics(List):
    a = List[0]
    b = List[1]
    c = List[2]
    d = List[3]
    gammafour = b + (1/2)*(a-1)*a
    betafour = c + a*b + (1/2)*a*(a+1) + (1/6)*(a-1)*a*(2*a-1)
    alphafour = d + a*(b+c+(1/2)*(a+1))
    + a*((1/2)*b*(a-1)+(1/3)*((2*(a**2))-(3*a)+1))
    + (1/24)*(a-2)*(a-1)*a*(3*a-1)
    return [gammafour,betafour,alphafour]
end function

function ObliterateCubics(List):
    a = List[0]
    b = List[1]
    c = List[2]
    betathree = b + (1/2)*(a-1)*a
    alphathree = c + a*b + (1/2)*a*(a+1) + (1/6)*a*((2*(a**2))-(3*a)+1)
    return [betathree,alphathree]
end function

function ObliterateQuadrics(List):
    a = List[0]
    b = List[1]
```

```
    alphatwo = b + (1/2)*a*(a+1)
    return [1,alphatwo]
end function
```

The MAIN procedure works very similarly to its counterpart in Algorithm A.1, with the only differences being the use of specialized functions to obliterate quartic, cubic, and quadric hypersurfaces.

```
procedure MAIN(DegreeList):
    if len(DegreeList) == 2: then
        FinalDegreeList = OBLITERATEQUADRICS(DegreeList)
        Sum = FinalDegreeList[0] = FinalDegreeList[1]
        return Sum
    else if len(DegreeList) == 3: then
        TempDegreeList = OBLITERATECUBICS(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = OBLITERATEQUADRICS(DegreeList)
        FinalDegreeList = TempDegreeList
        Sum = FinalDegreeList[0] = FinalDegreeList[1]
        return Sum
    else if len(DegreeList == 4: then
        TempDegreeList = OBLITERATEQUARTICS(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = OBLITERATECUBICS(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = OBLITERATEQUADRICS(DegreeList)
        FinalDegreeList = TempDegreeList
        Sum = FinalDegreeList[0] = FinalDegreeList[1]
        return Sum
    else:
        for index in range(1,len(DegreeList)-3):
            TempDegreeList = OBLITERATELARGESTDEGREEHYPER-
            SURFACES(DegreeList)
            DegreeList = TempDegreeList
        end for
        TempDegreeList = OBLITERATEQUARTICS(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = OBLITERATECUBICS(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = OBLITERATEQUADRICS(DegreeList)
        FinalDegreeList = TempDegreeList
        Sum = FinalDegreeList[0] = FinalDegreeList[1]
        return Sum
    end if
end procedure
```

## A.4. The geometric obliteration algorithm for $\mathcal{C}^{m-d-1}(\tau_{1,\dots,d}; \boldsymbol{P_0}, \dots, \boldsymbol{P_{m-d-1}})$.

**Algorithm A.6.** ***(The Geometric Obliteration Algorithm for***

$$\mathcal{C}^{m-d-1}(\tau_{1,\dots,d}; P_0, \dots, P_{m-d-1})\textbf{)}$$

- Imported Packages: scipy.special, math
- Input: A positive integer $d$ and and another positive integer $m \geq d+2$.
- Output: The optimal reduction bound of $\tau_{1,\dots,d}$ for $m$, $\Xi(m,d)$.

We will use the same functions ComputePolarCone and ObliterateLargestDegreeHypersurfaces which were originally defined in Algorithm A.1, as well as the functions ObliterateQuartics and ObliterateCubics which originally defined in Algorithm A.5.

We first implement a closed form for the type of an $(m-d-1)^{st}$ polar cone of $\tau_{1,\dots,d}$, which is Proposition 2.26 of [Sut2021C].

