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Star clusters in the matching, Morse, and generalized complex of discrete Morse functions

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ABSTRACT. In this paper, we determine the homotopy type of the complex of discrete Morse functions and matching complex of multiple families of complexes by utilizing star cluster collapses and the Cluster Lemma. We compute the homotopy type of the complex of discrete Morse functions of an extended notion of a star graph, as well as the homotopy type of the matching complex of a Dutch windmill graph. Additionally, we provide alternate computations of the homotopy type of the complex of discrete Morse functions of paths, the homotopy type of the matching complex of paths, and the homotopy type of the matching complex of cycles. We then use this same method of computing homotopy types to investigate the relationship between the homotopy type of the matching complex and the generalized Morse complex.

CONTENTS

1. Introduction

Let K be a simplicial complex. In [1], J. Barmak introduced the star cluster of a simplex in K and proved that if K is flag, then the star cluster is contractible. This provided a tool for studying the topology of independence complexes of

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graphs. For example, he used star clusters to show that the independence complex of any triangle-free graph has the homotopy type of a suspension and that the independence complex of a forest is either contractible or homotopy equivalent to a sphere. K. Iriye used star clusters to construct a matching tree for the independence complex of square grids with cyclic identification [8], and S. Goyal et al. used star clusters to compute the homotopy type of the independence complexes of generalised Mycielskian of complete graphs [6]. Another important tool in combinatorial topology is the Cluster lemma. This result was arrived at independently by both Jonsson [9, Lemma 4.2] and Hersh [7, Lemma 4.1], and it is an extremely convenient yet simple way to put a gradient vector field on a complex by gluing together gradient vector fields on a decomposition of the complex.

A goal of this paper will be to utilize both star clusters and the Cluster Lemma to study the homotopy type of the complex of Morse functions, generalized Morse functions, and matching complex. The complex of discrete Morse functions of K, denoted $\mathcal{M}(K)$, is the simplicial complex of all gradient vector fields on K (see Definition 3.2). Like the matching or independence complex of a graph, the complex of discrete Morse functions of a simplicial complex K is a complex that stores certain combinatorial information about K, and determining its homotopy type is an interesting question. Although the complex of discrete Morse functions of K in general is not a flag complex, it is a flag complex when $K = T$ is a tree. In this special case, we use star clusters and the Cluster lemma to show in Proposition 3.7 that $\mathcal{M}(T)$ has the homotopy type of a suspension. We also compute the homotopy type of the complex of discrete Morse functions on any number of paths of two different lengths joined at a single point (Theorem 3.12) and provide an alternate computation of the homotopy type of the complex of discrete Morse functions of a path (Proposition 3.8) originally computed by D. Kozlov in [10].

We next investigate the homotopy type of the generalized complex of discrete Morse functions, first introduced in [14]. From the perspective of star clusters, the generalized complex of discrete Morse functions has the advantage that it includes cycles and hence is a flag complex. We compute the homotopy type of the generalized complex of discrete Morse functions of a cycle (Theorem 4.3) and show that the complex of discrete Morse functions and Generalized complex of discrete Morse functions of a cycle with a single leaf have the same homotopy type. There seems to be further connections between the complex of discrete Morse functions and Generalized complex of discrete Morse functions, and we discuss some of these possibilities and open questions in the last section.

Another goal of this paper will be to utilize the idea of the star cluster and the Cluster Lemma to compute the homotopy type of the matching complex of certain complexes. If we consider a graph G, the matching complex, denoted $\mathrm{M}(G)$, is the complex constructed from all independent edge sets on G. It is easy to see that this is a flag complex. An interesting connection was made between the complex of discrete Morse functions and matching complex in [2][1], where the authors provide a natural filtration on the matching complex of a finite simplicial complex that allows us to relate the matching complex to the complex of discrete Morse functions. The authors observe that there is a one-to-one correspondence between elements in the generalized complex of discrete Morse functions of a graph G and matchings on the barycentric subdivision of G so that $\mathcal{GM}(G) \cong \mathrm{M}(\mathrm{sd}(G))$. Combining this with the fact mentioned above that the complex of discrete Morse functions of a tree is flag, we have the relationship $\mathcal{M}(T) \cong \mathcal{GM}(T) \cong \mathrm{M}(\mathrm{sd}(T))$.

Additionally, the homotopy type of the matching complexes for paths and cycles [10] and for forests [12] have been shown to be either contractible, a sphere, or a wedge of spheres. We provide alternate proofs of the computation of the homotopy type of the matching complex for paths and cycles (Proposition 5.3, Proposition 5.4) as well as provide a computation of the homotopy type of Dutch windmill graphs (Theorem 5.8).

2. Preliminaries

In this section, we establish the notation, terminology, and background results that will be needed throughout this paper. All simplicial complexes are assumed to be connected unless otherwise stated. We use $\simeq$ to denote a homotopy equivalence and $\cong$ to denote an isomorphism.

2.1. Background. Because we will be taking constructions on graphs, we adopt some graph theoretic language.

Definition 2.1. *A simplicial complex G such that $\dim(G) = 1$ is called a **graph**. If G is an acyclic graph, then we call G a **tree**. The number of edges of a vertex is the **degree** of the vertex. A **leaf** is any vertex of degree 1. The **path** P_n on n vertices is the simplicial complex with facets*

$$\{v_0, v_1\}, \{v_1, v_2\}, \cdots \{v_{n-2}, v_{n-1}\},$$

*The **length** of the path P_n is the number of edges $(n-1)$ in the path. A **cycle** of length $n \geq 3$ is the simplicial complex C_n with facets*

$$\{v_0, v_1\}, \{v_1, v_2\}, \cdots \{v_{n-2}, v_{n-1}\}, \{v_{n-1}, v_0\}.$$

Definition 2.2. *A simplicial complex K is a **flag complex** if for each non-empty set of vertices σ such that $\{v_i, v_j\} \in K$ for every $v_i, v_j \in \sigma$, we have that $\sigma \in K$.*

The flag complex of a graph G is the smallest flag complex that has G as a 1-skeleton.

[1]Because they are defining a matching complex on all simplicial complexes, the authors define the matching complex of a graph G to be comprised of elements of matchings on the order poset of G, i.e., a matching on the barycentric subdivision of G

Definition 2.3. *Let K be a simplicial complex and $v \in K$ be a vertex. The **star** of v in K, denoted by* $\mathrm{st}(v)$*, is the simplicial complex induced by the set of all simplices of K containing v. More generally, the star of a simplex s is the set of simplices having s as a face.*

Definition 2.4. [1, Definition 3.1] *Let σ be a simplex of a simplicial complex, K. We define the **star cluster** of σ in K as the subcomplex*

$$\mathrm{SC}_K(\sigma) = \bigcup_{v \in \sigma} \mathrm{st}_K(v) + \mathrm{lk}_K(v)$$

We denote the star cluster of σ by $\mathrm{SC}(\sigma)$ *when the context is clear.*

A simple but fundamental property of the star cluster of a simplex is that it is collapsible if the complex is flag.

Proposition 2.5. [1, Lemma 3.2] *The star cluster of a simplex in a flag complex is collapsible.*

Proposition 2.5 is one of the main tools we use in this paper. The other tool is the following Lemma:

Lemma 2.6. *(*[9, Lemma 4.2] *and* [7, Lemma 4.1]*) [Cluster Lemma] Let Δ be a simplicial complex which decomposes into collections Δ_σ of simplices, indexed by the elements σ in a partial order P which has a unique minimal element $\sigma_0 = \Delta_0$, Furthermore, assume that this decomposition is as follows:*

1. *Each simplex belongs to exactly one Δ_σ.*
2. *For each $\sigma \in P$, $\bigcup_{\tau \leq \sigma} \Delta_\tau$ is a subsimplicial complex of Δ.*

For each $\sigma \in P$, let M_σ be an acyclic matching in Δ_σ. Then $\bigcup_{\sigma \in P} M_\sigma$ is an acyclic matching on K.

