Probability Surveys
Vol. 11 (2014) 121–160
ISSN: 1549-5787
DOI: 10.1214/13-PS214

# Statistical properties of zeta functions' zeros

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**Abstract:** The paper reviews existing results about the statistical distribution of zeros for three main types of zeta functions: number-theoretical, geometrical, and dynamical. It provides necessary background and some details about the proofs of the main results.

**AMS 2000 subject classifications:** 11M26, 11M50, 62E20.
**Keywords and phrases:** Riemann's zeta, Selberg's zeta, Ruelle's zeta, Montgomery's conjecture, distribution of zeros.

Received February 2013.

## Contents

## 1. Introduction

The distribution of zeros of Riemann's zeta function is one of the central problems in modern mathematics. The famous Riemann conjecture states that all of these zeros are on the critical line $\mathrm{Re}s = 1/2$, and the accumulated numerical evidence supports this conjecture as well as a more precise statement that these zeros behave like eigenvalues of large random Hermitian matrices. While these statements are still conjectural, a great deal is known about the statistical properties of Riemann's zeros and the zeros of closely related functions. In this report we aim to summarize findings in this research area.

We give necessary background information, and we cover the three main types of zeta functions: number-theoretical, Selberg-type, and dynamical zeta functions.

Some interesting and important topics are left outside of the scope of this report. For example, we do not discuss quantum arithmetic chaos or characteristic polynomials of random matrices.

The paper is divided in three main sections according to the type of the zeta function we discuss. Inside each section we tried to separate the discussion of the properties of zeros at the global and local scales.

Let us briefly describe these types of zeta functions and their relationships. First, the number-theoretical zeta functions come from integers in number fields and their generalizations. Due to the additive and multiplicative structures of the integers, and in particular due to the unique decomposition in prime factors, the zeta functions have the Euler product formula $\zeta(s) = \prod_{\mathfrak{p}}(1 - (N\mathfrak{p})^s)^{-1}$ and a functional equation, $\zeta(1-s) = c(s)\zeta(s)$, with the multiplier $c(s)$ equal to a ratio of Gamma functions.

It is an important discovery of Hecke that one can define number-theoretical zeta functions in a different, and potentially more general way if one starts with modular forms, which are functions on the space of two-dimensional lattices that are invariant relative to the change of scale. If they are considered as functions of the basis $(z, 1)$ then they become functions of $z$ invariant relative to an action of the group $SL_2(\mathbb{Z})$. They are periodic and hence can be written as $\sum_{n\geq 0} c_n \exp(2\pi i n z)$, where $z$ is the ratio of the periods of the lattice. If $c_0 = 0$, then one can define a zeta function $\sum c_n n^{-s}$. It turns out that this zeta function satisfies a functional equation. In addition, if the original modular form is an eigenvector for certain operators (the Hecke operators), then the zeta function will have the Euler product property.

It appears that all number-theoretical zeta functions can be obtained by using this construction. However, the class of modular zeta functions is wider. Indeed, zeta functions with an arithmetic flavor can come not only from number fields but also from other algebraic objects, for example, from elliptic curves. A significant development occurred recently when it was proved that all zeta functions associated with elliptic curves come from modular zeta functions. Among other applications, this discovery was a key to the proof of Fermat's last theorem. (The existence of a non-trivial solution for $x^n + y^n = z^n$, $n > 2$, would imply the existence of a non-modular zeta function for an elliptic curve.)

An important representative of the second class of zetas is Selberg's zeta function, which is essentially a generating function for the lengths of closed geodesics on a surface with constant negative curvature. In more detail, let $\mathbb{H}$ be the upper half-plane with the hyperbolic metric and $\Gamma$ be a discrete subgroup of $SL_2(\mathbb{Z})$. Selberg showed that certain sums over eigenvalues of the Laplace operator on Riemann surface $\Gamma\backslash\mathbb{H}$ can be related to sums over closed geodesics of $\Gamma\backslash\mathbb{H}$. This relation is called Selberg's trace formula. Selberg's zeta function is constructed in such a way that it has the same relation to Selberg's trace formula as Riemann's zeta function has to the so-called "explicit formula" for sums over Riemann's zeros.

While Selberg's zeta function resembles Riemann's zeta in some features, there are significant differences. In particular, the statistical behavior of its zeros depends on the group $\Gamma$ and often it is significantly different from the behavior of Riemann's zeros.

The third class of zetas, the dynamical zeta functions, are generating functions for the lengths of closed orbits of a map $f$ that sends a set $M$ to itself. The most spectacular example of these functions is Weil's zeta functions of algebraic varieties over finite fields, which can be described as follows.

Let $M$ be the algebraic closure of an algebraic variety embedded in $\mathbb{F}_q^n$ (where $\mathbb{F}_q$ is a finite field), and let $f$ be the Frobenius map: $(x_1, \ldots, x_n) \to (x_1^q, \ldots, x_n^q)$. Then, the dynamic zeta associated to $f$ is called Weil's zeta function. These functions are remarkable since the Riemann conjecture is proved for them: It is known that their zeros are located on a circle that corresponds to the line $\mathrm{Re}z = 1/2$. Moreover, much is known about the statistical distribution of these zeros.

A particular case of these zeta functions, Weil's zeta functions for curves, can be understood as number-theoretic zeta functions for finite extensions of the field $\mathbb{F}_q(x)$. Hence, Weil's zeta functions work as a bridge between dynamic and number-theoretic zeta functions. In addition, Selberg's zeta function can be understood as the dynamic zeta function for the geodesic flow on the surface $\Gamma\backslash\mathbb{H}$. This shows that the three classes of zeta functions are intimately related to each other.

With this overview in mind, we now come to a more detailed description of available results about the statistics of zeta function zeros.

## 2. Number-theoretical zetas

### *2.1. Riemann's zeta*

There are several excellent sources on Riemann's zeta and Dirichlet L-functions, for example the books by Davenport [9] and Titchmarsh [59]. In addition, a very good reference for all topics in this report is provided by Iwaniec and Kowalski's book [26].

By definition Riemann's zeta function is given by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

for $\mathrm{Re} s > 1$. It can be analytically continued to a meromorphic function in the entire complex plane and it satisfies the functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{1}{2}s\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1}{2}(1-s)\right) \zeta(1-s). \tag{2}$$

Indeed, we can relate $\zeta(s)$ to the series $\theta(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$ :

$$\frac{\Gamma\left(\frac{1}{2}s\right)\zeta(s)}{\pi^{s/2}} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{s/2-1} e^{-n^2 \pi x} dx = \int_0^{\infty} x^{s/2-1} \theta(x)\, dx.$$

(As an aside remark, this representation can be used as a starting point for some surprising connections of the Riemann zeta function with the Brownian motion and Bessel processes, see [3].) From the identities for the Jacobi theta-functions, implied by the Poisson summation formula, it follows that

$$2\theta(x) + 1 = \frac{1}{\sqrt{x}} \left(2\theta\left(\frac{1}{\sqrt{x}}\right) + 1\right). \tag{3}$$

Writing

$$\int_0^{\infty} x^{s/2-1} \theta(x)\, dx = \int_0^{1} x^{s/2-1} \theta(x)\, dx + \int_1^{\infty} x^{s/2-1} \theta(x)\, dx,$$

and applying the identity to the integral from 0 to 1, we find that

$$\frac{\Gamma\left(\frac{1}{2}s\right)\zeta(s)}{\pi^{s/2}} = \frac{1}{s(s-1)} + \int_1^{\infty} \left(x^{-s/2-1/2} + x^{s/2-1}\right) \theta(x)\, dx, \tag{4}$$

which is symmetric relative to the change $s \to 1-s$.

In addition, formula (4) implies that the function $s(s-1)\pi^{-s/2}\Gamma(\frac{1}{2}s)\zeta(s)$ is entire. Hence, the only pole of $\zeta(s)$ is at $s = 1$, and $\zeta(s)$ has zeros at $-2$, $-4$, ..., that correspond to poles of $\Gamma(\frac{1}{2}s)$. These zeros are called *trivial*, while all others are called *non-trivial*. We will order the non-trivial zeros according to their imaginary part and denote them by $\rho_k$. By the functional equation, $\rho_k$ are located symmetrically relative to the line $\mathrm{Re} s = 1/2$, which is called *critical*, and it is known that $0 < \mathrm{Re}\rho_k < 1$. The Riemann's hypothesis asserts that all non-trivial zeros are on the critical line.

The second fundamental property of Riemann's zeta is the Euler product formula:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{5}$$

valid for $\mathrm{Re}s > 1$. It follows from the existence and uniqueness of prime factorization. This formula is a starting point for a very important idea which relates the sums over prime numbers and sums over zeta function zeros. For example, the Riemann-von Mangoldt formula says that

$$\sum_{n \le x} \Lambda(n) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n} \frac{x^{-2n}}{2n} - \frac{\zeta'}{\zeta}(0), \tag{6}$$

where $\Lambda(n) = \log p$, if $n$ is a prime $p$ or a power of $p$, and otherwise $\Lambda(n) = 0$. (The order of summation over zeros can be important here, so it is assumed that in computing $\sum_{\rho} \frac{x^{\rho}}{\rho}$, one takes all zeros with imaginary part between $-T$ and $T$, and then let $T \to \infty$. In addition, if $x$ is a prime power the formula has to be modified by subtracting $\frac{1}{2}\Lambda(x)$ on the left-hand side.)

The idea of the proof of (6) is to take the logarithmic derivative of (5):

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}, \tag{7}$$

and then to use the following formula with $y = x/n$ and $c > 0$:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$$

One integrates (7) against the test function $x^s/s$, in order to pick out the terms in the series in (7) with $n \le x$.

From (7), one gets

$$\sum_{n \le x} \Lambda(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[ -\frac{\zeta'}{\zeta}(s) \right] x^s \frac{ds}{s}.$$

Moving the line of integration away to infinity on the left and collecting the residues at the poles one finds formula (6). (See Chapter 17 in Davenport [9] for a detailed proof.)

By a similar method one can obtain:

$$\sum_{n \le x} \frac{\Lambda(n)}{n^s} = \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho-s}}{\rho - s} + \sum_{n} \frac{x^{-2n-s}}{2n+s} - \frac{\zeta'}{\zeta}(s). \tag{8}$$

While formula (6) is useful to study the distribution of primes if something is known about the distribution of zeros, formula (8) can be used in the reverse direction to study the behavior of $\frac{\zeta'}{\zeta}(s)$ if some information is known about primes.

Selberg discovered a variant of this formula that avoids the problem of conditional convergence in the sum over the zeta zeros. Define

$$\Lambda_x(n) = \begin{cases} \Lambda(n), & \text{for } 1 \le n \le x, \\ \Lambda(n) \left( \frac{\log^2 \frac{x^3}{n} - 2\log^2 \frac{x^2}{n}}{2\log^2 x} \right), & \text{for } x \le n \le x^2, \\ \Lambda(n) \frac{\log^2 \frac{x^3}{n}}{2\log^2 x}, & \text{for } x^2 \le n \le x^3. \end{cases}$$

Then, (Lemma 10 in [51] )

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\le x^3}\frac{\Lambda_x(n)}{n^s}+\frac{1}{\log^2 x}\sum_\rho \frac{x^{\rho-s}\left(1-x^{\rho-s}\right)^2}{(s-\rho)^3} + \frac{x^{1-s}\left(1-x^{1-s}\right)^2}{\log^2 x\cdot(1-s)^3}+\frac{1}{\log^2 x}\sum_{q=1}^{\infty}\frac{x^{-2q-s}\left(1-x^{-2q-s}\right)^2}{(2q+s)^3}. \tag{9}$$

Another useful variant is sometimes called Weil's and sometimes Delsarte's explicit formula (see [8] and [41]). Suppose that $H(s)$ is an analytic function in the strip $-c \le \operatorname{Im} s \le 1+c$ (for $c>0$) and that $|H(\sigma+it)| \le A(1+|t|)^{-(1+\delta)}$ uniformly in $\sigma$ in the strip. Let $h(t)=H(\frac{1}{2}+it)$ and define

$$\widehat{h}(x)=\int_{\mathbb{R}} h(t)\,e^{-2\pi i t x}dt.$$

(Note that analyticity of $H(s)$ implies that $\widehat{h}(x)$ has finite support.) Then,

$$\sum_\rho H(\rho) = H(0)+H(1)-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left[\widehat{h}\left(\frac{\log n}{2\pi}\right)+\widehat{h}\left(-\frac{\log n}{2\pi}\right)\right] - \frac{1}{2\pi}\int_{-\infty}^{\infty}h(t)\,\Psi(t)\,dt, \tag{10}$$

where

$$\Psi(t)=\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-it\right).$$

The idea of the proof is similar to the proof of the Riemann-von Mangoldt formula. One starts with the formula

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left[-\frac{\zeta'}{\zeta}(s)\right]H(s)\,ds = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2}}\frac{1}{2\pi}\int_{-\infty}^{\infty}h(t)\,e^{-i(\log n)t}dt = -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2}}\widehat{h}\left(\frac{\log n}{2\pi}\right).$$

Next, one can move the line of integration to $(-1-i\infty,-1+i\infty)$ and use the calculus of residues and the functional equation to obtain formula (10).

