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The de Rham cohomology of soft function algebras

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ABSTRACT. We study the dg-algebra $\Omega^\bullet_{A|\mathbb{R}}$ of algebraic de Rham forms of a real soft function algebra A, i.e., the algebra of global sections of a soft subsheaf of C_X, the sheaf of continuous functions on a space X. We obtain a canonical splitting $\mathrm{H}^n(\Omega^\bullet_{A|\mathbb{R}}) \cong \mathrm{H}^n(X, \mathbb{R}) \oplus V$, where V is some vector space. In particular, we consider the cases $A = C(X)$ for X a compact Hausdorff space and $A = C^\infty(X)$ for X a compact smooth manifold. For the algebra $\mathrm{PPol}_K(|K|)$ of piecewise polynomial functions on a polyhedron K the above splitting reduces to a canonical isomorphism $\mathrm{H}^*(\Omega^\bullet_{\mathrm{PPol}_K(|K|)|\mathbb{R}}) \cong \mathrm{H}^*(|K|, \mathbb{R})$. We also prove that the algebraic de Rham cohomology $\mathrm{H}^n(\Omega^\bullet_{C(X)|\mathbb{R}})$ is nontrivial for each $n \geq 1$ if X is an infinite compact Hausdorff space.

CONTENTS

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1. Introduction

All algebras are assumed to be commutative and all dg-algebras are assumed to be graded-commutative. To an algebra A over a field k one associates a dg-algebra $\Omega^{\bullet}_{A|k}$ with $\Omega^0_{A|k} = A$, called the de Rham dg-algebra, in a standard way (see Subsection 2.4 or [Kun86, §3]). Our main focus will be the cases of $\mathbb{R}$-algebras $A = C(X)$ for a compact Hausdorff space X, $A = C^\infty(M)$ for a smooth manifold M (possibly with boundary) and $A = \mathrm{PPol}_K(|K|)$, the algebra of piecewise polynomial functions on a polyhedron K. Here all functions are assumed to be real-valued, but our results will hold for complex-valued functions also. We study the cohomology groups $\mathrm{H}^*(\Omega^{\bullet}_{A|k})$.

There is a canonical morphism of dg-algebras $\pi : \Omega^{\bullet}_{C^\infty(M)|\mathbb{R}} \to \Omega^{\bullet}(M)$, where $\Omega^{\bullet}(M)$ is the dg-algebra of smooth differential $\mathbb{R}$-forms on M (see Subsection 4.2). The morphism π is the identity in degree 0. It is not an isomorphism. For example, the equality $df(t) = (\partial f/\partial t)dt$ holds in $\Omega^1_{C^\infty(\mathbb{R})|\mathbb{R}}$ if and only if $f(t)$ is an algebraic function of t ([Osb69, Corollary of Proposition 1]). Moreover, if M is of dimension ≥ 1, then the cardinality of any set of generators for $\Omega^1_{C^\infty(M)|\mathbb{R}}$ as a $C^\infty(M)$-module is at least that of $\mathbb{R}$ ([Gom90, Corollary 15]). The map on cohomology, induced by π, is not an isomorphism either, see [Osb69] for the proof that the closed form $dt/(1+t^2)$ is not exact in $\Omega^{\bullet}_{C^\infty(\mathbb{R})|\mathbb{R}}$. On the other hand, consider the algebra A of regular functions on a smooth affine variety V over $\mathbb{C}$. It is a subalgebra of $C^\infty(V,\mathbb{C})$ with inclusion $i : A \hookrightarrow C^\infty(V,\mathbb{C})$. Consider the composition

$$\Omega^{\bullet}_{A|\mathbb{C}} \xrightarrow{\Omega_i} \Omega^{\bullet}_{C^\infty(V)|\mathbb{C}} \xrightarrow{\pi_{\mathbb{C}}} \Omega^{\bullet}(V,\mathbb{C}),$$

where $\pi_{\mathbb{C}}$ is the analogue of π over $\mathbb{C}$. Grothendieck's comparison theorem states that the induced map $\mathrm{H}^*(\Omega^{\bullet}_{A|\mathbb{C}}) \to \mathrm{H}^*(\Omega^{\bullet}(V,\mathbb{C}))$ is an isomorphism ([Gro66, Theorem 1′]).

For a soft sheaf of k-algebras $\mathcal{F}$ on a compact Hausdorff space X we construct (see Section 3) a linear map

$$\Lambda_{\mathcal{F}} : \mathbb{H}^*(X, \underline{k}_X[0]) \to \mathrm{H}^*(\Omega^{\bullet}_{\mathcal{F}(X)|k}).$$

Here the domain is the cohomology of X with coefficients in the constant sheaf $\underline{k}_X$. This map is natural with respect to morphisms of ringed spaces (see Proposition 3.4). We are mostly interested in the case $k = \mathbb{R}$.

We prove (see Theorem 4.1) that for the sheaf of smooth functions C^∞_M on a smooth manifold M the following diagram is commutative:

$$\begin{tikzcd} \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[r, "\Lambda_{C^\infty_M}"'] \arrow[rr, bend left, "\Theta"] & \mathrm{H}^*(\Omega^{\bullet}_{C^\infty(M)|\mathbb{R}}) \arrow[r, "\mathrm{H}(\pi)"'] & \mathrm{H}^*(\Omega^{\bullet}(M)), \end{tikzcd}$$

where Θ is the canonical isomorphism (see Subsection 4.1). In particular, the map

$$\mathrm{H}(\pi) : \mathrm{H}^n(\Omega^{\bullet}_{C^\infty(M)|\mathbb{R}}) \to \mathrm{H}^n(\Omega^{\bullet}(M))$$

is surjective for $n \geq 0$. This is a generalization of a result obtained by Gómez, namely that $\mathrm{H}(\pi)$ is surjective for n even (see [Gom92, Section 4]).

Next, for an arbitrary space X and a subalgebra $i : A \hookrightarrow C(X)$, we construct a linear map (see Section 6)

$$\Psi_A : \mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \to \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]).$$

Our construction of Ψ_A relies on local Lipschitz contractibility of algebraic sets (Theorem 2.4) due to Shartser ([Sha11, Theorem 4.18]). This map is natural with respect to continuous maps of spaces covered by a homomorphism of algebras (see Proposition 6.6); in particular, $\Psi_A = \Psi_{C(X)} \circ \Omega_i$. We prove (see Theorem 8.4) that for a compact Hausdorff space X and a soft subsheaf $\mathcal{F}$ of C_X, the sheaf of continuous functions, the composition $\Psi_{\mathcal{F}(X)} \circ \Lambda_{\mathcal{F}}$ coincides with the identity map. Thus, the groups $\mathbb{H}^*(X, \underline{\mathbb{R}}_X[0])$ canonically split off of $\mathrm{H}^*(\Omega^\bullet_{\mathcal{F}(X)})$.

We also check (see Theorem 7.10) that for a smooth manifold M the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{H}^*(\Omega^\bullet_{C^\infty(M)|\mathbb{R}}) \arrow[r, "\Psi_{C^\infty(M)}"] \arrow[dr, "\mathrm{H}(\pi)"'] & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[d, "\Theta"] \\
& \mathrm{H}^*(\Omega^\bullet(M)).
\end{tikzcd}$$

For the sheaf of piecewise polynomial functions PPol_K on a polyhedron K (see Section 9) we prove that the morphisms $\Lambda_{\mathrm{PPol}_K}$ and $\Psi_{\mathrm{PPol}_K(|K|)}$ are isomorphisms (see Theorem 9.11).

In Section 10 we describe the group $\mathrm{H}^0(\Omega^\bullet_{A|k})$ for a general function algebra A over a field k of characteristic 0 (Corollary 10.5) calculated by Gómez in [Gom90]. The related result is [Osb69, Proposition 5] (note that the cohomology considered there is, in general, different from ours). We show that for a soft subsheaf of algebras $\mathcal{F} \subset C_X$ on a compact Hausdorff space X the morphisms $\Lambda_{\mathcal{F}}$ and $\Psi_{\mathcal{F}(X)}$ are isomorphisms in degree 0 (Proposition 10.6 and Corollary 10.7). We also prove that for an infinite compact Hausdorff space X the maps $\Lambda_{C_X} : \mathbb{H}^n(X, \underline{\mathbb{R}}_X[0]) \to \mathrm{H}^n(\Omega^\bullet_{C(X)|\mathbb{R}})$ and $\Psi_{C(X)} : \mathrm{H}^n(\Omega^\bullet_{C(X)|\mathbb{R}}) \to \mathbb{H}^n(X, \underline{\mathbb{R}}_X[0])$ are not isomorphisms in degrees $n > 0$. The same is true for the algebra of smooth functions on a smooth manifold, see Subsection 10.2.

From our results one can deduce the similar results for $\mathbb{C}$-algebras.

In the paper we only consider algebraic structures, however one can consider topological algebras. Using projective tensor products, one can define the dg-algebra $\tilde{\Omega}^\bullet_{\mathcal{A}|\mathbb{C}}$ for a Fréchet $\mathbb{C}$-algebra $\mathcal{A}$ (denoted by $\Omega^\bullet_{\mathrm{ab}}\mathcal{A}$ in [GVF01, §8]). This dg-algebra is a topological analogue of the de Rham dg-algebra.

Consider the Fréchet algebra $\mathcal{A} = C^\infty(M, \mathbb{C})$ for a compact smooth manifold M. The dg-algebra $\tilde{\Omega}^\bullet_{\mathcal{A}|\mathbb{C}}$ is isomorphic to $\Omega^\bullet(M, \mathbb{C})$, see [GVF01, Proposition 8.1].

Consider the Banach algebra $\mathcal{A} = C(X, \mathbb{C})$ for a compact Hausdorff space X. Then $\tilde{\Omega}^n_{\mathcal{A}|\mathbb{C}} = 0$ for $n \geq 1$. To see that, first note that by [Joh72, §8] or [Con85, Remarks 47, d], the continuous Hochschild cohomology $\mathrm{HH}^n(\mathcal{A}, \mathcal{A}^*)$ is zero for positive n. Then, by [Joh72, Corollary 1.3] the continuous Hochschild homology of $\mathcal{A}$ is zero in positive degrees, in particular, $\mathrm{HH}_1(\mathcal{A}) = 0$. By [GVF01, p. 346], we have $\mathrm{HH}_1(\mathcal{A}) \cong \tilde{\Omega}^1_{\mathcal{A}|\mathbb{C}}$. Hence, by the construction of $\tilde{\Omega}^n_{\mathcal{A}|\mathbb{C}}$ this space is zero for $n \geq 1$.

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2. Preliminaries

2.1. Sheaves. Here we outline the basic definitions and facts of sheaf theory needed in the paper. For this we follow the books of Godement [God58], Wedhorn [Wed16] and Bredon [Bre97].

We refer to [Wed16, Definition 10.2] for the definition of the *hypercohomology* groups $\mathbb{H}^*(X, \mathcal{F}^\bullet)$ of a complex of sheaves $\mathcal{F}^\bullet$ on a space X. By a complex we always mean a non-negative cochain complex. For a sheaf $\mathcal{F}$ we denote by $\mathcal{F}[0]$ the complex of sheaves with $\mathcal{F}$ in degree 0 and other terms zero.

Let $\mathcal{F}^\bullet$ be a complex of sheaves on X. Then there is a canonical homomorphism

$$\Upsilon : \mathrm{H}^*(\mathcal{F}^\bullet(X)) \to \mathbb{H}^*(X, \mathcal{F}^\bullet),$$

natural with respect to morphisms of complexes of sheaves. The map Υ is an isomorphism in degree 0. If the sheaves $\mathcal{F}^n$ are acyclic then Υ is an isomorphism, see [Wed16, Theorem and Definition 10.4, Proposition 10.8].

Lemma 2.1. *Take complexes of sheaves $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ on topological spaces X and Y, respectively. Suppose $f : X \to Y$ is a continuous map and $\varphi : \mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet$ is a morphism of complexes of sheaves. Then there is the induced map on hypercohomology $f^* : \mathbb{H}(Y, \mathcal{G}^\bullet) \to \mathbb{H}(X, \mathcal{F}^\bullet)$. Moreover, the following diagram is commutative:*

$$\begin{tikzcd}
\mathbb{H}^*(X, \mathcal{F}^\bullet) & \mathrm{H}^*(\mathcal{F}^\bullet(X)) \arrow[l, "\Upsilon"'] \\
\mathbb{H}^*(Y, \mathcal{G}^\bullet) \arrow[u, "f^*"] & \mathrm{H}^*(\mathcal{G}^\bullet(Y)). \arrow[l, "\Upsilon"'] \arrow[u, "\mathrm{H}(\varphi(Y))"']
\end{tikzcd}$$

Proof. See [Sta]. □

For a sheaf $\mathcal{F}$ on X and a closed subset $Z \subset X$ define

$$\mathcal{F}(Z) := \varinjlim_{\text{open } U \supset Z} \mathcal{F}(U).$$

For a sheaf $\mathcal{F}$ we fix the notation for the restriction map $\mathrm{res}_{W,W'} : \mathcal{F}(W) \to \mathcal{F}(W')$ for sets $W' \subset W$ open or closed.

A sheaf $\mathcal{F}$ is called *soft* if for every closed set $Z \subset X$ the restriction map $\mathrm{res}_{X,Z}$ is surjective. Any soft sheaf on a compact Hausdorff space is acyclic [Wed16, Proposition 10.17]. A sheaf $\mathcal{F}$ is called *flabby* if for every open set $U \subset X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. A flabby sheaf is acyclic.

For a presheaf F denote by ${}^+F$ the sheafification of F. Let $\mathrm{sh} : F \to {}^+F$ denote the sheafification map, i.e. the canonical map from a presheaf to the associated sheaf.

2.2. Algebraic sets.

An *algebraic set* in $\mathbb{R}^m$ is the set of solutions of a system of polynomial equations in $\mathbb{R}^m$.

Definition. Take B a finitely generated $\mathbb{R}$-algebra. Following [Nes03, 3.4], we define the *real spectrum* $\mathrm{spec}_{\mathbb{R}} B$ of B as the set of algebra homomorphisms $\psi : B \to \mathbb{R}$.

A set of generators $b_1, \dots, b_n$ of B gives rise to an injective map $i : \mathrm{spec}_{\mathbb{R}} B \hookrightarrow \mathbb{R}^m$, $i(\psi) := (\psi(b_1), \dots, \psi(b_n))$. We call such a map a *distinguished embedding*. The image of a distinguished embedding is an algebraic set. Equip $\mathrm{spec}_{\mathbb{R}} B$ with the induced topology under some distinguished embedding. This topology does not depend on the choice of a distinguished embedding because for any two distinguished embeddings $i : \mathrm{spec}_{\mathbb{R}} B \hookrightarrow \mathbb{R}^m$ and $j : \mathrm{spec}_{\mathbb{R}} B \hookrightarrow \mathbb{R}^l$ there exists a polynomial map $p : \mathbb{R}^m \to \mathbb{R}^l$ such that the following diagram is commutative

$$\begin{tikzcd} \mathrm{spec}_{\mathbb{R}} B \arrow[r, "i"] \arrow[dr, "j"'] & \mathbb{R}^m \arrow[d, "p"] \\ & \mathbb{R}^l. \end{tikzcd}$$

A polynomial map is smooth, hence, we can talk of maps into $\mathrm{spec}_{\mathbb{R}} B$ being smooth or locally Lipschitz. Hence, $\mathrm{spec}_{\mathbb{R}} B$ becomes a *predifferentiable space* (see [Che75, §1] for the definition).

If $\varphi : B' \to B$ is a homomorphism of finitely generated $\mathbb{R}$-algebras, then define $\mathrm{spec}_{\mathbb{R}} \varphi : \mathrm{spec}_{\mathbb{R}} B \to \mathrm{spec}_{\mathbb{R}} B'$ as $(\mathrm{spec}_{\mathbb{R}} \varphi)(\psi) := \psi \circ \varphi$. For $b \in B$ define the function $\hat{b} \in C(\mathrm{spec}_{\mathbb{R}} B)$ by $\hat{b}(\psi) := \psi(b)$. We have $\widehat{\varphi(c)} = \hat{c} \circ (\mathrm{spec}_{\mathbb{R}} \varphi)$ in $C(\mathrm{spec}_{\mathbb{R}} B)$ for $c \in B'$.

Now, let's assume that B is a finitely generated $\mathbb{R}$-subalgebra of $C(X)$ for some topological space X. To each point $x \in X$ we associate the homomorphism $B \to \mathbb{R}$, $b \mapsto b(x)$. We get a continuous map $\Gamma_B : X \to \mathrm{spec}_{\mathbb{R}} B$. If M is a smooth manifold and B is a finitely generated subalgebra of $C^\infty(M)$ then Γ_B is smooth.

