Dense Orbits in Orbital Varieties in $\mathfrak{sl}_n$

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Abstract

Let $\mathcal{O}$ be a nilpotent orbit in the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and let $\mathcal{V}$ be an orbital variety contained in $\mathcal{O}$. Let $\mathbf{P}$ be the largest parabolic subgroup of $\mathrm{SL}(n,\mathbb{C})$ stabilizing $\mathcal{V}$. We describe nilpotent orbits such that all the orbital varieties in them have a dense $\mathbf{P}$ orbit and show that for n big enough the majority of nilpotent orbits do not fulfill this.

Résumé

Soit $\mathcal{O}$ une orbite nilpotente dans l'algèbre de Lie $\mathfrak{sl}_n(\mathbb{C})$ et soit $\mathcal{V}$ une variété orbitale contenue dans $\mathcal{O}$. Soit $\mathbf{P}$ le plus grand sous-groupe parabolique de $\mathrm{SL}(n,\mathbb{C})$ stabilisant $\mathcal{V}$. Nous décrivons les orbites nilpotentes dont toutes les variétés orbitales contiennent une $\mathbf{P}$-orbite dense et montrons que pour n assez grand la majorité des orbites nilpotentes n'ont pas cette propriété.

1 Introduction

1.1 Let $\mathbf{G}$ be a connected semisimple finite dimensional complex algebraic group. Let $\mathfrak{g}$ be its Lie algebra and $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Consider the adjoint action of $\mathbf{G}$ on $\mathfrak{g}$. A $\mathbf{G}$ orbit $\mathbb{O}$ in $\mathfrak{g}$ is called nilpotent if it consists of nilpotent elements.

Fix some triangular decomposition $\mathfrak{g} = \mathfrak{n} \bigoplus \mathfrak{h} \bigoplus \mathfrak{n}^-$. An irreducible component of $\mathbb{O} \cap \mathfrak{n}$ is called an orbital variety contained in $\mathbb{O}$. Orbital varieties play a key role in Springer's Weyl group representations and in the primitive ideal theory of $U(\mathfrak{g})$.

The last can be detailed as follows. Since $\mathfrak{g}$ is semisimple we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form. This identification gives an adjoint orbit a symplectic structure. Let $\mathcal{V}$ be an orbital variety contained in $\mathbb{O}$. After N. Spaltenstein [Sp] and

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R. Steinberg [St] one has

$$(*) \qquad \dim \mathcal{V} = {}^1\!/_2 \dim \mathcal{O}.$$

Moreover it was pointed out by A. Joseph [J] that this implies that an orbital variety is a Lagrangian subvariety of its nilpotent orbit. According to the orbit method philosophy one would like to attach an irreducible representation of $U(\mathfrak{g})$ to $\mathcal{V}$. This can be naturally implemented in the case of $\mathfrak{g} = \mathfrak{sl}_n$ where there exists a one to one correspondence between the set of primitive ideals of $U(\mathfrak{g})$ containing the augmentation ideal of its centre and the set of orbital varieties in $\mathfrak{g}$. Moreover as it is shown in [M2] in this case $\overline{\mathcal{V}}$ is the associated variety of the corresponding simple highest weight module so that orbital varieties give a natural geometric understanding of the classification of primitive ideals. Hence the study of orbital varieties in $\mathfrak{sl}_n$ is especially interesting.

1.2 Orbital varieties remain rather mysterious objects. The only general description was given by R. Steinberg [St] and is as follows. Let $R \subset \mathfrak{h}^*$ be the set of roots, R^+ be the choice of positive roots defining $\mathfrak{n}$ and $\Pi \subset R^+$ be the corresponding set of simple roots. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ acting on R. Let $\mathbf{B}$ be the Borel subgroup of $\mathbf{G}$ corresponding to the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \bigoplus \mathfrak{n}$. Recall that $\mathfrak{n} = \bigoplus_{\alpha \in R^+} X_\alpha$ (resp.$\mathfrak{n}^- = \bigoplus_{\alpha \in -R^+} X_\alpha$) where X_α is the root subspace corresponding to α. For $w \in W$ set $\mathfrak{n} \cap^w \mathfrak{n} := \bigoplus_{\alpha \in R^+ \cap w(R^+)} X_\alpha$. For each subgroup $\mathbf{H}$ of $\mathbf{G}$ let $\mathbf{H}(\mathfrak{n} \cap^w \mathfrak{n})$ be the set of $\mathbf{H}$ conjugates of $\mathfrak{n} \cap^w \mathfrak{n}$. One easily sees that there exists a unique nilpotent orbit $\mathcal{O}$ such that $\mathbf{G}(\mathfrak{n} \cap^w \mathfrak{n}) = \overline{\mathcal{O}}$. Then $\mathcal{V}_w = \overline{\mathbf{B}(\mathfrak{n} \cap^w \mathfrak{n})} \cap \mathcal{O}$ is an orbital variety and the map $\phi : w \mapsto \mathcal{V}_w$ is a surjection of W onto the set of orbital varieties.

This description is not very satisfactory from the geometric point of view since a $\mathbf{B}$ invariant subvariety generated by a linear space is a very complex object. One of the attempts to give a reasonable description of an orbital variety is the following conjecture proposed by S. P. Smith.

Given an orbital variety $\mathcal{V}$ let $\mathbf{P}_\mathcal{V}$ be its stabilizer. This is a standard parabolic subgroup of $\mathbf{G}$. We say that an orbital variety $\mathcal{V}$ is of S type if there exists a dense $\mathbf{P}_\mathcal{V}$ orbit in it. We say that a nilpotent orbit $\mathcal{O}$ is of S type if all its orbital varieties are of S type.