Lemma 2.6 provides a way to put an acyclic matching on the entire complex by patching together acyclic matchings on parts of the complex. The key information is what is left unmatched or the critical simplices. In some cases, certain collections of critical simplices will uniquely determine the homotopy type of the original complex. This is given in the classical result of Forman.

Theorem 2.7. [5, Corollary 3.5] *Let K be a simplicial complex and M an acyclic matching on K with m_i critical simplices of dimension i. Then K has the homotopy type of a CW complex with exactly m_i cells of dimension i. In particular, if $m_0 = 1, m_n = k$, and $m_j = 0$ for all $j \neq 0, n$, then K has the homotopy type of a k-fold wedge of S^n.*

All of our computations below will in fact satisfy the stated special case of Theorem 2.7 and thus allow us to determine the homotopy type of the complex in question.

Definition 2.8. *Let K be a simplicial complex. A vertex v is said to **dominate** v' if every maximal simplex (facet) of v' also contains v.*

If v dominates v' in a simplicial complex K, the removal of v' from K is called an **elementary strong collapse** and is denoted by $K \searrow\searrow K - \{v'\}$. Conversely, the addition of a dominated vertex is an **elementary strong expansion** and is denoted by $\nearrow\nearrow$.

Definition 2.9. *Let K and L be simplicial complexes. If there is a sequence of strong collapses and expansions from K to L, then K and L are said to have the same **strong homotopy type**.*

*In the case where $L = *$, then K is said to have the **strong homotopy type of a point**. If there is a sequence of only strong collapses from K to a point, K is **strongly collapsible**.*

3. Homotopy type of the complex of discrete Morse functions

In order to describe the complex of discrete Morse functions and Generalized complex of discrete Morse functions, we will need the following.

Definition 3.1. *Let K be a simplicial complex. A **discrete vector field** V on K is defined by*

$$V := \{(\sigma^{(p)}, \tau^{(p+1)}) : \sigma < \tau, \text{ each simplex of } K \text{ in at most one pair}\}.$$

*Any pair in $(\sigma, \tau) \in V$ is called a **regular pair**, and σ, τ are called **regular simplices** or just **regular**. If $(\sigma^{(p)}, \tau^{(p+1)}) \in V$, we say that $p+1$ is the **index** of the regular pair. Any simplex in K which is not in V is called **critical**.*

Definition 3.2. *Let V be a discrete vector field on a simplicial complex K. A V-**path** or **gradient path** is a sequence of simplices*

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)} \dots, \beta_{k-1}^{(p+1)}, \alpha_k^{(p)}$$

*of K such that $(\alpha_i^{(p)}, \beta_i^{(p+1)}) \in V$ and $\beta_i^{(p+1)} > \alpha_{i+1}^{(p)} \neq \alpha_i^{(p)}$ for $0 \leq i \leq k-1$. If $k \neq 0$, then the V-path is called **non-trivial.** A V-path is said to be **closed** if $\alpha_k^{(p)} = \alpha_0^{(p)}$. A discrete vector field V which contains no non-trivial closed V-paths is called a **gradient vector field**.*

If the gradient vector field consists of only a single element, we say it is a **primitive** gradient vector field. We often denote a primitive gradient vector field $\{(u, uv)\}$ with $p = 0$ by $(u)v$.

Definition 3.3. *The **complex of discrete Morse functions** of K, denoted $\mathcal{M}(K)$, is the simplicial complex whose vertices are given by primitive gradient vector fields and whose n-simplices are given by gradient vector fields with $n+1$ regular pairs. A gradient vector field f is then associated with all primitive gradient vector fields $f := \{f_0, \dots, f_n\}$ with $f_i \leq f$ for all $0 \leq i \leq n$.*

One result about the complex of discrete Morse functions that we will utilize is that it is well-behaved with respect to disjoint unions of complexes.

Proposition 3.4. [4] *Let K, L be connected simplicial complexes each with at least one edge. Then $\mathcal{M}(K \sqcup L) = \mathcal{M}(K) * \mathcal{M}(L)$.*

Lemma 3.5. *The complex of discrete Morse functions $\mathcal{M}(K)$ is a flag complex if and only if K is a tree.*

Proof. Let T be a tree and $\mathcal{M}(T)$ the complex of discrete Morse functions of T. For $\mathcal{M}(T)$ to be a flag complex, each non-empty set of mutually compatible vertices needs to be all together compatible. In other words, for each non-empty set of vertices σ such that $\{v, w\} \subseteq \mathcal{M}(T)$ for every $v, w \in \sigma$, we have that $\sigma \in \mathcal{M}(T)$. Now the only case when a collection of pairwise compatible primitive gradient vector fields may not be compatible is when they form a cycle. But since trees are acyclic, a collection of pairwise compatible primitive gradient vector fields can never form a cycle so that $\mathcal{M}(T)$ is a flag complex.

Now suppose $\mathcal{M}(K)$ is a flag complex. Clearly neither K nor the 1-skeleton of K can contain a cycle since otherwise there would exist a collection of mutually compatible vertices on $\mathcal{M}(K)$ that are not all together compatible. Thus K must be a tree. $\square$

Although the flag condition greatly reduces the kind of complex of discrete Morse functions that we can study directly using star clusters, the following result of Barmak will allow us to say something general about the complex of discrete Morse functions of all trees.

Lemma 3.6. [1, Lemma 3.4] *Let K be a simplicial complex and K_1, K_2 be two collapsible subcomplexes such that $K = K_1 \cup K_2$. Then K is homotopy equivalent to $\Sigma(K_1 \cap K_2)$.*

We can now show that the complex of discrete Morse functions of all trees is a suspension.

Proposition 3.7. *Let T be a tree. Then $\mathcal{M}(T)$ has the homotopy type of a suspension.*

Proof. We apply Lemma 3.6 by constructing two collapsible subcomplexes of $\mathcal{M}(T)$ whose union is all of $\mathcal{M}(T)$. Pick any leaf $\{v_0, v_0v_1\}$ of T and consider the maximum gradient vector field σ_0 rooted in v_0 and the maximum gradient vector field rooted in v_1 [13, Proposition 3.3]. These corresponds to simplices $\sigma_0, \sigma_1 \in \mathcal{M}(T)$, respectfully. Define $\mathcal{M}_1(T) = \mathrm{SC}_{\mathcal{M}(T)}(\sigma_0)$ and $\mathcal{M}_2(T) = \mathrm{SC}_{\mathcal{M}(T)}(\sigma_1)$. Then $\mathcal{M}_1(T)$ and $\mathcal{M}_2(T)$ are collapsible subcomplexes of $\mathcal{M}(T)$ by Lemma 3.5. Furthermore, it is easy to see that $\mathcal{M}(T) = \mathcal{M}_1(T) \cup \mathcal{M}_2(T)$. Thus $\mathcal{M}(T) \simeq \Sigma(\mathcal{M}_1(T) \cap \mathcal{M}_2(T))$. $\square$

In addition to the general structure of the complex of discrete Morse functions of a tree, we can use Proposition 2.5 and Lemma 2.6 to compute the homotopy type of some specific classes of trees. Our first computation is the homotopy type of the complex of discrete Morse functions of a path. This was originally computed by Kozlov in [10]. Here we provide an alternate computation in a first illustration of our technique.