```
function PolarConeOfTschirnhausType(Type,Level):
    ReturnList = [1]
    for counter in range(1,Type):
        NewTerm = scipy.special.comb((Level+counter), counter, exact=True)
        OutputList.append(NewTerm)
    end for
    return ReturnList
end function
```

This function takes the type of an $(m-d-1)^{st}$ polar cone of $\tau_{1,\dots,d}$ as an input and outputs $\Xi(m,d)$.

```
function ObliterateAMinimalNumberOfQuadrics(List):
    a = List[0]
    b = List[1]
    Dimension = b + (1/2)*(a**2 + a - 2)
    NumberOfQuadrics = 1
    DimensionList = [Dimension]
    while  2**NumberOfQuadrics < Dimension:
        NumberOfQuadrics += 1
        Dimension = NumberOfQuadrics + (1/2)*(a**2 + a
          - NumberOfQuadrics**2 - NumberOfQuadrics)
        DimensionList.append(Dimension)
    end while
```

```
    MaxList1 = [2**(NumberOfQuadrics-1)+1, DimensionList[Number
    OfQuadrics-2]+m-d+1]
    MaxList2 = [2**NumberOfQuadrics+1, DimensionList[NumberOf
    Quadrics-1]+m-d+1]
    Max1 = max(MaxList1[0], MaxList1[1])
    Max2 = max(MaxList2[0], MaxList2[1])
    if  Max2 < Max1: then
        if  MaxList2[1] < MaxList2[0]:  then
            return MaxList2[0]
        else:
            return MaxList2[1]
        end if
    else:
        if  MaxList1[1] < MaxList1[0]:  then
            return MaxList1[0]
        else:
            return MaxList1[1]
        end if
    end if
end function
```

The MAIN procedure functions similarly to its counterpart in Algorithm A.5. The two differences are that the degree list is computed based on $m$ and $d$ and the use of OBLITERATEAMINIMALNUMBERQUADRICS instead of OBLITERATEQUADRICS.

```
procedure MAIN(m,d):
    PolarConeLevel = m-d-1
    DegreeList = POLARCONEOFTSCHIRNHAUSTYPE(d,PolarConeLev
    el)
    if len(DegreeList) == 2: then
        return OBLITERATEAMINIMALNUMBERQUADRICS(DegreeList)
    else if len(DegreeList) == 3: then
        TempDegreeList = OBLITERATECUBICS(DegreeList)
        DegreeList = TempDegreeList
        return OBLITERATEAMINIMALNUMBERQUADRICS(DegreeList)
    else if len(DegreeList == 4: then
        TempDegreeList = OBLITERATEQUARTICS(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = OBLITERATECUBICS(DegreeList)
        DegreeList = TempDegreeList
        return OBLITERATEAMINIMALNUMBERQUADRICS(DegreeList)
```