For $b \in B$ we have

$$\hat{b} \circ \Gamma_B = b. \tag{1}$$

Take $f : X \to Y$ a continuous map of topological spaces. Suppose that $B \subset C(X)$ and $B' \subset C(Y)$ are finitely generated subalgebras and $\varphi : B' \to B$ is a homomorphism such that the following diagram is commutative:

$$\begin{tikzcd} C(X) & B \arrow[l, hook'] \\ C(Y) \arrow[u, "f^*"] & B'. \arrow[l, hook'] \arrow[u, "\varphi"'] \end{tikzcd}$$

Then the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "\Gamma_B"] \arrow[d, "f"] & \operatorname{spec}_{\mathbb{R}} B \arrow[d, "\operatorname{spec}_{\mathbb{R}} \varphi"] \\ Y \arrow[r, "\Gamma_{B'}"] & \operatorname{spec}_{\mathbb{R}} B'. \end{tikzcd}$$

2.3. Sheaves of singular cochains. For a topological space X let $\mathrm{S}_n(X)$ denote the space of singular $\mathbb{R}$-chains in X and let $\mathrm{S}^n(X)$ denote the dual space, the space of singular cochains. We will need the following presheaves S^n_X for $n \geq 0$:

$$U \mapsto \mathrm{S}^n(U).$$

The standard differential on cochains gives rise to a complex of presheaves $\mathrm{S}^\bullet_X$ on X. For a smooth manifold M (possibly with boundary) let $\mathrm{S}_{\mathrm{sm},n}(M)$ denote the space of smooth singular $\mathbb{R}$-chains in M and let $\mathrm{S}^n_{\mathrm{sm}}(M)$ denote the dual space. We will need the following presheaves $\mathrm{S}^n_{\mathrm{sm},M}$ for $n \geq 0$:

$$U \mapsto \mathrm{S}^n_{\mathrm{sm}}(U).$$

The standard differential on cochains gives rise to a complex of presheaves $\mathrm{S}^\bullet_{\mathrm{sm},M}$ on M. More generally, the complex of presheaves $\mathrm{S}^\bullet_{\mathrm{sm},Q}$ is well defined for Q a predifferentiable space and functorial with respect to smooth maps between predifferentiable spaces. In particular, $\mathrm{S}^\bullet_{\mathrm{sm},\operatorname{spec}_{\mathbb{R}} B}$ are well defined.

For a set $V \subset \mathbb{R}^m$ let $\mathrm{S}_{\mathrm{Lip},n}(V)$ denote the space of Lipschitz singular $\mathbb{R}$-chains in V and let $\mathrm{S}^n_{\mathrm{Lip}}(V)$ denote the dual space. We will need the following presheaves $\mathrm{S}^n_{\mathrm{Lip},V}$ for $n \geq 0$:

$$U \mapsto \mathrm{S}^n_{\mathrm{Lip}}(U).$$

The standard differential on cochains gives rise to a complex of presheaves $\mathrm{S}^\bullet_{\mathrm{Lip},V}$ on V. The complex of presheaves $\mathrm{S}^\bullet_{\mathrm{Lip},\operatorname{spec}_{\mathbb{R}} B}$ is well defined and functorial with respect to algebra homomorphisms.

The sheaves ${}^+\mathrm{S}^n_X$ are flabby by [Wed16, proof of Theorem 11.13] and, hence, are acyclic. Similarly, one can prove that the sheaves ${}^+\mathrm{S}^n_{\mathrm{sm},Q}$ and ${}^+\mathrm{S}^n_{\mathrm{Lip},V}$ are flabby.

For the complex of sheaves ${}^+\mathrm{S}^\bullet_X$ we define the morphism of complexes of sheaves, called coaugmentation, $\epsilon : \underline{\mathbb{R}}_X[0] \to {}^+\mathrm{S}^\bullet_X$ by $\epsilon(1) := 1$. For the complexes ${}^+\mathrm{S}^\bullet_{\mathrm{sm},Q}$ and ${}^+\mathrm{S}^\bullet_{\mathrm{Lip},V}$ the coaugmentation is defined similarly.

Lemma 2.2.

(1) *For a topological space X the morphism of complexes*

$$\mathrm{sh} : \mathrm{S}^\bullet_X(X) \to {}^+\mathrm{S}^\bullet_X(X)$$

is a quasi-isomorphism.

(2) *For a predifferentiable space Q the morphism of complexes*

$$\mathrm{sh} : \mathrm{S}^\bullet_{\mathrm{sm},Q}(Q) \to {}^+\mathrm{S}^\bullet_{\mathrm{sm},Q}(Q)$$

is a quasi-isomorphism.

(3) *For a subset $V \subset \mathbb{R}^m$ the morphism of complexes*

$$\mathrm{sh} : \mathrm{S}^\bullet_{\mathrm{Lip},V}(V) \to {}^+\mathrm{S}^\bullet_{\mathrm{Lip},V}(V)$$

is a quasi-isomorphism.

See [Bre97, p. 26] for the proof of the first case. For the other two a similar proof applies.

Lemma 2.3.

(1) *For a locally contractible X the coaugmentation $\epsilon : \underline{\mathbb{R}}_X[0] \to {}^+\mathrm{S}^\bullet_X$ is a quasi-isomorphism.*

(2) *For a smooth manifold M the coaugmentation $\epsilon : \underline{\mathbb{R}}_M[0] \to {}^+\mathrm{S}^\bullet_{\mathrm{sm},M}$ is a quasi-isomorphism.*

(3) *For a locally Lipschitz contractible set $V \subset \mathbb{R}^m$ the coaugmentation $\epsilon : \underline{\mathbb{R}}_V[0] \to {}^+\mathrm{S}^\bullet_{\mathrm{Lip},V}$ is a quasi-isomorphism.*

See [Bre97, Example II.1.2] for the case of ${}^+\mathrm{S}^\bullet_X$, the rest is analogous.

An important result needed in this work is the local Lipschitz contractability of algebraic sets due to Shartser [Sha11, Theorem 4.1.8].

Theorem 2.4. *Take an algebraic set $V \subset \mathbb{R}^m$ and a point $v_0 \in V$. Then there exist an open set $U \subset \mathbb{R}^m$ with $v_0 \in U$ and a Lipschitz map $F : U \times [0,1] \to U$ such that*

(1) *$F(u,0) = u$ for $u \in U$;*
(2) *$F(u,1) = v_0$ for $u \in U$;*
(3) *$F(v,t) \in V$ for all $v \in V \cap U$ and $t \in [0,1]$.*

Corollary 2.5. *The coaugmentation $\epsilon : \underline{\mathbb{R}}_V[0] \to {}^+\mathrm{S}^\bullet_{\mathrm{Lip},V}$ is a quasi-isomorphism for V an algebraic set in $\mathbb{R}^m$.*

Proof. Follows directly from Lemma 2.3 and Theorem 2.4. □

2.4. The de Rham dg-algebra $\Omega^\bullet_{A|k}$ of A. To a k-algebra A one associates the dg-algebra $\Omega^\bullet_{A|k}$ ([Kun86, Theorem 3.2]) with $\Omega^0_{A|k} = A$. It has the following universal property: for any dg-algebra E and any algebra homomorphism f :

$A \to E^0$ there exists a unique morphism of dg-algebras $F : \Omega^\bullet_{A|k} \to E$ such that $F|_A = f$:

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "f"'] & \Omega^\bullet_{A|k} \arrow[d, dashed, "F"] \\ & E. \end{tikzcd}$$

The elements of $\Omega^n_{A|k}$ are called algebraic n-forms. The dg-algebra $\Omega^\bullet_{A|k}$ is covariant in the algebra A.

Lemma 2.6. *Suppose A and B are k-algebras and $\varphi : A \to B$ is a surjective homomorphism of algebras. Then the induced morphism $\Omega_\varphi : \Omega^\bullet_{A|k} \to \Omega^\bullet_{B|k}$ is surjective and its kernel is the ideal of $\Omega^\bullet_{A|k}$ generated by* $\operatorname{Ker}\varphi$ *and* $d(\operatorname{Ker}\varphi)$.

Proof. Set $T := \operatorname{Ker}\varphi$, then it is enough to consider the case $B = A/T$ and $\varphi : A \to B$ being the canonical projection. Take the ideal I of $\Omega^\bullet_{A|k}$ generated by T and dT. I is a dg-ideal, hence, $\Omega^\bullet_{A|k}/I$ is a dg-algebra and the canonical projection $\pi : \Omega^\bullet_{A|k} \to \Omega^\bullet_{A|k}/I$ is a morphism of dg-algebras. By the universal property there exists a unique morphism of dg-algebras $\mathcal{L} : \Omega^\bullet_{B|k} \to \Omega^\bullet_{A|k}/I$ such that $\mathcal{L}|_B$ is the identity map. On the other hand, the kernel of Ω_φ contains T and dT. Therefore, the morphism Ω_φ decomposes as the composition of dg-algebra morphisms $\Omega^\bullet_{A|k} \xrightarrow{\pi} \Omega^\bullet_{A|k}/I \xrightarrow{\mathcal{M}} \Omega^\bullet_{B|k}$. The composition

$$\Omega^\bullet_{B|k} \xrightarrow{\mathcal{L}} \Omega^\bullet_{A|k}/I \xrightarrow{\mathcal{M}} \Omega^\bullet_{B|k}$$

is clearly the identity map by the universal property of $\Omega^\bullet_{B|k}$. Consider the diagram

$$\begin{tikzcd} \Omega^\bullet_{A|k} \arrow[r, "\Omega_\varphi"'] \arrow[rr, bend left, "\pi"] \arrow[dr, "\pi"'] & \Omega^\bullet_{B|k} \arrow[r, "\mathcal{L}"'] & \Omega^\bullet_{A|k}/I \\ & \Omega^\bullet_{A|k}/I. \arrow[u, "\mathcal{M}"'] \arrow[ur, "="'] & \end{tikzcd}$$

The projection map π is surjective and the left triangle is clearly commutative. The upper triangle is commutative by the universal property of $\Omega^\bullet_{A|k}$. Hence, the right triangle is commutative. Therefore, the morphisms $\mathcal{L}$ and $\mathcal{M}$ are mutually inverse. As the morphism π is surjective and its kernel is generated by T and dT, we have Ω_f is surjective and its kernel is generated by T and dT. $\square$

Lemma 2.7. *Suppose a k-algebra A is the filtered colimit of k-algebras A_i. Then* $\Omega^n_{A|k} \cong \varinjlim \Omega^n_{A_i|k}$.

The proof can be found in [Kun86, Proposition 4.1].

We can generalize the notion of algebraic forms to the case of sheaves. For a topological space X and a sheaf of k-algebras $\mathcal{F}$ on X consider the following presheaf $\Omega^n_{\mathcal{F}|k}$ of $\mathcal{F}$-modules:

$$U \mapsto \Omega^n_{\mathcal{F}(U)|k}.$$

For every open U we obtain the complex $\Omega^\bullet_{\mathcal{F}(U)|k}$ and, hence, we have the complex of presheaves $\Omega^\bullet_{\mathcal{F}|k}$. The associated sheaves form the complex of sheaves ${}^+\Omega^\bullet_{\mathcal{F}|k}$.

For sets $W' \subset W$ open or closed we fix the notations for the restriction map $\mathrm{res}_{W,W'} : \Omega^\bullet_{\mathcal{F}(W)|k} \to \Omega^\bullet_{\mathcal{F}(W')|k}$ and $\mathrm{res}_{W,W'} : {}^+\Omega^\bullet_{\mathcal{F}|k}(W) \to {}^+\Omega^\bullet_{\mathcal{F}|k}(W')$ induced by the restriction map $\mathrm{res}_{W,W'} : \mathcal{F}(W) \to \mathcal{F}(W')$.

Consider the morphism of complexes of presheaves $\mathbb{R}_X[0] \to \Omega^\bullet_{\mathcal{F}|k}$, $1 \mapsto 1$, where $\mathbb{R}_X$ is the constant presheaf. The sheafification functor gives rise to a morphism of complexes of sheaves, the coaugmentation, $\epsilon : \underline{\mathbb{R}}_X[0] \to {}^+\Omega^\bullet_{\mathcal{F}|k}$.

Lemma 2.8. *For a soft sheaf of algebras $\mathcal{F}$ on X the sheaves ${}^+\Omega^n_{\mathcal{F}|k}$ are soft.*

Proof. The associated sheaf ${}^+\Omega^n_{\mathcal{F}|k}$ has the natural structure of an $\mathcal{F}$-module. So it is a sheaf of modules over a soft sheaf and, therefore, is soft by [God58, Theorem II.3.7.1]. □

3. The map $\Lambda_{\mathcal{F}} : \mathbb{H}^*(X, \underline{k}_X[0]) \to \mathbf{H}^*(\Omega^\bullet_{\mathcal{F}(X)|k})$

3.1. The global sections of ${}^+\Omega^n_{\mathcal{F}|k}$. In this subsection we take $\mathcal{F}$ to be a soft sheaf of k-algebras on a compact Hausdorff space X.

Lemma 3.1. *Suppose that $S, F \subset X$ are closed sets such that $S \cap F = \emptyset$. Then there exists a section $g \in \mathcal{F}(X)$ such that $\mathrm{res}_{X,S}(g) = 0$ and $\mathrm{res}_{X,F}(g) = 1$.*

The proof can be found in [God58, Theorem II.3.7.2].

Lemma 3.2. *Suppose that $U \subset X$ is an open set. Take $\omega \in \Omega^n_{\mathcal{F}(X)|k}$ such that $\omega|_U = 0$ in $\Omega^n_{\mathcal{F}(U)|k}$. Take also a section $\varphi \in \mathcal{F}(X)$ such that $\operatorname{supp}\varphi \subset U$. Then $\varphi\omega = 0$.*

Proof. Put $S := \operatorname{supp}\varphi$. The restriction homomorphism $\mathrm{res}_{X,S} : \mathcal{F}(X) \to \mathcal{F}(S)$ is surjective as $\mathcal{F}$ is soft. By Lemma 2.6 the kernel of $\mathrm{res}_{X,S} : \Omega^\bullet_{\mathcal{F}(X)|k} \to \Omega^\bullet_{\mathcal{F}(S)|k}$ is the ideal generated by $\operatorname{Ker}\mathrm{res}_{X,S}$ and $d(\operatorname{Ker}\mathrm{res}_{X,S})$. By assumption $\omega \in \operatorname{Ker}\mathrm{res}_{X,S}$ and, hence, it is enough to prove the statement for ω of the form

$$\omega = dt \wedge \lambda + u\eta,$$

where $t, u \in \mathcal{F}(X)$, $\lambda, \eta \in \Omega^\bullet_{\mathcal{F}(X)|k}$ and $\mathrm{res}_{X,S}(t) = \mathrm{res}_{X,S}(u) = 0$. We have

$$\varphi\omega = (\varphi dt) \wedge \lambda + (\varphi u)\eta.$$

We have $\varphi u = 0$ as the supports of φ and u do not intersect. It remains to prove that $\varphi dt = 0$. By Lemma 3.1 there exists a section $g \in \mathcal{F}(X)$ such that $\mathrm{res}_{X,S}(g) = 0$ and $\mathrm{res}_{X,\mathrm{supp}\, t}(g) = 1$. We have

$$\varphi dt = \varphi d(tg) = (\varphi t)dg + (\varphi g)dt = 0$$

as $t = tg$, $\varphi t = 0$ and $\varphi g = 0$. □

Lemma 3.3. *Suppose $\mathcal{F}$ is a soft sheaf of algebras on X. Then the sheafification* $\mathrm{sh} : \Omega^n_{\mathcal{F}(X)|k} = \Omega^n_{\mathcal{F}|k}(X) \to {}^+\Omega^n_{\mathcal{F}|k}(X)$ *is an isomorphism.*

Proof. First we prove injectivity. Take an n-form $\omega \in \Omega^n_{\mathcal{F}|k}(X)$ such that its image in ${}^+\Omega^n_{\mathcal{F}|k}(X)$ is 0. Then there is a finite open cover (U_i) of X such that the restrictions $\omega|_{U_i}$ are 0 for all i. Choose a partition of unity ($\varphi_i \in \mathcal{F}(X)$) subordinate to the cover (U_i). It can always be done by [God58, Theorem II.3.6.1]. By Lemma 3.2, $\varphi_i\omega = 0$ for all i. Now, $\omega = \sum_i \varphi_i\omega = 0$.

For surjectivity, take a global section of the sheaf ${}^+\Omega^n_{\mathcal{F}|k}$. It can be represented by a set of pairs (U_i, ω_i) where the sets U_i form a finite open cover of X and $\omega_i \in \Omega^n_{\mathcal{F}(U_i)|k}$ such that the germs of ω_i agree at every point. We seek a form $\omega \in \Omega^n_{\mathcal{F}(X)|k}$ such that $\omega_x = (\omega_i)_x$ in $\Omega^n_{\mathcal{F}_x|k}$ for each i and $x \in U_i$. Take a partition of unity (φ_i) subordinate to the cover (U_i) and denote $F_i := \mathrm{supp}\, \varphi_i$. The restriction map $\mathrm{res}_{X,S} : \Omega^n_{\mathcal{F}(X)|k} \to \Omega^n_{\mathcal{F}(S)|k}$ is surjective for any closed S by Lemma 2.6. Hence, we can extend the forms $\mathrm{res}_{X,F_i}(\omega_i)$ to some forms $\bar{\omega}_i \in \Omega^n_{\mathcal{F}(X)|k}$. Now, put $\omega := \sum_i \varphi_i\bar{\omega}_i \in \Omega^n_{\mathcal{F}(X)|k}$. Take i and a point $x \in U_i$. Introduce $J := \{j \mid x \in F_j\}$. We have

$$\omega_x = (\sum_j \varphi_j\bar{\omega}_j)_x = (\sum_{j\in J} \varphi_j\bar{\omega}_j)_x = (\sum_{j\in J} \varphi_j\omega_j)_x = (\sum_{j\in J} \varphi_j)_x(\omega_i)_x = (\omega_i)_x.$$

□

3.2. The construction and naturality of $\boldsymbol{\Lambda_{\mathcal{F}} : \mathbb{H}^*(X, \underline{k}_X[0]) \to \mathrm{H}^*(\Omega^\bullet_{\mathcal{F}(X)|k})}$.