Conjecture 1.1 (S. P. Smith) — *In $\mathfrak{sl}_n$ all orbital varieties are of S type.*

The truth of this conjecture would give a more elegant and simple description of orbital varieties. For a given orbital variety closure it would provide a way to construct a resolution of its singularities and be the first step towards a description of its ideal of definition. It could also provide a natural way to define orbital varieties in the case of quantum groups. These implications made the conjecture (suggested by S.P. Smith some ten years ago) quite attractive.

1.3 The conjecture is true for $\mathfrak{sl}_n$ when $n \leq 8$ as shown by E. Benlolo in [B]. Yet here we show that the conjecture is false in general.

In 2.5 we give the first counter-example to the Smith conjecture which appears in $\mathfrak{sl}_9$ and is the only counter-example for $n \leq 9$. We give some other counter-examples which we use in what follows.

Further we investigate the situation for $n \gg 0$. In § 3 we give sufficient conditions for an orbit to be not of S type. This can be explained as follows.

Take $\mathfrak{g} = \mathfrak{sl}_n$. Consider $\mathfrak{sl}_i$ for $i < n$ as a Levi subalgebra $\mathfrak{l}_{1,i}$ of $\mathfrak{g}$ (cf. 3.2). Set $\mathfrak{n}_{1,i} = \mathfrak{n} \cap \mathfrak{l}_{1,i}$ and define the projection $\pi_{1,i} : \mathfrak{n} \to \mathfrak{n}_{1,i}$. A result of [M1] is that $\pi_{1,i}$ takes an orbital variety closure in $\mathfrak{sl}_n$ to an orbital variety closure in $\mathfrak{sl}_i$.

Given an orbital variety $\mathcal{V}$ let $\tau(\mathcal{V})$ be its τ-invariant (cf. 2.4). As we explain in 3.2 if $\mathcal{V}$ is of S type and $\alpha_i \notin \tau(\mathcal{V})$ then $\pi_{1,i}(\mathcal{V})$ must be of S type. From this given an orbital variety not of S type in $\mathfrak{sl}_i$ we show how to construct orbital varieties not of S type in $\mathfrak{sl}_n$ for $n > i$.

1.4 In § 4 we give sufficient conditions for an orbit to be of S type. This can be explained as follows.

Orbital varieties are irreducible components of $\mathcal{O} \cap \mathfrak{n}$. Yet they are as far as possible of being disjoint. Indeed after N. Spaltenstein [Sp] for any two orbital varieties $\mathcal{V}, \mathcal{V}' \subset \mathcal{O}$ there exist a chain of orbital varieties $\mathcal{V} = \mathcal{V}_1, \cdots, \mathcal{V}_k = \mathcal{V}' \subset \mathcal{O}$ with $\mathrm{codim}(\mathcal{V}_i \cap \mathcal{V}_{i+1}) = 1$ for all $i \in \{1, 2, \cdots, k-1\}$.

In $\mathfrak{sl}_n$ if a nilpotent orbit is neither regular nor minimal it contains more than one orbital variety. Following A. Joseph we apply Vogan's analysis [V] to orbital varieties. For a given orbital variety $\mathcal{V}$ this defines the orbital variety $\mathcal{T}_{\alpha\beta}(\mathcal{V})$ (cf. 4.2). One has $\mathrm{codim}(\mathcal{T}_{\alpha\beta}(\mathcal{V}) \cap \mathcal{V}) = 1$ and for any given pair of orbital varieties $\mathcal{V}, \mathcal{V}' \subset \mathcal{O}$ one may pass from $\mathcal{V}$ to $\mathcal{V}'$ by a sequence of $\mathcal{T}_{\alpha\beta}$ operations. This refines Spaltenstein's result.

In each nilpotent orbit there exists a Bala-Carter component (cf. 4.3). As shown by R. Carter in [C] a Bala-Carter component contains a dense $\mathbf{B}$ orbit. One can use such orbital varieties and Vogan's analysis to construct other orbital varieties of S type; but this does not lead to all orbital varieties of S type. The problem is that the dimension of $\mathbf{P}_{\mathcal{T}_{\alpha\beta}(\mathcal{V})}$ can differ by more than one from the dimension of $\mathbf{P}_{\mathcal{V}}$ and then we cannot conclude that $\mathcal{V}$ of S type implies $\mathcal{T}_{\alpha\beta}(\mathcal{V})$ of S type. Generally speaking this is the reason that the orbital varieties not of S type appear. However the algorithm we obtain is not decisive; but it helps to construct orbital varieties of S type and to give counter-examples to conjecture 1.1.

To show that a specific nilpotent orbit is of S type we find in it enough orbital varieties with a dense $\mathbf{B}$ orbit so that applying Vogan's analysis we get all the orbital varieties in the given orbit. These computations compose the main part of § 4 and are technically the most difficult part of the work. A few orbits described at the end of § 4 stay unclassified. These cases apparently require more subtle computations.

2 Counter-examples

Lemma 2.1 — *Fix $w \in W$. If the orbital variety $\mathcal{V}_w$ has a dense $\mathbf{P}_{\mathcal{V}_w}$ orbit $\mathcal{P}$ then*

$$\mathcal{P} \cap (\mathfrak{n} \cap^w \mathfrak{n}) \neq \emptyset.$$

It is convenient to replace $\mathfrak{sl}_n$ by $\mathfrak{g} = \mathfrak{gl}_n$. This obviously makes no difference. Note that the adjoint action of $\mathbf{G} = GL_n$ on $\mathfrak{g}$ is just a conjugation.