```
    else:
        for index in range(1,len(DegreeList)-3):
            TempDegreeList = ObliterateLargestDegreeHyper-
            surfaces(DegreeList)
            DegreeList = TempDegreeList
        end for
        TempDegreeList = ObliterateQuartics(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = ObliterateCubics(DegreeList)
        DegreeList = TempDegreeList
        return ObliterateAMinimalNumberQuadrics(DegreeList)
    end if
end procedure
```

## References


[AS1976] Arnol'd, Vladimir I.; Shimura, Goro. Superpositions of algebraic functions. In: Mathematical developments arising from Hilbert problems. *Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society held at Northern Illinois University*, De Kalb, Ill., (May, 1974), 45–46. Proceedings of Symposia in Pure Mathematics, Vol. XXVIII. *American Mathematical Society, Providence, R.I.*, 1976. xii+628 pp. MR0419125, Zbl 0326.00002, doi: 10.1090/pspum/028.1. 107

[Ber1923] Bertini, Eugenio. Introduzione alla geometria projettiva degli iperspazi con appendice sulle curve algebriche e loro singolarità. Seconda edizione riveduta ed ampliata. *Messina, G. Principato,* 1923. JFM 49.0484.08 111, 112

[Bra1975] Brauer, Richard. On the resolvent problem. *Ann. Mat. Pura Appl. (4)*, **102** (1975) 45–55. MR0371854 (51 #8071), Zbl 0299.12105, doi: 10.1007/BF02410594. 107, 125, 132

[Bri1786] Bring, Erland S. Meletemata quædam Mathematica circa Transformationem Æquationum Algebraicarum ("Some Selected Mathematics on the Transformation of Algebraic Equations"). *University of Lund*, 1786. 132

[Che1954] Chebotarev, G. N. On the problem of resolvents. *Kazan. Gos. Univ. Uč. Zap.* **114** (1954), no. 2, 189–193. MR0077496 (17,1045c). 108, 126, 132

[CHM2017] Chen, Alexander; He, Yang-Hui; McKay, John. Erland Samuel Bring's "Transformation of Algebraic Equations". Preprint, 2017. arXiv:1711.09253.

[DM1998] Debarre, Oliver; Manivel, Laurent. Sur la variété des espaces linéaires contenus dans une intersection complète. *Math. Ann.* **312** (1998), no. 3, 549–574. MR1654757 (99j:14048), Zbl 0913.14015, doi: 10.1007/s002080050235. 108, 127, 133, 134

[Dix1993] Dixmier, Jacques. Histoire du 13e problème de Hilbert. *Analyse diophantienne et géométrie algébrique*, 85–94. *Cahiers Sém. Hist. Math. Sér. 2,* **3**, *Univ. Paris VI, Paris*, 1993. MR1240756 (94j:12005), Zbl 0795.01012. 108

[Dol2012] Dolgachev, Igor V. Classical algebraic geometry. A modern view. *Cambridge University Press, Cambridge,* 2012. xii+639 pp. ISBN: 978-1-107-01765-8. MR2964027, Zbl 1252.14001, doi: 10.1017/CBO9781139084437. 111

[FW2019] Farb, Benson; Wolfson, Jesse. Resolvent degree, Hilbert's 13th problem and geometry. *Enseign. Math.* **65** (2019), no. 3-4, 303–376. MR4113045, Zbl 1460.14104, arXiv:1803.04063, doi: 10.4171/LEM/65-3/4-2. 107, 110, 111, 133

[Ham1836] HAMILTON, WILLIAM R. Inquiry into the validity of a method recently proposed by George B. Jerrard, Esq., for transforming and resolving equation of elevated degrees. *Report of the Sixth Meeting of the British Assocation for the Advancement of Science*, (Bristol, 1836), 295-348. *John Murray, London*, 1837. 132

[Har1992] HARRIS, JOE. Algebraic geometry. A first course. Graduate Texts in Mathematics, 133. *Springer-Verlag, New York,* 1992. xx+328 pp. ISBN: 0-387-97716-3. MR1182558 (93j:14001), Zbl 0779.14001, doi: 10.1007/978-1-4757-2189-8. 109

[Heb2021] HEBERLE, CURTIS R. Removal of 5 terms from a degree 21 polynomial. Preprint, 2021. arXiv:2112.08586. 108