Definition. For a soft sheaf of algebras $\mathcal{F}$ on a compact Hausdorff space X we define the map $\Lambda_{\mathcal{F}}$ by the following diagram:

$$\begin{tikzcd}
\mathbb{H}^*(X, \underline{k}_X[0]) \arrow[r, "\mathbb{H}(\varepsilon)"] \arrow[ddr, dashed, "\Lambda_{\mathcal{F}}"'] & \mathbb{H}^*(X, {}^+\Omega^\bullet_{\mathcal{F}|k}) \\
& \mathrm{H}^*({}^+\Omega^\bullet_{\mathcal{F}|k}(X)) \arrow[u, "\cong"', "\Upsilon"] \\
& \mathrm{H}^*(\Omega^\bullet_{\mathcal{F}(X)|k}). \arrow[u, "\cong"', "\mathrm{H}(\mathrm{sh})"]
\end{tikzcd}$$

Here the map H(sh) is an isomorphism by Lemma 3.3 and the map Υ is an isomorphism as the sheaves ${}^+\Omega^\bullet_{\mathcal{F}|k}$ are acyclic by Lemma 2.8.

A morphism of ringed spaces $(f, \varphi) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves $\varphi : \mathcal{G} \to f_* \mathcal{F}$.

Proposition 3.4. *The linear map Λ_- is natural with respect to morphisms of ringed spaces in the following sense: for a morphism of ringed spaces $(f, \varphi) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ the following diagram is commutative:*

$$\begin{tikzcd}
\mathbb{H}^*(X, \underline{k}_X[0]) \arrow[r, "\Lambda_{\mathcal{F}}"] & \mathrm{H}^*(\Omega^\bullet_{\mathcal{F}(X)|k}) \\
\mathbb{H}^*(Y, \underline{k}_Y[0]) \arrow[r, "\Lambda_{\mathcal{G}}"] \arrow[u, "f^*"] & \mathrm{H}^*(\Omega^\bullet_{\mathcal{G}(Y)|k}). \arrow[u, "\mathrm{H}(\Omega_{\varphi(Y)})"']
\end{tikzcd}$$

Proof. The morphism of sheaves $\varphi : \mathcal{G} \to f_* \mathcal{F}$ defines the morphism of complexes of sheaves ${}^+\Omega_\varphi : {}^+\Omega^\bullet_{\mathcal{G}|k} \to f_* {}^+\Omega^\bullet_{\mathcal{F}|k}$.

Consider the diagram

$$\begin{tikzcd}
\mathbb{H}^*(X, \underline{k}_X[0]) \arrow[r, "\mathbb{H}(\epsilon)"] \arrow[rrr, bend left, "\Lambda_{\mathcal{F}}"] & \mathbb{H}^*(X, {}^+\Omega^\bullet_{\mathcal{F}|k}) & \mathrm{H}^*({}^+\Omega^\bullet_{\mathcal{F}|k}(X)) \arrow[l, "\Upsilon"', "\cong"] & \mathrm{H}^*(\Omega^\bullet_{\mathcal{F}(X)|k}) \arrow[l, "\mathrm{H}(\mathrm{sh})"', "\cong"] \\
\mathbb{H}^*(Y, \underline{k}_Y[0]) \arrow[r, "\mathbb{H}(\epsilon)"] \arrow[u, "f^*"] \arrow[rrr, bend right, "\Lambda_{\mathcal{G}}"'] & \mathbb{H}^*(Y, {}^+\Omega^\bullet_{\mathcal{G}|k}) \arrow[u, "f^*"] & \mathrm{H}^*({}^+\Omega^\bullet_{\mathcal{G}|k}(Y)) \arrow[l, "\Upsilon"', "\cong"] \arrow[u, "\mathrm{H}({}^+\Omega_\varphi(Y))"'] & \mathrm{H}^*(\Omega^\bullet_{\mathcal{G}(Y)|k}). \arrow[l, "\mathrm{H}(\mathrm{sh})"', "\cong"] \arrow[u, "\mathrm{H}(\Omega_{\varphi(Y)})"']
\end{tikzcd}$$

The middle square is commutative by Lemma 2.1. The other squares are commutative for obvious reasons. □

4. Splitting for the algebra of smooth functions

In this section M is a compact smooth manifold. We prove that for a smooth manifold M the groups $\mathbb{H}^*(M, \underline{\mathbb{R}}_M[0])$ canonically split off of $\mathrm{H}^*(\Omega^\bullet_{C^\infty(M)|\mathbb{R}})$.

4.1. Canonical isomorphism Θ. Denote by $\Omega^\bullet_M$ the complex of sheaves of smooth differential forms on M. We often denote by $\Omega^\bullet(M)$ the complex $\Omega^\bullet_M(M)$. Consider the morphism of complexes, the coaugmentation, $\epsilon : \underline{\mathbb{R}}_M[0] \to \Omega^\bullet_M$ defined by $\epsilon(1) := 1$. We define Θ by the following commutative diagram

$$\begin{tikzcd}
\mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[r, "\mathbb{H}(\epsilon)", "\cong"'] \arrow[rr, bend left, "\Theta"] & \mathbb{H}^*(M, \Omega^\bullet_M) & \mathrm{H}^*(\Omega^\bullet(M)). \arrow[l, "\Upsilon"', "\cong"]
\end{tikzcd}$$

The map $\mathbb{H}(\epsilon)$ in this case is an isomorphism (as ϵ is a quasi-isomorphism) and Υ is an isomorphism and so Θ is an isomorphism.

4.2. Morphisms π and $\underline{\pi}$.

(1) Take an open set $U \subset M$; the identity map $C^\infty(U) \to \Omega^0(U)$ can be uniquely extended to a morphism of dg-algebras $\pi : \Omega^\bullet_{C^\infty(U)|\mathbb{R}} \to \Omega^\bullet(U)$ by the universal property of $\Omega^\bullet_{C^\infty(U)|\mathbb{R}}$. The following diagram is commutative:

$$\begin{tikzcd} C^\infty(U) \arrow[r] \arrow[dr] & \Omega^\bullet_{C^\infty(U)|\mathbb{R}} \arrow[d, "\pi"] \\ & \Omega^\bullet(U). \end{tikzcd}$$

(2) This way we obtain a morphism of complexes of presheaves $\pi : \Omega^\bullet_{C^\infty_M|\mathbb{R}} \to \Omega^\bullet_M$.

(3) By the universal property of sheafification we have the morphism of complexes of sheaves $\underline{\pi} : {}^+\Omega^\bullet_{C^\infty_M|\mathbb{R}} \to \Omega^\bullet_M$.

The following diagrams are commutative:

$$\begin{tikzcd} \underline{\mathbb{R}}_M[0] \arrow[r, "\epsilon"] \arrow[dr, "\epsilon"'] & {}^+\Omega^\bullet_{C^\infty_M|\mathbb{R}} \arrow[d, "\underline{\pi}"] \\ & \Omega^\bullet_M, \end{tikzcd} \qquad \begin{tikzcd} \Omega^\bullet_{C^\infty_M|\mathbb{R}} \arrow[r, "\pi"] \arrow[d, "\mathrm{sh}"'] & \Omega^\bullet_M \\ {}^+\Omega^\bullet_{C^\infty_M|\mathbb{R}}. \arrow[ur, "\underline{\pi}"'] & \end{tikzcd} \tag{2}$$

4.3. Main splitting theorem.

Theorem 4.1. *The following diagram is commutative:*

$$\begin{tikzcd} \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[r, "\Lambda_{C^\infty_M}"] \arrow[rr, bend left, "\Theta"] & \mathrm{H}^*(\Omega^\bullet_{C^\infty(M)|\mathbb{R}}) \arrow[r, "\mathrm{H}(\pi)"] & \mathrm{H}^*(\Omega^\bullet(M)). \end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd} \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[r, "\mathbb{H}(\epsilon)"] \arrow[rr, bend left, "\mathbb{H}(\epsilon)"] \arrow[drr, "\Theta"'] \arrow[ddr, bend right, "\Lambda_{C^\infty_M}"'] & \mathbb{H}^*(M, {}^+\Omega^\bullet_{C^\infty_M|\mathbb{R}}) \arrow[r, "\mathbb{H}(\underline{\pi})"] & \mathbb{H}^*(M, \Omega^\bullet_M) \\ & \mathrm{H}^*({}^+\Omega^\bullet_{C^\infty_M|\mathbb{R}}(M)) \arrow[u, "\Upsilon"] \arrow[r, "\mathrm{H}(\underline{\pi})"'] & \mathrm{H}^*(\Omega^\bullet(M)) \arrow[u, "\Upsilon"', "\cong"] \\ & \mathrm{H}^*(\Omega^\bullet_{C^\infty(M)|\mathbb{R}}). \arrow[u, "\mathrm{H}(\mathrm{sh})"] \arrow[ur, bend right, "\mathrm{H}(\pi)"'] & \end{tikzcd}$$

The left quadrangle is commutative by the definition of $\Lambda_{C^\infty_M}$. The upper triangle and the bottom right triangles are commutative by Diagrams 2. The upper right square is commutative by the naturality of Υ. The upper triangle with Θ is commutative by the definition of Θ. As the right arrow Υ is an isomorphism, the whole diagram is commutative. $\square$

5. Simplicial dg-algebra of flat cochains $\Omega^{\bullet}_{\flat}(\Delta^{-})$

5.1. Flat cochains. We present the dg-algebra $\Omega^{\bullet}_{\flat}(\Delta^n)$ of flat cochains on the closed n-simplex. We follow [Whi57], see also [Hei05]. The dg-algebras $\Omega^{\bullet}_{\flat}(\Delta^n)$, $n \geq 0$, will form the simplicial dg-algebra $\Omega^{\bullet}_{\flat}(\Delta^{-})$.

For a convex set $V \subset \mathbb{R}^m$ we denote by $\operatorname{aff} V$ the affine hull of V. We denote by $\operatorname{relint} V$ the interior of V relative to $\operatorname{aff} V$.

Definition. For a convex set $V \subset \mathbb{R}^m$ denote by $\mathrm{S}^{\mathrm{aff}}_k(V)$ the vector space of affine singular k-chains in V with coefficients in $\mathbb{R}$. The boundary map $\partial : \mathrm{S}^{\mathrm{aff}}_k(V) \to \mathrm{S}^{\mathrm{aff}}_{k-1}(V)$ is defined in the usual manner: we denote by $\gamma_i : \Delta^{k-1} \to \Delta^k$ the i-th face embedding, then for a simplex $\sigma : \Delta^k \to V$ we denote by $\partial_i\sigma : \Delta^{k-1} \to V$ the composition $\sigma \circ \gamma_i$ and by $\partial\sigma$ the singular chain $\sum_{i=0}^{k}(-1)^i\partial_i\sigma$. The vector spaces $\mathrm{S}^{\mathrm{aff}}_k(V)$ together with the boundary map form the chain complex $\mathrm{S}^{\mathrm{aff}}_{\bullet}(V)$.

An affine map $f : V \to V'$ induces the morphism of complexes

$$\mathrm{S}^{\mathrm{aff}}(f) : \mathrm{S}^{\mathrm{aff}}_{\bullet}(V) \to \mathrm{S}^{\mathrm{aff}}_{\bullet}(V').$$

Hence, the correspondence $V \mapsto \mathrm{S}^{\mathrm{aff}}_{\bullet}(V)$ is covariant with respect to affine maps.

Define the *mass* $|\alpha|$ of an affine k-chain $\alpha = \sum \lambda_i\sigma_i$, where σ_i are distinct singular simplices, as

$$|\alpha| := \sum |\lambda_i||\sigma_i|_{\mathrm{L}}$$

where $|\sigma_i|_{\mathrm{L}}$ denotes the Lebesgue k-measure of σ_i. Define the flat seminorm $|\cdot|^{\flat}$ on $\mathrm{S}^{\mathrm{aff}}_k(V)$ as

$$|\alpha|^{\flat} := \inf_{\beta \in \mathrm{S}^{\mathrm{aff}}_{k+1}(V)} \{|\alpha - \partial\beta| + |\beta|\}.$$

Lemma 5.1. *The map $\mathrm{S}^{aff}_k(V) \to \mathrm{S}^{aff}_k(V')$ induced by an inclusion of convex sets $V \hookrightarrow V'$ preserves the flat seminorm.*

Proof. Follows from [Whi57, Lemma V.2b]. $\square$

Lemma 5.2. *Let V be a convex set. The map $\mathrm{S}^{aff}_k(\operatorname{relint} V) \to \mathrm{S}^{aff}_k(V)$ induced by the inclusion* $\operatorname{relint} V \hookrightarrow V$ *has image dense with respect to the flat seminorm.*

Proof. Any singular affine simplex in V can be approximated by one in $\operatorname{relint} V$. Cf. [Whi57, VIII.1(h)]. $\square$

Definition. If V is a convex set we define $\Omega^k_{\flat}(V)$ as the vector space of linear functionals $\mathrm{S}^{\mathrm{aff}}_k(V) \to \mathbb{R}$ bounded with respect to the seminorm $|\cdot|^{\flat}$. We call the elements of $\Omega^k_{\flat}(V)$ *flat cochains* on V. We define the differential dX of a cochain X by the formula $\langle dX, \alpha\rangle := \langle X, \partial\alpha\rangle$. We obtain the complex $\Omega_\flat^\bullet(V)$. This definition is equivalent to the one given by Whitney in [Whi57, VIII.1(b) and VII.10].

The complex $S_\bullet^{\mathrm{aff}}(V)$ was covariant in V with respect to affine maps, so the complex $\Omega_\flat^\bullet(V)$ is contravariant in V with respect to affine maps.

We refer the reader to [Whi57, IX.14] where Whitney defines the graded-commutative multiplication of flat cochains in open sets. The multiplication is natural with respect to affine maps [Whi57, X.11]. For an open convex set V the complex $\Omega_\flat^\bullet(V)$ becomes a dg-algebra, which is contravariant with respect to affine maps. Hence, the multiplication is well defined in relatively open convex sets. Next we wish to define the product of two flat cochains on a closed convex set V. For this we need the following lemma.

Lemma 5.3. *For a closed convex set $V \subset \mathbb{R}^m$ the inclusion* $\operatorname{relint} V \hookrightarrow V$ *induces an isomorphism $\rho : \Omega_\flat^\bullet(V) \to \Omega_\flat^\bullet(\operatorname{relint} V)$.*

Proof. Follows from Lemmas 5.1 and 5.2. □

For a closed convex set $V \subset \mathbb{R}^m$ we introduce the multiplication in $\Omega_\flat^\bullet(V)$ in the way that the isomorphism of complexes $\rho : \Omega_\flat^\bullet(V) \to \Omega_\flat^\bullet(\operatorname{relint} V)$ from Lemma 5.3 becomes an isomorphism of dg-algebras. This multiplication and its naturality are implicit in [Whi57, VII.12]. In Proposition 5.6 we show that the multiplication is natural with respect to affine maps.

Take V a closed convex set. The inclusion $V \hookrightarrow \operatorname{aff} V$ induces the map $\pi : \Omega_\flat^\bullet(\operatorname{aff} V) \to \Omega_\flat^\bullet(V)$.