Let $\mathfrak{n}$ be the subalgebra of strictly upper-triangular matrices in $\mathfrak{g}$ and $\mathbf{B}$ be the (Borel) subgroup of upper-triangular matrices in $\mathbf{G}$. All parabolic subgroups we consider further are standard, that is contain $\mathbf{B}$.

Let e_{ij} be the matrix having 1 in the ij entry and 0 elsewhere. Set $\Pi := \{\alpha_i\}_{i=1}^{n-1}$. Take $i \leq j$. Then for $\alpha = \sum_{k=i}^{j} \alpha_k$, the root space $X_\alpha = \mathbb{C}e_{i,j+1}$ and the root space $X_{-\alpha} = \mathbb{C}e_{j+1,i}$.

We identify W with the permutation subgroup $\mathbf{S}_n$ of GL_n. For $\alpha \in \Pi$ let s_α be the corresponding fundamental reflection and set $s_i = s_{\alpha_i}$.

Let $[\ ,\ ]$ denote the Lie product on $\mathfrak{g}$ given here by commutation in $\operatorname{End} V$. For a standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ we set $\mathfrak{p} := \operatorname{Lie}\mathbf{P}$ which is a standard parabolic subalgebra of $\mathfrak{g}$, that is contains $\mathfrak{b}$.

Lemma 2.2 — *Take $M \in \mathfrak{g}$ and a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. One has*

$$\dim \mathbf{P}M = \dim[\mathfrak{p}, M].$$

Combining these two lemmas we obtain

Corollary 2.3 — *Fix $w \in W$. The orbital variety $\mathcal{V}_w$ is of S type if and only if for some $M \in \mathfrak{n} \cap^w \mathfrak{n}$ one has*

$$\dim[\mathfrak{p}, M] = \dim \mathcal{V}_w.$$

2.2 Nilpotent orbits in $\mathfrak{sl}_n$ are parameterized by Young diagrams. Orbital varieties are parameterized by standard Young tableaux. Let us explain these parameterizations.

In $\mathfrak{sl}_n$ or $\mathfrak{gl}_n$ each nilpotent orbit $\mathcal{O}$ is described by its Jordan form. A Jordan form in turn is parameterized by a partition $\lambda = (\lambda_1 \geq \lambda_2 \cdots \lambda_k > 0)$ of n giving the length of Jordan blocks. We denote by $\mathcal{O}_\lambda$ the nilpotent orbit determined by λ.

It is convenient to represent a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ of n as a Young diagram D_λ, that is an array of k rows of boxes starting on the left with the i-th row containing λ_i boxes. The dual partition $\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2 \cdots)$ is defined by setting $\hat{\lambda}_i$ equal to the length of the $i-$th column of the diagram D_λ that is $\hat{\lambda}_i = \sharp\{j : \lambda_j \geq i\}$.

One has (cf. [H] § 3.8)

$$(**) \qquad \dim \mathcal{O}_\lambda = n^2 - \sum_{i=1}^{k} \hat{\lambda}_i^2.$$

Define a partial order on partitions as follows. Given two partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots \mu_j)$ of n we set $\lambda \geq \mu$ if

$$\sum_{l=1}^{i} \lambda_\ell \geq \sum_{\ell=1}^{i} \mu_\ell, \text{ for all } i = 1, 2, \cdots, k.$$

The following result of M. Gerstenhaber (cf. [H] § 3.10) shows that this order corresponds to inclusion of nilpotent orbit closures:

Theorem 2.4 — *Given two partitions λ and μ of n one has $\lambda \geq \mu$ if and only if $\overline{\mathcal{O}}_\lambda \supset \overline{\mathcal{O}}_\mu$.*

2.3 Given a partition λ of n fill the boxes of D_λ with n distinct positive integers. If the entries increases in rows from left to right and in columns from top to bottom we call such an array a Young tableau. If the numbers in Young tableau form a set of integers from 1 to n we call it standard. Let $\mathbf{T}_n$ be the set of standard Young tableaux of size n.

The shape of a Young tableau T is defined to be a Young diagram from which T was built. It defines a partition of n which we denote $\operatorname{sh} T$.

The Robinson - Schensted correspondence $w \mapsto (Q(w), R(w))$ gives a bijection (see, for example [Kn]) from the symmetric group $\mathbf{S}_n$ onto the pairs of standard Young tableaux of the same shape. By R. Steinberg [St1] for all $w, y \in \mathbf{S}_n$ one has $\mathcal{V}_w = \mathcal{V}_y$ iff $Q(w) = Q(y)$. This parameterizes the set of orbital varieties by $\mathbf{T}_n$. Moreover $\operatorname{sh} Q(w) = \lambda$ if and only if $\mathcal{V}_w$ is contained in $\mathcal{O}_\lambda$.

We set $\mathcal{V}_T := \mathcal{V}_w$ if $Q(w) = T$, $\quad \mathbf{P}_T := \mathbf{P}_{\mathcal{V}_T}$ and $\mathfrak{p}_T := \mathfrak{p}_{\mathcal{V}_T}$.

Let T be some Young tableaux with $\operatorname{sh} T = \lambda = (\lambda_1, \cdots)$. Denote by T_j^i its ij-th entry. If k is the entry T_j^i of T, set $r_T(k) = i$ and $c_T(k) = j$.

For $i : 1 \leq i \leq \hat{\lambda}_1$ set $T^i := (T_1^i, \cdots, T_{\lambda_i}^i)$. This is the ordered set of entries of the i-th row. For each $T \in \mathbf{T}_n$ we define $w_r(T) \in \mathbf{S}_n$ through

$$w_r(T) := \begin{pmatrix} 1 \cdots & \cdots & \cdots n \\ T^{\hat{\lambda}_1} & \cdots & T^1 \end{pmatrix}.$$

By [M3], § 3.2.2 $Q(w_r(T)) = T$.