A wonderful illustration for the significance of explicit formulas is given by Landau's formula:

$$\sum_{0<\operatorname{Im}\rho<T} x^\rho = -\frac{T}{2\pi}\Lambda\left(\sqrt{x}\right)+O(\log T).$$

Assuming Riemann's hypothesis, we write $\rho=\frac{1}{2}+i\gamma$, and then

$$\sum_{0<\gamma<T}\cos(\gamma\log x) = -\frac{T}{2\pi}\frac{\Lambda(x)}{x}+O(\log T).$$

Note that the right hand side has a spike when $x$ is a prime power. This is illustrated in Figure 1.

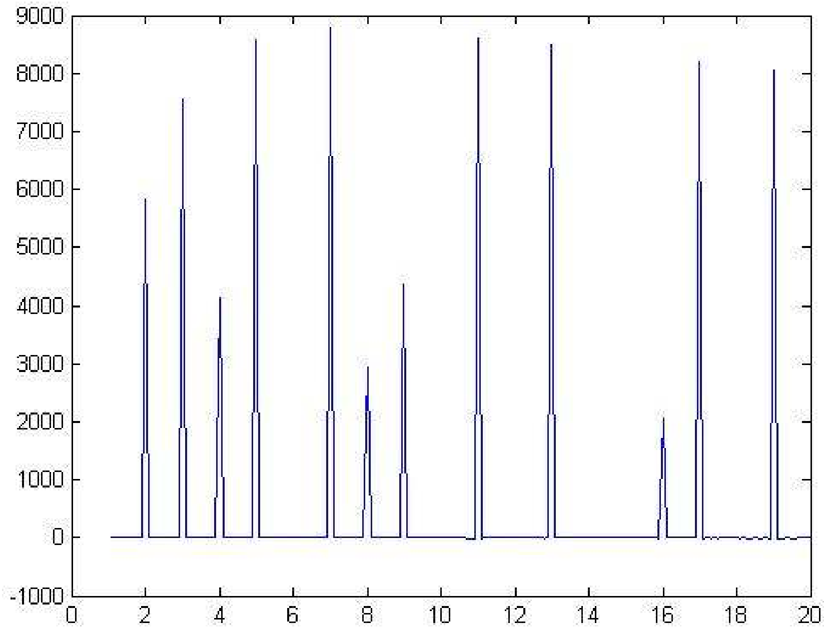


FIG 1. *Sums of* $\cos(\gamma \log(x))$ *over the first* 100,000 *zeros (*$T \approx 75 \times 10^3$*). The horizontal axis shows* $x$.

The most general explicit formula was derived by Weil (see [61]). We will not present it here since it involves adelic language and this would take us too far afield.

### 2.1.1. Statistics of zeros on global scale

Let $\mathcal{N}(T)$ denote the number of zeros with the imaginary part strictly between 0 and $T$. If there is a zero with imaginary part equal to $T$, then we count this zero as $1/2$. Define

$$S(T) := \frac{1}{\pi}\mathrm{Im}\log\zeta\left(\frac{1}{2}+iT\right),$$

where the logarithm is calculated by continuous variation along the contour $\sigma + iT$, with $\sigma$ changing from $+\infty$ to $1/2$.

By applying the argument principle to $\zeta$ and utilizing the functional equation, it is possible to show (see Chapter 15 in [9]) that

$$\mathcal{N}(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + S(T) + O\left(\frac{1}{1+T}\right).$$

Let

$$X(t) := \frac{\sqrt{2}\pi S(t)}{\sqrt{\log\log t}}.$$

Then, we have the following theorem by Selberg. (See Theorem 6 in [51].)

**Theorem 2.1 (Selberg)** *Let $T^a \le H \le T^2$, where $\frac{1}{2} < a \le 1$. Then for every $k \ge 1$*

$$\frac{1}{H}\int_T^{T+H} |X(t)|^{2k}\, dt = \frac{2k!}{k!2^k} + O((\log\log T)^{-1/2}),$$

*with the constant in the remainder term depending on $k$ and $a$ only.*

In other words, if $t$ is chosen randomly in the interval $[0, T]$ and $T \to \infty$, then $X(t)$ converges in distribution to a Gaussian random variable. Note that the Riemann Hypothesis is not assumed in this result. Under the assumption of the Riemann Hypothesis, Selberg proved an analogous result with a better error term assuming only $a > 0$ (Theorem 3 in [52]).

Recently, Selberg's result was generalized by Bourgade, who determined the correlation structure of $X(t)$ on relatively small scales. The basis for this development is the following result (cf. Theorem 1.1 in [7]). Take a large $t > 0$ and small $\varepsilon_t \ge 0$ and look at $l$ points on the line $\mathrm{Re} z = \frac{1}{2} + \varepsilon_t$ with imaginary parts $\omega t + f_t^{(i)}$, $1 \le i \le l$, where $\omega$ is a random variable uniformly distributed on $[0, 1]$. Note that the randomness $\omega$ is the same for all points so the distance between the points is measured by offsets $f_t^{(i)}$. We assume that $\varepsilon_t \to 0$, so the points approach the critical line as $t \to \infty$. The first case is when $\varepsilon_t \gg 1/\log t$ and we look at the scale $|f_t^{(j)} - f_t^{(i)}| \approx \varepsilon_t^{c_{ij}}$, with $c_{ij} \ge 0$. The second case is when $\varepsilon_t \ll 1/\log t$, for example, $\varepsilon_t = 0$. In this case, we look at the scale $|f_t^{(j)} - f_t^{(i)}| \approx (1/\log t)^{c_{ij}}$. It turns out that after a proper normalization the values of the logarithm of the Riemann zeta function at this cluster of points converge to a non-trivial multivariate Gaussian distribution.

**Theorem 2.2 (Bourgade)** *Let $\omega$ be uniform on $(0, 1)$, $\epsilon_t \to 0$, $t \to \infty$, $\epsilon_t \gg 1/\log t$, and functions $0 \le f_t^{(1)} < \cdots < f_t^{(l)} < c < \infty$. Suppose that for all $i \ne j$,*

$$\frac{\log\left|f_t^{(j)} - f_t^{(i)}\right|}{\log \epsilon_t} \to c_{i,j} \in [0, \infty]\,. \tag{11}$$

*Then the vector*

$$\frac{1}{\sqrt{-\log \epsilon_t}}\left(\log\zeta\left(\frac{1}{2} + \varepsilon_t + i f_t^{(1)} + i\omega t, \ldots, \frac{1}{2} + \varepsilon_t + i f_t^{(l)} + i\omega t\right)\right) \tag{12}$$

*converges in law to a complex Gaussian vector $(Y_1, \ldots, Y_l)$ with the zero mean and covariance function*

$$\mathrm{Cov}\,(Y_i, Y_j) = \begin{cases} 1 & \textit{if } i = j, \\ 1 \wedge c_{i,j} & \textit{if } i \ne j. \end{cases}$$

*Moreover, the above result remains true if $\epsilon_t \ll 1/\log t$, replacing the normalization $-\log \varepsilon_t$ with $\log\log t$ in (11) and (12).*

This theorem implies the following result for the zeros of the Riemann zeta function (cf. Corollary 1.3 in [7]). Let

$$\Delta(t_1, t_2) = \left(\mathcal{N}(t_2) - \frac{t_2}{2\pi}\log\frac{t_2}{2\pi e}\right) - \left(\mathcal{N}(t_1) - \frac{t_1}{2\pi}\log\frac{t_1}{2\pi e}\right),$$

which represents the number of zeros with the imaginary part between $t_1$ and $t_2$ minus its deterministic prediction. Then the claim is that this excess number of zeros in an interval with the length of order $(\log t)^{-\delta}$ $(0 < \delta < 1)$ is a Gaussian variable with the variance proportional to $(1-\delta)\log\log t$. Moreover, the limiting Gaussian process has an interesting covariance structure.

**Corollary 2.3 (Bourgade)** *Let $K_t$ be such that, for some $\varepsilon > 0$ and all $t$, $K_t > \varepsilon$. Suppose $\log K_t/\log\log t \to \delta \in [0,1)$, as $t \to \infty$. Then the finite-dimensional distributions of the process*

$$\frac{\Delta\left(\omega t + \alpha/K_t, \omega t + \beta/K_t\right)}{\frac{1}{\pi}\sqrt{(1-\delta)\log\log t}},\ 0 \le \alpha < \beta < \infty,$$

*converge to those of a centered Gaussian process $(\widetilde{\Delta}(\alpha,\beta),\ 0 \le \alpha < \beta < \infty)$ with the covariance structure*

$$\mathbb{E}\left(\widetilde{\Delta}\left(\alpha,\beta\right)\widetilde{\Delta}\left(\alpha',\beta'\right)\right) = \begin{cases} 1 & \text{if } \alpha = \alpha',\ \text{and } \beta = \beta', \\ 1/2 & \text{if } \alpha = \alpha',\ \text{and } \beta \neq \beta', \\ 1/2 & \text{if } \alpha \neq \alpha',\ \text{and } \beta = \beta', \\ -1/2 & \text{if } \beta = \alpha', \\ 0 & \text{elsewhere.} \end{cases}$$

Note that since the average spacing between zeros is $1/\log t$, hence the number of zeros in the interval $(\omega t + \alpha/K_t, \omega t + \beta/K_t)$ is of order $(\log t)^{1-\delta}$.

This result perfectly corresponds to a result of Diaconis and Evans about eigenvalue fluctuations of random unitary matrices (cf. Theorem 6.3 in [10]).

The key to both Selberg and Bourgade's results is Selberg's approximation for the function $S(t)$ (cf. Theorem 4 in [51]).

**Proposition 2.4 (Selberg)** *Suppose $k \in \mathbb{Z}^+$, $0 < a < 1$. Then there exists $c_{a,k} > 0$ such that for any $1/2 \le \sigma \le 1$ and $t^{a/k} \le x \le t^{1/k}$, it is true that*

$$\frac{1}{t}\int_1^t \left|\log\zeta\left(\sigma + is\right) - \sum_{p \le x^3} \frac{1}{p^{\sigma+is}}\right|^{2k} ds \le c_{a,k}.$$

The proof of this statement, in turn, depends on Selberg's formula (9).

### *2.1.2. Statistics of zeros on local scale*

Now, assume the Riemann Hypothesis and suppose that we are interested in calculating local statistics for the pairs of the zeta zeroes, for example, in $\sum_{0<\gamma,\gamma'\le T} r((\gamma-\gamma')\frac{\log T}{2\pi})$, where $r(x)$ is a test function. By representing $r(x)$ as the Fourier transform, we can write this statistic differently,

$$\sum_{0<\gamma,\gamma'\le T} r\left(\left(\gamma-\gamma'\right)\frac{\log T}{2\pi}\right) w\left(\gamma-\gamma'\right) = \frac{T}{2\pi}\log T\int_{-\infty}^{\infty} F\left(\alpha\right)\widehat{r}\left(\alpha\right)d\alpha, \qquad (13)$$

where we added a convenient weighting function $w(u) = 4/(4+u^2)$. Here $\widehat{r}(\alpha) = \int_{-\infty}^{\infty} r(u)e^{-2\pi i\alpha u}du$, and

$$F(\alpha) = F(\alpha, T) := \frac{2\pi}{T\log T} \sum_{0<\gamma,\gamma'<T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma').$$

Consequently, all information about local statistics is encoded in the function $F(\alpha)$. Montgomery proved the following result [40]. (In fact, the estimate holds uniformly throughout $0 \le \alpha \le 1$, as was later proved by Goldston in [15].)