[Hil1927] HILBERT, DAVID. Über die Gleichung neunten Grades, *Math. Ann.,* **97** (1927), no. 1, 243–250. MR1512361, Zbl 52.0103.02, doi: 10.1007/BF01447867. 132

[Kle1884] KLEIN, FELIX. Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade. Teubner, Leipzig, 1884. JFM 16.0061.01 132

[Kle1887] KLEIN, FELIX. Zur Theorie der allgemeinen Gleichungen sechsten und siebenten Grades. *Math. Ann.* **28** (1887), no. 4, 499–532. MR1510392, JFM 19.0082.03, doi: 10.1007/BF01447913.

[Kle1905] KLEIN, FELIX. Über die Auflösung der allgemeinen Gleichungen fünften und sechsten Grades. *Math. Ann* **61**, (1905), no. 1, 50–71. MR1511331, JFM 36.0132.03. 132

[Mor1956] KLEIN, FELIX. Lectures on the icosahedron and the solution of equations of the fifth degree. Translated into English by George Gavin Morrice. Second and revised edition. *Dover Publications, Inc., New York, N.Y.,* 1956. xvi+289 pp. MR0080930 (18,329c), Zbl 0072.25901. 132

[Ost1959] OSTROWSKI, ALEXANDER M. A quantitative formulation of Sylvester's law of inertia. *Proc. Natl. Acad. Sci. U.S.A.* **45** (1959), 740–744. MR0110720 (22 #1588), Zbl 0087.01802, doi: 10.1073/pnas.45.5.740. 121

[Rei2022] REICHSTEIN, ZINOVY. Hilbert's 13th problem for algebraic groups. Preprint, 2022. arXiv:2204.13202. 111, 134

[Seg1945] SEGRE, BENIAMINO. The algebraic equations of degrees 5, 9, 157 ... , and the arithmetic upon an algebraic variety. *Ann. of Math. (2)* **46** (1945), 287–301. MR0013126 (7,108h), Zbl 0061.01807 doi: 10.2307/1969024. 108, 112, 132

[Sut2019] SUTHERLAND, ALEXANDER J. Felix Klein's "About the solution of general equations of fifth and sixth degree (excerpt from a letter to Mr. K. Hensel)". Preprint, 2019. arXiv:1911.02358. 132

[Sut2021A] SUTHERLAND, ALEXANDER J. Anders Wiman's "On the application of Tschirnhaus transformations to the reduction of algebraic equations". Preprint, 2021. arXiv:2106.09247.

[Sut2021B] SUTHERLAND, ALEXANDER J. G. N. Chebotarev's "On the problem of resolvents". Preprint, 2021. arXiv:2107.01006.

[Sut2021C] SUTHERLAND, ALEXANDER J. Upper bounds on resolvent degree and its growth rate. Preprint, 2021. arXiv:2107.08139. 108, 109, 111, 112, 113, 122, 125, 126, 127, 130, 131, 132, 133, 134, 142

[Syl1887] SYLVESTER, JAMES J. On the so-called Tschirnhausen transformation. *J. Reine Angew. Math.* **100** (1887), 465–486. MR1580113, JFM 19.0079.02. 108, 114, 117, 119, 121, 122, 132

[SH1887] SYLVESTER, JAMES J.; HAMMOND, JAMES. On Hamilton's numbers. *Philos. Trans. Roy. Soc. A* **178** (1887), 285–312. JFM 19.0080.02, doi: 10.1098/rsta.1887.0010. 132

[SH1888] SYLVESTER, JAMES J.; HAMMOND, JAMES. On Hamilton's numbers. II. *Philos. Trans. Roy. Soc. A* **179** (1888), 65–72. JFM 20.0081.01, doi: 10.1098/rsta.1888.0004. 132

[Tsc1683] VON TSCHIRNHAUS, EHRENFRIED W.. Methodus auferendi omnes terminos intermedios ex data aeqvatione (Method of eliminating all intermediate terms from a given equation). *Acta Eruditorum* (1683), 204–207. 132

[Wim1927] WIMAN, ANDERS. Über die Anwendung der Tschirnhausen-transformation auf die reduktion algebraischer Gleichungen. *Nova Acta R. Soc. Scient. Uppsala,* **4**, no. 16, (1927). JFM 55.0073.04 108, 126, 132

[Wol2021] WOLFSON, JESSE. Tschirnhaus transformations after Hilbert. *Enseign. Math.* **66** (2020), no. 3-4, 489–540. MR4254270, Zbl 7461693, arXiv:2001.06515, doi: 10.4171/LEM/66-3/4-9. 107, 108, 113, 121, 125, 126, 132, 134

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