2.4 Let $\mathbf{P}_{\alpha_i}$ be the standard parabolic subgroup with Lie algebra $\mathfrak{p}_{\alpha_i} := \mathfrak{b} \oplus X_{-\alpha_i}$. Take $w \in W, T \in \mathbf{T}_n$, an orbital variety $\mathcal{V}$ and a standard parabolic subgroup $\mathbf{P}$. Define their τ-invariants to be

$$\begin{aligned}
\tau(w) &:= \Pi \cap w(-R^+),\\
\tau(T) &:= \{\alpha_i : r_T(i+1) > r_T(i)\},\\
\tau(\mathbf{P}) &:= \{\alpha_i : \mathbf{P}_{\alpha_i} \subset \mathbf{P}\},\\
\tau(\mathcal{V}) &:= \{\alpha_i : \mathbf{P}_{\alpha_i}(\mathcal{V}) = \mathcal{V}\}.
\end{aligned}$$

Note that $\mathbf{P}$ is uniquely determined by its τ-invariant.

One has (cf.[J],§ 9)

Lemma 2.5 — $\tau(w) = \tau(\mathcal{V}_w) = \tau(Q(w)) = \tau(\mathbf{P}_{\mathcal{V}_w})$.

2.5 Our first counter-example to conjecture 1.1 is constructed in $\mathfrak{sl}_n$ where $n = 9$. By [B] this is the smallest possible value of n. Set

$$Q = \begin{array}{|ccccc|}\hline 1 & 2 & 3 & 6 & 9 \\ 4 & 5 & 8 & & \\ 7 & & & & \\ \hline\end{array}\ .$$

Using $(*)$ and $(**)$ we get $\dim \mathcal{V}_Q = 31$. On the other hand one can show (using "Mathematica") that $\dim[\mathfrak{p}_Q, M] \leq 30$ for all $M \in \mathfrak{n} \cap^{w_r(Q)} \mathfrak{n}$. By corollary 2.3 this means that $\mathcal{V}_Q$ is not of S type.

We show in 4.4 that $\mathcal{O}_\lambda$ for $\lambda \geq (n-3, 3)$ is of S type for all n. We show in 3.3 that $\mathcal{O}_\lambda$ for $\lambda = (n-4, 4)$ is not of S type for all n sufficiently large. The first case of the latter occurs when $n = 10$. Take

$$R = \begin{array}{|cccccc|}\hline 1 & 2 & 3 & 6 & 7 & 10 \\ 4 & 5 & 8 & 9 & & \\ \hline\end{array}\ .$$

Again using $(*)$ and $(**)$ we get $\dim \mathcal{V}_R = 41$ and again we get that $\dim[\mathfrak{p}_R, M] \leq 40$ for all $M \in \mathfrak{n} \cap^{w_r(R)} \mathfrak{n}$. Thus the orbital variety $\mathcal{V}_R$ is not of S type.

In what follows we use one more example of orbital variety not of S type. This is $\mathcal{V}_U$ where

$$U = \begin{array}{|ccccc|}\hline 1 & 2 & 3 & 6 & 9 \\ 4 & 5 & 8 & & \\ 7 & 10 & & & \\ \hline\end{array}\ .$$

3 Orbits not of S type in $\mathfrak{sl}_n$ for $n \geq 13$.

3.1 In this section we show that in $\mathfrak{sl}_n$ for $n \geq 13$ one has

Proposition 3.1 — *For each* $\lambda = (\lambda_1, \lambda_2, \cdots)$ *such that* $(5,3,1,\cdots) \leq \lambda \leq (n-4,4)$ *and* $\lambda_2 > 2$ *the corresponding* $\mathcal{O}_\lambda$ *is not of* S *type.*

To show this we need a few facts about projections on Levi factors.

3.2 Take $J \subset \Pi$ and let W_J be the subgroup of W generated by $\{s_\alpha : \alpha \in J\}$. Let w_{OJ} be the longest element of W_J.

Let D_J be the set of minimal length representatives of right cosets of W_J. Each $w \in W$ has a unique expression of the form $w = w_J d_J$ where $w_J \in W_J$ and $d_J \in D_J$. We define the projection $\pi_J : W \to W_J$ by $\pi_J(w) = w_J$.

Set $\mathbf{P}_J := \mathbf{B} W_J \mathbf{B}$. This is the standard parabolic subgroup of $\mathbf{G}$ with $\tau(\mathbf{P}_J) = J$. It has the Levi decomposition $\mathbf{P}_J = \mathbf{L}_J \ltimes \mathbf{M}_J$ where $\mathbf{L}_J$ is its Levi factor and $\mathbf{M}_J$ is its unipotent radical. Let $\mathfrak{p}_J$ be the corresponding standard parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{m}_J$ where $\mathfrak{l}_J$ is the Levi subalgebra of $\mathfrak{p}_J$ and $\mathfrak{m}_J$ is its nilpotent radical.

Define the projection $\pi_J : \mathbf{P}_J \to \mathbf{L}_J$ by Levi decomposition. Given $\mathbf{P} \subset \mathbf{P}_J$ a parabolic subgroup of $\mathbf{G}$ then $\pi_J(\mathbf{P})$ is a parabolic subgroup of $\mathbf{L}_J$.