**Theorem 2.5 (Montgomery)** *For real $\alpha$, the function $F(\alpha)$ is real and $F(\alpha) = F(-\alpha)$. If $T > T_0(\varepsilon)$, then $F(\alpha) \ge -\varepsilon$ for all $\alpha$. For fixed $\alpha$ satisfying $0 \le \alpha < 1$, we have*

$$F(\alpha) = (1 + o(1))\, T^{-2\alpha} \log T + \alpha + o(1),$$

*as $T$ tends to infinity; this holds uniformly for $0 \le \alpha \le 1 - \varepsilon$.*

This result allows us to calculate the local statistics for smooth test functions that have their Fourier transforms compactly supported on the interval $[-1, 1]$.

Montgomery conjectures that

$$F(\alpha) = 1 + o(1)$$

for $\alpha \ge 1$, uniformly in bounded intervals. If it is true, then it can be used to calculate the local statistics for a much larger class of test functions. In particular, Montgomery's conjecture can also be formulated as follows.

**Conjecture 2.6 (Montgomery)** *For fixed $\alpha < \beta$,*

$$\sum_{\substack{0<\gamma,\gamma'<T \\ 2\pi\alpha/\log T<\left|\gamma-\gamma'\right|<2\pi\beta/\log T}} 1 \sim \left( \int_{\alpha}^{\beta} \left[ 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 \right] du + \delta(\alpha, \beta) \right) \frac{T}{2\pi} \log T,$$

*as $T$ goes to infinity. Here $\delta(\alpha, \beta) = 1$ if $0 \in [\alpha, \beta]$, $\delta(\alpha, \beta) = 0$ otherwise.*

The proof of Montgomery's theorem is based on the analysis of the following variant of the explicit formula.

**Lemma 2.7 (Montgomery)** *If $1 < \sigma < 2$ and $x \ge 1$ then*

$$\sum_{\gamma} \frac{(2\sigma - 1)x^{i\gamma}}{\left(\sigma - \frac{1}{2}\right) + (t - \gamma)^2} = -x^{-1/2} \left( \sum_{n \le x} \Lambda(n) \left( \frac{x}{n} \right)^{1-\sigma+it} + \sum_{n>x} \Lambda(n) \left( \frac{x}{n} \right)^{\sigma+it} \right)$$
$$+ x^{1/2-\sigma+it} (\log \tau + O_\sigma(1)) + O_\sigma\left( x^{1/2} \tau^{-1} \right),$$

*where $\tau = |t| + 2$. The implicit constants depend only on $\sigma$.*

Set $\sigma = \frac{3}{2}$ and $x = T^{\alpha}$. One computes the integral of the left-hand side:

$$\int_0^T \left| 2 \sum_{\gamma} \frac{x^{i\gamma}}{1 + (t-\gamma)^2} \right|^2 dt = 2\pi F(\alpha, T) T \log T + O\left( (\log T)^3 \right).$$

For the corresponding integral of the right-hand side, one uses the Montgomery-Vaughan formula,

$$\int_0^T \left| \sum_{n=1}^{\infty} \frac{a_n}{n^{it}} \right|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 (T + O(n))$$

and finds that the integral of the right-hand side equals

$$T \log x + O(x \log x) + \frac{T}{x^2} \left[ (\log T)^2 + O(\log T) \right],$$

which gives the claim of the theorem when one substitutes $x = T^{\alpha}$.

For more information about Montgomery conjecture, see [14].

Montgomery's result allows us to compute the statistic (13) for pairs of Riemann's zeta zeros, provided that the Fourier transform of the test function $r$ is supported on the interval $[-1, 1]$. What about statistics of other $k$-tuples of the zeros? First of all, the case of linear statistics ($k = 1$) is similar to the questions considered by Selberg and Bourgade except that now we allow for more general test functions. The main interest here is to see how far we can go in localizing this functions.

In this directions Hughes and Rudnick in [22] studied the distribution of

$$\mathcal{N}_f(t, T) := \sum_{\gamma_j} f\left( \frac{\log T}{2\pi} (\gamma_j - t) \right),$$

where $\gamma_j = \frac{1}{i}(\rho_i - \frac{1}{2})$ and $\gamma_j$ are not assumed real. The function $f$ is a real-valued even function with the smooth compactly-supported Fourier transform. (If $f$ is the indicator function of an interval $[-a, a]$ and if all $\gamma_j$ are real, then $\mathcal{N}_f(t, T)$ counts number of zeros in the interval $[t - a\frac{2\pi}{\log T}, t + a\frac{2\pi}{\log T}]$. However, the Fourier transform of the indicator function does not have compact support.)

Choose a weight function $w(x)$, such that $w \geq 0$, $\int w(x)dx = 1$, and $\widehat{w}(x)$ is compactly supported, and define an averaging operator

$$\langle F \rangle_{T,H} := \int_{\mathbb{R}} F(t) w\left( \frac{t - T}{H} \right) \frac{dt}{H}.$$

**Theorem 2.8 (Hughes-Rudnick)** *Let the averaging window $H = T^a$ for $0 < a \leq 1$, and let be such that $\widehat{f}(u) = \int f(x) e^{-2\pi i x u} dx \in C_c^{\infty}(\mathbb{R})$ and* Supp $f \subset$ *$(-2a/m, 2a/m)$ with integer $m \geq 1$. Then, as $T \to \infty$, the first $m$ moments of $\mathcal{N}_f$, $\langle \mathcal{N}_f^m \rangle_{T,H}$ converge to those of a Gaussian random variable with expectation $\int f(x)dx$ and variance*

$$\sigma_f^2 = \int \min(|u|, 1) \widehat{f}(u)^2 du.$$

Hence, if the frequency of the test function oscillations is bounded (and therefore the function is very smooth and well delocalized in the $x$-space), then the first moments of the linear statistic converge to those of the Gaussian variable. What about higher moments? Hughes and Rudnick show (Theorem 6.5 in [23]) that a similar result holds in the random matrix theory for eigenvalues of a unitary random matrix. For random matrices, higher moments do not converge to Gaussian values (Theorem 7.4 in [23]) Based on this analogy, they conjecture that it is the same for the linear statistics of the zeta function zeros.

We will describe the ideas of the proof of Theorem 2.8 below in the case when they are applied to the zeros of Dirichlet's L-functions.

What about statistics of $k$-tuples of zeros when $k > 2$? This case was considered by Rudnick and Sarnak in [47]. Their results hold for a quite large class of $L-$functions, and we will postpone their discussion to a later section. Briefly, they are similar to Montgomery's results since they show that the behavior of the zeta zeros is very similar to the behavior of eigenvalues of random unitary matrices. Another similarity is that the results are proven under some restrictive conditions on the Fourier transform of the test function. It is an outstanding problem to prove that all results about correlations of zeros hold without these restrictive hypotheses.

### *2.2. Dirichlet's L-functions*

In order to understand the behavior of the Riemann zeta zeros, it is worthwhile to check for which functions their zeros have similar behavior. The simplest example of such a family of functions is provided by Dirichlet's L-functions.

Let $\chi(n)$ denote a multiplicative *character* modulo a positive integer $q$. That is, the function $\chi$ maps integers to the unit circle; it is multiplicative, $\chi(nm) = \chi(n)\chi(m)$, and $\chi(n) = 0$ if $n$ and $q$ are not relatively prime. The character which sends every integer relatively prime to $q$ to 1 is called the *principal character* modulo $q$. The *conductor* of the character is the minimal integer $N$ such that the character is periodic modulo $N$. For simplicity, let $q$ be a prime in the following. In this case the conductor equals $q$. A character is *odd* if $\chi(-1) = -1$, and *even* if $\chi(-1) = 1$.

*The Dirichlet L-function* corresponding to the character $\chi$ is defined by the series

$$L\left(s,\chi\right) = \sum_{n=1}^{\infty} \frac{\chi\left(n\right)}{n^s}.$$

This function has the Euler product representation:

$$\zeta\left(s\right) = \prod_{p} \left(1 - \frac{\chi\left(p\right)}{p^s}\right)^{-1} \tag{14}$$

because of the multiplicativity of $\chi(n)$. Moreover, the argument behind the relation to theta functions (4) can be repeated and as a consequence, one finds that series (2.2)can be continued to a function which is meromorphic in the whole complex plane and satisfies a functional equation. Namely, let $\mu = 0$ if $\chi$ is even and $\mu = 1$ if $\chi$ is odd. Define

$$\Phi(s,\chi) = q^{\frac{1}{2}(s+\mu)}\pi^{-\frac{1}{2}(s+\mu)}\Gamma\left(\frac{1}{2}(s+\mu)\right)L(s,\chi),$$

then the functional equation has the form

$$\Phi(1-s,\overline{\chi}) = \frac{i^{\mu}\sqrt{q}}{\tau(\chi)}\Phi(s,\chi),$$

where $\tau(\chi)$ is the Gauss sum:

$$\tau(\chi) = \sum_{m=1}^{q}\chi(m)\, e^{2\pi i m/q}.$$

The reason for appearance of $\tau(\chi)$ is that the modularity relation (3) becomes more complicated in this case. For proofs, see Chapter 9 in Davenport [9].

Many other properties of the Dirichlet L-functions is similar to that of the Riemann zeta functions. In particular, one can establish similar explicit formulas.

Two notable differences from the Riemann zeta is that (i) if $\chi$ is not principal, then $L(s,\chi)$ is entire; and (ii) if $\chi$ is not even, then a complex conjugate of a zero is not necessarily a zero.

### *2.2.1. Global scale*

For $T > 0$, let $N(T,\chi)$ denote the number of zeros of $L(s,\chi)$ with $0 < \sigma < 1$ and $0 \le t \le T$, counting possible zeros with $t = 0$ or $t = T$ as one half only. Let

$$S(t,\chi) = \frac{1}{\pi}\mathrm{Im}\log L\left(\frac{1}{2}+it,\chi\right).$$

Then it can be shown that

$$\mathcal{N}(T,\chi) = \frac{T}{2\pi}\log\frac{Tq}{2\pi e} - \frac{\chi(-1)}{8} + S(T,\chi) - S(0,\chi) + O\left(\frac{1}{1+T}\right).$$

See formula (1.3) in Selberg's paper [50] and Chapter 16 in Davenport [9].

If the character is fixed and $T$ is large, then the results for $\mathcal{N}(T,\chi)$ are quite similar to results for $\mathcal{N}(T)$.

A different situation arises when the interval $[0,T]$ is fixed and the character $\chi$ varies (in particular, if $\chi$ is random). This situation was studied by Selberg, who proved the following result (cf. Theorem 9 in [50]).

**Theorem 2.9 (Selberg)** *For $|t| \le q^{1/4-\varepsilon}$, we have*

$$\frac{1}{q-2}\sum_{\chi}|S(t,\chi)|^{2r} = \frac{(2r)!}{r!\,(2\pi)^{2r}}(\log\log q)^{r} + O((\log\log q)^{r-1}),$$

*where the summation if over all non-principal characters over the base $q$.*

In other words, if $t$ is fixed and $q$ grows to infinity, then the distribution of $S(t,\chi)$ approaches the distribution of a Gaussian random variable with the variance $\frac{1}{2\pi^2}\log\log q$.

One is naturally let to the question of correlations between $S(t,\chi)$ for different $\chi$. One result in this direction is stated by Fujii (see p. 233 in [13]). Namely,

$$\int_0^T S(t,\chi_1)\,S(t,\chi_2)\,dt = \frac{\delta_{\chi_1,\chi_2}}{2\pi}T\log T + A(\chi_1,\chi_2)\,T + O\left(\frac{T}{\sqrt{\log T}}\right),$$

where $A(\chi_1,\chi_2)$ is a constant that depends on $\chi_1$ and $\chi_2$, which basically says that $S(t,\chi_1)$ and $S(t,\chi_2)$ are uncorrelated as functions of a random $t$ if $\chi_1 \neq \chi_2$. (See, however, a critique of Fujii's proof on p. 4 in [31].)

Apparently, the question of correlations between $S(t_1,\chi)$ and $S(t_2,\chi)$ as functions of a random $\chi$ has not yet been investigated.