Lemma 5.4. *The map $\pi : \Omega_\flat^\bullet(\operatorname{aff} V) \to \Omega_\flat^\bullet(V)$ is surjective.*

Proof. Follows from Lemma 5.1 by the Hahn-Banach theorem. Alternatively, see [Whi57, VIII.1(h)] and apply Lemma 5.3. □

Lemma 5.5. *The map $\pi : \Omega_\flat^\bullet(\operatorname{aff} V) \to \Omega_\flat^\bullet(V)$ is a morphism of dg-algebras.*

Proof. Consider the following diagram

$$\Omega_\flat^\bullet(\operatorname{aff} V) \xrightarrow{\pi} \Omega_\flat^\bullet(V) \xrightarrow{\rho} \Omega_\flat^\bullet(\operatorname{relint} V).$$

The composition $\rho \circ \pi$ is induced by the inclusion $\operatorname{relint} V \hookrightarrow \operatorname{aff} V$ and, hence, is a morphism of dg-algebras. Since ρ is an isomorphism of dg-algebras, π is a morphism of dg-algebras. □

Proposition 5.6. *Consider $f : V \to V'$ an affine map of closed convex sets. Then the induced morphism $\Omega_\flat(f) : \Omega_\flat^\bullet(V') \to \Omega_\flat^\bullet(V)$ preserves multiplication.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\Omega_\flat^\bullet(\operatorname{aff} V') \arrow[r, "\Omega_\flat(\operatorname{aff} f)"] \arrow[d, "\pi"'] & \Omega_\flat^\bullet(\operatorname{aff} V) \arrow[d, "\pi"] \\
\Omega_\flat^\bullet(V') \arrow[r, "\Omega_\flat(f)"] & \Omega_\flat^\bullet(V).
\end{tikzcd}$$

Here $\operatorname{aff} f : \operatorname{aff} V \to \operatorname{aff} V'$ is the affine extension of f. The left vertical arrow is surjective by Lemma 5.4. The vertical arrows are morphisms of dg-algebras by Lemma 5.5. The map $\Omega_\flat(\operatorname{aff} f)$ is a morphism of dg-algebras as it is induced by an affine map of relatively open sets. Therefore, $\Omega_\flat(f)$ is a morphism of dg-algebras. □

An order-preserving map $[n] \to [l]$ induces an affine map $\Delta^n \to \Delta^l$. Hence, we obtain the simplicial dg-algebra $\Omega_\flat^\bullet(\Delta^-)$ defined as $[n] \mapsto \Omega_\flat^\bullet(\Delta^n)$. In particular, there are face maps $\partial_i : \Omega_\flat^\bullet(\Delta^n) \to \Omega_\flat^\bullet(\Delta^{k-1})$ induced by the face embeddings $\gamma_i : \Delta^{k-1} \to \Delta^n$.

5.2. The attributes of the simplicial dg-algebra $\Omega_\flat^\bullet(\Delta^-)$. (1) We define the linear maps $\mathfrak{J}_n : \Omega_\flat^n(\Delta^n) \to \mathbb{R}$ as

$$\mathfrak{J}_n(\omega) := \langle \omega, \mathrm{id}_{\Delta^n} \rangle .$$

Proposition 5.7 (Stokes' formula)**.** *For every $\eta \in \Omega_\flat^{n-1}(\Delta^n)$ the following formula holds*

$$\mathfrak{J}_n(d\eta) = \sum_{i=0}^{n} (-1)^i \mathfrak{J}_{n-1}(\partial_i \eta).$$

Proof. We have

$$\mathfrak{J}_n(d\eta) = \langle d\eta, \mathrm{id}_{\Delta^n} \rangle = \langle \eta, \partial \, \mathrm{id}_{\Delta^n} \rangle = \sum_{i=0}^{n} (-1)^i \langle \eta, \gamma_i \rangle =$$

$$= \sum_{i=0}^{n} (-1)^i \langle \partial_i \eta, \mathrm{id}_{\Delta^{n-1}} \rangle = \sum_{i=0}^{n} (-1)^i \mathfrak{J}_{n-1}(\partial_i \eta).$$

□

(2) The Lipschitz functions on Δ^n form the algebra $\mathrm{Lip}(\Delta^n)$. The correspondence $[n] \mapsto \mathrm{Lip}(\Delta^n)$ gives rise to the simplicial algebra $\mathrm{Lip}(\Delta^-)$. Every Lipschitz function $f \in \mathrm{Lip}(\Delta^n)$ defines a flat 0-cochain $\zeta(f)$ on Δ^n in the following way. Take an affine simplex $\sigma : \{v_0\} = \Delta^0 \to \Delta^n$ and set

$$\big\langle \zeta(f), \sigma \big\rangle := f(\sigma(v_0)).$$

It is easy to check that 0-cochain $\zeta(f)$ is flat (see [Whi57, Theorem VII.4B]. We obtain the linear map $\zeta : \mathrm{Lip}(\Delta^n) \to \Omega_\flat^\bullet(\Delta^n)$ with image in $\Omega_\flat^0(\Delta^n)$. The composition $\rho \circ \zeta : \mathrm{Lip}(\Delta^n) \to \Omega_\flat^\bullet(\Delta^n) \xrightarrow{\cong} \Omega_\flat^\bullet(\operatorname{relint} \Delta^n)$ is a homomorphism of algebras (by the definition of multiplication on open sets [Whi57, IX.14]). Hence, ζ is a homomorphism of algebras. The map ζ clearly preserves the simplicial structure and we obtain the morphism of simplicial algebras

$$\zeta : \mathrm{Lip}(\Delta^-) \to \Omega^0_\flat(\Delta^-).$$

(3) Consider the simplicial dg-algebra $\Omega^\bullet(\Delta^-)$ of smooth differential forms. Every smooth form $\theta \in \Omega^k(\Delta^n)$ gives rise to a flat cochain $\nabla(\theta) \in \Omega^k_\flat(\Delta^n)$ by

$$\langle \nabla(\theta), \sigma \rangle := \int_{\Delta^k} \sigma^* \theta$$

for affine $\sigma : \Delta^k \to \Delta^n$. The flatness can be easily checked (see [Whi57, V.14 and Theorem V.10A]). This correspondence gives rise to a simplicial linear map

$$\nabla : \Omega^\bullet(\Delta^-) \to \Omega^\bullet_\flat(\Delta^-).$$

By Stokes' formula for smooth forms the differential is preserved. The composition

$$\Omega^\bullet(\Delta^n) \xrightarrow{\nabla} \Omega^\bullet_\flat(\Delta^n) \xrightarrow{\rho} \Omega^\bullet_\flat(\mathrm{relint}\,\Delta^n)$$

preserves multiplication by the definition of multiplication of flat cochains [Whi57, X.14]. Hence, by the definition of multiplication in $\Omega^\bullet_\flat(\Delta^n)$ the map ∇ preserves multiplication. Therefore, the map

$$\nabla : \Omega^\bullet(\Delta^-) \to \Omega^\bullet_\flat(\Delta^-)$$

is a morphism of simplicial dg-algebras.

(4) We define the linear maps $\mathfrak{X}_n : \Omega^n(\Delta^n) \to \mathbb{R}$ as

$$\mathfrak{X}_n(\omega) := \int_{\Delta^n} \omega.$$

The diagram

$$\begin{tikzcd}
\Omega^n(\Delta^n) \arrow[r, "\mathfrak{X}_n"] \arrow[d, "\nabla"] & \mathbb{R} \\
\Omega^n_\flat(\Delta^n) \arrow[ur, "\mathfrak{I}_n"'] &
\end{tikzcd}$$

is commutative by construction.

(5) The smooth functions form the simplicial algebra $C^\infty(\Delta^-)$. It embeds via $i : C^\infty(\Delta^-) \hookrightarrow \mathrm{Lip}(\Delta^-)$ in the simplicial algebra $\mathrm{Lip}(\Delta^-)$. The following diagram of morphisms of simplicial algebras is commutative:

$$\begin{tikzcd}
C^\infty(\Delta^-) \arrow[r, hook] \arrow[d, hook, "i"'] & \Omega^\bullet(\Delta^-) \arrow[d, "\nabla"] \\
\mathrm{Lip}(\Delta^-) \arrow[r, "\zeta"] & \Omega^\bullet_\flat(\Delta^-).
\end{tikzcd}$$

6. The map $\Psi_A : \mathbf{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \to \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0])$

6.1. The pullback of an algebraic de Rham form. Take B a finitely generated $\mathbb{R}$-algebra. We construct a morphism of dg-algebras $\mu(\sigma) : \Omega^\bullet_{B|\mathbb{R}} \to \Omega^\bullet_\flat(\Delta^n)$ for every singular Lipschitz simplex $\sigma : \Delta^n \to \mathrm{spec}_\mathbb{R} B$. For such σ we define an algebra homomorphism $\theta(\sigma) : B \to \mathrm{Lip}(\Delta^n)$ as $b \mapsto \hat{b}\circ\sigma$. By the universal property of $\Omega^\bullet_{B|\mathbb{R}}$ there exists a unique morphism of dg-algebras $\mu(\sigma) : \Omega^\bullet_{B|\mathbb{R}} \to \Omega^\bullet_\flat(\Delta^n)$ making the following diagram commute:

$$\begin{tikzcd}
\Omega^\bullet_{B|\mathbb{R}} \arrow[r, dashed, "\mu(\sigma)"] & \Omega^\bullet_\flat(\Delta^n) \\
B \arrow[u] \arrow[r, "\theta(\sigma)"] & \mathrm{Lip}(\Delta^n). \arrow[u, "\zeta"]
\end{tikzcd}$$

Lemma 6.1. *The following diagram is commutative:*

$$\begin{tikzcd}
\Omega^\bullet_{B|\mathbb{R}} \arrow[d, "\mu(\sigma)"'] \arrow[dr, "\mu(\partial_i\sigma)"] & \\
\Omega^\bullet_\flat(\Delta^n) \arrow[r, "\partial_i"] & \Omega^\bullet_\flat(\Delta^{n-1}).
\end{tikzcd}$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
\Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\mu(\sigma)"'] \arrow[rr, bend left, "\mu(\partial_i\sigma)"] & \Omega^\bullet_\flat(\Delta^n) \arrow[r, "\partial_i"'] & \Omega^\bullet_\flat(\Delta^{n-1}) \\
B \arrow[u] \arrow[r, "\theta(\sigma)"] \arrow[rr, bend right, "\theta(\partial_i\sigma)"'] & \mathrm{Lip}(\Delta^n) \arrow[u, "\zeta"] \arrow[r, "\partial_i"] & \mathrm{Lip}(\Delta^{n-1}). \arrow[u, "\zeta"']
\end{tikzcd}$$

The right square obviously commutes. By the definition of μ the left square and the outer contour commute. The bottom triangle commutes by the definition of θ. Hence, by the universal property of $\Omega^\bullet_{B|\mathbb{R}}$ the upper triangle also commutes. $\square$

The map $\mu(\sigma)$ is natural in algebra, namely:

Lemma 6.2. *Suppose $\varphi : B' \to B$ is a homomorphism of finitely generated $\mathbb{R}$-algebras. Consider a commutative diagram*

$$\begin{tikzcd}
\Delta^n \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] & \mathrm{spec}_\mathbb{R} B \arrow[d, "\mathrm{spec}_\mathbb{R}\varphi"] \\
& \mathrm{spec}_\mathbb{R} B'
\end{tikzcd}$$

with σ and σ' Lipschitz. Then the following diagram commutes:

$$\begin{tikzcd} \Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\mu(\sigma)"] & \Omega^\bullet_\flat(\Delta^n) \\ \Omega^\bullet_{B'|\mathbb{R}}. \arrow[u, "\Omega_\varphi"] \arrow[ur, "\mu(\sigma')"'] & \end{tikzcd}$$

Proof. Consider the following diagram:

$$\begin{tikzcd} \Omega^\bullet_{B'|\mathbb{R}} \arrow[r, "\Omega_\varphi"'] \arrow[rr, bend left, "\mu(\sigma')"] & \Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\mu(\sigma)"'] & \Omega^\bullet_\flat(\Delta^n) \\ B' \arrow[u] \arrow[r, "\varphi"] \arrow[rr, bend right, "\theta(\sigma')"'] & B \arrow[u] \arrow[r, "\theta(\sigma)"] & \mathrm{Lip}(\Delta^n). \arrow[u, "\zeta"'] \end{tikzcd}$$

The left square obviously commutes. By the definition of μ the right square and the outer contour commute. The bottom triangle commutes: for $c \in B'$

$$\theta(\sigma)(\varphi(c)) = \widehat{\varphi(c)}\circ\sigma = \hat{c}\circ(\mathrm{spec}_{\mathbb{R}}\varphi)\circ\sigma = \hat{c}\circ\sigma' = \theta(\sigma')(c).$$

Hence, by the universal property of $\Omega^\bullet_{B'|\mathbb{R}}$ the upper triangle also commutes. □

6.2. The map $\xi_B : \Omega^\bullet_{B|\mathbb{R}} \to \mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B)$. Let B be a finitely generated $\mathbb{R}$-algebra. To an algebraic n-form $\omega \in \Omega^n_{B|\mathbb{R}}$ we associate a Lipschitz singular cochain $\xi_B(\omega)$. On a Lipschitz singular simplex $\sigma : \Delta^n \to \mathrm{spec}_{\mathbb{R}} B$ we define $\xi_B(\omega)$ as

$$\langle \xi_B(\omega), \sigma \rangle := \mathfrak{I}_n(\mu(\sigma)(\omega)) \in \mathbb{R}.$$

Proposition 6.3. *The above gives a morphism of complexes*

$$\xi_B : \Omega^\bullet_{B|\mathbb{R}} \to \mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B).$$

Proof. The only part that needs to be checked is that the map preserves d. Take a Lipschitz singular simplex $\sigma : \Delta^n \to \mathrm{spec}_{\mathbb{R}} B$ and $\eta \in \Omega^{n-1}_{B|\mathbb{R}}$, then

$$\langle \xi_B(d\eta), \sigma \rangle = \mathfrak{I}_n(\mu(\sigma)(d\eta)) \overset{(2)}{=} \mathfrak{I}_n(d(\mu(\sigma)(\eta))) \overset{(3)}{=} \sum_{i=0}^{n} (-1)^i \mathfrak{I}_{n-1}(\partial_i(\mu(\sigma)(\eta))) \overset{(4)}{=}$$

$$\overset{(4)}{=} \sum_{i=0}^{n} (-1)^i \mathfrak{I}_{n-1}(\mu(\partial_i\sigma)(\eta)) = \sum_{i=0}^{n} (-1)^i \langle \xi_B(\eta), \partial_i\sigma \rangle = \langle \xi_B(\eta), \partial\sigma \rangle = \langle d\xi_B(\eta), \sigma \rangle .$$

The second equality follows from the fact that $\mu(\sigma)$ is a morphism of complexes. The third equality follows from Proposition 5.7. The fourth equality follows from Lemma 6.1. □

Lemma 6.2 allows us to prove the naturality of ξ in algebras:

Lemma 6.4. *Suppose $\varphi : B' \to B$ is a homomorphism of finitely generated $\mathbb{R}$-algebras. Then the following diagram is commutative*

$$\begin{tikzcd}
\Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\xi_B"] & \mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B) \\
\Omega^\bullet_{B'|\mathbb{R}} \arrow[u, "\Omega_\varphi"] \arrow[r, "\xi_{B'}"] & \mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B'). \arrow[u, "\mathrm{S}_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}}\varphi)"']
\end{tikzcd}$$

6.3. The map $\Phi_B : \mathrm{H}^*(\Omega^\bullet_{B|\mathbb{R}}) \to \mathbb{H}^*(\mathrm{spec}_{\mathbb{R}} B, \underline{\mathbb{R}}_{\mathrm{spec}_{\mathbb{R}} B}[0])$.

We define the homomorphism Φ_B as the vertical map making the following diagram commutative:

$$\begin{tikzcd}
\mathbb{H}^*(\mathrm{spec}_{\mathbb{R}} B, \underline{\mathbb{R}}_{\mathrm{spec}_{\mathbb{R}} B}[0]) \arrow[r, "\mathbb{H}(\epsilon)", "\cong"'] & \mathbb{H}^*(\mathrm{spec}_{\mathbb{R}} B, {}^+\mathrm{S}^\bullet_{\mathrm{Lip},\mathrm{spec}_{\mathbb{R}} B}) \\
\mathrm{H}^*(\Omega^\bullet_{B|\mathbb{R}}) \arrow[u, dashed, "\Phi_B"] \arrow[r, "\mathrm{H}(\xi_B)"] & \mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B)). \arrow[u, "\cong", "\mathrm{Y}\circ\mathrm{H}(\mathrm{sh})"']
\end{tikzcd}$$

Here $\mathbb{H}(\epsilon)$ is an isomorphism by Corollary 2.5.

Lemma 6.5. *Let $\varphi : B' \to B$ be a homomorphism of finitely generated $\mathbb{R}$-algebras. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{H}^*(\Omega^\bullet_{B|\mathbb{R}}) \arrow[r, "\Phi_B"] & \mathbb{H}^*(\mathrm{spec}_{\mathbb{R}} B, \underline{\mathbb{R}}_{\mathrm{spec}_{\mathbb{R}} B}[0]) \\
\mathrm{H}^*(\Omega^\bullet_{B'|\mathbb{R}}) \arrow[u, "\mathrm{H}(\Omega_\varphi)"] \arrow[r, "\Phi_{B'}"'] & \mathbb{H}^*(\mathrm{spec}_{\mathbb{R}} B', \underline{\mathbb{R}}_{\mathrm{spec}_{\mathbb{R}} B'}[0]). \arrow[u, "(\mathrm{spec}_{\mathbb{R}}\varphi)^*"']
\end{tikzcd}$$

Proof. It follows directly from Lemma 6.4 and naturality of $\mathrm{Y}\circ\mathrm{H}(\mathrm{sh})$. □

Remark. There are other ways to define a map, analogous to Φ_B. One way is to consider semi-algebraic cochains, instead of Lipschitz ones, see [HLTV11]. Another way would be to use the *filtered de Rham complex*, see [PS08, Proposition 7.24]. Our construction of Φ_B allows us to relate it to the classical de Rham complex $\Omega^\bullet(M)$ in the case $B \subset C^\infty(M)$, see Lemma 7.9.