Set $\mathbf{B}_J := \mathbf{L}_J \cap \mathbf{B}$ and $\mathfrak{n}_J := \mathfrak{l}_J \cap \mathfrak{n}$. One has $\mathbf{B} = \mathbf{B}_J \ltimes \mathbf{M}_J$ and $\mathfrak{n} = \mathfrak{n}_J \oplus \mathfrak{m}_J$. Let $\pi_J : \mathbf{B} \to \mathbf{B}_J$ and $\pi_J : \mathfrak{n} \to \mathfrak{n}_J$ be the projections onto the first factor.

Note that these projections are compatible in the sense that for any $M \in \mathfrak{n}$ and any parabolic subgroup $\mathbf{P} \subset \mathbf{P}_J$

$$\pi_J(\mathbf{P}M) = \pi_J(\mathbf{P})\pi_J(M).$$

***Proposition 3.2* ([M3, 4.1.2])** — *For every* $w \in W$ *one has*

(i) $\pi_J(\mathfrak{n} \cap^w \mathfrak{n}) = \mathfrak{n}_J \cap^{\pi_J(w)} \mathfrak{n}_J$.

(ii) $\pi_J(\overline{\mathcal{V}}_w) = \overline{\mathcal{V}}_{\pi_J(w)}$.

Given $J = \{\alpha_k\}_{k=i}^{j-1}$ we set $\pi_{i,j} := \pi_J$, $\mathbf{L}_{i,j} := \mathbf{L}_J$ etc. Set

$$\mathbf{L}'_{i,j} = \{M : M \in \mathbf{L}_{i,j}, (M)_{kk} = 1 \text{ if } k < i \text{ or } k > j\}.$$

Then $\mathbf{L}'_{i,j}$ is a general linear subgroup of $\mathbf{L}_{i,j}$. For $T \in \mathbf{T}_n$ let $T_{1,i} := \pi_{1,i}(T)$ be a tableau obtained from T by successive elimination of boxes containing $n, n-1, \cdots, i+1$ As it is shown in [M3], 4.6.3

$$Q(\pi_{1,i}(w)) = Q_{1,i}(w).$$

Take $J = \Pi \setminus \{\alpha_i\}$. In that case $\mathbf{P}_J = (\mathbf{L}'_{1,i} \times \mathbf{L}'_{i+1,n}) \ltimes \mathbf{M}_J$. Let $\pi_{1,i} : \mathbf{P}_J \to \mathbf{L}'_{1,i}$ be the projection onto the first factor.

Lemma 3.3 — *Take $J = \Pi \setminus \{\alpha_i\}$*

(i) *Let $\mathbf{P}$ be a standard parabolic subgroup contained in $\mathbf{P}_J$. For every $M \in \mathfrak{n}$ one has $\pi_{1,i}(\mathbf{P}M) = \pi_{1,i}(\mathbf{P})\,\pi_{1,i}(M)$.*

(ii) *Let $T \in \mathbf{T}_n$ be such that $\alpha_i \notin \tau(T)$ then $\pi_{1,i}(\mathbf{P}_T) = \mathbf{P}_{T_{1,i}}$.*

Combining proposition 3.2 and lemma 3.3 we obtain

Corollary 3.4 — *Fix $T \in \mathbf{T}_n$ such that $\alpha_i \notin \tau(T)$. If the orbital variety $\mathcal{V}_T$ is of S type then the orbital variety $\mathcal{V}_{T_{1,i}}$ is of S type.*

3.3 Now we can show proposition 3.1. Suppose $n \geq 13$. If λ is such that $(5,3,1,\cdots) \leq \lambda \leq (n-4,4)$ and $\lambda_2 > 2$ then it must satisfy one of the following

(i) $\hat{\lambda}_1 = 2$.

(ii) $\lambda_1 \geq 6$ and $\hat{\lambda}_1 \geq 3$

(iii) $\lambda_1 = 5$ and $\hat{\lambda}_1 = 3$ (this occurs only if $13 \leq n \leq 15$).

(iv) $\lambda_1 = 5, \hat{\lambda}_1 > 3$ and $\lambda_2 + \lambda_3 \geq 5$.

(v) $\lambda_1 = 5, \lambda_2 = 3$ and $\hat{\lambda}_1 = n - 6$.

Further we consider each case separately using one of the examples from 2.5. and corollary 3.4. Since all the proofs are similar let us show only the proof of (i).

Note that if $\hat{\lambda}_1 = 2$ then $\lambda_1 \geq [\frac{n+1}{2}] \geq 7$. Consider T with $\operatorname{sh} T = \lambda$ such that $\pi_{1,10}(T)$ is R from 2.5 and $T_7^1 = 11$. By lemma 2.5 one has $\alpha_{10} \notin \tau(\mathcal{V}_T)$ and by 2.5, $\mathcal{V}_{\pi_{1,10}(T)}$ is not of S type. Hence by corollary 3.4, $\mathcal{V}_T$ is not of S type.

4 Orbits of S type in $\mathfrak{sl}_n$

4.1 Now let us consider the cases $\lambda \notin \{\mu : (n-4,4) \leq \mu \leq (5,3,1,\cdots), \mu_2 > 2\}$. The main purpose of this section is to show the

Proposition 4.1 — *$\mathcal{O}_\lambda$ is of S type if λ is one of the following*

(i) $\lambda > (n-4,4)$.

(ii) $\lambda = (\lambda_1, \lambda_2, 1, \cdots, 1)$ *where* $\lambda_2 \leq 2$.

(iii) $\lambda = (\lambda_1, \cdots)$ *where* $\lambda_1 = 2$ *and* $\lambda_i \leq 2$ *for all* $i : 2 \leq i \leq \hat{\lambda}_1$.