### *2.2.2. Local scale*

In [23], Hughes and Rudnick study the linear statistics of low-lying zeros of $L$-functions on the local scale. Hughes and Rudnick order the zeros $\rho_{i,\chi} = \frac{1}{2} + i\gamma_{i,\chi}$ as follows:

$$\cdots \le \mathrm{Re}\gamma_{-2,\chi} \le \mathrm{Re}\gamma_{-1,\chi} < 0 \le \mathrm{Re}\gamma_{1,\chi} \le \mathrm{Re}\gamma_{2,\chi} \le \cdots$$

and define

$$x_{i,\chi} = \frac{\log q}{2\pi}\gamma_{i,\chi}.$$

Then they define

$$W_f(\chi) = \sum_{i=-\infty}^{\infty} f(x_{i,\chi})$$

where $f$ is a rapidly decaying test function.

The question is to understand the behavior of the averages

$$\left\langle W_f^m \right\rangle = \frac{1}{q-2}\sum_{\chi\neq\chi_0} W_f(\chi).$$

The basis for their analysis is a variant of the explicit formula (10) that relates a sum over zeros of $L(s,\chi)$ to a sum over prime powers. This formula is a particular version of the formula from Rudnick and Sarnak [47], which is valid for a more general class of zeta functions. Let $h(r)$ be any even analytic function in the strip $-c \le \mathrm{Im} r \le 1+c$ (for $c>0$) such that $|h(r)| \le A(1+|r|)^{-(1+\delta)}$ (for $r\in\mathbb{R}$, $A>0$, $\delta>0$). Then (cf. formula (2.1) in [23]),

$$\begin{aligned}\sum_j h(\gamma_{j,x}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\left(\log q + G_\chi(r)\right)dr \\ &\quad - \sum_n \frac{\Lambda(n)}{\sqrt{n}}\widehat{h}(\log n)\left(\chi(n)+\overline{\chi}(n)\right),\end{aligned} \tag{15}$$

where $\widehat{h}(u) = \frac{1}{2\pi}\int h(r)e^{-iru}dr$, and

$$G_{\chi}(r) = \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\mu(\chi)+ir\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\mu(\chi)-ir\right) - \frac{1}{2}\log\pi.$$

(Recall that $\mu(\chi) = 0$, if $\chi$ is even, and $= 1$, if $\chi$ is odd.)

Take $h(r) = f(\frac{\log q}{2\pi}r)$, so that $\widehat{h}(u) = \frac{1}{\log q}\widehat{f}(\frac{u}{\log q})$. We say that $f$ is *admissible*, if it is a real, even function whose Fourier transform $\widehat{f}(u) := \int f(r)e^{-2\pi iru}dr$ is compactly supported, and such that $|f(r)| \le A(1+|r|)^{-(1+\delta)}$. Then, from (15) we get the following decomposition:

$$W_f(\chi) = \overline{W_f}(\chi) + W_f^{osc}(\chi),$$

where

$$\overline{W_f}(\chi) := \int_{-\infty}^{\infty} f\left(\frac{\log q}{2\pi}r\right)(\log q + G_{\chi}(r))\,dr,$$

and

$$W_f^{osc}(\chi) := -\frac{1}{\log q}\sum_n \frac{\Lambda(n)}{\sqrt{n}}\widehat{f}\left(\frac{\log n}{\log q}\right)(\chi(n)+\overline{\chi}(n)).$$

For large $q$,

$$\overline{W_f}(\chi) := \int_{-\infty}^{\infty} f(x)\,dx + O\left(\frac{1}{\log q}\right),$$

which is asymptotically independent of $\chi$.

For the oscillating part one has the following result (cf. Theorem 5.1 in [23]).

**Theorem 2.10 (Hughes and Rudnick)** *Let $f$ be an admissible function and assume that*

$$\mathrm{Supp}\left(\widehat{f}\right) \subseteq [-\alpha,\alpha],\ \ \alpha > 0.$$

*If $m < 2/\alpha$, then the $m$-th moment of $W_f^{osc}$ is*

$$\lim_{q\to\infty}\left\langle\left(W_f^{osc}\right)^m\right\rangle_q = \begin{cases} \frac{m!}{2^{m/2}(m/2)!}\sigma(f)^m, & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd,}\end{cases}$$

*where*

$$\sigma(f)^2 = \int_{-1}^{1}|u|\,\widehat{f}(u)^2\,du.$$

In other words, the first several moments of the statistic $W_f$ converge to the corresponding moments of a Gaussian random variable. A similar situation holds for an eigenvalue statistic of random unitary matrices. Hughes and Rudnick show (Theorem 7.4) that higher moments for this eigenvalue statistic are not Gaussian (using results from the work of Diaconis and Shahshahani [11]). They conjecture that the same result should hold for the statistic $W_f$.

As an application, Hughes and Rudnick derived some results for the smallest zero of $L(s,\chi)$. In particular, they showed that for infinitely many $q$ there are characters $\chi$ such that the imaginary part of the zero is between 0 and 1/4 (Corollary 8.2 in [23]). They conjecture that 1/4 can be substituted with arbitrary positive constant.

Moreover, if $\beta > 6.333$, then a proportion of characters $\chi$ whose $L$ function has a zero with imaginary part between 0 and $\beta$ is greater than $c(\beta) > 0$ for all sufficiently large $q$. (Theorem 8.3 in [23]). The conjecture is that this is in fact true for every $\beta > 0$.

### 2.3. L-functions for modular forms (The Hecke L-functions)

Hecke generalized the Riemann zeta function by using ideals in an imaginary quadratic field $K$ instead of integers:

$$L_K(s) = \sum_{\mathfrak{a}} (N(\mathfrak{a}))^{-s}.$$

This function has a Euler product formula and a functional equation, although the latter is somewhat different: let

$$\Lambda_K(s) = \left(\frac{\sqrt{|D|}}{2\pi}\right)^s \Gamma(s) L_K(s),$$

where $D$ is the discriminant of the imaginary quadratic field $K$. Then

$$\Lambda_K(1-s) = \Lambda_K(s).$$

(Compare this with (2).) The proof of this functional equation is similar to Riemann's proof and relies on a modularity property of a certain complex-analytic function, that is, on its behavior relative to the change of variable $z \to 1/z$.

Motivated by this example, Hecke had a fruitful idea of obtaining L-functions from complex-analytic functions that transform well under the action of the modular group, and then checking which additional conditions are needed to ensure that a functional equation and a Euler product formula holds. The objects constructed in this way are called Hecke L-functions.

In order to illustrate, let

$$\begin{aligned} f(z) &= \sum_{n>0} c_n e^{2\pi i n z}, \text{ and} \\ L(f,s) &= \sum_{n>0} c_n n^{-s}, \end{aligned}$$

Define

$$\Lambda(f,s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(f,s).$$

Since

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) n^{-s} = \int_0^\infty e^{-2\pi n y/\sqrt{N}} y^s \frac{dy}{y},$$

we have the representation

$$\Lambda(f,s) = \int_0^\infty f\left(\frac{iy}{\sqrt{N}}\right) y^s \frac{dy}{y},$$

which implies that

$$\Lambda(f,s) = \int_1^{\infty} f\left(\frac{iy}{\sqrt{N}}\right) y^s \frac{dy}{y} + i^k \int_1^{\infty} Wf\left(\frac{iy}{\sqrt{N}}\right) y^{k-s} \frac{dy}{y}, \tag{16}$$

where

$$Wf(z) = \left(\sqrt{N}z\right)^{-k} f\left(\frac{-1}{Nz}\right). \tag{17}$$

Consequently, any eigenfunction of the operator $W$ will have a functional equation similar to the functional equation for the Riemann zeta function. The idea is to find a suitable finite-dimensional space of functions $f$, which is invariant under the action of $W$ and diagonalize $W$ in this finite-dimensional space.

Below, we give a brief outline of these ideas. A good source for this material is Chapter 14 in [26] and Chapter V in [38].

### *2.3.1. L-functions from modular forms*

Let $\Gamma$ is a subgroup of finite index in $SL_2(\mathbb{Z})$ and let $\mathbb{H}$ denote the upper half-plane $\{z|\mathrm{Im} z > 0\}$. The set $\mathbb{H}^* = \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$ can be made into a Hausdorff topological space and one can define a continuous action of $SL_2(\mathbb{Z})$ on $\mathbb{H}^*$ as an extension of the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ :

$$\text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } \gamma z = \frac{az+b}{cz+d}.$$

The *cusps* are points in $\mathbb{H}^* \backslash \mathbb{H}$. One can also show that $\Gamma \backslash \mathbb{H}^*$ is a compact Hausdorff space (that is, a compact space in which every two points have disjoint open neighbourhoods), which is a Riemann surface (that is, it admits a complex structure).

Next, we define an action of $SL_2(\mathbb{Z})$ on functions $f : \mathbb{H}^* \to \mathbb{C}$ :

$$\text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } f \circ [\gamma]_k := (cz+d)^{-k} f(\gamma z).$$

In fact if $\gamma \in GL_2(\mathbb{Z})$, then one can define its action by

$$f \circ [\gamma]_k := f \circ \left[\det(\gamma)^{-1/2} \gamma\right]_k.$$

It can be checked that this is indeed a group action of $GL_2(\mathbb{Z})$ on functions.

While modular forms can be defined for any discrete subgroup $\Gamma$, the most studied are subgroups $\Gamma_0(N)$:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \, (\mathrm{mod} N) \right\}.$$

For these subgroups, we have the following definition.

**Definition 2.11** *Let $\chi$ be a Dirichlet character modulo $N$. A* modular form *for $\Gamma_0(N)$ of weight $k \geq 1$ and character $\chi$ is a function $\mathbb{H}^* \to \mathbb{C}$ such that*

*(a) $f$ is holomorphic on $\mathbb{H}$;*
*(b) for any $\gamma \in \Gamma_0(N)$, $f \circ [\gamma]_k = \chi(d) f$;*
*(c) $f$ is holomorphic at the cusps.*

*A* cusp form *is a modular form which is zero at all the cusps.*

In particular, the complex analyticity at infinity implies that a modular form for $\Gamma_0(N)$ can be written as

$$f(z) = \sum_{n \geq 0} c_n e^{2\pi i n z}.$$

For a cusp form, $c_0 = 0$.

For simplicity, *we will only consider the forms with the principal character $\chi$* and we will write $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$ to denote the linear spaces of the modular and cusp forms. One can show that these spaces are finite dimensional for each $k$. Note that if $k$ is odd then $\mathcal{M}_k(N)$ is zero since $f \circ [-I]_k = (-1)^k f$, hence we should have $f = -f$.

**Definition 2.12** *Let $f$ be a cusp form of weight $2k$ for $\Gamma_0(N)$,*

$$f(z) = \sum_{n > 0} c_n e^{2\pi i n z}.$$

*The $L$-series of the cusp form $f$ is the Dirichlet series*

$$L(f, s) = \sum_{n > 0} c_n n^{-s}.$$

It is possible to estimate that $|c_n| \leq C n^k$, and therefore this series is convergent for $\mathrm{Re} s > k + 1$.

The crucial fact is that the space of modular forms is invariant under the action of operator $W$ from (17). Indeed, $Wf = f \circ [\omega]_k$, where $\omega = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then

$$\omega\gamma\omega^{-1} = \begin{pmatrix} d & -c/N \\ -Nb & a \end{pmatrix} \in \Gamma_0(N).$$

Hence

$$Wf \circ [\gamma]_k = f \circ \left[\omega\gamma\omega^{-1}\right]_k \circ [\omega]_k = f \circ [\omega]_k = Wf.$$

where the second step is by modularity of $f$. One can also check that $W$ preserves $\mathcal{S}_{2k}(N)$. Since $W^2 = 1$, hence the only eigenvalues of $w_N$ are $\pm 1$, and $\mathcal{S}_{2k}(N)$ is a direct sum of the corresponding eigenspaces, $\mathcal{S}_{2k} = \mathcal{S}_{2k}^{+1} + \mathcal{S}_{2k}^{-1}$.

By the argument in the beginning of this section (see formula (16)), one can infer the following result.