6.4. The map $\Psi_A : \mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \to \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0])$. For a topological space X and a subalgebra $A \subset C(X)$ write A as the filtered colimit of its finitely generated subalgebras: $A = \varinjlim B$. The functors Ω and H preserve filtered colimits. For an inclusion of finitely generated subalgebras $i : B' \hookrightarrow B$ of $C(X)$ consider the diagram

$$\begin{tikzcd}
\mathrm{H}^*(\Omega^\bullet_{B|\mathbb{R}}) \arrow[r, "\Phi_B"] & \mathbb{H}^*(\mathrm{spec}_\mathbb{R} B, \underline{\mathbb{R}}_{\mathrm{spec}_\mathbb{R} B}[0]) \arrow[r, "\Gamma_B^*"] & \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]) \\
\mathrm{H}^*(\Omega^\bullet_{B'|\mathbb{R}}) \arrow[u, "\mathrm{H}(\Omega_i)"] \arrow[r, "\Phi_{B'}"] & \mathbb{H}^*(\mathrm{spec}_\mathbb{R} B', \underline{\mathbb{R}}_{\mathrm{spec}_\mathbb{R} B'}[0]). \arrow[u, "(\mathrm{spec}_\mathbb{R} i)^*"] \arrow[ur, "\Gamma_{B'}^*"'] &
\end{tikzcd}$$

The left square is commutative by Lemma 6.5 and the right triangle is commutative by the naturality of Γ_B (see Subsection 2.2). We pass to the colimit of $\Gamma_B^* \circ \Phi_B$ over all finitely generated subalgebras $B \subset A$ and obtain the map

$$\Psi_A : \mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \to \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]).$$

Proposition 6.6. *Take $f : X \to Y$ a continuous map of topological spaces. Suppose that $A \subset C(X)$ and $A' \subset C(Y)$ are subalgebras and $\varphi : A' \to A$ is a homomorphism such that the following diagram is commutative:*

$$\begin{tikzcd}
C(X) & A \arrow[l, hook'] \\
C(Y) \arrow[u, "f^*"] & A'. \arrow[l, hook'] \arrow[u, "\varphi"']
\end{tikzcd}$$

Then the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \arrow[r, "\Psi_A"] & \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]) \\
\mathrm{H}^*(\Omega^\bullet_{A'|\mathbb{R}}) \arrow[u, "\mathrm{H}(\Omega_\varphi)"] \arrow[r, "\Psi_{A'}"] & \mathbb{H}^*(Y, \underline{\mathbb{R}}_Y[0]). \arrow[u, "f^*"']
\end{tikzcd}$$

Proof. It follows directly from Lemma 6.5 and naturality of Γ. □

7. Identifying the map $\Psi_{C^\infty(M)}$

7.1. The map $\xi_B^{\mathrm{sm}} : \Omega^\bullet_{B|\mathbb{R}} \to \mathrm{S}^\bullet_{\mathrm{sm}}(\mathrm{spec}_\mathbb{R} B)$. This subsection mirrors the Subsections 6.1 and 6.2, so the proofs will be omitted.

Take B a finitely generated $\mathbb{R}$-algebra. We construct a morphism of dg-algebras $\mu^{\mathrm{sm}}(\sigma) : \Omega^\bullet_{B|\mathbb{R}} \to \Omega^\bullet(\Delta^n)$ for every smooth singular simplex $\sigma : \Delta^n \to \mathrm{spec}_\mathbb{R} B$. For such σ we define an algebra homomorphism $\theta^{\mathrm{sm}}(\sigma) : B \to C^\infty(\Delta^n)$ as $b \mapsto \hat{b} \circ \sigma$. By the universal property of $\Omega^\bullet_{B|\mathbb{R}}$ there exists a unique morphism of dg-algebras $\mu^{\mathrm{sm}}(\sigma) : \Omega^\bullet_{B|\mathbb{R}} \to \Omega^\bullet(\Delta^n)$ making the following diagram commute:

$$\begin{tikzcd}
\Omega^\bullet_{B|\mathbb{R}} \arrow[r, dashed, "\mu^{\mathrm{sm}}(\sigma)"] & \Omega^\bullet(\Delta^n) \\
B \arrow[u] \arrow[r, "\theta^{\mathrm{sm}}(\sigma)"] & C^\infty(\Delta^n). \arrow[u, hook]
\end{tikzcd}$$

Lemma 7.1 (cf. Lemma 6.1)**.** *The following diagram is commutative:*

$$\begin{tikzcd} \Omega^\bullet_{B|\mathbb{R}} \arrow[d, "\mu^{\mathrm{sm}}(\sigma)"'] \arrow[dr, "\mu^{\mathrm{sm}}(\partial_i\sigma)"] & \\ \Omega^\bullet(\Delta^n) \arrow[r, "\partial_i"] & \Omega^\bullet(\Delta^{n-1}). \end{tikzcd}$$

The map $\mu^{\mathrm{sm}}(\sigma)$ is natural in algebra, namely:

Lemma 7.2 (cf. Lemma 6.2)**.** *Suppose $\varphi : B' \to B$ is a homomorphism of finitely generated $\mathbb{R}$-algebras. Consider a commutative diagram*

$$\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] & \mathrm{spec}_{\mathbb{R}} B \arrow[d, "\mathrm{spec}_{\mathbb{R}}\varphi"] \\ & \mathrm{spec}_{\mathbb{R}} B' \end{tikzcd}$$

with σ and σ' smooth. Then the following diagram commutes:

$$\begin{tikzcd} \Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\mu^{\mathrm{sm}}(\sigma)"] & \Omega^\bullet_\flat(\Delta^n) \\ \Omega^\bullet_{B'|\mathbb{R}}. \arrow[u, "\Omega_\varphi"] \arrow[ur, "\mu^{\mathrm{sm}}(\sigma')"'] & \end{tikzcd}$$

Next we construct the map $\xi^{\mathrm{sm}}_B : \Omega^\bullet_{B|\mathbb{R}} \to \mathrm{S}^\bullet_{\mathrm{sm}}(\mathrm{spec}_{\mathbb{R}} B)$ for a finitely generated $\mathbb{R}$-algebra B. For an algebraic n-form $\omega \in \Omega^n_{B|\mathbb{R}}$ and a smooth simplex $\sigma : \Delta^n \to \mathrm{spec}_{\mathbb{R}} B$ we set

$$\left\langle \xi^{\mathrm{sm}}_B(\omega), \sigma \right\rangle := \mathfrak{T}_n(\mu^{\mathrm{sm}}(\sigma)(\omega)) \in \mathbb{R}$$

(the map $\mathfrak{T}_n$ was defined in Paragraph 5.2(4)).

Proposition 7.3 (cf. Proposition 6.3)**.** *The above gives a morphism of complexes*

$$\xi^{\mathrm{sm}}_B : \Omega^\bullet_{B|\mathbb{R}} \to \mathrm{S}^\bullet_{\mathrm{sm}}(\mathrm{spec}_{\mathbb{R}} B).$$

Lemma 7.2 allows us to prove the naturality of ξ^{sm} in algebras:

Lemma 7.4 (cf. Lemma 6.4)**.** *Suppose $\varphi : B' \to B$ is a homomorphism of finitely generated $\mathbb{R}$-algebras. Then the following diagram is commutative:*

$$\begin{tikzcd} \Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\xi^{\mathrm{sm}}_B"] & \mathrm{S}^\bullet_{\mathrm{sm}}(\mathrm{spec}_{\mathbb{R}} B) \\ \Omega^\bullet_{B'|\mathbb{R}} \arrow[u, "\Omega_\varphi"] \arrow[r, "\xi^{\mathrm{sm}}_{B'}"] & \mathrm{S}^\bullet_{\mathrm{sm}}(\mathrm{spec}_{\mathbb{R}} B'). \arrow[u, "\mathrm{S}_{\mathrm{sm}}(\mathrm{spec}_{\mathbb{R}}\varphi)"'] \end{tikzcd}$$

7.2. Comparing ξ_B and ξ_B^{sm}. Take B a finitely generated $\mathbb{R}$-algebra.

Lemma 7.5. *Take $\sigma : \Delta^n \to \operatorname{spec}_{\mathbb{R}} B$ a smooth simplex. Then the following diagram is commutative:*

$$\begin{tikzcd} \Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\mu^{\mathrm{sm}}(\sigma)"] \arrow[dr, "\mu(\sigma)"'] & \Omega^\bullet(\Delta^n) \arrow[d, "\nabla"] \\ & \Omega^\bullet_\flat(\Delta^n). \end{tikzcd}$$

Proof. Consider the following diagram:

$$\begin{tikzcd} \Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\mu^{\mathrm{sm}}(\sigma)"'] \arrow[rr, bend left, "\mu(\sigma)"] & \Omega^\bullet(\Delta^n) \arrow[r, "\nabla"'] & \Omega^\bullet_\flat(\Delta^n) \\ B \arrow[u] \arrow[r, "\theta^{\mathrm{sm}}(\sigma)"] \arrow[rr, bend right, "\theta(\sigma)"'] & C^\infty(\Delta^n) \arrow[u] \arrow[r, hook] & \mathrm{Lip}(\Delta^n). \arrow[u, "\zeta"'] \end{tikzcd}$$

The left square and the outer contour commute by the definitions of μ and μ^{sm}. The right square commutes by Paragraph 5.2(5). The bottom triangle commutes by the definitions of θ and θ^{sm}. Hence, by the universal property of $\Omega^\bullet_{B|\mathbb{R}}$ the upper triangle also commutes. □

Lemma 7.6. *The following diagram is commutative:*

$$\begin{tikzcd} \Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\xi_B^{\mathrm{sm}}"] \arrow[dr, "\xi_B"'] & S^\bullet_{\mathrm{sm}}(\operatorname{spec}_{\mathbb{R}} B) \\ & S^\bullet_{\mathrm{Lip}}(\operatorname{spec}_{\mathbb{R}} B). \arrow[u, "\mathrm{res}"'] \end{tikzcd} \tag{3}$$

Here res *is the restriction of Lipschitz cochains to smooth chains.*

Proof. Consider the diagram:

$$\begin{tikzcd} \Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\mu^{\mathrm{sm}}(\sigma)"] \arrow[dr, "\mu(\sigma)"'] & \Omega^\bullet(\Delta^n) \arrow[r, "\mathfrak{T}_n"] \arrow[d, "\nabla"'] & \mathbb{R} \\ & \Omega^\bullet_\flat(\Delta^n). \arrow[ur, "\mathfrak{I}_n"'] & \end{tikzcd} \tag{4}$$

The left triangle is commutative by Lemma 7.5 The right triangle is commutative by Paragraph 5.2(4). Hence, the statement follows by the definitions of ξ and ξ^{sm}. □

7.3. Morphisms $\varkappa$ and $\underline{\varkappa}$. Let M be a smooth manifold. Take an open set $U \subset M$, a smooth simplex $\sigma : \Delta^n \to U$ and a smooth n-form $\omega \in \Omega^n(U)$. We define $\langle \varkappa(\omega), \sigma\rangle := \mathfrak{T}_n(\sigma^*(\omega))$. By Stokes' formula we obtain the morphism of complexes

$$\varkappa : \Omega^\bullet(U) \to \mathrm{S}^\bullet_{\mathrm{sm}}(U).$$

This way we obtain a morphism of complexes of presheaves $\varkappa : \Omega^\bullet_M \to \mathrm{S}^\bullet_{\mathrm{sm},M}$. Taking the composition with the sheafification map we get the morphism of complexes of sheaves

$$\underline{\varkappa} : \Omega^\bullet_M \to {}^+\mathrm{S}^\bullet_{\mathrm{sm},M}.$$

The following diagram is commutative:

$$\begin{tikzcd} \underline{\mathbb{R}}_M[0] \arrow[r, "\epsilon"] \arrow[dr, "\epsilon"'] & \Omega^\bullet_M \arrow[d, "\underline{\varkappa}"] \\ & {}^+\mathrm{S}^\bullet_{\mathrm{sm},M}. \end{tikzcd} \tag{5}$$

The following diagram is also commutative:

$$\begin{tikzcd} \mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{sm}}(M)) \arrow[r, "\mathrm{H(sh)}"] & \mathrm{H}^*({}^+\mathrm{S}^\bullet_{\mathrm{sm}}(M)) \arrow[r, "\Upsilon"] & \mathbb{H}^*(M, {}^+\mathrm{S}^\bullet_{\mathrm{sm},M}) \\ & \mathrm{H}^*(\Omega^\bullet(M)) \arrow[ul, "\mathrm{H}(\varkappa)"] \arrow[u, "\mathrm{H}(\underline{\varkappa}(M))"'] \arrow[r, "\Upsilon"] & \mathbb{H}^*(M, \Omega^\bullet_M) \arrow[u, "\mathbb{H}(\underline{\varkappa})"'] \end{tikzcd} \tag{6}$$

7.4. Identifying the map Ψ_B. In this subsection we take M a smooth manifold.

Lemma 7.7. *For the inclusion of a finitely generated subalgebra $i : B \hookrightarrow C^\infty(M)$ and a smooth singular simplex $\sigma : \Delta^n \to M$ the following diagram commutes:*

$$\begin{tikzcd} \Omega^\bullet_{B|\mathbb{R}} \arrow[d, "\Omega_i"'] \arrow[drr, "\mu^{\mathrm{sm}}(\Gamma_B\circ\sigma)"] & & \\ \Omega^\bullet_{C^\infty(M)|\mathbb{R}} \arrow[r, "\pi"'] & \Omega^\bullet(M) \arrow[r, "\sigma^*"'] & \Omega^\bullet(\Delta^n). \end{tikzcd}$$

Proof. Consider the diagram:

$$\begin{tikzcd} \Omega^\bullet_{B|\mathbb{R}} \arrow[r, "\Omega_i"'] \arrow[rrr, bend left, "\mu^{\mathrm{sm}}(\Gamma_B\circ\sigma)"] & \Omega^\bullet_{C^\infty(M)|\mathbb{R}} \arrow[r, "\pi"'] & \Omega^\bullet(M) \arrow[r, "\sigma^*"'] & \Omega^\bullet(\Delta^n) \\ B \arrow[u] \arrow[r, hook, "i"] \arrow[rrr, bend right, "\theta^{\mathrm{sm}}(\Gamma_B\circ\sigma)"'] & C^\infty(M) \arrow[u] \arrow[r, equal] & C^\infty(M) \arrow[u] \arrow[r, "\sigma^*"] & C^\infty(\Delta^n). \arrow[u] \end{tikzcd}$$

The left and right squares clearly commute. The middle square commutes by the definition of π. The outer contour commutes by the definition of μ^{sm}. For $b \in B$ we have

$$\theta^{\mathrm{sm}}(\Gamma_B \circ \sigma)(b) = \hat{b} \circ \Gamma_B \circ \sigma = b \circ \sigma = \sigma^*(b)$$

by the definition of θ^{sm} and Equation 1. Hence, the bottom quadrangle commutes. By the universal property of $\Omega^\bullet_{B|\mathbb{R}}$ the upper quadrangle commutes. $\square$

Lemma 7.8. *For the inclusion of a finitely generated subalgebra $i : B \hookrightarrow C^\infty(M)$ the following diagram is commutative:*

$$\begin{tikzcd}
\Omega^\bullet_{B|\mathbb{R}} \arrow[rr, "\xi^{\mathrm{sm}}_B"] \arrow[d, "\Omega_i"'] & & \mathrm{S}^\bullet_{\mathrm{sm}}(\operatorname{spec}_{\mathbb{R}} B) \arrow[d, "\mathrm{S}_{\mathrm{sm}}(\Gamma_B)"] \\
\Omega^\bullet_{C^\infty(M)|\mathbb{R}} \arrow[r, "\pi"] & \Omega^\bullet(M) \arrow[r, "\varkappa"] & \mathrm{S}^\bullet_{\mathrm{sm}}(M)
\end{tikzcd}$$

Proof. For a form $\omega \in \Omega^\bullet_{B|\mathbb{R}}$ we have

$$\begin{aligned}
\left\langle \mathrm{S}_{\mathrm{sm}}(\Gamma_B)(\xi^{\mathrm{sm}}_B(\omega)), \sigma \right\rangle &= \left\langle \xi^{\mathrm{sm}}_B(\omega), \Gamma_B \circ \sigma \right\rangle = \mathfrak{T}_n(\mu^{\mathrm{sm}}(\Gamma_B \circ \sigma)(\omega)) \overset{(3)}{=} \\
&\overset{(3)}{=} \mathfrak{T}_n(\sigma^*(\pi(\Omega_i(\omega)))) = \langle \varkappa(\pi(\Omega_i(\omega))), \sigma \rangle .
\end{aligned}$$

The equality (3) follows from Lemma 7.7. $\square$

Lemma 7.9. *For the inclusion of a finitely generated subalgebra $i : B \hookrightarrow C^\infty(M)$ the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{H}^*(\Omega^\bullet_{B|\mathbb{R}}) \arrow[rr, "\Phi_B"] \arrow[d, "\mathrm{H}(\Omega_i)"'] & & \mathbb{H}^*(\operatorname{spec}_{\mathbb{R}} B, \underline{\mathbb{R}}_{\operatorname{spec}_{\mathbb{R}} B}[0]) \arrow[d, "\Gamma_B^*"] \\
\mathrm{H}^*(\Omega^\bullet_{C^\infty(M)|\mathbb{R}}) \arrow[r, "\mathrm{H}(\pi)"] & \mathrm{H}^*(\Omega^\bullet(M)) & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]). \arrow[l, "\Theta"]
\end{tikzcd}$$