In each case there are specific reasons for $\mathcal{O}_\lambda$ to be of S type. Our strategy is as follows. First we consider some special orbital varieties $\mathcal{V}$ in $\mathcal{O}$ for which we are constructing $M \in \mathcal{V}$ such that $\mathbf{B}M$ is dense in $\mathcal{V}$. Then we use a technique related to Vogan's $\mathcal{T}_{\alpha\beta}$ operation [V] to construct $M \in \mathcal{V}$ for an arbitrary $\mathcal{V}$ in $\mathcal{O}$ such that $\mathbf{P}_{\mathcal{V}}M$ is dense in $\mathcal{V}$.

Here we will only underline the general ideas and describe $M \in \mathcal{V}$ without proofs since they involve heavy (although straightforward) computations. The full description can be found in [M4].

4.2 Let us first develop a technique related to $\mathcal{T}_{\alpha\beta}$.

Let $\alpha, \beta \in \Pi$ be the adjacent roots in the Dynkin diagram, that is such that $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$. Set

$$D(\mathcal{T}_{\alpha\beta}) = \{w \in W : \alpha \notin \tau(w), \beta \in \tau(w)\}.$$

and define a map $\mathcal{T}_{\alpha\beta} : D(\mathcal{T}_{\alpha\beta}) \to D(\mathcal{T}_{\beta\alpha})$ by

$$\mathcal{T}_{\alpha\beta}(w) = \begin{cases} s_\alpha w & \text{if } \beta \notin \tau(s_\alpha w) \\ s_\beta w & \text{otherwise} \end{cases}$$

Let $l(w)$ denote the length of a minimal decomposition of w into fundamental reflections. Note that for $w \in D(\mathcal{T}_{\alpha\beta})$ one has $l(s_\alpha w) = l(w) + 1$ and $l(s_\beta w) = l(w) - 1$.

Set

$$\begin{aligned} D_{\mathcal{V}}(T_{\alpha\beta}) &= \{\mathcal{V} - \text{orbital varieties}: \ \alpha \notin \tau(\mathcal{V}), \beta \in \tau(\mathcal{V})\}; \\ D_T(T_{\alpha\beta}) &= \{T \in \mathbf{T}_n : \alpha \notin \tau(T), \beta \in \tau(T)\}; \\ D_{\mathbf{P}}(T_{\alpha\beta}) &= \{\mathbf{P} - \text{parabolic} : \alpha \notin \tau(\mathbf{P}), \beta \in \tau(\mathbf{P})\}; \\ D_{\mathfrak{p}}(T_{\alpha\beta}) &= \{\mathfrak{p} - \text{parabolic} : \alpha \notin \tau(\mathfrak{p}), \beta \in \tau(\mathfrak{p})\}. \end{aligned}$$

In what follows we will omit the subscript since it is obvious which set among those mentioned above we are considering. By [V, 3.7], [J, 9.11] one has $\mathcal{V}_{\mathcal{T}_{\alpha\beta}(w)} = \mathcal{V}_{\mathcal{T}_{\alpha\beta}(y)}$ iff $\mathcal{V}_w = \mathcal{V}_y$ hence

$$\mathcal{T}_{\alpha\beta}(\mathcal{V}_w) := \mathcal{V}_{\mathcal{T}_{\alpha\beta}(w)} \quad \text{and} \quad \mathcal{T}_{\alpha\beta}(Q(w)) = Q(\mathcal{T}_{\alpha\beta}(w)).$$

Given $\mathbf{P}_J \in D(\mathcal{T}_{\alpha\beta}), \mathfrak{p}_J \in D(\mathcal{T}_{\alpha\beta})$ set

$$\mathcal{T}_{\alpha\beta}(\mathbf{P}_J) = \mathbf{P}_{\{J\setminus\beta,\alpha\}} \quad \text{and} \quad \mathcal{T}_{\alpha\beta}(\mathfrak{p}_J) = \mathfrak{p}_{\{J\setminus\beta,\alpha\}}.$$

If $l(w) > l(\mathcal{T}_{\alpha\beta}(w))$ then $\mathbf{P}_{\mathcal{T}_{\alpha\beta}(\mathcal{V}_w)} \supseteq \mathcal{T}_{\alpha\beta}(\mathbf{P}_{\mathcal{V}_w})$.

Let T be a tableau satisfying one of the following conditions

(I) $r_T(i) \le r_T(i-1) < r_T(i+1)$

(II) $r_T(i) < r_T(i+2) \le r_T(i+1)$

By lemma 2.5 these conditions imply $T \in D(\mathcal{T}_{\alpha\beta})$ where $\beta = \alpha_i$ and $\alpha = \alpha_{i-1}$ in case (I) or $\alpha = \alpha_{i+1}$ in case (II). If T satisfy both (I), (II) fix α either α_{i-1} or α_{i+1}. Set $U = \mathcal{T}_{\alpha\beta}(T)$. In both cases

$$U_m^j = \begin{cases} i+1 & \text{if } T_m^j = i \\ i & \text{if } T_m^j = i+1 \\ T_m^j & \text{otherwise} \end{cases}$$

Recall the definition of $w_r(T)$ from 2.3. Note that

$$w_r(U) = \mathcal{T}_{\alpha\beta}(w_r(T)) = s_i w_r(T).$$

Let $s_i : \mathfrak{gl}_n \to \mathfrak{gl}_n$ be the transformation changing the i-th row with the $i+1$-th row and the i-th column with $i+1$-th column.