Proposition 3.8. *Let P_t be the path on t vertices, $t \geq 3$. Then*

$$\mathcal{M}(P_t) \simeq \begin{cases} * & \text{if } t = 3n \\ \mathbb{S}^{2n-1} & \text{if } t = 3n+1 \\ \mathbb{S}^{2n} & \text{if } t = 3n+2 \end{cases}$$

Proof. We apply the Cluster Lemma. In order to do so, we decompose $\mathcal{M}(P_t)$ into collections Δ_k. First, we construct collections of sub-simplices σ_i for $i = 0, \ldots n$. We construct collections as follows:

(1) Let $\sigma_0 := \mathrm{SC}((v_0)v_1, (v_1)v_2, \ldots, (v_{t-3})v_{t-2}, (v_{t-2})v_{t-1})$
(2) For $1 \leq j \leq n$, we define the following:
  (a) When $j = 2k-1$, let $\sigma_j := \mathrm{st}((v_{t-(3k-1)})v_{t-3k})$
  (b) When $j = 2k$, let $\sigma_j := \mathrm{st}((v_{3k})v_{3k-1})$
(3) Let $\sigma_{n+1} := \mathcal{M}(P_t) - \bigcup_{i=0}^{n} \sigma_i$

Now define $\Delta_0 := \sigma_0$ and $\Delta_k := \sigma_k - \bigcup_{j=0}^{k-1} \sigma_j$, and observe that $\bigcup_{k=0}^{n+1} \Delta_k = \mathcal{M}(P_t)$. Define an acyclic matching on each Δ_j as follows:

We know that Δ_0 is collapsible by Proposition 2.5 and Lemma 3.5 so Δ_0 has an acylcic matching with a single unmatched 0-simplex.

Let $j = 2k-1$, $1 \leq j \leq n$. Any simplex $V \in \Delta_j$ by definition contains $(v_{t-(3k-1)})v_{t-1}$. Match V with $V \cup \{(v_{t-(3k-2)})v_{t-(3k-1)}\}$ (or $V - \{(v_{t-(3k-2)})v_{t-(3k-1)}\}$ if V already contains this vector). In this way, all simplices in Δ_{2k-1} are matched with no unmatched simplices. Furthermore, since this matching is a subset of the matching on the cone on $(v_{t-(3k-1)})v_{t-1}$, it is acyclic.

Let $j = 2k$, $2 \leq j \leq n$. Any simplex $V \in \Delta_j$ by definition contains $(v_{3k})v_{3k-1}$. Match V with $V \cup \{(v_{3k-1})v_{3k-2}\}$ (or V with this vector removed, as above). In this way, all simplices in Δ_{2k} are matched with no unmatched simplices. Again, this matching is a subset of the matching on a cone so it is acyclic.

Now consider Δ_{n+1}. We have three cases:

(Note: When considering $n = 1$ in cases 2 and 3, disregard matchings containing vertices with negative indices e.g. v_{-1})

Case 1: Let $t = 3n$. Then $\Delta_{n+1} = \emptyset$, and thus $\mathcal{M}(P_t) \simeq *$.

Case 2: Let $t = 3n+1$. Then Δ_{n+1} contains a single simplex V of dimension $(2n-1)$ satisfying

$$(v_{3\lfloor \frac{n}{2} \rfloor})v_{3\lfloor \frac{n}{2} \rfloor - 1}, (v_{3\lfloor \frac{n}{2} \rfloor + 2})v_{3\lfloor \frac{n}{2} \rfloor + 1} \notin V.$$

Thus by Theorem 2.7, $\mathcal{M}(P_t) \simeq \mathbb{S}^{2n-1}$.

Case 3: Let $t = 3n+2$. Then Δ_{n+1} contains a single simplex V of dimension $2n$ satisfying

$$(v_{3\lfloor \frac{n}{2} \rfloor})v_{3\lfloor \frac{n}{2} \rfloor - 1}, (v_{3\lfloor \frac{n}{2} \rfloor + 3})v_{3\lfloor \frac{n}{2} \rfloor + 2} \notin V.$$

Thus by Theorem 2.7, $\mathcal{M}(P_t) \simeq \mathbb{S}^{2n}$. □

We can prove something a bit stronger in the case of a path on $3t$ vertices. We first prove a lemma.

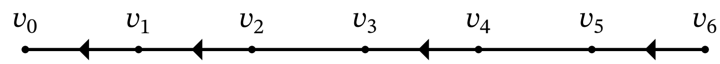


FIGURE 1. For $t = 7 = 3(2)+1$, Case 2 of the proof of Theorem 3.8 implies that $V \in \Delta_3$ will result in the gradient vector field (critical simplex) in $\mathcal{M}(P_7)$ pictured above.

Lemma 3.9. *Let K be a simplicial complex with leaf $\{a, ab\}$ and c a neighbor of b not equal to a. Then $(b)c$ is dominated in $\mathcal{M}(K)$ by $(a)b$.*

Proof. Consider any facet of $(b)c$ in $\mathcal{M}(K)$. A facet of $\mathcal{M}(K)$ is a maximal gradient vector field on K, and since $(b)a$ is not compatible with $(b)c$ and $\{a, ab\}$ is a leaf, $(a)b$ must be in any maximal gradient vector field containing $(b)c$. Thus $(a)b$ dominates $(b)c$ in $\mathcal{M}(K)$. $\square$

Proposition 3.10. *Let P_{3n} be the path on $3n$ vertices, $n \geq 1$. Then $\mathcal{M}(P_t) \searrow\searrow *$*

Proof. By Lemma 3.9, $(v_1)v_2$ dominates $(v_2)v_3$. After removing $(v_2)v_3$, we see that $(v_3)v_2$ dominates $(v_4)v_3$, and so we remove $(v_4)v_3$. Continuing in this manner, we see that $(v_{3k-2})v_{3k-1}$ dominates $(v_{3k-1})v_{3k}$ for all $1 \leq k \leq n$, and $(v_{3k})v_{3k-1}$ dominates $(v_{3k+1})v_{3k}$ for for all $1 \leq k < n$. Hence we may remove each of these primitive gradient vector fields.

Now the last primitive gradient vector field removed is $(v_{3n-1})v_{3n}$ since it was dominated by $(v_{3n-2})v_{3n-1}$. We now claim that $(v_{3n})v_{3n-1}$ dominates every remaining vertex. To see this, observe that because $(v_{3n-1})v_{3n}$ has been removed, $(v_{3n})v_{3n-1}$ is compatible with all remaining vertices $(v_i)v_j$, and no $(v_i)v_j$ can exist in a facet of the remaining Morse complex without $(v_{3n})v_{3n-1}$. We remove all $(v_i)v_j$ until we are only left with $(v_{3n})v_{3n-1}$. Thus $\mathcal{M}(P_{3n-1})$ is strongly collapsible. $\square$

Recall that the **star graph** S_n on $n+1$ vertices is the complete bipartite graph $K_{1,n}$. Alternatively, we may view S_n as the result of taking n paths of length 1 and gluing them to a common endpoint (the so-called wedge product). We generalize S_n in the following definition.