Proof. Consider the diagram:

$$\begin{tikzcd}
\mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{Lip}}(\operatorname{spec}_{\mathbb{R}} B)) \arrow[r, "\Upsilon \circ \mathrm{H}(\mathrm{sh})", "\cong"'] \arrow[d, "\mathrm{H}(\mathrm{res})"'] & \mathbb{H}^*(\operatorname{spec}_{\mathbb{R}} B, {}^+\mathrm{S}^\bullet_{\mathrm{Lip},\operatorname{spec}_{\mathbb{R}} B}) \arrow[d, "\mathbb{H}(\mathrm{res})"'] & \\
\mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{sm}}(\operatorname{spec}_{\mathbb{R}} B)) \arrow[r, "\Upsilon \circ \mathrm{H}(\mathrm{sh})", "\cong"'] \arrow[d, "\mathrm{H}(\mathrm{S}_{\mathrm{sm}}(\Gamma_B))"'] & \mathbb{H}^*(\operatorname{spec}_{\mathbb{R}} B, {}^+\mathrm{S}^\bullet_{\mathrm{sm},\operatorname{spec}_{\mathbb{R}} B}) \arrow[d, "\Gamma_B^*"'] & \mathbb{H}^*(\operatorname{spec}_{\mathbb{R}} B, \underline{\mathbb{R}}_{\operatorname{spec}_{\mathbb{R}} B}[0]) \arrow[l, "\mathbb{H}(\epsilon)"'] \arrow[ul, bend right, "\mathbb{H}(\epsilon)"', "\cong"] \arrow[d, "\Gamma_B^*"] \\
\mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{sm}}(M)) \arrow[r, "\Upsilon \circ \mathrm{H}(\mathrm{sh})", "\cong"'] & \mathbb{H}^*(M, {}^+\mathrm{S}^\bullet_{\mathrm{sm},M}) & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[l, "\mathbb{H}(\epsilon)", "\cong"'] \arrow[d, "\mathbb{H}(\epsilon)", "\cong"'] \\
\mathrm{H}^*(\Omega^\bullet(M)) \arrow[u, "\mathrm{H}(\varkappa)"] \arrow[rr, "\Upsilon"] & & \mathbb{H}^*(M, \Omega^\bullet_M) \arrow[ul, "\mathbb{H}(\underline{\varkappa})"]
\end{tikzcd} \tag{7}$$

The diagram without the bottom row commutes for obvious reasons. The bottom left quadrangle commutes by Diagram 6. The bottom right triangle commutes by Diagram 5. We do not know if the morphism $\mathbb{H}(\epsilon)$ in the middle row is an isomorphism.

Next, we consider the following diagram:

$$
\begin{tikzcd}
\mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{sm}}(M)) \arrow[r, "\mathbb{H}(\epsilon)^{-1}\circ \mathrm{Y}\circ \mathrm{H}(\mathrm{sh})"] & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[d, "\mathbb{H}(\epsilon)", "\cong"'] \\
\mathrm{H}^*(\Omega^\bullet(M)) \arrow[u, "\mathrm{H}(\varkappa)"] \arrow[ur, "\Theta^{-1}"] \arrow[r, "\mathrm{Y}"'] & \mathbb{H}^*(M, \Omega^\bullet_M).
\end{tikzcd}
\tag{8}
$$

By the bottom stage of the previous diagram the outer contour commutes. The bottom triangle commutes by the definition of Θ, Subsection 4.1. The vertical map $\mathbb{H}(\epsilon)$ is an isomorphism. Therefore, the upper triangle is commutative.

Consider the following diagram:

$$
\begin{tikzcd}
\mathrm{H}^*(\Omega^\bullet_{B|\mathbb{R}}) \arrow[r, "\mathrm{H}(\xi_B)"] \arrow[rr, bend left, "\Phi_B"] \arrow[dr, "\mathrm{H}(\xi^{\mathrm{sm}}_B)"'] \arrow[dd, "\mathrm{H}(\Omega_i)"] & \mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{Lip}}(\mathrm{spec}_{\mathbb{R}} B)) \arrow[r, "\mathbb{H}(\epsilon)^{-1}\circ \mathrm{Y}\circ \mathrm{H}(\mathrm{sh})"'] \arrow[d, "\mathrm{H}(\mathrm{res})"] & \mathbb{H}^*(\mathrm{spec}_{\mathbb{R}} B, \underline{\mathbb{R}}_{\mathrm{spec}_{\mathbb{R}} B}[0]) \arrow[dd, "\Gamma_B^*"] \\
 & \mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{sm}}(\mathrm{spec}_{\mathbb{R}} B)) \arrow[d, "\mathrm{H}(\mathrm{S}_{\mathrm{sm}}(\Gamma_B))"] & \\
\mathrm{H}^*(\Omega^\bullet_{C^\infty(M)|\mathbb{R}}) \arrow[dr, "\mathrm{H}(\pi)"'] & \mathrm{H}^*(\mathrm{S}^\bullet_{\mathrm{sm}}(M)) \arrow[r, "\mathbb{H}(\epsilon)^{-1}\circ \mathrm{Y}\circ \mathrm{H}(\mathrm{sh})"'] & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \\
 & \mathrm{H}^*(\Omega^\bullet(M)) \arrow[u, "\mathrm{H}(\varkappa)"] \arrow[ur, "\Theta^{-1}"'] & .
\end{tikzcd}
$$

The upper triangle is commutative by the definition of Φ_B. The left pentagon is commutative by Lemma 7.8. The left triangle is commutative by Diagram 3. The right pentagon is commutative by Diagram 7. The bottom right triangle commutes by Diagram 8 and Θ is an isomorphism. Thus, the whole diagram is commutative. Therefore, the claim follows. □

7.5. The calculation of $\boldsymbol{\Psi_{C^\infty(M)}}$. In the previous section we have constructed a morphism $\Psi_A : \mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \to \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0])$ for every topological space X and subalgebra $A \subset C(X)$. In case $X = M$ a smooth manifold and $A = C^\infty(M)$ we would like to calculate this map explicitly.

By passing in Lemma 7.9 to colimit over all finitely generated subalgebras $B \subset C^\infty(M)$ we get

Theorem 7.10. *The following diagram is commutative:*

$$
\begin{tikzcd}
\mathrm{H}^*(\Omega^\bullet_{C^\infty(M)|\mathbb{R}}) \arrow[r, "\mathrm{H}(\pi)"'] \arrow[rr, bend left, "\Psi_{C^\infty(M)}"] & \mathrm{H}^*(\Omega^\bullet(M)) & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]). \arrow[l, "\Theta"]
\end{tikzcd}
$$

8. The composition $\mathbb{H}^*(\boldsymbol{X}, \underline{\mathbb{R}}_{\boldsymbol{X}}[\mathbf{0}]) \xrightarrow{\boldsymbol{\Lambda}_{\mathcal{F}}} \mathbf{H}^*(\boldsymbol{\Omega}^{\bullet}_{\mathcal{F}(\boldsymbol{X})|\mathbb{R}}) \xrightarrow{\boldsymbol{\Psi}_{\mathcal{F}(\boldsymbol{X})}} \mathbb{H}^*(\boldsymbol{X}, \underline{\mathbb{R}}_{\boldsymbol{X}}[\mathbf{0}])$

Lemma 8.1. *Take M a compact smooth manifold. Then the following diagram is commutative:*

$$\begin{tikzcd} & \mathrm{H}^*(\Omega^{\bullet}_{C^\infty(M)|\mathbb{R}}) \arrow[dr, "\Psi_{C^\infty(M)}"] & \\ \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[ur, "\Lambda_{C^\infty_M}"] \arrow[rr, "\mathrm{id}"] & & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]). \end{tikzcd}$$

Proof. Consider the following diagram:

$$\begin{tikzcd} \mathrm{H}^*(\Omega^{\bullet}_{C^\infty(M)|\mathbb{R}}) \arrow[r, "\mathrm{H}(\pi)"] \arrow[dr, "\Psi_{C^\infty(M)}"] & \mathrm{H}^*(\Omega^{\bullet}(M)) \\ \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[u, "\Lambda_{C^\infty_M}"] \arrow[r, "\mathrm{id}"] & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]). \arrow[u, "\Theta"'] \end{tikzcd}$$

The right triangle commutes by Theorem 7.10. The outer contour is commutative by Theorem 4.1 . Since Θ is an isomorphism, the left triangle also commutes. $\square$

Corollary 8.2. *For a compact smooth manifold M the following diagram is commutative:*

$$\begin{tikzcd} & \mathrm{H}^*(\Omega^{\bullet}_{C(M)|\mathbb{R}}) \arrow[dr, "\Psi_{C(M)}"] & \\ \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[ur, "\Lambda_{C_M}"] \arrow[rr, "\mathrm{id}"] & & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]). \end{tikzcd}$$

Proof. The inclusion morphism of sheaves $C^\infty_M \hookrightarrow C_M$ allows us to consider the diagram

$$\begin{tikzcd} \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[r, "\Lambda_{C_M}"] \arrow[rr, bend left, "\Lambda_{C^\infty_M}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{H}^*(\Omega^{\bullet}_{C(M)|\mathbb{R}}) \arrow[d, "\Psi_{C(M)}"] & \mathrm{H}^*(\Omega^{\bullet}_{C^\infty(M)|\mathbb{R}}) \arrow[l] \arrow[dl, "\Psi_{C^\infty(M)}"] \\ & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]). & \end{tikzcd}$$

The outer contour is commutative by Lemma 8.1. The right triangle is commutative by Proposition 6.6. The upper triangle is commutative by Proposition 3.4. Hence, the left triangle is commutative. $\square$

Corollary 8.3. *For a compact Hausdorff space X the following diagram is commutative:*

$$\begin{tikzcd} & H^*(\Omega^\bullet_{C(X)|\mathbb{R}}) \arrow[dr, "\Psi_{C(X)}"] & \\ \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]) \arrow[ur, "\Lambda_{C_X}"] \arrow[rr, "\mathrm{id}"] & & \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]). \end{tikzcd}$$

Proof. Choose a cohomology class $\lambda \in \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0])$. We show that

$$(\Psi_{C(X)} \circ \Lambda_{C_X})(\lambda) = \lambda.$$

First, by [God58, II.5.10] there is a polyhedron N (the geometric realization of some nerve) and a continuous map $f : X \to N$, such that $\lambda = f^*(\delta)$ for some $\delta \in \mathbb{H}^*(N, \underline{\mathbb{R}}_N[0])$. Second, there exists a compact smooth manifold (with boundary) M such that $N \subset M$ and N is a deformation retract of M (see [Hir62, Theorem 1]). There exists $\gamma \in \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0])$ such that $\gamma|_N = \delta$. Consider the composition $g : X \xrightarrow{f} N \hookrightarrow M$. We have $\lambda = g^*(\gamma)$. Consider the diagram

$$\begin{tikzcd} \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]) \arrow[r, "\Lambda_{C_X}"] & H^*(\Omega^\bullet_{C(X)|\mathbb{R}}) \arrow[r, "\Psi_{C(X)}"] & \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]) \\ \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]) \arrow[r, "\Lambda_{C_M}"] \arrow[u, "g^*"] & H^*(\Omega^\bullet_{C(M)|\mathbb{R}}) \arrow[r, "\Psi_{C(M)}"] \arrow[u, "H(\Omega_{g^*})"] & \mathbb{H}^*(M, \underline{\mathbb{R}}_M[0]). \arrow[u, "g^*"] \end{tikzcd}$$

This diagram is commutative by Propositions 3.4 and 6.6. By Corollary 8.2 the equality $(\Psi_{C(X)} \circ \Lambda_{C_X})(\lambda) = \lambda$ follows. □

Theorem 8.4. *For a compact Hausdorff space X and a soft subsheaf of algebras $\mathcal{F} \hookrightarrow C_X$ the following diagram is commutative*

$$\begin{tikzcd} & H^*(\Omega^\bullet_{\mathcal{F}(X)|\mathbb{R}}) \arrow[dr, "\Psi_{\mathcal{F}(X)}"] & \\ \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]) \arrow[ur, "\Lambda_{\mathcal{F}}"] \arrow[rr, "\mathrm{id}"] & & \mathbb{H}^*(X, \underline{\mathbb{R}}_X[0]). \end{tikzcd}$$

Proof. It immediately follows from naturality of Λ and Ψ (Propositions 3.4 and 6.6) and Corollary 8.3. □

9. Piecewise polynomial functions

9.1. Polyhedra and rectilinear maps. A *polyhedron* K is a finite set of affine simplices in $\mathbb{R}^m$ such that

(1) for any $a \in K$ and any face $b \subset a$ we have $b \in K$;
(2) if $a, b \in K$ then $a \cap b$ is either a common face of a and b or empty.

A subset P of K that is also a polyhedron is called a *subpolyhedron* of K. Define the space $|K| \subset \mathbb{R}^m$ as the union of all simplices of K.

We say that a function $f : |K| \to |K'|$ is a *rectilinear map* if for any $a \in K$ there exists $b \in K'$ such that $f(a) \subset b$ and f maps a to b affinely. We call two rectilinear maps $f, g : |K| \to |K'|$ *adjacent* if for every simplex $a \in K$ the set $f(a) \cup g(a)$ is contained in a simplex of K'.

We say that a polyhedron P is a *minor* of a polyhedron K if $|P| \subset |K|$ and $|P| \hookrightarrow |K|$ is a rectilinear map (in other words, for any $a \in P$ there is $b \in K$ such that $a \subset b$). We call a minor P of K a *subdivision* if $|P| = |K|$.

We call a polyhedron S a *star* with the center $x \in \mathbb{R}^m$ if $\{x\}$ is a vertex of S and each maximal by inclusion simplex of S has $\{x\}$ as a vertex. Take a polyhedron K with $x \in |K|$, then a minor S of K is called a *star neighborhood* of x if S is a star with center x and $x \in \operatorname{int}_{|K|} |S|$.

For a polyhedron K consider the algebra $\operatorname{Pol}(K)$ of functions $\varphi : |K| \to \mathbb{R}$ such that for each simplex $a \in K$ $\varphi|_a$ is a polynomial. This algebra was considered, for example, in [Bil89]. The algebra $\operatorname{Pol}(K)$ is contravariant with respect to rectilinear maps, in particular, if $i : |P| \hookrightarrow |K|$ is an inclusion of a minor we have the restriction homomorphism $\operatorname{Pol}(i) : \operatorname{Pol}(K) \to \operatorname{Pol}(P)$.

Lemma 9.1. *Suppose $K, P_1, \dots, P_l$ are polyhedra such that P_i are minors of K. Then there exist subdivisions K' of K and P_i' of P_i such that P_i' is a subpolyhedron of K' for each i.*

Proof. See [RS72, Addendum 2.12]. □

Lemma 9.2. *Take K a polyhedron, $U \subset |K|$ a set open in $|K|$ and $D \subset U$ a set closed in $|K|$. Then there exists a minor P of K such that $|P| \subset U$ and $D \subset \operatorname{int}_{|K|} |P|$.*

Proof. For each $x \in D$ choose a cube Q_x with center x such that $|Q_x| \subset U$. By compactness of D choose a finite set $(Q_{x_i})_{i=1}^l$ of cubes such that

$$S \subset \bigcup_{i=1}^{l} \operatorname{int}_{|K|} |Q_{x_i}|.$$

By Lemma 9.1 there exist subdivisions K' of K and Q'_{x_i} of Q_{x_i} such that Q'_{x_i} is a subpolyhedron of K' for each i. Therefore, the union of all Q'_{x_i} forms a subpolyhedron of K' and hence a minor of K. □

Definition. Let us define the sheaf of piecewise polynomial functions on a polyhedron.

(1) For a polyhedron K and a subset $U \subset |K|$ open in $|K|$ we call a function $s : U \to \mathbb{R}$ *piecewise polynomial* if for each point $x \in U$ there exists a minor K_x of K, such that $|K_x| \subset U$ with $x \in \operatorname{int}_{|K|} |K_x|$ and $s|_{|K_x|} \in \operatorname{Pol}(K_x)$.
(2) The set $\operatorname{PPol}(U)$ of piecewise polynomial functions on U forms an algebra (use Lemma 9.1).

(3) Take two sets $V \subset U$ open in $|K|$. It is not hard to see that the restriction of a piecewise polynomial function $s \in \mathrm{PPol}(U)$ to V is piecewise polynomial. Hence, the correspondence $U \mapsto \mathrm{PPol}(U)$ defines a sheaf on $|K|$ which we denote by PPol_K.