Proposition 4.2 — *Let $T \in \mathbf{T}_n$ satisfy one of conditions* (I), (II). *Let $\beta = \alpha_i$ and $\alpha = \alpha_{i-1}$ (or α_{i+1}). Let $M \in \mathfrak{n} \cap^{w_r(T)} \mathfrak{n}$ be such that $(M)_\alpha \neq 0$. Let $\mathbf{P} \subset \mathbf{P}_T$ be a parabolic subgroup such that both $\mathbf{P}_{\alpha_{i-1}}, \mathbf{P}_{\alpha_{i+1}} \not\subset \mathbf{P}$.*

(i) *If $M \in \mathcal{V}_T$ then $s_i(M) \in \mathcal{V}_T \cap \mathcal{T}_{\alpha\beta}(\mathcal{V}_T)$*

(ii) *If $\mathbf{P}M$ is dense in $\mathcal{V}_T$ then $\mathcal{T}_{\alpha\beta}(\mathbf{P})s_i(M)$ is dense in $\mathcal{T}_{\alpha\beta}(\mathcal{V}_T)$. In particular, in that case, $\mathcal{T}_{\alpha\beta}(\mathcal{V}_T)$ is of S type.*

We also need the following observation

Lemma 4.3 — *Let $\mathcal{V}_w$ have a dense $\mathbf{P}_{\mathcal{V}_w}$ orbit $\mathcal{P}$. Let $M \in \mathfrak{n} \cap^w \mathfrak{n} \cap \mathcal{P}$ and let $X_{ij} \subset \mathfrak{n} \cap^w \mathfrak{n}$. Set $M^{ij}(a) = M + ae_{ij}$. Then $\mathbf{P}_{\mathcal{V}_w} M^{ij}(a)$ is dense in $\mathcal{V}_w$ for all but finite number of $a \in \mathbb{C}$. In particular if $\alpha_i \notin \tau(w)$ and $(M)_{ii+1} = 0$ then the $\mathbf{P}_{\mathcal{V}_w}$ orbit of $M + ae_{ii+1}$ is dense in $\mathcal{V}$ for all but finite number of $a \in \mathbb{C}$.*

4.3 We call an orbital variety $\mathcal{V}$ a Richardson component if $\mathcal{V} = \mathcal{V}_{w_{0J}}$ for some $J \subset \Pi$. By the Richardson theorem [R] Richardson components are of S type.

Let w_0 be the longest element of W. We call an orbital variety $\mathcal{V}$ a Bala-Carter component if $\mathcal{V} = \mathcal{V}_{w_{0J}w_0}$. By § 5.9 of [C] every Bala-Carter component has a dense $\mathbf{B}$ orbit. Moreover let $\mathcal{V}$ be Bala-Carter then

$$(M_{\mathcal{V}})_{ij} = \begin{cases} 1 & \text{if } j = i+1, \alpha_i \notin \tau(\mathcal{V}), \\ 0 & \text{otherwise.} \end{cases}$$

is an element of $\mathcal{V}$ whose $\mathbf{B}$ orbit is dense in $\mathcal{V}$.

One can verify using proposition 4.2 that if $\mathcal{V} \in D(\mathcal{T}_{\alpha\beta})$ is Bala-Carter then $\mathcal{T}_{\alpha\beta}(\mathcal{V})$ is of S type. Moreover using lemma 4.3 we can continue this process. Unfortunately this algorithm is not decisive, that is we cannot claim that an orbital variety which is not obtained in such a manner from the known orbital varieties of S type is not of S type. Yet this algorithm gives orbital varieties of S type which are neither Bala-Carter nor Richardson.

4.4 To show (i) of proposition 4.1 we first show

Proposition 4.4 — *Let $\lambda = (m, k)$ and let $\mathcal{V}_T$ be an orbital variety contained in $\mathcal{O}_\lambda$ such that*

$$T = \begin{array}{|ccccccc|} \hline 1 & \dots & \dots & i\text{-}1 & i{+}k & \dots & n \\ \hline i & \dots & i{+}k\text{-}1 & & & & \\ \hline \end{array}$$

Define M_T by

$$(M_T)_{jl} = \begin{cases} 1 & \textit{if } j \neq i-1, l = j+1 \\ 1 & \textit{if } j = i-1, l = i+k \\ 0 & \textit{otherwise.} \end{cases}$$

Then $M_T \in \mathfrak{n} \cap^{w_r(T)} \mathfrak{n} \cap \mathcal{V}_T$ and $\mathbf{B}M_T$ is dense in $\mathcal{V}_T$.

Now if $\lambda = (n-1, 1)$ then every orbital variety is of form described in proposition 14, hence it has a dense $\mathbf{B}$ orbit.

Let $\lambda = (n-2, 2)$. Let T be a tableau of shape λ. Set $T = T(i, j)$ if $T_1^2 = i, T_2^2 = j$. If $j \neq i+1$ let us define $M(i, j)$ by

$$(M(i,j))_{lm} = \begin{cases} 1 & \text{if } m = l+1 \text{ and } l \neq i-1, i, j-1 \\ 1 & \text{if } l = i-1, m = i+1 \\ 1 & \text{if } l = i, m = j \\ 1 & \text{if } l = j-1, m = j+1 \\ 0 & \text{otherwise.} \end{cases}$$

For example

$$T(3,6) = \begin{array}{|cccccc|} \hline 1 & 2 & 4 & 5 & 7 & 8 \\ \cline{3-6} 3 & 6 & \multicolumn{4}{|c}{} \\ \cline{1-2} \end{array} \qquad M(3,6) = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

We define $M(i, i+1)$ as in proposition 14. It is immediate that $M(i, j) \in \mathfrak{n} \cap^{w(i,j)} \mathfrak{n}$. One can show using proposition 4.2 that $\mathbf{P}_{\alpha_{i-1}} M(i, j)$ is dense in $\mathcal{V}_{T(i,j)}$.