**Theorem 2.13 (Hecke)** *Let $f \in \mathcal{S}_{2k}(\Gamma_0(N))$ be a cusp form in the $\varepsilon$-eigenspace, $\varepsilon = 1$ or $-1$. Then the function $\Lambda(s,f) := N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$ extends analytically to a holomorphic function on the whole complex plane, and satisfies the functional equation*

$$\Lambda(s,f) = \varepsilon(-1)^k \Lambda(k-s,f).$$

The natural question is what about the Euler product formula?

By studying the properties of the modular forms that arise from the $L$-functions of the quadratic imaginary fields (and that have the product formula almost by definition), Hecke was able to formulate a list of properties which should be imposed on the modular form $f$, so that $L(f,z)$ had a product formula.

Namely, define the Hecke operators (cf. formula (14.46) in [26])

$$[T(n)f](z) := \frac{1}{n}\sum_{ad=n} a^k \sum_{0\le b<d} f\left(\frac{az+b}{d}\right).$$

It can be checked that these are linear operators on $\mathcal{S}_{2k}(\Gamma_0(N))$. (See Section IX.6 in Knapp [30] or Section V.4 of Milne [38] or Section VII.5 of Serre [53] for details). They have the following properties:

**Theorem 2.14 (Hecke)** *The maps $T(n)$ have the following properties:*

*(a)* $T(mn) = T(m)T(n)$ *if* $\gcd(m,n) = 1$;
*(b)* $T(p)T(p^r) = T(p^{r+1}) + p^{2k-1}T(p^{r-1})$ *if $p$ does not divide $N$*;
*(c)* $T(p^r) = T(p)^r$, $r \ge 1$, *if* $p|N$;
*(d) all $T(n)$ commute.*

Moreover, by acting on the Fourier expansion, one finds that the first Fourier coefficient in the expansion of $T(n)f$ is $c_n$. Hence, if $f$ is an eigenfunction of $f$ with eigenvalue $\lambda_n$, then $c_n = \lambda_n c_1$.

Since the Hecke operators commute we can look for the modular functions $f$ which are eigenfunctions for all of them. Then, the multiplicativity properties of $T(n)$ imply the corresponding properties for coefficients $c_n$, which leads to a Euler product formula for $L(f,s)$. This is formalized in the following result.

**Theorem 2.15 (Hecke)** *Let $f$ be a cusp form of weight $2k$ for $\Gamma_0(N)$ that is simultaneously an eigenvector for all $T(n)$, say $T(n)f = \lambda_n f$, and let*

$$f = \sum_{n\ge 1} c_n q^n,\ \ q = e^{2\pi i z}.$$

*Let $c_1 = 1$. Then, (i) coefficients of the series are eigenvalues of the Hecke operators,*

$$c_n = \lambda_n$$

*and (ii)*

$$L(f,s) = \prod_{p|N}\frac{1}{1-c_p p^{-s}}\prod_{p\nmid N}\frac{1}{1-c_p p^{-s}+p^{2k-1-s}}.$$

For example, $\mathcal{S}_{12}(\Gamma_0(1))$ has dimension 1, and therefore it is generated by a single function, which is called the $\Delta$-function:

$$\Delta(q) = q\prod_{1}^{\infty}(1-q^n)^{24} = \sum \tau(n)\, q^n,$$

where $\tau(n)$ is the Ramanujan $\tau$-function. It follows that the $L$-function associated to the $\Delta$-function has both a functional equation and the Euler product property.

In a more general situation, if we wish to find forms that have both a functional equation and a Euler product, then we must overcome the obstacle that in some exceptional cases operators $W$ and $T(n)$ do not commute. However, this obstacle can be circumvented and it can be proved that such good modular forms do exist. They are called *primitive forms* or *newforms.*

In summary, the L-functions of primitive forms have both a functional equation and the Euler product property. As a consequence, one can write explicit formulas for these L-functions.

### *2.3.2. L-functions from Maass forms*

A nice source for the material in this section is the lecture notes by Liu [33]. A lot of additional information about Maass forms can be found in the book by Iwaniec [25].

Modular forms are holomorphic and they are not easy to construct or compute. One can try to use Hecke ideas for a different class of functions that satisfy a modularity condition. In this way one comes to the concept of a Maass form.

**Definition 2.16** *A smooth function $f \neq 0$ is called a* Maass form *for group $\Gamma$, if*

*(i) for all $g \in \Gamma$ and all $z \in \mathbb{H}$, $f(gz) = f(z)$;*

*(ii) $f$ is an eigenfunction of the non-Euclidean Laplace operator:*

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = \lambda f,$$

*and*

*(iii) there exists a positive integer $N$, such that*

$$f(z) \ll y^N,\ y \to \infty.$$

*A Maass form $f$ is said to be a* cusp form *if the equality*

$$\int_0^1 f(z+b)\, db = 0$$

*holds for all $z \in \mathbb{H}$.*

A Maass form $f$ is call *odd* if $f(-x+iy) = -f(x+iy)$, and *even* if $f(-x+iy) = f(x+iy)$.

Note that it is relatively easy to generate Maass forms as eigenfunctions of the Laplace operator on a fundamental domain of the group $\Gamma$. By expanding a Maass form in Fourier series and taking the Fourier coefficients as the coefficients of a Dirichlet series, one can construct new L-functions. Precisely, let $f$ be either an even or an odd cusp Maass form with eigenvalue $1/4+r^2$. Then, one can write:

$$f(x+iy) = \sqrt{y} \sum_{n \neq 0} c_n K_{ir} (2\pi |n| y) e^{2\pi i n x}, \tag{18}$$

where $K_{ir}$ are Bessel functions, and we define

$$L(f,s) = \sum_{n>0} c_n n^{-s}. \tag{19}$$

The key idea here is the fact that the Laplace operator commutes with Hecke operators, and therefore all these operators can be simultaneously diagonalized. By a computation, the first Fourier coefficient of $T(n)f$ is $c_n c_1$. As a consequence, $L$-functions corresponding to Maass forms have a product formula.

What about the functional equation? It holds. However, instead of the standard formula for the Gamma function one needs the following integral:

$$\int_0^\infty K_{ir}(y) y^s \frac{dy}{y} = \Gamma\left(\frac{s+ir}{2}\right) \Gamma\left(\frac{s-ir}{2}\right).$$

**Theorem 2.17** *Let $f$ be a Maass form with eigenvalue $1/4+r^2$. Let $\varepsilon = 0$ or $1$ depending on whether $f$ is even or odd. Let*

$$\Lambda(f,s) = \pi^{-s} \Gamma\left(\frac{s+\varepsilon+ir}{2}\right) \Gamma\left(\frac{s+\varepsilon-ir}{2}\right) L(f,s).$$

*Then $\Lambda(s,f)$ is an entire function that satisfies*

$$\Lambda(f,s) = (-1)^\varepsilon \Lambda(f,1-s).$$

### *2.3.3. Statistical properties of zeros: Global scale*

Let

$$S(f,t) := \frac{1}{\pi} \arg L\left(f, \frac{1}{2}+it\right),$$

where $f$ is the Maass form with an eigenvalue $\lambda$ and $L$ is the corresponding $L$-function. $S(f,t)$ is related to the number of zeros of $L$ in the critical strip in the same way as the usual $S(t)$ function is related to the number of zeros of Riemann's zeta function.

We are interested here in the distribution of $S(f,t)$ with respect to the random choice of $f$.

Of course one need to explain what is meant by the random choice of $f$. Let $S_j(t) := S(f_j,t)$ where $f_j$ has an eigenvalue $\lambda_j = \frac{1}{4} + r_j^2$. Define $\nu_j(n) := c_j(n)/\sqrt{\cosh \pi r_j}$, where $c_j(n)$ are coefficients in the expansion (18) for the Maass form $f_j$. The numbers $\nu_j(1)$ will be used as weights in the limiting procedure. (We assume that $f_j$ are normalized to have a unit norm as $L^2$-functions. Therefore, $\nu_j(1)$ are not necessarily equal to 1.) One knows that $\nu_j(1)$ are $O(\sqrt{r_j})$ and

$$\frac{1}{T^2}\sum_{r_j\leq T}\left|\nu_j\left(1\right)\right|^2=\frac{1}{\pi^2}+O\left(\frac{\log T}{T}\right).$$

Since by the Weyl law, the number of $r_j$ below $T$ is proportionate to $T^2$, these weights can be thought as having bounded magnitude and not too sparse. Here is one of the results about the randomness of $S_j(t)$ (Theorem 3 in [21]).

**Theorem 2.18 (Hejhal-Luo)** *Let $h>0$ and $t>0$ be fixed. Then we have*

$$\lim_{T\to\infty}\frac{1}{(2HT)}\sum_{|r_j-T|\leq H}\frac{\pi^2\left|\nu_j\left(1\right)\right|^2}{2}\frac{\left(S_j\left(t\right)\right)^n}{\left(\log\log T\right)^{n/2}}=C_n,$$

*where $C_n$ are moments of the Gaussian distribution.*

### *2.3.4. Local scale*

Rudnick and Sarnak [47] extended the results of Montgomery to zeta functions that arise from modular and Maass forms. In fact, they work in greater generality and study the zeta functions that arise from the automorphic cuspidal representations of $GL_m$.The Hecke modular L-functions correspond to the case $m=2$. Their main tool is the following explicit formula, which we formulate for the case of the Hecke L-functions. Let

$$\begin{aligned} L\left(s,f\right) &= \prod_{p|N}\frac{1}{1-c\left(p\right)p^{-s}}\prod_{p\nmid N}\frac{1}{1-c\left(p\right)p^{-s}+p^{2k-1-s}} \\ &= \prod L_p\left(s,f\right). \end{aligned}$$

where

$$L_p\left(s,f\right)=\frac{1}{\left(1-\alpha_1(p)p^{-s}\right)\left(1-\alpha_2(p)p^{-s}\right)},$$

with the convention that for $p|N$ one of $\alpha_i(p)$ is zero. Let $a(p^k)=\alpha_1(p)^k+\alpha_2(p)^k$, and define $b(n)=\Lambda(n)a(n)$. Then

$$\frac{L'}{L}=-\sum_{n=1}^{\infty}\frac{b\left(n\right)}{n^s}.$$

**Theorem 2.19 (Rudnick and Sarnak)** *Let $\widehat{h}\in C_c^\infty(\mathbb{R})$ be a smooth compactly supported function, and let $h(r)=\int\widehat{h}(u)e^{iru}du$. Then*

$$\begin{aligned} \sum h(\gamma) &= \frac{\log Q}{2\pi}\int_{-\infty}^{\infty}h(r)dr \\ &\quad+\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\left(\sum_j\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\mu_j+ir\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\mu_j-ir\right)\right]\right)dr \end{aligned}$$

$$-\sum_{n=1}^{\infty}\left(\frac{b(n)}{\sqrt{n}}\widehat{h}(\log n)+\frac{\overline{b(n)}}{\sqrt{n}}\widehat{h}(-\log n)\right), \tag{20}$$

*where $\mu_j$ are some parameters that depend on the form $f$, and $Q$ is the conductor of the form.*

By using this result and estimates on the size of coefficients $b(n)$, Rudnick and Sarnak proved a generalization of the Montgomery theorem. Their result is valid not only for the Riemann zeta function, but also for Dirichlet $L$-functions, for Hecke modular $L$-functions and for $L$-functions that correspond to automorphic cuspidal representations of $GL_3$. We formulate it for Hecke modular $L$-functions.

Consider the class of smooth test functions $F(x_1, \ldots, x_n)$ that satisfy the following conditions:

**TF 1** $F(x_1, \ldots, x_n)$ is symmetric.
**TF 2** $F(x + t(1, \ldots, 1)) = F(x)$ for all $t \in \mathbb{R}$.
**TF 3** $F(x) \to 0$ rapidly as $|x| \to \infty$ in the hyperplane $\sum x_j = 0$.

If $B_N$ is a set of $N$ numbers $x_1, \ldots, x_N$, then the $n$-level correlation sum is defined by

$$R_n\left(B_N, F\right) = \frac{n!}{N} \sum_{\substack{S \subset B_N \\ |S|=n}} F\left(S\right).$$

Define the $n$-level correlation density by

$$W_n\left(x_1, \ldots, x_n\right) = \det\left(K\left(x_i - x_j\right)\right),\ K\left(x\right) = \frac{\sin \pi x}{\pi x}.$$

Then the following result holds (cf. Theorem 1.2 in [47]).