Definition 3.11. *An* ***extended star graph****, denoted S_{v_1,v_2,v_3}, is the graph obtained by starting with v_1 paths of length 1, v_2 paths of length 2, and v_3 paths of lengths 3 and identifying an endpoint of each path with a fixed vertex c called the* ***center****. By an* ***extended leaf of length*** *k, we mean a path of length k from the center vertex, c, to a vertex, v_k, of degree 1.*

Clearly $S_k = S_{k,0,0}$ recovers the star graph. It was shown in [4, Proposition 3.5] that not only is $\mathcal{M}(S_n)$ (strongly) collapsible for $n \geq 2$, but that any complex with at least two leaves sharing a common vertex is strongly collapsible. Hence, we let $v_0 = 0$ in our computation below.

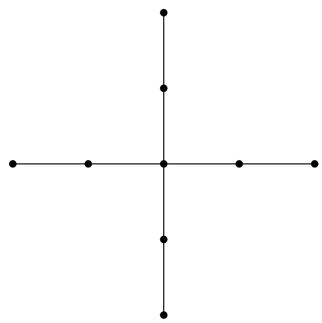


FIGURE 2. The extended star graph, $S_{0,4,0}$. By corollary [3.14], we see that $\mathcal{M}(S_{0,4}) \simeq \mathbb{S}^4 \vee \mathbb{S}^4 \vee \mathbb{S}^4$.

Theorem 3.12. *Let $S_{0,n,m}$ be an extended star graph. Then,*

$$\mathcal{M}(S_{0,n,m}) \simeq \vee^{n-1}\mathbb{S}^{2m+n}$$

Proof. Define a collection of subsimplices σ_i for $i = 0, \dots n$ on $\mathcal{M}(S_{0,n,m})$ as follows:

Let c be the center vertex of $S_{0,n,m}$ and let $\{v_{a_i}v_{b_i}, v_{b_i}\}$ be the leaf of each extended leaf of length 2, $i = 1, 2, \dots, n$, and $\{v_{\alpha_j}v_{\beta_j}, v_{\beta_j}\}$ the leaf of each extended leaf of length 3, $j = 1, 2, \dots, m$ with $v_{\gamma_j} \neq v_{\beta_j}$ the other neighbor of v_{α_j}.

(1) Let σ_0 be the star cluster of the gradient vector field rooted in c. Such a gradient vector field exists and is unique by [13, Proposition 3.3].
(2) Let $\sigma_1 := \cup_{i=1}^{m} \mathrm{st}(\{(c)v_{\gamma_i}\})$

Now define $\Delta_0 := \sigma_0, \Delta_1 := \sigma_1 - \sigma_0$, and $\Delta_2 := \mathcal{M}(S_{0,n,m}) - (\sigma_0 \cup \sigma_1)$. Clearly $\Delta_0 \cup \Delta_1 \cup \Delta_2 = \mathcal{M}(S_{0,n,m})$ so we can apply the Cluster Lemma. We define an acyclic matching on each Δ_j as follows:

First, Δ_0 is collapsible by Proposition 2.5 and Lemma 3.5 so there is an acyclic matching on Δ_0 with a single critical 0-simplex.

To construct a matching on Δ_1, we first observe that a typical element of Δ_1 is of the form $(c)v_{\gamma_i}$ along with other arrows pointing away from the center vertex c. Furthermore, because σ_0 contains all gradient vector fields with any arrow pointing towards c, all elements of Δ_1 are not compatible with any arrow pointing towards c. Upon inspection, there are exactly $2m$ such gradient vector fields. Match the $(2m+n-1)$-simplex of Δ_1 containing $(c)v_{\gamma_i}$ but not containing $(v_{\gamma_i})v_{\alpha_i}$ to the corresponding $(2m+n)$-simplex containing both $(c)v_{\gamma_i}$ and $(v_{\gamma_i})v_{\alpha_i}$. This produces an acyclic matching on all elements in Δ_1.

Lastly, observe that Δ_2 contains $n+1$ elements. We will create a single matching, leaving $n-1$ unmatched $(2m+n)$-simplices and hence critical. A typical element of Δ_2 is of the form $(\bigcup_{i=1}^{n}(v_{a_i})v_{b_i}) \cup (\bigcup_{i=1}^{m}(v_{\gamma_i})v_{\alpha_i}) \cup (\bigcup_{i=1}^{m}(v_{\alpha_i})v_{\beta_1})$ along with possibly one of $(c)v_{a_i}$. Match the $(2m+n-1)$-simplex of Δ_2 containing none of the $(c)v_{a_i}$ with the $(2m+n)$-simplex containing $(c)v_{a_1}$.

If $n > 1$ then, there are $n-1$ unmatched $(2m+n)$-simplices τ_i, where each τ_i contains $(c)v_{a_i}$ for $i = 2, 3, \dots, n-1$. Thus $\mathcal{M}(S_{0,n,m}) \simeq \vee^{n-1}\mathbb{S}^{2m+n}$. □

We obtain several special cases which we list as corollaries below.

Corollary 3.13. *Let $S_{0,1,n}$ be an extended star graph. Then,*

$$\mathcal{M}(S_{0,1,n}) \simeq *$$

Corollary 3.14. *Let $S_{0,n}$ be an extended star graph. Then,*

$$\mathcal{M}(S_{0,n}) \simeq \vee^{n-1}\mathbb{S}^n$$

Corollary 3.15. *Let $S_{0,0,n}$ be an extended star graph. Then,*

$$\mathcal{M}(S_{0,0,n}) \simeq \mathbb{S}^{2n-1}$$

3.1. Strongly collapsing to suspensions. Although we showed earlier that the complex of discrete Morse functions of all trees are suspensions (Proposition 3.7), in general, it is unknown when a complex of discrete Morse functions is a suspension. Here we provide some convenient results for showing the complex of discrete Morse functions of many simplicial complexes are suspensions. Lemma 3.18 and Corollary 3.19 is for all simplicial complexes, while Proposition 3.21 is for cycles.

Definition 3.16. *Let $\mathbb{P}$ be the set of all (finite) posets, and $\mathbb{K}$ be the set of all simplicial complexes. Define a function $f : \mathbb{P} \to \mathbb{K}$ as follows: for each $P \in \mathbb{P}$, construct a simplicial complex $f(P)$ whose vertex set is the edge set of P. Then let $\sigma = e_1e_2\cdots e_k$ be a simplex of $f(P)$ if and only if the edges $e_1, e_2, \cdots e_k$ oriented upward and all other edges oriented downward form an acyclic matching of P.*

Remark 3.17. *For any simplicial complex K, $\mathcal{M}(K) \simeq f(\mathcal{H}(K))$. Our definition 3.16 generalizes the notion of taking the complex of discrete Morse functions to degenerate Hasse diagrams. We will similarly call $f(P)$ the complex of discrete Morse functions of the poset P.*

Lemma 3.18. *Let K be a simplicial complex. Then, $\mathcal{M}(K \vee_v P_{3i+1}) \simeq \Sigma^{2i}\mathcal{M}(K)$*

Proof. Suppose $v_0, v_1, \dots, v_{3i} \in V(P_{3i+1})$ such that v_0 is our leaf vertex and v_{3i} is wedged with some $v \in V(K)$. Consider a sequence of strong collapses starting with vertex $(v_0)v_1$ dominating vertex $(v_1)v_2$ in the complex of discrete Morse functions. This can then be immediately followed by a strong collapse of $(v_2)v_1$ dominating $(v_3)v_2$ in the complex of discrete Morse functions. Upon inspection, each vertex $(v_{3t})v_{3t+1}$ dominates $(v_{3t+1})v_{3t+2}$ and each $(v_{3t+2})v_{3t+1}$ dominates $(v_{3(t+1)})v_{3t+2}$ along our wedged path. Note that this is the same sequence of strong collapses seen in Proposition 3.10.