Set $\lambda = (n-3, 3)$. Let T be a tableau of shape λ. Set $T = T(i, j, k)$ if $T_1^2 = i, T_2^2 = j, T_3^2 = k$. Set $w(i, j, k) = w_r(T(i, j, k))$. Here we distinguish 2 cases

(a) $k = j+1$;

(b) $k > j+1$.

In case (a) if $j \neq i+1$ we define $M(i,j,j+1)$ by

$$(M(i,j,j+1))_{lm} = \begin{cases} 1 & \text{if } m = l+1 \text{ and } l \neq i-1, i, j-1 \\ 1 & \text{if } l = i-1, m = i+1 \\ 1 & \text{if } l = i, m = j \\ 1 & \text{if } l = j-1, m = j+2 \\ 0 & \text{otherwise} \end{cases}$$

We define $M(i,i+1,i+2)$ as in proposition 4.4. Again using proposition 4.2 one can show that $M(i,j,j+1) \in \mathfrak{n} \cap^{w(i,j,j+1)} \mathfrak{n}$ and $\mathbf{P}_{\alpha_{i-1}} M(i,j,j+1)$ is dense in $\mathcal{V}_{T(i,j,j+1)}$.

In case (b) if $j \neq i+1$ we define $M(i,j,k)$ by

$$(M(i,j,k))_{lm} = \begin{cases} 1 & \text{if } m = l+1 \text{ and } l \neq i-1, i, j-1, k-1, k \\ 1 & \text{if } l = i-1, m = i+1 \\ 1 & \text{if } l = i, m = j \\ 1 & \text{if } l = j-1, m = j+1 \\ 1 & \text{if } l = j, m = k \\ 1 & \text{if } l = k-1, m = k+1 \\ 0 & \text{otherwise} \end{cases}$$

Define $M(i,i+1,k)$ by

$$(M(i,i+1,k))_{lm} = \begin{cases} 1 & \text{if } m = l+1 \text{ and } l \neq i-1, k-1, k \\ 1 & \text{if } l = i-1, m = i+2 \\ 1 & \text{if } l = i+1, m = k \\ 1 & \text{if } l = k-1, m = k+1 \\ 0 & \text{otherwise} \end{cases}$$

Again one can show that $M(i,j,k) \in \mathfrak{n} \cap^{w_r(i,j,k)} \mathfrak{n}$ and that $\mathbf{P}_{\alpha_{i-1},\alpha_{k-1}} M(i,j,k)$ is dense in $\mathcal{V}_{T(i,j,k)}$.

4.5 To show (ii) of proposition 4.1 we first show

Proposition 4.5 — *Let $\lambda = (\lambda_1, 1, \cdots)$. Let T be a Young tableau of shape λ. Set*

$$(M_T)_{ij} = \begin{cases} 1 & \textit{if } j = T_k^1, i = T_{k-1}^1, k > 2 \\ 1 & \textit{if } j = T_k^1, i = j-1, k > 1 \\ 0 & \textit{otherwise} \end{cases}$$

Then $\mathbf{B}M_T$ is dense in $\mathcal{V}_T$.

Then once more for each T of shape $(i, 2, 1, \cdots)$ we construct $M_T \in \mathfrak{n} \cap^{w_r(T)} \mathfrak{n}$ and show using propositions 4.2, 4.5 and lemma 4.3 that $\mathbf{P}_T M_T$ is dense in $\mathcal{V}_T$.

4.6 Now we show (iii) of proposition 4.1. Recall notation $c_T(k)$ from 2.3. Let $\lambda = (2, 2, \cdots)$. Let T be a Young tableau of shape λ. Let $T' = \pi_{1,n-1}(T)$. We define M_T inductively as follows

(i) If $c_T(n) = 1$ set

$$(M_T)_{ij} = \begin{cases} (M_{T'})_{ij} & \text{if } i, j < n \\ 0 & \text{otherwise} \end{cases}$$

(ii) If $c_T(n) = 2$ set $m = \max\{i = {T'}^k_1 : (M_{T'})_{ij} = 0 \text{ for all } j\}$ and set

$$(M_T)_{ij} = \begin{cases} (M_{T'})_{ij} & \text{if } i, j < n \\ 1 & \text{if } i = m, j = n \\ 0 & \text{otherwise} \end{cases}$$

For example

$$T = \begin{array}{|c|c|} \hline 1 & 2 \\ 3 & 6 \\ 4 & 7 \\ \cline{2-2} 5 & \\ \cline{1-1} \end{array} \qquad M_T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

One can show

Proposition 4.6 — *Let $\lambda = (2, 2, \cdots)$. Let T be a Young tableau of shape λ and M_T defined as above. Then $M_T \in \mathcal{V}_T$ and $\mathbf{B}M_T$ is dense in $\mathcal{V}_T$.*

4.7 We finish the discussion on orbits of S type with two general comments:

(i) If $\lambda = (n-1, 1)$ then all the orbital varieties contained in $\mathcal{O}_\lambda$ are Richardson. If $\lambda = (2, 1, 1, \cdots)$ then all the orbital varieties contained in $\mathcal{O}_\lambda$ are Bala-Carter. So the results were known in these cases.

(ii) The orbits $\mathcal{O}_\lambda$ with $\lambda = (\lambda_1, \lambda_2, \cdots)$ satisfying one of the following conditions

a. $\lambda_1 = 4, \lambda_2 + \lambda_3 \geq 4$;

b. $\lambda_1 = 3, \lambda_2 + \lambda_3 \geq 4$;

c. $\lambda_1 > 2, l_2 = l_3 = 2$;

are still unclassified. Perhaps they are of S type but the proof requires more subtle computations.

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