**Theorem 2.20** *Assume the Riemann hypothesis for the zeros of $L(s, f)$. Let $F(x_1, \ldots, x_N)$ satisfy **TF 1, 2, 3** and in addition assume that $\widehat{F}(\xi)$ is supported in $\sum_j |\xi_j| < 1$. Then,*

$$R_n\left(B_N, F\right) \to \int F\left(x\right) W_n\left(x\right) \delta\left(\frac{x_1 + \cdots + x_n}{n}\right) dx_1 \ldots dx_n$$

*as $N \to \infty$.*

Rudnick and Sarnak mention that the result can probably be proven for functions $F$ with the Fourier transform supported in $\sum_j |\xi_j| < 2$ by an improvement of their method, and conjecture that it holds without any assumption on the support of $\widehat{F}(\xi)$.

### *2.4. Elliptic curve zeta functions*

The main source for this section is the book [38] by Milne. Consider an elliptic curve

$$E : Y^2 Z = X^3 + aXZ^2 + bZ^3,$$

where $a$ and $b$ are integer, and assume that $|\Delta| = |4a^3 + 27b^2|$ cannot be made smaller by a change of variable $X \to X/c^2$, $Y \to Y/c^3$. This equation is called *minimal*. The equation

$$\overline{E} : Y^2 Z = X^3 + \overline{a} X Z^2 + \overline{b} Z^3,$$

with $\overline{a}$ and $\overline{b}$ the images of $a$ and $b$ in $\mathbb{F}_p$ (the finite field with $p$ elements) is called *the reduction of* $E$ *modulo* $p$. (It is assumed here that $p \neq 2, 3$. In the case when $p$ is 2 or 3, a somewhat different notion of the minimal equation is needed.) Let $N_p$ is the number of solutions of this equation in $\mathbb{F}_p$.

There are four possible cases:

(a) **Good reduction**. $\overline{E}$ is an elliptic curve. (That is, the determinant does not vanish and therefore the curve is smooth.) This happens if $p \neq 2$ and $p$ does not divide $\Delta$.

(b) **Cuspidal, or additive, reduction.** This is the case in which the reduced curve has a cusp. For $p \neq 2, 3$, this case occurs exactly when $p | 4a^3 + 27b^2$ and $p | -2ab$.

(c) **Nodal, or multiplicative, reduction.** The reduced curve has a node. For $p \neq 2, 3$, it occurs exactly when $p | 4a^3 + 27b^2$, $p \nmid -2ab$.

 (c1) **Split case.** The tangents at the node are rational over $\mathbb{F}_p$. This happens when $-2ab$ is a square in $\mathbb{F}_p$.

 (c2) **Non-split case.** The tangents at the node are not rational over $\mathbb{F}_p$.This occurs when $-2ab$ is not a square in $\mathbb{F}_p$.

The names additive and multiplicative refer to the group of points on the reduced curve, which in these cases is isomorphic either to $(\mathbb{F}_p, +)$, or $(\mathbb{F}_p^*, \times)$.

We define the $L$ function associated with the elliptic curve $E$ as follows.

$$L(s, E) := \prod_p \frac{1}{L_p(p^{-s})}.$$

Here, the local factors $L_p(T)$ are defined as follows:

$$L_p(T) = \begin{cases} 1 - a_p T + p T^2, & \text{if } p \text{ is good, with } a_p = p + 1 - N_p, \\ 1 - T & \text{if } E \text{ has split multiplicative reduction,} \\ 1 + T & \text{if } E \text{ has non-split multiplicative reduction,} \\ 1 & \text{if } E \text{ has additive reduction.} \end{cases}$$

Let $S$ be the (finite) set of primes with bad reduction. Then we can also write

$$\begin{aligned} L(s, E) &:= \prod_{p \in S} \frac{1}{L_p(p^{-s})} \prod_{p \notin S} \frac{1}{1 + (N_p - p - 1) p^{-s} + p^{1-2s}} \\ &= \prod_{p \in S} \frac{1}{L_p(p^{-s})} \prod_{p \notin S} \frac{1}{(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})}. \end{aligned}$$

The Hasse-Weil conjecture says that $L(s, E)$ can be analytically continued to a meromorphic function on the whole of $\mathbb{C}$ and satisfies a functional equation.

A recent work by Wiles and others confirmed this conjecture by showing that all elliptic curves are "modular", in particular, their $L$-functions arise from modular forms. To a certain extent, this result reduces the study of the elliptic $L$-functions to the study of the Hecke modular $L$-functions.

It is known that the numbers $a_p$ do not exceed $2\sqrt{p}$ in absolute value. For a fixed elliptic curve and different primes $p$, these numbers are believed to be distributed on the interval $[-2\sqrt{p}, 2\sqrt{p}]$ according to the semicircle distribution but this is not proven. In fact, this conjecture is related to the Birch-Swinnerton conjecture that states that

$$L(s, E) \sim C(s-1)^r \text{ as } s \to 1,$$

where $r$ is the rank of the group of rational points on $E$, and $C$ is a certain predicted constant. (see Chapter 10 in [38] for more information.)

It is possible to construct zeta functions for other nonsingular projective varieties and the conjecture by Hasse and Weil states that these zeta functions satisfy a functional equation (and the Riemann hypothesis). However, apparently not much is known beyond the cases of projective spaces and elliptic curves.

## 3. Selberg's zeta functions for compact and non-compact manifolds

It is useful to keep in mind that we will now talk about a new type of zeta functions, which is significantly different from number-theoretical zeta functions. While there is an explicit formula, it relates Laplace eigenvalues and geodesics, not zeta zeros and primes. The possibility of a relation between these two types of zetas is only conjectural.

The main source for this section is Hejhal's book [19].

### *3.1. Selberg's zeta function and trace formula*

Let $M$ be a compact Riemann surface of genus $g \geq 2$. Then, $M$ can be identified with a quotient space $\Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half-plane and $\Gamma$ is a discrete subgroup of $SL_2(R)$. We assume that $\mathbb{H}$ has the Poincare metric $|dz|/y$ with the area element $dxdy/y^2$, and therefore it has the constant negative curvature. This metric is naturally projected on the surface $M$.

This is not the most general situation of interest since most of the quotient spaces $\Gamma\backslash\mathbb{H}$ occurring in arithmetic applications have cusps and therefore are non-compact. However, the theory is most clear and transparent for the compact surfaces.

The Laplace operator on $M$ can be defined by the following formula.

$$-\Delta : u \to -y^2 (u_{xx} + u_{yy}).$$

It can be shown that this operator has a discrete set of non-positive eigenvalues:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots,$$

and the only point of accumulation of these eigenvalues is $\infty$.

Let us define

$$r_n = \begin{cases} \sqrt{\lambda_n - \frac{1}{4}}, & \text{if } \lambda_n \geq \frac{1}{4}, \\ i\sqrt{-\lambda_n + \frac{1}{4}}, & \text{if } \lambda_n < 1/4, \end{cases}$$

so that $\lambda_n = \frac{1}{4} + r_n^2$.

Also let $\overline{m} = \max\{k : \lambda_k < 1/4\}$.

Let $\mathcal{G}(M)$ be the set of all closed geodesics on $M$, and let $\mathcal{P}(M)$ be the subset of all prime closed geodesics (that is, the closed geodesics that cannot be represented as a non-trivial multiple of another closed geodesic). It is known that $\mathcal{G}(M)$ is a countable set, which we can order by the lengths of its elements. Closed geodesics correspond to hyperbolic elements of the group $\Gamma$ (that is, the elements of $\Gamma$ with the trace outside of $[-2, 2]$) up to conjugacy of these elements. If $P \in \Gamma$ corresponds to a geodesic $\gamma$, then $\gamma$ is prime if and only if there is no $P_0 \in \Gamma$ such that $P = P_0^k$ for an integer $k > 1$.

If $l(\gamma)$ denotes the length of the geodesic $\gamma$, corresponding to $P \in \Gamma$, then we set

$$|\gamma| := e^{l(\gamma)},$$

and note that

$$|\gamma|^{1/2} + |\gamma|^{-1/2} = |\mathrm{Tr} P| .$$

We will also write $|\gamma| = N[P]$ (meaning norm of $P$).

The Selberg trace formula relates sums over eigenvalues $\lambda_k$ to sums over hyperbolic elements (geodesics) $[P]$. Let $h(u)$ be a function which (i) is analytic in the strip $|\mathrm{Im} u| \leq 1/2 + \delta$, (ii) is even: $h(u) = h(-u)$, and (iii) declines sufficiently fast in the strip: $|h(u)| = O((1 + |\mathrm{Re} u|)^{-2-\delta})$.

Let $\widehat{h}(t) = \frac{1}{2\pi} \int h(u) e^{-itu} du$. Then the Selberg trace formula holds (cf. Theorem I.7.5 in Hejhal [19]),

$$\begin{aligned} \sum_{n=0}^{\infty} h(r_n) &= \frac{\mu(F)}{2\pi} \int_{\mathbb{R}} r h(r) \tanh(\pi r)\, dr \\ &\quad + \sum_{[T]} \frac{\ln N[T_0]}{N[T]^{1/2} - N[T]^{-1/2}} \widehat{h}(\ln N[T]), \end{aligned} \tag{21}$$

where the sum is over all distinct conjugacy classes of hyperbolic elements $[T]$, $[T_0]$ is the primitive element for $T$, $T = T_0^k$, and $\mu(F)$ is the area of the fundamental region of the group $\Gamma$.

It is instructive to compare this formula with formula (20). Since (21) resembles the explicit formulas from number theory, it is natural to define *Selberg's zeta function* as follows (cf. Definition II.4.1 in [19]):

$$Z(s) = \prod_{\gamma \in \mathcal{P}(M)} \prod_{k=0}^{\infty} \left(1 - |\gamma|^{-s-k}\right), \ \mathrm{Re} s > 1. \tag{22}$$

It turns out that Selberg's zeta function is closely related to the eigenvalues of the Laplace operator on $M$ (cf. Theorem II.4.10 and II.4.11 in [19]).

**Theorem 3.1 (Hejhal-Selberg)** *(a) $Z(s)$ is an entire function;*

*(b) Let $\beta$ be a real number $\geq 2$. For all $s$, the following identity holds:*

$$\begin{aligned}\frac{1}{2s-1}\frac{Z'(s)}{Z(s)} &= \frac{1}{2\beta}\frac{Z'\left(\frac{1}{2}+\beta\right)}{Z\left(\frac{1}{2}+\beta\right)}+\sum_{n=0}^{\infty}\left[\frac{1}{r_n^2+\left(s-\frac{1}{2}\right)^2}-\frac{1}{r_n^2+\beta^2}\right] \\ &\quad +\frac{\mu(F)}{2\pi}\sum_{k=0}^{\infty}\left[\frac{1}{\beta+\frac{1}{2}+k}-\frac{1}{s+k}\right].\end{aligned}$$

*(c) $Z(s)$ has "trivial" zeros $s=-k$, $k\geq 1$, with multiplicity $(2g-2)(2k+1)$;*
*(d) $s=0$ is a zero of multiplicity $2g-1$;*
*(e) $s=1$ is a zero of multiplicity $1$;*
*(f) the nontrivial zeros of $Z(s)$ are located at $\frac{1}{2}\pm ir_n$.*

Since all but a finite number of eigenvalues are greater than $1/4$ hence all but a finite number of $r_n$ is real and therefore the claim (f) implies that all but a finite number of zeros of $Z(s)$ are located on the line $\mathrm{Im} z = 1/2$.

The formula in claim (b) of this theorem follows from Selberg's trace formula and it can be thought as a functional equation for the logarithmic derivative of $Z(s)$. In particular, it implies the functional equation for the zeta functions itself (cf. Theorem 4.12 in [19]).

**Theorem 3.2 (Hejhal-Selberg)** *Selberg's zeta function satisfies the following functional equation:*

$$Z(s)=Z(1-s)\exp\left[\mu(F)\int_0^{s-\frac{1}{2}}v\tan(\pi v)\,dv\right].$$

### *3.2. Statistics of zeros*

The number of zeta zeros in a long interval has the following asymptotic expression:

$$N[k:0\leq r_k\leq T]=\frac{\mu(F)}{4\pi}T^2+S(T)+E(T),$$

where

$$S(T)=\frac{1}{\pi}\arg Z\left(\frac{1}{2}+iT\right),$$

and

$$E(T)=O(1)=2c\int_0^T t\left[\tanh(\pi t)-1\right]dt-(\overline{m}+1).$$

In other words, the number of zeros in a unit interval is $\sim cT$. In comparison, for Riemann's zeta function we have $\sim c\log T$ zeros in the unit interval.