These strong collapses correspond to the removal of the edges between each v_{3t+1} and $v_{3t+1}v_{3t+2}$ on the Hasse diagram, along with the edges between $v_{3(t+1)}$ and $v_{3(t+1)}v_{3t+2}$. We can quickly notice that this yields two components on our degenerate Hasse diagram per index i; all of which are Hasse diagrams of $P_2 = \ell$, either "right-side-up" or "upside-down."

The v_{3i} is wedged to K, and those $2i$ components are separated from $\mathcal{H}(K)$. In other words, they are not effected the strong collapses. Thus,

$$\mathcal{M}(K \vee_v P_{3i+1}) \searrow\searrow f((H(K) \sqcup H(\ell_1) \sqcup \dots \sqcup H(\ell_{2i}))$$

So, by Proposition 3.4 and Remark 3.17,

$$\begin{aligned} f((H(K) \sqcup H(\ell_1) \sqcup \ldots \sqcup H(\ell_{2i})) &\simeq f((H(K)) * f(H(\ell_1)) * \ldots * f(H(\ell_{2i})) \\ &\simeq \mathcal{M}(K) * \mathcal{M}(\ell_1) * \ldots * \mathcal{M}(\ell_{2i}) \\ &\simeq \Sigma^{2i}\mathcal{M}(K) \end{aligned}$$ □

The following corollaries are immediate.

Corollary 3.19. *Let K be a simplicial complex. Then, for $v_1, v_2, \ldots, v_m \in V(K)$,*

$$\mathcal{M}(K \vee_{v_1} P_{3n_1+1} \vee_{v_2} \ldots \vee_{v_m} P_{3n_m+1}) \simeq \Sigma^{2(n_1+\ldots+n_m)}\mathcal{M}(K).$$

Corollary 3.20. *Let K be a simplicial complex on n vertices. Then, $\mathcal{M}(L_{3i+1}(K)) \simeq \Sigma^{2i(n)}\mathcal{M}(K)$.*

Proposition 3.21. *Let C_n be a cycle on n vertices. Then, $\mathcal{M}(C_n \vee_v P_{3i+2}) \simeq \Sigma^{2i+1}\mathcal{M}(P_{n-1})$*

Proof. Firstly, it is clear to see that $\mathcal{M}(C_n \vee_v P_{3i+2}) = \mathcal{M}(C_n \vee_v \ell \vee_{v'} P_{3i+1})$ when the path is wedged onto the leaf vertex, v'.

By [4, Lemma 5.4], we know that for any simplicial complex, K, and vertex $v \in V(K)$, the complex of discrete Morse functions $\mathcal{M}(K \vee_v \ell)$ strongly collapses to $f((H(K) - v \sqcup H(\ell))$. It can be shown similarly, using the sequence of strong collapses seen in the proof of Proposition 3.10, that

$$\mathcal{M}(C_n \vee_v \ell \vee_{v'} P_{3i+1}) \searrow\searrow f((H(C_n) - v) \sqcup H(\ell_0) \sqcup H(\ell_1) \sqcup \ldots \sqcup H(\ell_{2i}))$$

Therefore, by Proposition 3.4 and Remark 3.17,

$$\begin{aligned} &f((H(C_n) - v) \sqcup H(\ell_0) \sqcup H(\ell_1) \sqcup \ldots \sqcup H(\ell_{2i})) \\ &\quad \simeq f((H(C_n) - v) * f(H(\ell_0)) * \ldots * f(H(\ell_{2i})) \\ &\quad \simeq \mathcal{M}(P_{n-1}) * \mathcal{M}(\ell_0) * \ldots * \mathcal{M}(\ell_{2i}) \\ &\quad \simeq \Sigma^{2i+1}\mathcal{M}(P_{n-1}). \end{aligned}$$ □

3.2. Sufficient condition for strong collapsibility. The following results were proved in [4]:

Proposition 3.22. [4, Proposition 3.5] *If a simplicial complex K has two leaves sharing a vertex, then $\mathcal{M}(K)$ is strongly collapsible.*

The next theorem gives another condition under which the complex of discrete Morse functions is strongly collapsible.

Theorem 3.23. *Suppose a simplicial complex, K, has two paths, of length $3n+2$ and $3m+2$ respectively, wedged at $v \in V(K)$, then $\mathcal{M}(K)$ is strongly collapsible.*

Proof. We proceed by induction. Consider the base case where both paths wedged at $v \in V(K)$ are of of length 2. Then two leaves are sharing a vertex, and so $\mathcal{M}(K) \searrow\searrow *$ by Proposition 3.22. Suppose this holds true for paths length 2 up to paths of length $3(n-1)+2$ and $3(m-1)+2$. Now, we will show that this holds for paths of length $3n+2$ and $3m+2$ respectively:

Consider the sequence of strong collapses used in Proposition 3.10 starting at the leaf vertex, v_1 of P_{3n+2}. Then, $(v_1)v_2$ dominates $(v_2)v_3$, $(v_3)v_2$ dominates $(v_4)v_3$, and then we are on to considering the remaining path of length $3(n-1)+2$. Similarly, we can consider the remaining path of length $3(m-1)+2$ for the other path. By assumption, $\mathcal{M}(K) \searrow\searrow *$ for paths of lengths $3(n-1)+2$ and $3(m-1)+2$.

It is clear to see that the sequence of strong collapses for paths of lengths $3(n-1)+2$ and $3(m-1)+2$ is not hindered by the remaining primitive gradient vector fields from v_0 to v_4, as each $(v_{3k-2})v_{3k-1}$ dominates $(v_{3k-1})v_{3k}$ for all $1 \leq k \leq n, m$, and each $(v_{3k})v_{3k-1}$ dominates $(v_{3k+1})v_{3k}$ for all $1 \leq k \leq n, m$.

So, using the same sequence of strong collapses we would for paths of lengths $3(n-1)+2$ and $3(m-1)+2$, $\mathcal{M}(K) \searrow\searrow *$ when paths of length $3n+2$ and $3m+2$ are both wedged at the same $v \in V(K)$. □

4. Homotopy type of the generalized complex of discrete Morse functions

The following was defined in [14] in order to estimate the connectivity of the complex of discrete Morse functions.

Definition 4.1. *The* ***generalized complex of discrete Morse functions*** $\mathcal{GM}(K)$ *of a simplicial complex, K, is the simplicial complex whose vertices are the primitive gradient vector fields on K, with a finite collection of vertices spanning a simplex whenever the primitive gradient vector fields are pairwise compatible. Reworded, the simplices of* $\mathcal{GM}(K)$ *are the discrete vector fields on K, with face relation given by inclusion.*

Note that $\mathcal{GM}(K)$ is a flag complex since it allows closed V-paths on K. We now compute the homotopy type of the generalized complex of discrete Morse functions for cycles. First, a definition.