Proposition 9.3. *For a polyhedron K the sheaf PPol_K is soft.*

Proof. Take a subset $D \subset |K|$ closed in $|K|$ and an element of $\mathrm{PPol}_K(D)$, which is represented by a set $U \subset |K|$ open in $|K|$ with $D \subset U$ and a section $s \in \mathrm{PPol}_K(U)$. By Lemma 9.2 there exists a minor P of K such that $D \subset \mathrm{int}_{|K|}|P|$ and $|P| \subset U$. Apply Lemma 9.2 again to obtain a minor P_2 of K such that $|P| \subset \mathrm{int}_{|K|}|P_2|$ and $|P_2| \subset U$. By Lemma 9.1 there exist subdivisions K' of K, P_2' of P_2 and P' of P such that P' and P_2' are subpolyhedra of K'. Take an element $t \in \mathrm{Pol}(K')$ such that $t|_{|P'|} \equiv 1$ and $t|_{|K'|-|P_2'|} \equiv 0$. The function ts is a global section of PPol_K and ts and s coincide in $\mathrm{PPol}_K(D)$. □

9.2. The maps $\Lambda_{\mathrm{PPol}_K}$ and $\Psi_{\mathrm{PPol}_K(|K|)|\mathbb{R}}$. Here we prove that for the sheaf of piecewise polynomial functions on K the maps $\Lambda_{\mathrm{PPol}_K}$ and $\Psi_{\mathrm{PPol}_K(|K|)|\mathbb{R}}$ are in fact isomorphisms.

The following notion can be found in [Ger71]. Two morphisms of $\mathbb{R}$-algebras $\varphi_0, \varphi_1 : A \to B$ are called *simply homotopic* if there exists a homomorphism $H : A \to B \otimes \mathbb{R}[t]$ such that the following diagram is commutative for $\lambda = 0, 1$:

$$\begin{tikzcd} A \arrow[r, "H"'] \arrow[rr, bend left, "\varphi_\lambda"] & B \otimes \mathbb{R}[t] \arrow[r, "t \mapsto \lambda"'] & B. \end{tikzcd} \tag{9}$$

The following is a well known definition of homotopic morphisms of dg-algebras and can be found in [Leh90, II.1].

Definition. Two morphisms of dg-algebras $\varphi_0, \varphi_1 : E \to E'$ are called *homotopic* if there exists a morphism $H : E \to E' \otimes \Omega^\bullet_{\mathbb{R}[t]|\mathbb{R}}$ such that the following diagram is commutative for $\lambda = 0, 1$:

$$\begin{tikzcd} E \arrow[r, "H"'] \arrow[rr, bend left, "\varphi_\lambda"] & E' \otimes \Omega^\bullet_{\mathbb{R}[t]|\mathbb{R}} \arrow[r, "t \mapsto \lambda"'] & E'. \end{tikzcd}$$

Here $t \mapsto \lambda$ is the dg-algebra morphism that is the identity on E' and sends t to λ.

Lemma 9.4. *Homotopic morphisms of dg-algebras induce equal maps on the cohomology groups.*

Proof. See [Leh90, Lemma II.1]. □

Proposition 9.5. *Suppose the morphisms $\varphi_0, \varphi_1 : A \to B$ of algebras are simply homotopic. Then the induced morphisms $\mathrm{H}(\Omega_{\varphi_\lambda}) : \mathrm{H}^*(\Omega^\bullet_{A|\mathbb{R}}) \to \mathrm{H}^*(\Omega^\bullet_{B|\mathbb{R}})$ for $\lambda = 0, 1$ are equal.*

Proof. We prove that the morphisms of dg-algebras $\Omega_{\varphi_\lambda} : \Omega^\bullet_{A|\mathbb{R}} \to \Omega^\bullet_{B|\mathbb{R}}$ are homotopic for $\lambda = 0, 1$. By Lemma 9.4 it will imply that the maps on the cohomology are equal. As φ_λ are simply homotopic for $\lambda = 0, 1$, we have a homomorphism $H : A \to B \otimes \mathbb{R}[t]$ such that Diagram 9 commutes. The obvious morphisms $\Omega^\bullet_{B|\mathbb{R}} \to \Omega^\bullet_{B\otimes\mathbb{R}[t]|\mathbb{R}}$ and $\Omega^\bullet_{\mathbb{R}[t]|\mathbb{R}} \to \Omega^\bullet_{B\otimes\mathbb{R}[t]|\mathbb{R}}$ form the canonical morphism $u : \Omega^\bullet_{B|\mathbb{R}} \otimes \Omega^\bullet_{\mathbb{R}[t]|\mathbb{R}} \to \Omega^\bullet_{B\otimes\mathbb{R}[t]|\mathbb{R}}$, which is an isomorphism by [Kun86, Corollary 4.2].

Consider the following commutative diagram:

$$\begin{tikzcd}
 & \Omega^\bullet_{A|\mathbb{R}} \arrow[dl, "u^{-1}\circ\Omega_H"'] \arrow[d, "\Omega_H"] \arrow[dr, "\Omega_{\varphi_\lambda}"] & \\
\Omega^\bullet_{B|\mathbb{R}} \otimes \Omega^\bullet_{\mathbb{R}[t]|\mathbb{R}} \arrow[r, "u", "\cong"'] \arrow[rr, bend right, "t\mapsto\lambda"'] & \Omega^\bullet_{B\otimes\mathbb{R}[t]|\mathbb{R}} \arrow[r, "t\mapsto\lambda"] & \Omega^\bullet_{B|\mathbb{R}}.
\end{tikzcd}$$

We obtain that Ω_{φ_λ} are homotopic for $\lambda = 0, 1$. □

Lemma 9.6. *Suppose $f_0, f_1 : |K| \to |K'|$ are two adjacent rectilinear maps of polyhedra. Then the induced homomorphisms $f_0^*, f_1^* : \mathrm{Pol}(K') \to \mathrm{Pol}(K)$ are simply homotopic.*

Proof. Consider the algebra T of functions s on $|K| \times [0, 1]$ such that for each $a \in K$ the restriction $s|_{a\times[0,1]}$ is a polynomial function. The map $\mathrm{Pol}(K) \otimes \mathbb{R}[t] \to T$ sending $\beta \otimes p(t)$ to the function $(x, t) \mapsto \beta(x)p(t)$ is an isomorphism. Construct the homomorphism $H : \mathrm{Pol}(K') \to T$ as $H(\alpha)(x, t) := \alpha(f_0(x)(1 - t) + f_1(x)t)$. As f_0 and f_1 are adjacent, $H(\alpha) \in T$. Also, $H(\alpha)|_{t=\lambda} = \alpha(f_\lambda(x)) = f_\lambda^*(\alpha)$ for $\lambda = 0, 1$. The needed homotopy map is the lift of the homomorphism H to $\mathrm{Pol}(K) \otimes \mathbb{R}[t]$. □

For a dg-algebra E we consider the morphism of complexes, the coaugmentation, $\epsilon : \mathbb{R}[0] \to E$ defined by $\epsilon(1) = 1$.

Corollary 9.7 (Poincaré lemma)**.** *Suppose S is a star with center x. Then the coaugmentation $\epsilon : \mathbb{R}[0] \to \Omega^\bullet_{\mathrm{Pol}(S)|\mathbb{R}}$ is a quasi-isomorphism.*

Proof. We denote by Q the one-point polyhedron $\{\{x\}\}$. Consider the rectilinear maps $\mathrm{col} : |S| \to |Q|$ and $i : |Q| \hookrightarrow |S|$. By the definition of a star the composition $i\circ\mathrm{col}$ is adjacent to the identity $\mathrm{id} : |S| \to |S|$. Consider the following diagram:

$$\begin{tikzcd}
 & \mathrm{H}^*(\mathbb{R}[0]) \arrow[dl, "\mathrm{H}(\epsilon)"'] \arrow[d, "\mathrm{H}(\epsilon)"', "\cong"] \arrow[dr, "\mathrm{H}(\epsilon)"] & \\
\mathrm{H}^*(\Omega^\bullet_{\mathrm{Pol}(S)|\mathbb{R}}) \arrow[r, "\mathrm{H}(\Omega_{\mathrm{Pol}(i)})"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{H}^*(\Omega^\bullet_{\mathrm{Pol}(Q)|\mathbb{R}}) \arrow[r, "\mathrm{H}(\Omega_{\mathrm{Pol}(\mathrm{col})})"] & \mathrm{H}^*(\Omega^\bullet_{\mathrm{Pol}(S)|\mathbb{R}})
\end{tikzcd}$$

By Proposition 9.5 and Lemma 9.6 the bottom triangle is commutative. The rest of the diagram commutes for obvious reasons. One can easily see that the morphism of complexes $\epsilon : \mathbb{R}[0] \to \Omega^\bullet_{\mathrm{Pol}(Q)|\mathbb{R}}$ is an isomorphism. Hence, $\mathrm{H}(\epsilon) : \mathrm{H}^*(\mathbb{R}[0]) \to \mathrm{H}^*(\Omega^\bullet_{\mathrm{Pol}(K)|\mathbb{R}})$ is an isomorphism. □

Lemma 9.8. *Take K a polyhedron and $x \in |K|$. Suppose S_0, S_1 are star neighborhoods of x in K, then there exists a star neighborhood S' of x in K which is a minor of S_i for $i = 0, 1$.*

Proof. Follows from Lemma 9.1. □

Consider a polyhedron K and a point $x \in |K|$. Let $\Sigma(x)$ be the set of all star neighborhoods S of x in K. Define a partial order on $\Sigma(x)$ as $S \le S'$ if S' is a minor of S. By Lemma 9.8 the set $\Sigma(x)$ is directed. Each star neighborhood S of x in K gives rise to a homomorphism $\mathrm{Pol}(S) \to (\mathrm{PPol}_K)_x$. By taking the colimit over the directed set $\Sigma(x)$ we obtain the homomorphism

$$\chi : \varinjlim_{S \in \Sigma(x)} \mathrm{Pol}(S) \to (\mathrm{PPol}_K)_x.$$

Lemma 9.9. *The map χ is an isomorphism.*

Proof. For injectivity, take a star neighborhood S of x in K and $t \in \mathrm{Pol}(S)$ such that $\chi(t) = 0$. There exists a set U open in $|K|$ with $x \in U$ such that $t|_U = 0$ as a function. Take a star neighborhood S' of x such that $|S'| \subset U$ and S' is a minor of S. We observe that $t|_{|S'|} = (t|_U)|_{|S'|} = 0$ and, hence, t is zero in $\varinjlim_{S \in \Sigma(x)} \mathrm{PPol}_K(S)$.

For surjectivity, take a germ $g \in (\mathrm{PPol}_K)_x$, a set $U \subset |K|$ open in $|K|$ and a section $\tilde{g} \in \mathrm{PPol}_K(U)$ such that $\tilde{g}_x = g$. As $\tilde{g} \in \mathrm{PPol}_K(U)$ there is a minor P of K such that $|P| \subset U$, $x \in \mathrm{int}_{|K|} |P|$ and $\tilde{g}|_{|P|} \in \mathrm{Pol}(P)$. Take a star neighborhood S of x in P. Then χ maps $\tilde{g}|_{|S|}$ to g. □

Proposition 9.10. *For a polyhedron K the coaugmentation*

$$\epsilon : \underline{\mathbb{R}}_{|K|}[0] \to {}^+\Omega^\bullet_{\mathrm{PPol}_K|\mathbb{R}}$$

is a quasi-isomorphism.

Proof. We prove that ϵ is a quasi-isomorphism on stalks. We have

$$(\underline{\mathbb{R}}_{|K|}[0])_x \to ({}^+\Omega^\bullet_{\mathrm{PPol}_K|\mathbb{R}})_x = (\Omega^\bullet_{\mathrm{PPol}_K|\mathbb{R}})_x = \Omega^\bullet_{(\mathrm{PPol}_K)_x|\mathbb{R}} \cong \varinjlim_{S \in \Sigma(x)} \Omega^\bullet_{\mathrm{Pol}(S)|\mathbb{R}}.$$

The last isomorphism follows from Lemma 9.9. By Corollary 9.7 the morphism $(\underline{\mathbb{R}}_{|K|}[0])_x \to \varinjlim_{S \in \Sigma(x)} \Omega^\bullet_{\mathrm{Pol}(S)|\mathbb{R}}$ is a quasi-isomorphism. □

Theorem 9.11. *The maps*

$$\Lambda_{\mathrm{PPol}_K} : \mathbb{H}^*(|K|, \underline{\mathbb{R}}_{|K|}[0]) \to \mathrm{H}^*(\Omega^\bullet_{\mathrm{PPol}_K(|K|)|\mathbb{R}})$$

and

$$\Psi_{\mathrm{PPol}_K(|K|)} : \mathrm{H}^*(\Omega^\bullet_{\mathrm{PPol}_K(|K|)|\mathbb{R}}) \to \mathbb{H}^*(|K|, \underline{\mathbb{R}}_{|K|}[0])$$

are isomorphisms.

Proof. By Theorem 8.4 it is enough to prove that $\Lambda_{\mathrm{PPol}_K}$ is an isomorphism. Recall that $\Lambda_{\mathrm{PPol}_K}$ is defined as the diagonal map in the following diagram

$$\begin{tikzcd}
\mathbb{H}^*(|K|, \underline{\mathbb{R}}_{|K|}[0]) \arrow[r, "\mathbb{H}(\epsilon)"] \arrow[ddr, dashed, "\Lambda_{\mathrm{PPol}_K}"'] & \mathbb{H}^*(|K|, {}^+\Omega^\bullet_{\mathrm{PPol}_K|\mathbb{R}}) \\
& \mathrm{H}^*({}^+\Omega^\bullet_{\mathrm{PPol}_K|\mathbb{R}}(|K|)) \arrow[u, "\Upsilon", "\cong"'] \\
& \mathrm{H}^*(\Omega^\bullet_{\mathrm{PPol}_K(|K|)|\mathbb{R}}). \arrow[u, "\mathrm{H}(\mathrm{sh})", "\cong"']
\end{tikzcd}$$

By Proposition 9.10 the map $\mathbb{H}(\epsilon)$ is an isomorphism, hence, $\Lambda_{\mathrm{PPol}_K}$ is an isomorphism. □

10. When are the maps Λ and Ψ isomorphisms?

10.1. $\mathrm{H}^0(\Omega^\bullet_{A|k})$ for a function algebra A.
Let k be a field.

Definition. We call the elements $a_1, \dots, a_n$ of a k-algebra A algebraically independent over k if there is no non-zero polynomial q over k such that $q(a_1, \dots, a_n) = 0$. We call an element a algebraic over k if there is a non-zero polynomial p over k such that $p(a) = 0$.

Lemma 10.1. *Let A be a k-algebra. Suppose the elements $a_1, \dots, a_n$ of A are algebraically independent over k. Then there exists a prime ideal $\mathfrak{p} \subset A$ such that the images of $a_1, \dots, a_n$ in $A/\mathfrak{p}$ are algebraically independent over k.*

Proof. See [Gom90, Proposition 6]. □

Lemma 10.2. *Suppose K/k is a field extension for k of characteristic 0. Consider the differential $d : K \to \Omega^1_{K|k}$. Then for $x \in K$ we have $dx = 0$ iff x is algebraic over k.*

Proof. See [Mat89, §26]. □

Lemma 10.3. *Consider a k-algebra A for k of characteristic 0 and take $a \in A$ such that $da = 0$ in $\Omega^1_{A|k}$. Then a is algebraic.*

This is [Gom90, Proposition 7] where the assumption on characteristic being zero is implicit.

Proof. Assume a is transcendental. By Lemma 10.1 there exists a prime $\mathfrak{p} \subset A$ such that the image $\bar{a}$ of a in $A/\mathfrak{p}$ is transcendental. Hence, $\bar{a}$ is transcendental in the field of fractions $\mathrm{Frac}(A/\mathfrak{p})$. By Lemma 10.2 $d\bar{a} \neq 0$ in $\Omega^1_{\mathrm{Frac}(A/\mathfrak{p})|k}$, which is a contradiction.

$\square$

We call a subalgebra A of $\mathrm{Maps}(X, k)$ a *function algebra* on a set X.

Theorem 10.4. *Take A a function algebra on X and $a \in A$. Then $da = 0$ in $\Omega^1_{A|k}$ if and only if a takes a finite number of values.*

This theorem is proved in [Gom90, proof of Theorem 8]. We give a proof, which is different in the "if" direction. A related result is [Osb69, Proposition 3].

Proof. Assume first $da = 0$. By Lemma 10.3 there exists a non-zero polynomial p such that $p(a) = 0$. Hence, a takes a finite number of values.

Conversely, assume a attains distinct values $r_1, \dots, r_m \in k$ and consider the polynomial $q(t) = (t - r_1) \dots (t - r_m)$. Clearly $q(a) = 0$ and $\gcd(q, q') = 1$. Hence, there exist polynomials $h_0, h_1 \in k[t]$ such that $h_0 q + h_1 q' = 1$. Substitute a into this equality and obtain $h_1(a)q'(a) = 1$; hence, $q'(a)$ is invertible. Take the equality $q(a) = 0$ and apply d to both sides, we get $q'(a)da = 0$ and subsequently $da = 0$. $\square$

Corollary 10.5. *Take A a function algebra on a space X. Then $\mathrm{H}^0(\Omega^\bullet_{A|k})$ is the vector space of functions $a \in A$ that take a finite number of values.*

A related result is [Osb69, Proposition 5].