It is known (cf. Theorems 8.1 and 17.1 in [19]) that

$$S\left(T\right)=O\left[\frac{T}{\log T}\right],\ \text{and}\ S\left(T\right)=\Omega_{\pm}\left[\left(\frac{\log T}{\log\log T}\right)^{1/2}\right].$$

(Recall that the notation $f(x)=\Omega_{+}[g(x)]$ means that $\limsup(\frac{f(x)}{g(x)})>0$, and $f(x)=\Omega_{-}[g(x)]$ means that $\liminf(\frac{f(x)}{g(x)})<0$.)

It was found that it is difficult to generalize the results concerning the moments of Riemann's $S(x)$ function to the case of Selberg's zeta. Since these results are essential for the study of statistical properties of zeta zeros, there is a stumbling block here.

Selberg managed to resolve this problem partially for a particular choice of the group $\Gamma$.

Let $p\geq 3$ be a prime and $A$ be a quadratic non-residue modulo $p$. Define

$$\Gamma=\Gamma\left(A,p\right)=\left\{\begin{pmatrix} y_0+y_1\sqrt{A} & y_2\sqrt{p}+y_3\sqrt{Ap}\\ y_2\sqrt{p}-y_3\sqrt{Ap} & y_0-y_1\sqrt{A}\end{pmatrix};\ y_0,y_1,y_2,y_3\ \text{are integer.}\right\}$$

and call it a *quaternion group.*

Let $S(t)=S^{+}(t)-S^{-}(t)$, where $S^{+}(t)=\max\{0,S(t)\}$ and $S^{-}(t)=\max\{0,-S(t)\}$. Then the following theorem holds (cf. Theorem 18.8 in [19]).

**Theorem 3.3 (Hejhal-Selberg)** *Let* $\Gamma=\Gamma(A,p)$ *with* $p\equiv 1 \pmod 4$. *Then (for large* $T$*):*

$$\frac{1}{T}\int_{T}^{qT}S^{+}\left(t\right)^{2}dt\geq c_1\frac{T}{\left(\log T\right)^{2}}$$

*where* $c_1$ *is a positive constant that depends only on* $\Gamma$. *A similar inequality holds for* $S^{-}(t)$.

In order to appreciate this result note that it suggests that the average deviation of $S(T)$ from its mean is of the order larger than $\sqrt{T}/(\log T)$ which should be compared with the number of zeros in the interval $[0,T]$, that is, $cT^2$. In other words, the deviation is larger than $(\mathcal{N}(T))^{1/4-\varepsilon}$. To put it in prospective note that the average deviation of the zeros of $S(T)$ for Riemann's zeta function is of the order $(\log\log T)^{1/2}$ which is smaller than $\log\log\mathcal{N}(T)$, where $\mathcal{N}(T)\sim cT\log T$ is the number of Riemann's zeros in $[0,T]$. These situations appear to be quite different.

Moreover, recently there was some numeric and theoretical work on the eigenvalues of the Laplace operator on manifolds $\Gamma\backslash\mathbb{H}$ for arithmetic groups $\Gamma$. First, numeric and heuristic analysis showed that the spacings between eigenvalues resemble spacings between points from a Poisson point process rather than spacings between eigenvalues of a random matrix ensemble (see Bogomolny et al. [5] and references wherein). Next some rigorous explanations of this finding have been given that relate it to large multiplicities of closed geodesics with the same length. See Luo and Sarnak ([35] and [36]) and Bogomolny et al. [6].

There is also some work on correlations of closed geodesics – see Pollicott and Sharp [43].

### *3.3. Comparison with the circle problem*

The Selberg zeta function is closely related to counting geodesics on a space $\Gamma\backslash\mathbb{H}$, in the same way as the Riemann zeta function is related to counting primes. It is natural to look at Laplace eigenvalues and geodesic counting problem in a simpler situation, such as a compact Riemann surface of genus 1. Such a surface can be represented as a quotient space $\Lambda\backslash\mathbb{C}$, where $\Lambda$ is a lattice. Consider, for concreteness, $\Lambda = [1, i]$. Then the eigenvalues of the Laplace operator are $4\pi^2(m^2 + n^2)$, where $m$ and $n$ are integer, and the number of the eigenvalues below $t$ equals the number of integer points in the circle $t/\pi$. Let

$$r(n) = N\left\{(a,b) \in \mathbb{Z}\times\mathbb{Z} : a^2 + b^2 = n\right\},$$

and

$$A(x) = \sum_{0\leq n\leq x} r(n) = \pi x + R(x).$$

The function $A(x)$ can be thought as the counting function both for eigenvalues of the Laplace operator and for closed geodesics of bounded length.

Then by using the Poisson summation formula it is possible to derive the following result (cf. Theorem 4.1 in [20] and Theorem 559 in [32]).

$$\sum r(n) f(n) = \pi \sum r(n) \int_0^\infty f(x) J_0\left(2\pi\sqrt{nx}\right) dx.$$

Informally, if one uses this identity with the indicator function for $f(x)$ (which is, in fact, not allowed under the conditions of the theorem), then one obtains the following formula (cf. formula (4.10) in [20] )

$$R(x) = \sqrt{x}\sum_{n=1}^{\infty} \frac{r(n)}{\sqrt{n}} J_1\left(2\pi\sqrt{nx}\right).$$

Rigorous variants of this formula lead to various estimates on $R(x)$, in particular it is known (cf. Theorems 509, 542, and 548 in [32]) that

$$R = O\left(x^{1/3}\right) \text{ and } R = \Omega_\pm\left(x^{1/4}\right),$$

and that

$$\frac{1}{x}\int_0^x R(t)^2\, dt = cx^{1/2} + O\left[(\log x)^3\right].$$

This suggest that the "standard deviation" of $R(t)$ is $x^{1/4}$. Similar to the case with Laplacian eigenvalues on $\Gamma\backslash\mathbb{H}$, the statistical behavior of eigenvalues does not resemble the behavior of random matrix eigenvalues or Riemann's zeta zeros.

Some more details about this problem can be found in [29], which considers the question about the number of points inside a random circle. More recent research can be found in [18], where it is shown that the distribution of the error term $R(x)$ converges to a non-Gaussian distribution as $x \to \infty$, and in [4], where this result is extended to circles with the center at a point $(\alpha, 0)$, and it is shown that the nature of the resulting distribution depends strongly on $\alpha$.

## 4. Zeta functions of dynamical systems

Dynamical zeta functions are closely related to Selberg's zeta function which can be thought as a dynamical zeta function for the geodesic flow on a Riemann surface. At the same time, there is a connection to number-theoretical zeta functions, namely, to the zeta functions of curves over finite fields. The main sources for this section are reviews by Ruelle ([48] and [49]) and Pollicott ([44] and [45]).

### *4.1. Zetas for maps*

Let $f$ be a map of a set $M$ to itself, let the periodic orbits of $f$ be denoted by $P$, and let $|P|$ denote the period of the orbit $P$. Then, we can define the zeta of $f$ by the following formula:

$$\begin{aligned} \zeta(z) &= \prod_P \left(1 - z^{|P|}\right)^{-1} \\ &= \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \left|\mathrm{Fix}\ f^m\right|, \end{aligned} \tag{23}$$

where $|\mathrm{Fix}\ f^m|$ denote the number of fixed points of $f^m$.

#### *4.1.1. Permutations*

Let $M$ be a finite set, and let $f$ be given by a permutation matrix $A$. Then the number of fixed points of $f^m$ is given by $\mathrm{Tr}A^m$. Hence, we have

$$\begin{aligned} \zeta(z) &= \exp \mathrm{Tr} \sum_{m=1}^{\infty} \frac{(zA)^m}{m} \\ &= \exp\left(-\mathrm{Tr}\log\left(1 - zA\right)\right) \\ &= 1/\det\left(1 - zA\right), \end{aligned}$$

which is closely related to the characteristic polynomial of matrix $A$. In particular, all poles of the zeta are on the unit circle.

#### *4.1.2. Smooth mappings of compact manifolds*

Let $f$ be a differentiable mapping of a compact orientable smooth manifold $X$ to itself. Assume that that $f$ is non-singular at all fixed points. Recall that the degree of $f$ at a fixed point $x$ equals to $+1$ if the map preserves orientation at the fixed point, and to $-1$ if it reverses the orientation, that is, $\deg_x(f) := \operatorname{sign}\det(df_x - I)$. We define the *Lefschetz zeta function* as

$$\zeta_L(z) = \exp\sum_{m=1}^{\infty}\frac{z^m}{m}\sum_{x\in\mathrm{Fix}(f^m)} d_x(f^m).$$

In this case one can use the Lefschetz fixed point formula that says:

$$\sum_{x\in\mathrm{Fix}(f^m)} d_x(f^m) = \sum_{i=0}^{\dim M}(-1)^i\,\mathrm{Tr}\left((f^m)_{*i} : H_i \to H_i\right),$$

where $H_i$ is the $i$-th homology group of the compact manifold $M$ with real coefficients, and $(f^m)_{*i}$ is the map induced by $f^m$ on $H_i$.

In particular, if $\lambda_{ij}$ are eigenvalues of $f_{*i}$, then we get

$$\begin{aligned}
\zeta_L(z) &= \exp\sum_{m=1}^{\infty}\frac{1}{m}\sum_{i=0}^{\dim M}(-1)^i\sum_{j=1}^{\dim H_i}(z\lambda_{ij})^m \\
&= \prod_{i=0}^{\dim M}\left(\prod_{j=1}^{\dim H_i}(1 - z\lambda_{ij})^{-1}\right)^{(-1)^i}, \\
&= \prod_{i=0}^{\dim M}\det(1 - zf_{*i})^{(-1)^{i+1}}
\end{aligned}$$

which is a rational function. If the map $f$ is a complex-analytic map of two complex compact manifold then this calculation can be refined by using the holomorphic Lefschetz formula that relates a sum over the fixed points of such a map to a sum over its Dolbeault cohomology groups. This often leads to additional information about $\lambda_{ij}$.

As an example, let $M$ be a torus $\mathbb{R}^2/\mathbb{Z}^2$ and let $f$ be induced by a linear transformation $A \in SL_2(\mathbb{Z})$. Assume that the eigenvalues of $A$ are positive and not on the unit circle: $\lambda_1 > 1 > \lambda_2 > 0$. Then $\sum_x \deg_x(f^m) = \det(A^m - I)$. Hence, we have

$$\begin{aligned}
\zeta_L(z) &= \exp\sum_{m=1}^{\infty}\frac{z^m}{m}\det(A^m - I) \\
&= \frac{(1 - z\lambda_1)(1 - z\lambda_2)}{(1 - z)(1 - z\lambda_1\lambda_2)} \\
&= \frac{(1 - z\lambda_1)(1 - z\lambda_2)}{(1 - z)^2}.
\end{aligned}$$

The original dynamical zeta of continuous maps (in which fixed points are counted without taking into account the degree of $f^m$) is often called the *Artin-Mazur zeta function* (see Artin-Mazur [1]). If this map is a diffeomorphism (a bijection smooth in both directions) and if it satisfies some additional conditions (hyperbolicity or Axiom A), then it is known that this function is rational. (This was conjectured by Smale [55], and proved by Guckenheimer [16] and Manning [37].)

#### *4.1.3. Subshifts*

Suppose next that $A$ is an $N$-by-$N$ matrix of zeros and ones, and that the set $M$ consists of doubly infinite sequences $\{x_i\}$ of symbols $1, \ldots, N$, which satisfy the following criterion. A sequence $\{x_i\}$ belongs to $M$ if and only if $A_{x_i x_{i+1}} = 1$ for every $i$. In other words, the symbol $x_i$ determines which of the other symbols are possible candidates for $x_{i+1}$. The map $f$ is simply a shift on this set $M$: $\{x_i\} \to \{x_{i+1}\}$. In this case, the number of fixed points of $f^m$ is $\mathrm{Tr}(A^m)$, and we have

$$\xi(z) = 1/\det(1 - zA). \tag{24}$$

#### *4.1.4. Ihara's zeta function*

A basic reference for this section is a book by Terras [58].