Definition 4.2. *Let* C_t *be a cycle,* $t \geq 3$*, with vertices* $v_0, \dots, v_{t-1}$*. Let* $V_k := \{(v_{i+1})v_i : k \leq i \leq t-1\}$*. We define* $\mathrm{st}_{\mathrm{mod}}(V_k) := \{\sigma \in \mathrm{st}(V_k) : (v_k)v_{k-1} \notin \sigma\}$*.*

Theorem 4.3. *Let* C_t *be the cycle on t vertices,* $t > 3$*. Then*

$$\mathcal{GM}(C_t) \simeq \begin{cases} \mathbb{S}^{2n-1} \vee \mathbb{S}^{2n-1} & \text{if } t = 3n \\ \mathbb{S}^{2n} & \text{if } t = 3n+1 \\ \mathbb{S}^{2n} & \text{if } t = 3n+2 \end{cases}$$

Proof. We decompose $\mathcal{GM}(C_t)$ into collections Δ_k. We begin by constructing the following collections:

(1) Let $\sigma_0 := \mathrm{SC}(\{(v_0)v_1, (v_1)v_2, \dots, (v_{t-1})v_0\})$
(2) For $1 \leq j \leq t-2$, let $\sigma_j := \mathrm{st}_{\mathrm{mod}}(\{(v_j)v_{j-1}(v_{j+2})v_{j+1}\})$

Define $\Delta_0 := \sigma_0$ and $\Delta_k := \sigma_k - \bigcup_{j=0}^{k-1} \sigma_j$. Then $\bigcup_{k=0}^{t-2} \Delta_k = \mathcal{GM}(C_t)$. Clearly Δ_0 is collapsible. Now match k-simplex of the form $\{(v_j)v_{j-1}(v_{j+2})v_{j+1} \dots\}$ with the $(k+1)$-simplex of the form $\{(v_j)v_{j-1}(v_{j+1})v_j(v_{j+2})v_{j+1} \ldots\}$. There are three cases to consider.

Case 1: Let $t = 3n$. Then there will be two critical $(2n-1)$-simplices, both of which were excluded from every σ_j by the definition of $\mathrm{st}_{\mathrm{mod}}$. However, all other simplices have been matched. Thus $\mathcal{GM}(C_{3n}) \simeq \mathbb{S}^{2n-1} \vee \mathbb{S}^{2n-1}$.

Case 2: Let $t = 3n+1$. Then there will be one critical $(2n)$-simplex while all other simplices have been matched. Thus $\mathcal{GM}(C_{3n+1}) \simeq \mathbb{S}^{2n}$.

Case 3: Let $t = 3n+2$. There is a single critical $(2n)$-simplex which was excluded from every σ_j by the definition of $\mathrm{st}_{\mathrm{mod}}$ with all other simplices matched. Thus $\mathcal{GM}(C_{3n+2}) \simeq \mathbb{S}^{2n}$. □

Now we investigate the generalized complex of discrete Morse functions of a cycle with a leaf attached. We use the notation $C_t \vee \ell$ to denote the cycle of length t with a leaf ℓ joined to some vertex of C_t.

Theorem 4.4. *Let C_t be the path on t vertices, $t > 3$. Then*

$$\mathcal{GM}(C_t \vee \ell) \simeq \begin{cases} * & \text{if } t = 3n \\ \mathbb{S}^{2n} & \text{if } t = 3n+1 \\ \mathbb{S}^{2n+1} & \text{if } t = 3n+2 \end{cases}$$

Proof. Let $\{v_1, v_0v_1\}$ be the leaf attached to $v_1 \in C_t$.

To apply the Cluster lemma, we first construct collections as follows:

(1) Let $\sigma_0 := \mathrm{SC}(\{(v_0)v_1(v_1)v_2(v_2)v_3 \ldots (v_n)v_1\})$

(2) For $1 \leq j \leq n$,

  (a) Let $j = 2k-1$ and define

$$\sigma_j := \mathrm{st}(\{(v_{1+3(k-1)})v_{0+3(k-1)}(v_{3+3(k-1)})v_{2+3(k-1)}\})$$

  (b) Let $j = 2k$ and define

$$\sigma_j := \mathrm{st}(\{(v_{t-3(k-1)})v_{(t-1)-3(k-1)}(v_{(t-2)-3(k-1)})v_{(t-3)-3(k-1)}\})$$

(3) For $j = n+1$,

  (a) if $n+1 = 2k-1$, then

$$\sigma_{n+1} := \mathrm{st}(\{(v_{t-3(k-1)})v_{(t-1)-3(k-1)}(v_{(t-1)-3(k-1)})v_{(t-2)-3(k-1)}\})$$

  (b) if $n+1 = 2k$, then

$$\sigma_{n+1} := \mathrm{st}(\{(v_{1+3(k-1)})v_{0+3(k-1)}(v_{2+3(k-1)}))v_{1+3(k-1)}\})$$

Let $\Delta_0 := \sigma_0$ and $\Delta_k := \sigma_k - \bigcup_{j=0}^{k-1} \sigma_j$. Then $\bigcup_{k=0}^{n+1} \Delta_k = \mathcal{GM}(C_t \vee l)$. Clearly Δ_0 is collapsible. Now match each Δ_j for $1 < j < n$ by the following:

If $j = 2k-1$, match each m-simplex of the form

$$\{(v_{1+3(k-1)})v_{0+3(k-1)}(v_{3+3(k-1)})v_{2+3(k-1)} \ldots\}$$

to the corresponding $m+1$-simplex of the form

$$\{(v_{1+3(k-1)})v_{0+3(k-1)}(v_{2+3(k-1)})v_{1+3(k-1)}(v_{3+3(k-1)})v_{2+3(k-1)} \ldots\}$$

If $j = 2k$, match each m-simplex of the form

$$\{(v_{t-3(k-1)})v_{(t-1)-3(k-1)}(v_{(t-2)-3(k-1)})v_{(t-3)-3(k-1)} \ldots\}$$

to the corresponding $m+1$-simplex of the form

$$\{(v_{t-3(k-1)})v_{(t-1)-3(k-1)}(v_{(t-1)-3(k-1)})v_{(t-2)-3(k-1)}(v_{(t-2)-3(k-1)})v_{(t-3)-3(k-1)} \ldots\}$$

Thus all simplicies in Δ_j for $1 < j < n$ have been matched.

Now we must match simplices in Δ_{n+1}. We consider three cases:

Case 1: Let $t = 3n$. Then $\Delta_{n+1} = \emptyset$, and thus $\mathcal{GM}(C_t \vee l) \simeq *$.

Case 2: Let $t = 3n+1$. Then Δ_{n+1} only contains one $2n$-simplex. Thus $\mathcal{GM}(C_t \vee l) \simeq \mathbb{S}^{2n}$.

Case 3: Let $t = 3n+2$. Then Δ_{n+1} only contains one $2n+1$-simplex. Thus $\mathcal{GM}(C_t \vee l) \simeq \mathbb{S}^{2n+1}$. □

The homotopy type of the complex of discrete Morse functions of $C_t \vee \ell$ was computed in [4, Proposition 5.6]. It turns out to be the same as the homotopy type of the Generalized complex of discrete Morse functions of $C_t \vee \ell$. We thus have

Corollary 4.5. *Let $C_t \vee \ell$ be a cycle with a leaf. Then,*

$$\mathcal{GM}(C_t \vee \ell) \simeq \mathcal{M}(C_t \vee \ell).$$

A collapse of $\mathcal{GM}(C_t \vee \ell)$ onto $\mathcal{M}(C_t \vee \ell)$ can be seen by considering the closed V-paths in $\mathcal{GM}(C_t \vee \ell)$ that are added to $\mathcal{M}(C_t \vee \ell)$. We see that there are four such V-paths: a clockwise cycle, a counterclockwise cycle, a clockwise cycle with an inward facing arrow on the leaf, and a counterclockwise cycle with an inward facing arrow on the leaf. By matching the clockwise cycle with the clockwise cycle with an inward facing arrow on the leaf and also matching the counterclockwise cycle to the counterclockwise cycle with an inward facing arrow on the leaf, we have collapsed $\mathcal{GM}(C_t \vee \ell)$ back into $\mathcal{M}(C_t \vee \ell)$, showing a homotopy equivalence.

5. Homotopy type of the matching complex

A well-known complex associated to a graph is the matching complex.