Proposition 10.6. *For a soft subsheaf of algebras $\mathcal{F} \hookrightarrow C_X$ on a compact Hausdorff space X the map $\Lambda_{\mathcal{F}} : \mathbb{H}^0(X, \underline{\mathbb{R}}[0]) \to \mathrm{H}^0(\Omega^\bullet_{\mathcal{F}(X)|\mathbb{R}})$ is an isomorphism.*

Proof. By Corollary 10.5 the space $\mathrm{H}^0(\Omega^\bullet_{\mathcal{F}(X)|\mathbb{R}})$ consists of functions $f \in \mathcal{F}(X)$ that take a finite number of values. As $\mathcal{F}$ is a subsheaf of C_X and X is compact, the functions in $\mathcal{F}(X)$ that take a finite number of values are exactly the locally constant functions from $\mathcal{F}(X)$. The morphism of sheaves of algebras $\underline{\mathbb{R}}_X \to \mathcal{F}$ induces a homomorphism $\underline{\mathbb{R}}_X(X) \to \mathcal{F}(X) = \Omega^0_{\mathcal{F}(X)|\mathbb{R}}$. By above, this homomorphism can be extended to the morphism of complexes $\tilde{\epsilon} : \underline{\mathbb{R}}_X(X)[0] \to \Omega^\bullet_{\mathcal{F}(X)|\mathbb{R}}$ and $\mathrm{H}(\tilde{\epsilon})$ is an isomorphism in degree 0.

The coaugmentation $\epsilon : \underline{\mathbb{R}}_X[0] \to {}^+\Omega^\bullet_{\mathcal{F}|\mathbb{R}}$ induces a morphism of complexes $\epsilon(X) : \underline{\mathbb{R}}_X(X)[0] \to {}^+\Omega^\bullet_{\mathcal{F}|\mathbb{R}}(X)$. The following diagram commutes:

$$\begin{tikzcd}
\underline{\mathbb{R}}_X(X)[0] \arrow[d, "\epsilon(X)"] \arrow[dr, "\tilde{\epsilon}"] & \\
{}^+\Omega^\bullet_{\mathcal{F}|\mathbb{R}}(X) & \Omega^\bullet_{\mathcal{F}(X)|\mathbb{R}}. \arrow[l, "\mathrm{sh}"]
\end{tikzcd}$$

Consider the diagram:

$$\begin{tikzcd}
\mathbb{H}^0(X, \underline{\mathbb{R}}_X[0]) \arrow[d, "\mathbb{H}(\epsilon)"'] \arrow[rrr, bend left, "\Lambda_{\mathcal{F}}"] & \mathrm{H}^0(\underline{\mathbb{R}}_X(X)[0]) \arrow[l, "\cong", "\Upsilon"'] \arrow[d, "\mathrm{H}(\epsilon(X))"] \arrow[drr, "\mathrm{H}(\tilde{\epsilon})", "\cong"'] & & \\
\mathbb{H}^0(X, {}^{+}\Omega^{\bullet}_{\mathcal{F}|\mathbb{R}}) & \mathrm{H}^0({}^{+}\Omega^{\bullet}_{\mathcal{F}|\mathbb{R}}(X)) \arrow[l, "\cong", "\Upsilon"'] & & \mathrm{H}^0(\Omega^{\bullet}_{\mathcal{F}(X)|\mathbb{R}}). \arrow[ll, "\cong", "\mathrm{H}(\mathrm{sh})"']
\end{tikzcd}$$

The outer contour commutes by the definition of $\Lambda_{\mathcal{F}}$. The right triangle commutes by the above diagram. The left square obviously commutes. Hence, the whole diagram is commutative. The upper map Υ is an isomorphism by a property of Υ (see Subsection 2.1). Therefore, $\Lambda_{\mathcal{F}}$ is an isomorphism. □

It turns out that the group $\mathrm{H}^0(\Omega^{\bullet}_{C(X)|\mathbb{R}})$ behaves just as we expect.

Corollary 10.7. *For a soft subsheaf of algebras $\mathcal{F}$ of C_X on a compact Hausdorff space X the map*

$$\Psi_{\mathcal{F}(X)} : \mathrm{H}^0(\Omega^{\bullet}_{\mathcal{F}(X)|\mathbb{R}}) \to \mathbb{H}^0(X, \underline{\mathbb{R}}[0])$$

is an isomorphism.

Proof. It follows directly from Proposition 10.6 and Theorem 8.4. □

10.2. Λ and Ψ are not isomorphisms in general. Take k a field.

Lemma 10.8. *Suppose k is of characteristic 0. Suppose F/k is a field extension and L/F is a finite field extension. Then the map*

$$\mathrm{H}(\Omega_i) : \mathrm{H}^*(\Omega^{\bullet}_{F|k}) \to \mathrm{H}^*(\Omega^{\bullet}_{L|k})$$

induced by the inclusion $i : F \hookrightarrow L$ is injective.

Proof. There is a morphism of complexes $\Sigma : \Omega^{\bullet}_{L|k} \to \Omega^{\bullet}_{F|k}$ with the following properties (see [Kun86, §16]):

(1) if we consider $\Omega^{\bullet}_{L|k}$ as a $\Omega^{\bullet}_{F|k}$-module, then Σ is $\Omega^{\bullet}_{F|k}$-linear;
(2) the restriction of Σ to the elements of degree 0 coincides with the trace $\sigma : L \to F$.

Therefore, the composition

$$\Omega^{\bullet}_{F|k} \xrightarrow{\Omega_i} \Omega^{\bullet}_{L|k} \xrightarrow{\Sigma} \Omega^{\bullet}_{F|k}$$

is the multiplication by $\sigma(1) = [L : F]$. As k is of characteristic zero the map $\mathrm{H}(\Omega_i)$ is an injection. □

Lemma 10.9. *Let K/k be a field extension, where K is an infinite field, and $K(\Gamma)/K$, where $\Gamma = \{\gamma_1, \dots, \gamma_l\}$, be a purely transcendental extension. Then the map*

$$\mathrm{H}(\Omega_i) : \mathrm{H}^*(\Omega^{\bullet}_{K|k}) \to \mathrm{H}^*(\Omega^{\bullet}_{K(\Gamma)|k}),$$

induced by the inclusion $i : K \hookrightarrow K(\Gamma)$, is injective.

Proof. The field $K(\Gamma)$ is the colimit over finite sets $S \subset K[\Gamma] - \{0\}$ of the localizations of $K[\Gamma]$:

$$K(\Gamma) \cong \varinjlim_{S \subset K[\Gamma]-\{0\}} K[\Gamma][S^{-1}].$$

Therefore, it is enough to prove that the map

$$\mathrm{H}(\Omega_{i_S}) : \mathrm{H}^*(\Omega^\bullet_{K|k}) \to \mathrm{H}^*(\Omega^\bullet_{K[\Gamma][S^{-1}]|k})$$

induced by the inclusion $i_S : K \hookrightarrow K[\Gamma][S^{-1}]$ is injective for each S.

There exists a point $(\bar{\gamma}_1, \dots, \bar{\gamma}_l) \in K^l$ which is not a zero of any element of S. Consider the homomorphism $K[\Gamma] \to K, \gamma_j \mapsto \bar{\gamma}_j$. By the universal property of localization it can be extended to a homomorphism $\tau : K[\Gamma][S^{-1}] \to K$ such that $\tau|_K$ is the identity. Therefore, the composition

$$\mathrm{H}^*(\Omega^\bullet_{K|k}) \xrightarrow{\mathrm{H}(\Omega_{i_S})} \mathrm{H}^*(\Omega^\bullet_{K[\Gamma][S^{-1}]|k}) \xrightarrow{\mathrm{H}(\Omega_\tau)} \mathrm{H}^*(\Omega^\bullet_{K|k})$$

is the identity map, which suffices. □

Lemma 10.10. *Suppose the characteristic of k is zero. The equation*

$$\sum_{i=1}^{n} x_i(\partial/\partial x_i)F_i = c \tag{10}$$

where $c \in k$ and $c \neq 0$, has no solutions in rational functions $F_i \in k(x_1, \dots, x_n)$.

Proof. We proceed by induction on n. For $n = 0$ the claim is obvious. Take $l \in \mathbb{N} \cup \{0\}$ such that, for each i, x_1 is not a factor in the denominators of $x_1^l F_i$. Multiply the equation (10) by x_1^l and apply $(\partial/\partial x_1)^l$ to both sides of the equation. We get

$$\sum_{i=1}^{n} (\partial/\partial x_1)^l (x_1^l x_i(\partial/\partial x_i)F_i) = l!c.$$

Notice that the operators $x(\partial/\partial x)$ and $(\partial/\partial x)^l x^l$ commute. That can be easily checked by induction. Hence, the operators $x_i(\partial/\partial x_i)$ and $\left(\partial/\partial x_j\right)^l x_j^l$ commute for any i and j. We get

$$\sum_{i=1}^{n} x_i(\partial/\partial x_i)(\partial/\partial x_1)^l (x_1^l F_i) = l!c.$$

Put $G_i := (\partial/\partial x_1)^l (x_1^l F_i)$. We obtain

$$\sum_{i=1}^{n} x_i(\partial/\partial x_i)G_i = l!c.$$

But none of G_i has x_1 as a factor in the denominator so we can substitute $x_1 := 0$ in the above equation and obtain

$$\sum_{i=2}^{n} x_i(\partial/\partial x_i)(G_i|_{x_1=0}) = l!c.$$

By the induction hypothesis this equation has no solutions in rational functions. □

Lemma 10.11. *Suppose the characteristic of k is zero. The closed form*

$$\omega = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \in \Omega^n_{k(x_1,\dots,x_n)|k}$$

is not exact.

Proof. Suppose there exists $\eta \in \Omega^{n-1}_{k(x_1,\dots,x_n)|k}$ such that $d\eta = \omega$. Write η as

$$\eta = \sum_{i=1}^{n} F_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

where $F_i \in k(x_1, \dots, x_n)$. Then

$$d\eta = \left[\sum_{i=1}^{n}(-1)^{i+1}(\partial/\partial x_i)F_i\right] dx_1 \wedge \cdots \wedge dx_n.$$

The equality $d\eta = \omega$ then takes the form

$$\left[\sum_{i=1}^{n}(-1)^{i+1}(\partial/\partial x_i)F_i\right] dx_1 \wedge \cdots \wedge dx_n = \frac{1}{x_1 \dots x_n} dx_1 \wedge \cdots \wedge dx_n.$$

The vector $dx_1 \wedge \cdots \wedge dx_n$ of the vector space $\Omega^n_{k(x_1,\dots,x_n)|k}$ over the field $k(x_1, \dots, x_n)$ is not zero [Mat89, p. 201]. Hence,

$$\sum_{i=1}^{n}(-1)^{i+1}(\partial/\partial x_i)F_i = \frac{1}{x_1 \dots x_n}.$$

Put $G_i := (-1)^{i+1} x_1 \dots \hat{x}_i \dots x_n F_i$. Then the above equation takes the form

$$\sum_{i=1}^{n} x_i(\partial/\partial x_i)G_i = 1.$$

By Lemma 10.10 this equation has no solutions in rational functions. □

Theorem 10.12. *Suppose k is of characteristic* 0*. Take A a k-algebra and a set of invertible elements $a_1, \dots, a_n$ in A that are algebraically independent over k. Then the closed form*

$$\omega = \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \in \Omega^n_{A|k}$$

is not exact.

Proof. By Lemma 2.7 it is enough to consider A being finitely generated. By Lemma 10.1 there is a prime ideal $\mathfrak{p} \subset A$ such that the elements $a_1, \dots, a_n$ are still algebraically independent in $A/\mathfrak{p}$. Consider the field of fractions $\mathrm{Frac}(A/\mathfrak{p})$ which is finitely generated over k as a field. The elements $a_1, \dots, a_n$ are algebraically independent in $\mathrm{Frac}(A/\mathfrak{p})$.

Take a finite transcendental basis

$$\hat{\Gamma} = \{a_1, \dots, a_n, \gamma_1, \dots, \gamma_l\}$$

of $\mathrm{Frac}(A/\mathfrak{p})$ over k. The field $\mathrm{Frac}(A/\mathfrak{p})$ is a finite extension of $k(\hat{\Gamma})$. By Lemma 10.8 and Lemma 10.9 the composition

$$k(a_1, \dots, a_n) \to k(\hat{\Gamma}) \to \mathrm{Frac}(A/\mathfrak{p})$$

induces an injection on the de Rham cohomology.

The image of ω in $\Omega^n_{\mathrm{Frac}(A/\mathfrak{p})|k}$ coincides with the image of the closed form

$$\omega_0 = \frac{da_1}{a_1} \wedge \dots \wedge \frac{da_n}{a_n} \in \Omega^n_{k(a_1,\dots,a_n)|k},$$

which is not exact by Lemma 10.11. □

Theorem 10.13. *Let X be a topological space and $f \in C(X)$ take an infinite number of distinct values. Take a subalgebra $A \subset C(X)$ such that $e^{\lambda f} \in A$ for all $\lambda \in \mathbb{R}$. Then for each $n \geq 1$ $\mathrm{H}^n(\Omega^\bullet_{A|\mathbb{R}}) \neq 0$, moreover, the map*

$$\Psi_A : \mathrm{H}^n(\Omega^\bullet_{A|R}) \to \mathbb{H}^n(X, \underline{\mathbb{R}}_X[0])$$

is not injective.

Proof. Choose a set of linearly independent over $\mathbb{Q}$ numbers $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and consider the functions $a_i := e^{\lambda_i f(x)} \in A$. These functions are algebraically independent and invertible, hence, the closed form

$$\omega = \frac{da_1}{a_1} \wedge \dots \wedge \frac{da_n}{a_n} \in \Omega^n_{A|\mathbb{R}}$$

is not exact by Theorem 10.12.

For the second part, we prove that $\Psi_A([\omega]) = 0$. Denote by $i : A \hookrightarrow C(X)$ the inclusion. Consider the algebra $E \subset C(\mathbb{R})$ generated as an algebra by the functions $e^{\lambda t} \in C(\mathbb{R})$. We have the commutative diagram

$$\begin{tikzcd} C(X) & A \arrow[l, hook'] \\ C(\mathbb{R}) \arrow[u, "f^*"] & E, \arrow[l, hook'] \arrow[u, "\varphi"'] \end{tikzcd}$$

where the homomorphism φ is induced by the homomorphism f^*. Then the following diagram is commutative by Proposition 6.6:

$$\begin{array}{ccc} \mathrm{H}^n(\Omega^{\bullet}_{A|\mathbb{R}}) & \xrightarrow{\Psi_A} & \mathbb{H}^n(X, \underline{\mathbb{R}}_X[0]) \\ {\scriptstyle \mathrm{H}(\Omega_\varphi)}\uparrow & & \uparrow {\scriptstyle f^*} \\ \mathrm{H}^n(\Omega^{\bullet}_{E|\mathbb{R}}) & \xrightarrow{\Psi_E} & \mathbb{H}^n(\mathbb{R}, \underline{\mathbb{R}}_{\mathbb{R}}[0]){=}0. \end{array}$$

The equality $\mathbb{H}^n(\mathbb{R}, \underline{\mathbb{R}}_{\mathbb{R}}[0]) = 0$ is well known.

Consider the functions $\tilde{a}_i := e^{\lambda_i t} \in E$ and the closed form

$$\tilde{\omega} = \frac{d\tilde{a}_1}{\tilde{a}_1} \wedge \cdots \wedge \frac{d\tilde{a}_n}{\tilde{a}_n} \in \Omega^n_{E|\mathbb{R}}.$$

We have $\omega = \Omega_\varphi(\tilde{\omega})$. By the commutativity of the above diagram, $\Psi_A([\omega]) = 0$. Since $[\omega] \neq 0$, the map Ψ_A is not injective. □

Corollary 10.14. *Suppose X is a compact Hausdorff space and $\mathcal{F}$ is a soft subsheaf of C_X such that $\mathcal{F}(X)$ satisfies the conditions imposed on the algebra A in Theorem 10.13. Then $\Lambda_{\mathcal{F}} : \mathbb{H}^n(X, \underline{\mathbb{R}}_X[0]) \to \mathrm{H}^n(\Omega^{\bullet}_{\mathcal{F}|\mathbb{R}})$ is not surjective.*

Proof. By Theorem 8.4 the composition $\Psi_{\mathcal{F}(X)} \circ \Lambda_{\mathcal{F}} = \mathrm{id}$. By Theorem 10.13 $\Psi_{\mathcal{F}(X)}$ is not injective and, hence, $\Lambda_{\mathcal{F}}$ is not surjective. □

As the first example one can consider a smooth manifold M of positive dimension and $A = C^\infty(M)$. Then $\mathrm{H}^n(\Omega^{\bullet}_{C^\infty(M)|\mathbb{R}}) \neq 0$ for $n \geq 1$. Also, one can consider an infinite compact Hausdorff space X and $A = C(X)$. Then $\mathrm{H}^n(\Omega^{\bullet}_{C(X)|\mathbb{R}}) \neq 0$ for $n \geq 1$.

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