Let $G$ be a finite graph. *Ihara's zeta function* of $G$ is a dynamical zeta function for a subshift associated with this graph. Namely, orient edges of $G$ arbitrarily. Let the $2|E|$ oriented edges be denoted $e_1, e_2, \ldots, e_n, e_{n+1} = e_1^{-1}, \ldots, e_{2n} = e_n^{-1}$. The subshift matrix is a $2n$-by-$2n$ edge adjacency matrix $W_G$ which is defined as follows.

**Definition 4.1** *The* edge adjacency matrix $W_G$ *is a $2n$-by-$2n$ matrix with the rows and columns corresponding to oriented edges such that its $(i, j)$ entry equals $1$ if the terminal vertex of edge $i$ equals the initial vertex of edge $j$ and edge $j$ is not the inverse of edge $i$.*

In particular, from (24) we have a determinantal formula:

$$\zeta_G(u)^{-1} = \det(I - uW_G).$$

It is possible to define Ihara's zeta function more directly. The finite points of $f^m$ in this example are closed non-backtracking tailless paths of length $m$, where a *path* is a sequence of oriented edges such that the end of one edge equals the beginning of the next edge. A path $(e_1, e_2, \ldots, e_m)$ is *closed* if the end of $e_m$ corresponds to the beginning of $e_1$. It is *non-backtracking* if $e_{i+1} \neq e_i^{-1}$ for any $i$, and it is *tailless* if $e_m \neq e_1^{-1}$.

Hence, according to (23),

$$\zeta_G(u) = \prod_{[P]} \left(1 - u^{l(P)}\right)^{-1}, \tag{25}$$

where the product is over all primes, that is, all equivalence classes of primitive closed non-backtracking tailless paths. (A closed path $P$ is *primitive* if $P \neq D^m$ for $m \geq 2$ and any other path $D$; and two paths are called equivalent if they can be obtained from each other by a cyclic permutation of edges.)

For arbitrary regular finite graphs, this definition of Ihara's zeta function was introduced by Sunada ([56] and [57])) following a suggestion in the book "Trees" by Serre.

Ihara's zeta function has another representation as a determinant which was first discovered by Ihara for regular graphs [24] and then by Bass [2] and Hashimoto [17] for arbitrary finite graphs:

$$\zeta_G(u)^{-1} = \left(1-u^2\right)^{|E|-|V|} \det\left(I - A_G u + Q_G u^2\right),$$

where $A_G$ is the adjacency matrix of $G$, and $Q_G$ is the diagonal matrix whose $j$-th diagonal entry is ($-1$ + degree of $j$-th vertex).

If the graph is $q+1$ regular, that is, if every vertex has degree $q+1$, then $Q_G$ is scalar and we can see that poles of $\zeta_G(u)$ are related to the zeros of the characteristic polynomial of $A_G$, that is to the eigenvalues of the matrix $A_G$. Precisely, the poles $u_i$ are related to the eigenvalues $\lambda_i$ by the formula:

$$u_i = \frac{\lambda_i \pm \sqrt{\lambda_i^2 - 4d}}{2d}.$$

Since the eigenvalues are always real, we find that a pole $u_i$ is on the circle $|u| = 1/\sqrt{d}$ if and only if the corresponding eigenvalue is sufficiently small, $|\lambda_i| \leq 2\sqrt{d}$.

There are two trivial poles at 1 and $1/q$ corresponding to the largest eigenvalue $\lambda = d+1$. The Riemann hypothesis for regular graphs says that all nontrivial poles are on this circle. This is not always true, and it holds if and only if $|\lambda_1| \leq 2\sqrt{q}$, where $\lambda_1$ is the second largest in magnitude eigenvalue of $A_G$. The graphs that satisfy this condition are often called Ramanujan graphs following a paper by Lubotsky, Phillips, and Sarnak [34], which constructed an infinite family of such graphs by using the Ramanujan conjecture from the theory of modular forms.

A random regular graph is approximately Ramanujan with high probability. This means that for arbitrary $\varepsilon > 0$, the probability that $|\lambda_1| \geq 2\sqrt{q}+\varepsilon$ becomes arbitrarily small as the size of the graph grows. (This is known as Alon's conjecture, and was proved in a lengthy paper by Friedman [12]). The distribution properties of the largest eigenvalue are still unknown. It is also unknown what proportion of the eigenvalues exceed the threshold $2\sqrt{q}$.

In his Ph.D. thesis [42], Derek Newland studied spacings of eigenvalues of random regular graphs and spacings of the Ihara zeta zeros and found numerically that they resemble spacings in the Gaussian Orthogonal Ensemble. See also an earlier paper by Jacobson, Miller, Rivin and Rudnick [27].

For Ihara'z zeta function, there is an analog of Selberg's trace formula which we formulate for the case of a $q+1$ regular graph.

First, note that formula (25) implies

$$\begin{aligned} u\frac{\zeta_G'}{\zeta_G}(u) &= \sum_{[P]} l(P)\left(u^{l(P)} + u^{2l(P)} + \cdots\right) \\ &= \sum N_m u^m, \end{aligned}$$

where $N_m$ be the number of all non-backtracking tailless closed paths of length $m$. Then the following explicit formula holds (cf. Proposition 25.1 in [58]).

**Theorem 4.2 (Terras)** *Suppose $0 < a < 1/q$. Assume that $h(u)$ is meromorphic in the plane and holomorphic for $|u| > a - \varepsilon$, $\varepsilon > 0$ Assume that $h(u) = O(|u|^{-1-\alpha})$, $\alpha > 0$. Let $\widehat{h}(n) := \frac{1}{2\pi i}\int_{|u|=a} u^n h(u)du$ and assume that $\widehat{h}(n)$ decays rapidly enough. Then,*

$$\sum_{\rho} \rho h(\rho) = \sum_{n\geq 1} N_n \widehat{h}(n),$$

*where the sum on the left is over the poles of $\zeta_G(u)$.*

The book [58] mentions that this formula can be used to derive the limit law for eigenvalues of a large random regular graph (the McKay-Kesten law) and give references to this and some other applications.

#### *4.1.5. Frobenius maps*

Let $M$ be the set of solutions of a system of algebraic equations in $r$ variables over the algebraic closure of the finite field $\mathbb{F}_q$ and let $f$ be the Frobenius map Frob: $(x_1, \ldots, x_r) \to (x_1^q, \ldots, x_r^q)$. In the case of curves, the dynamic zeta function defined by (23) is equivalent to a number-theoretic zeta function introduced by Artin.

Namely, let an affine curve $C$ be given by the equation $f(X,Y) = 0$ over the finite field $\mathbb{F}_q$. Let $\mathfrak{p}$ denote a prime ideal of the field $\mathbb{F}_q[X,Y]/f(X,Y)$ and let the order of $\mathbb{F}_q[X,Y]/\mathfrak{p}$ be denoted by $N\mathfrak{p}$. Then, by analogy with Riemann's zeta function we can define

$$\zeta_C(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}}.$$

It turns out that if one uses the change of variable $u = q^{-s}$, then this function is equivalent to the dynamical zeta function for the Frobenius map acting on the algebraic closure of the curve $C$.

Here is an example. Let $C$ be an elliptic curve, then

$$\zeta_C(u) = \frac{1 - u(q + 1 - N_1) + u^2 q}{(1-u)(1-uq)},$$

where $N_1$ is the number of the points of $C$ in $\mathbb{F}_q$. One proof of this fact is based on the Riemann-Roch theorem that allows to count the ideals with a given norm more or less explicitly. (See [38].) Another proof uses an analogy with the smooth maps of compact manifolds. It proceeds by constructing a theory of cohomologies for algebraic varieties over finite fields, which has a suitable Lefschetz fixed point formula. See the book by Silverman [54] for more details about this proof.

The Riemann hypothesis in this example is equivalent to the statement that

$$|N_1 - q - 1| \leq 2\sqrt{q},$$

since this implies that the roots of the polynomial in the numerator are on the circle with radius $q^{-1/2}$. For elliptic curves this was proved by Hasse.

In [60] Weil conjectured that the zeta function of every algebraic variety over finite fields is rational, that it has a functional equation, and that it satisfies the Riemann hypothesis. The proof of this general conjecture led to an introduction of many new ideas in algebraic geometry by Dwork, Grothendick, and Deligne.

The statistical properties of the zeta's zeros were investigated in the case of curves over finite fields by Katz and Sarnak [28]. They have studied zeros' distribution when the genus of the corresponding curve grows to infinity and found that for "most" of the curves the local statistics of the zeros approach those of the eigenvalues of random matrix ensembles.

### *4.1.6. Maps of an interval*

For yet another example consider the map $x \to 1 - \mu x^2$ of the interval $[-1, 1]$ to itself. For a special value of $\mu \approx 1.401155\ldots$ (the Feingenbaum value), this map has one periodic orbit of period $2^n$ for every integer $n \geq 0$. Therefore, for the dynamic zeta function we have:

$$\zeta(z) = \prod_{n=0}^{\infty} \left(1 - z^{2^n}\right)^{-1} = \prod_{n=0}^{\infty} \left(1 + z^{2^n}\right)^{n+1}.$$

This $\zeta$ satisfies the functional equation $\zeta(z^2) = (1 - z)\zeta(z)$. More generally, the piecewise monotone maps of the interval $[-1, 1]$ to itself correspond to the *Milnor-Thurston zeta functions*. See [39]. Apparently, so far there have been no systematic study of the statistical properties of their poles and zeros.

## ***4.2. Zetas for flows***

If $f$ is a flow on $M$, that is, a map $M \times \mathbb{R}^+ \to M$, then we can define the zeta function of this flow as

$$\zeta(s) = \prod_{\omega} \left(1 - e^{-sl(\omega)}\right)^{-1},$$

where $\omega$ denotes a periodic orbit of $f$, and $l(\omega)$ is its length. It is a more general case than the case of maps since there is a construction ("suspension") that allows us to realize maps as flows (see Example 5 on p. 70 in [46] or Section 4.3 in [45]) but not vice versa. Unsurprisingly, it turns out that zeta functions for flows are more difficult to investigate than zeta functions for maps.

If we imagine that prime numbers correspond to periodic orbits of a flow and that the length of the orbit indexed by $p$ is given by $\log p$, then the zeta function of the flow will coincide with Riemann's zeta function.

One particularly important example of a flow is the geodesic flow on a smooth manifold $M$. In the case when $M$ has a constant negative curvature, the corresponding dynamical zeta function is closely related to the Selberg zeta function.

Namely,

$$\zeta(s) = \frac{Z(s+1)}{Z(s)} \text{ and } Z(s) = \prod_{n=0}^{\infty} \zeta(s+n)^{-1},$$

where $Z(s)$ is as defined in (22) (see Remark 2.5 in [45]). Another important example is the geodesic flow for billiards on polygons.

I am not aware about the functional equation for zeta functions that comes from general geodesic flows (other than flows on manifolds of a constant negative curvature). Also, it appears that not much is known about the statistical properties of the distribution of zeros and poles of these zeta functions.

On the other hand one can study the location of the pole with the maximal real part and the results of this study give valuable information about the distribution of closed geodesics. In this way, one can study the distribution of closed geodesics on spaces of variable curvature (see Corollary 6.11 in [45]).

## 5. Conclusion

We considered the statistical properties of various zeta functions. For the zeros of number-theoretical zeta functions, the main observation is that they satisfies many properties which are true for eigenvalues of random matrices. The main outstanding problem (besides the Riemann hypothesis) is to push this similarity to its natural limits and, in particular, show that the Montgomery conjecture about the correlations of the Riemann zeros is true.

Next, we observed that for some of groups $\Gamma \subset SL_2(\mathbb{Z})$, the statistical properties of Selberg's zeta zeros are different from those of the random matrix eigenvalues. The exact description of these properties is not known.

The statistical properties of the zeros of dynamical zeta functions are not investigated in many cases. Two notable exceptions are the zeta functions of curves over finite fields and Ihara's zeta functions. However, even in this case there are many unsolved problems. For example, it is not known whether the distribution of local statistics for the zeros of Ihara's zeta functions coincide with the corresponding distribution for the random matrix eigenvalues.

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