Definition 5.1. *Let the **matching complex** of a graph, G, denoted* $\mathrm{M}(G)$*, is a simplicial complex with vertices given by edges of G and faces given by matchings of G, where a matching is a subset of edges $H \subseteq E(G)$ such that any vertex $v \in V(H)$ has degree at most* 1*.*

The homotopy types of the matching complexes of the path and cycle were computed in [10]. As in the case of the complex of discrete Morse functions for a path, we provide an alternate proof of these computations using discrete Morse theory, the cluster lemma, and star clusters. Then we provide a new result, computing the homotopy type of the matching complex for Dutch windmill graphs. We first make the following simple but useful observation. As observed in [2], $\mathcal{GM}(G) \cong \mathrm{M}(\mathrm{sd}(G))$ for G any graph. Thus the results in section 4 hold for the matching complex on the barycentric subdivision of the graph in question. It was furthermore proved in [4, Proposition 3.5] that if a graph G has two leaves sharing a common vertex, then the complex of discrete Morse functions is contractible. The same result holds for the generalized complex of discrete Morse functions. We thus have the following:

Corollary 5.2. *If a graph G has two leaves sharing a common vertex, then* $\mathrm{M}(\mathrm{sd}(G))$ *is contractible.*

Proposition 5.3. *Let P_t be a path on $t \geq 3$ vertices. Then*

$$\mathrm{M}(P_t) \simeq \begin{cases} \mathbb{S}^{n-1} & \text{if } t = 3n \\ \mathbb{S}^{n-1} & \text{if } t = 3n+1 \\ * & \text{if } t = 3n+2 \end{cases}$$

Proof. We apply the Cluster Lemma. In order to do so, we decompose $\mathrm{M}(P_t)$ into collections Δ_k. First, we construct collections of sub-simplices σ_i. We construct collections as follows:

(1) Let $\sigma_0 := \mathrm{SC}(\{\bigcup_{i=0}^{k}(v_{3i}v_{3i+1})\})$, $k \leq n$
(2) Let $\sigma_1 := \mathrm{st}\{(v_1v_2)\}$

Let $\Delta_0 := \sigma_0$ and $\Delta_1 := \sigma_1 - \sigma_0$. Now any maximal matching of P_t contains either v_0v_1 or v_1v_2. If it contains v_0, then it is in Δ_0. If it contains v_1v_2, then it is in Δ_1. Hence $\Delta_0 \cup \Delta_1 = \mathrm{M}(P_t)$ so that we define an acyclic matching on Δ_0, Δ_1 and apply the Cluster Lemma.

Now Δ_0 is flag so it is collapsible by Proposition 2.5 and Lemma 3.5. To construct a matching on Δ_1, we consider three cases:

Case 1: Let $t = 3n$. Then Δ_1 is a single simplex given by $\{\bigcup_{i=0}^{n-1}(v_{3i+1}v_{3i+2})\}$. Hence this corresponds to an $(n-1)$-simplex in the complex of discrete Morse functions and thus is critical so that $\mathrm{M}(P_{3n}) \simeq \mathbb{S}^{n-1}$.

Case 2: Let $t = 3n+1$. As in Case 1, Δ_1 is a single matching given by $\{\bigcup_{i=0}^{n-1}(v_{3i+1}v_{3i+2})\}$. This matching corresponds to a critical $(n-1)$-simplex in the complex of discrete Morse functions and thus $\mathrm{M}(P_{3n+1}) \simeq \mathbb{S}^{n-1}$.

Case 3: Let $t = 3n+2$. Then $\Delta_1 = \emptyset$. Thus $\mathrm{M}(P_{3n+2}) \simeq *$. □

We also provide an alternate proof for computing the homotopy type of the matching complex of the cycle using the same technique and a similar matching.

Proposition 5.4. *Let C_t be a cycle on $t \geq 3$ vertices. Then*

$$\mathrm{M}(C_t) \simeq \begin{cases} \mathbb{S}^{n-1} \vee \mathbb{S}^{n-1} & \text{if } t = 3n \\ \mathbb{S}^{n-1} & \text{if } t = 3n+1 \\ \mathbb{S}^{n} & \text{if } t = 3n+2 \end{cases}$$

Proof. As usual, we apply the Cluster Lemma by first constructing collections of subsimplices σ_i.

(1) Let $\sigma_0 := \mathrm{SC}(\{\bigcup_{i=0}^{k}(v_{3i}v_{3i+1})\})$, $k \leq n$ ($k \leq n-1$ when $t = 3n+1$)
(2) Let $\sigma_1 := \mathrm{st}\{(v_{(t-1)}v_0)\}$
(3) Let $\sigma_2 := \mathrm{st}\{(v_1v_2)\}$

Define $\Delta_0 := \sigma_0, \Delta_1 := \sigma_1 - \sigma_0$, and $\Delta_2 := \sigma_2 - (\sigma_0 \cup \sigma_1)$. Since every matching of C_t is in one of the σ_i, it follows that $\Delta_0 \cup \Delta_1 \cup \Delta_2 = \mathrm{M}(C_t)$. To define an acyclic matching on the Δ_i, we first observe that Δ_0 is collapsible.

The matchings on both Δ_1 and Δ_2 are considered in three cases:

Case 1: Let $t = 3n$. In Δ_1, there exists one $(n-1)$-simplex, $\{\bigcup_{i=0}^{k}(v_{2+3i}v_{3+3i})\}$. Thus it cannot be matched so it it critical. In Δ_2, there exists one $(n-1)$-simplex of the form $\{\bigcup_{i=0}^{k}(v_{1+3i}v_{2+3i})\}$ which also cannot be matched. Thus $\mathrm{M}(C_{3n}) \simeq \mathbb{S}^{n-1} \vee \mathbb{S}^{n-1}$.

Case 2: Let $t = 3n+1$. Any $(n-1)$-simplex V in Δ_1 does not contain $\{(v_{t-3}v_{t-2})\}$ so we match V with $V \cup \{(v_{t-3}v_{t-2})\}$. This yields a perfect acyclic matching on Δ_1. Now there is only one simplex in Δ_2; namely, the $(n-1)$-simplex $\{\bigcup_{i=0}^{n-1}(v_{3i+1}v_{3i+2})\}$. This $(n-1)$-simplex is critical, hence $\mathrm{M}(C_{3n+1}) \simeq \mathbb{S}^{n-1}$.

Case 3: Let $t = 3n+2$. For each $(n-1)$-simplex V of Δ_1, there is exactly one k, $0 \leq k \leq n-1$, such that both $v_{3k+1}v_{3k+2}$ and $v_{3k+2}v_{3k+3}$ are not in V. Match this V with $V \cup \{v_{3k+2}v_{3k+3}\}$. Then there is one n-simplex left unmatched, namely, $\{\bigcup_{i=0}^{n}(v_{3i+1}v_{3i+2})\}$. Observe that Δ_2 is empty, and thus $\mathrm{M}(C_{3n+2}) \simeq \mathbb{S}^n$. □

Definition 5.5. *A **centipede graph**, $\mathscr{C}_t$ is a graph obtained by adding a leaf to each vertex on a path P_t. If $v_0, \dots, v_{t-1}$ are the vertices of P_t, denote the vertex of the leaf added to v_i by v_i'.*

Proposition 5.6. *Let $\mathscr{C}_t$ be a centipede graph. Then*

$$\mathrm{M}(\mathscr{C}_t) \simeq \begin{cases} \mathbb{S}^{n-1} & \text{if } t = 2n \\ * & \text{if } t = 2n+1 \end{cases}$$

Proof. Let $\mathscr{C}_t$ be a centipede graph. We apply the Cluster Lemma and construct collections as follows:

(1) Let $\sigma_0 := \mathrm{SC}(\{\bigcup_{i=0}^{t-1}(v_iv_i')\})$
(2) Let $\sigma_1 := \mathrm{st}((v_0v_1))$

Define $\Delta_0 := \sigma_0$ and $\Delta_1 := \sigma_1 - \sigma_0$ so that $\Delta_0 \cup \Delta_1 = \mathrm{M}(\mathscr{C}_t)$. Define an acyclic matching on each Δ_i as follows:

We know Δ_0 is collapsible by Proposition 2.5 and Lemma 3.5.

For Δ_1, we have two cases:

Case 1: Let $t = 2n$. Then the only element in Δ_1 is $\{\bigcup_{j=0}^{n-1}(v_{2j}v_{2j+1})\}$, an $(n-1)$-simplex. Hence $\mathrm{M}(\mathscr{C}_t) \simeq \mathbb{S}^{n-1}$.

Case 2: Let $t = 2n+1$. Then $\Delta_1 = \emptyset$. Thus $\mathrm{M}(\mathscr{C}_t) \simeq *$. □

Definition 5.7. *Let D_m^n be a **Dutch windmill graph**. D_m^n is obtained by taking n copies of the cycle C_m and joining them at a common vertex.*

Theorem 5.8. *Let D_m^n be a Dutch windmill graph. Then*

$$\mathrm{M}(D_m^n) \simeq \begin{cases} * & \text{if } m = 3k \\ \mathbb{S}^{nk-1} & \text{if } m = 3k+1 \\ \bigvee^{2n-1}\mathbb{S}^{nk} & \text{if } m = 3k+2 \end{cases}$$

Proof. Let D_m^n be the Dutch windmill graph with center vertex v_0 and for each of the n cycles C_m, let $v_{(j)_i}$ denote vertex j of cycle i, $0 \le j \le m-1$ and $1 \le i \le n$. We apply the Cluster lemma by defining the following collections:

(1) Let $\sigma_0 := \mathrm{SC}\{\bigcup_{i=1}^{n}(\bigcup_{j=0}^{k-1}(v_{(3j+1)_i}v_{(3j+2)_i}))\}$
(2) For $1 \le \nu \le k-1$, let $\sigma_\nu := \bigcup_{i=1}^{n}(\mathrm{st}(\bigcup_{j=1}^{k-\nu}(v_{(3j)_i}v_{(3j+1)_i})))$
(3) Let $\sigma_k := \bigcup_{i=1}^{n}(\mathrm{st}(\bigcup_{j=0}^{k-1}(v_{(3j+2)_i}v_{(3j+3)_i})))$

Define $\Delta_0 := \sigma_0$ and $\Delta_\beta := \sigma_\beta - \bigcup_{\alpha=0}^{\beta-1}\sigma_\alpha$. Then $\bigcup_{\beta=0}^{k}\Delta_\beta = \mathrm{M}(D_m^n)$. We now define an acyclic matching on each Δ_β as follows:

We know Δ_0 is collapsible by Proposition 2.5 and Lemma 3.5. Observe that $\Delta_1, \dots, \Delta_k = \emptyset$ for $m = 3k$, which implies that $\mathrm{M}(D_m^n) \simeq *$.

Hence, suppose $m \ne 3k$. Let $1 \le \nu \le k-1$ and consider Δ_ν. For each $1 \le i \le n$, we match $\bigcup_{j=1}^{k-\nu}(v_{(3j)_i}v_{(3j+1)_i})$ with $(v_{(3(k-\nu)+2)_i}v_{(3(k-\nu)+3)_i}) \cup \bigcup_{j=1}^{k-\nu}(v_{(3j)_i}v_{(3j+1)_i})$. This produces an acyclic matching for all gradient vector fields in Δ_ν. It remains to put a matching on to Δ_k.

For Δ_k, we consider cases:

Case 1: Let $m = 3k+1$. Then Δ_k has one element, namely,

$$\bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i}).$$

This is an $(nk-1)$-unmatched simplex, so it is critical, and thus $\mathrm{M}(D_m^n) \simeq \mathbb{S}^{nk-1}$.

Case 2: Let $m = 3k+2$. Then Δ_k has $2n+1$ elements which are given by

$$\bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i})$$

For each $1 \le \ell \le n$, $(v_{(0)_\ell}v_{(1)_\ell}) \cup \bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i})$

For each $1 \le \ell \le n$, $(v_{(0)_\ell}v_{(m-1)_\ell}) \cup \bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i})$

We can only create one matching, namely, we match

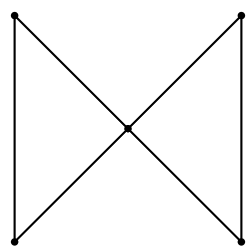
$$\bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i}) \text{ with } (v_{(0)_1}v_{(1)_1}) \cup \bigcup_{i=1}^{n}\bigcup_{j=0}^{k-1}(v_{(2+3j)_i}v_{(3+3j)_i}).$$

This leaves $2n-1$ (nk)-simplices unmatched. Thus, $\mathrm{M}(D^n_m) \simeq \vee^{2n-1}\mathbb{S}^{nk}$.

□

6. Future directions and potential pursuits

Open Question 1. *One direction that seems to hold great potential for computing the homotopy type of the complex of discrete Morse functions concerns the relationship between the homotopy type of the complex of discrete Morse functions and the generalized complex of discrete Morse functions. We argued using two elementary collapses that $\mathcal{GM}(C_t \vee l) \simeq \mathcal{M}(C_t \vee l)$. Because the former is a flag complex, its homotopy type should theoretically be easier to compute. Consider another example*



Using the Cluster Lemma starting with a star collapse, we can apply a matching to $\mathcal{M}(D^2_3)$, finding that its homotopy type is collapsible. Additionally, using the Cluster Lemma starting with a star cluster collapse, we can apply a matching to $\mathcal{GM}(D^2_3)$ to compute the homotopy type of a point. We would like to question whether there is a way to use the generalized complex of discrete Morse functions as a tool for computing the homotopy type of the complex of discrete Morse functions for certain complexes.

Open Question 2. *One way to use the homotopy type of the generalized complex of discrete Morse functions to determine the homotopy type of the complex of discrete Morse functions is to show that the former collapses to the later. This is in general not always possible since, the homotopy type of the Generalized complex of discrete Morse functions of a cycle computed in Theorem 4.3, does not agree with the homotopy type of the cycle of the complex of discrete Morse functions. However, one can use the matching found in the proof of Theorem 4.3, throw out the closed V-paths in the matching, and obtain a matching on the complex of discrete Morse functions. In this case, the critical cells occur in different dimensions so the homotopy type is not uniquely determined. However, there may be special cases where the homotopy type can be recovered from knowledge of the critical cells* *and some other information. See, for example,* [11, Theorem 2.2]. *In particular, the homotopy type of the complex of discrete Morse functions of the* $3-$*simplex remains unknown. Chari and Joswig* [3] *showed that the* $3-$*simplex satisfies* $\{b_0 = 1, b_5 = 99\}$ *using software. While we cannot use star clusters to collapse the complex of discrete Morse functions of the n-simplex, can we create a matching on the generalized complex of discrete Morse functions and then remove the cyclic gradient vector fields from the matching? Or, can a similar matching strategy provide further insight on how to apply a matching to the complex of discrete Morse functions of the* $n-$*simplex?*

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