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Resurgence of Refined Topological Strings and Dual Partition Functions

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Abstract. We study the resurgent structure of the refined topological string partition function on a non-compact Calabi–Yau threefold, at large orders in the string coupling constant g_s and fixed refinement parameter b. For $\mathsf{b} \neq 1$, the Borel transform admits two families of simple poles, corresponding to integral periods rescaled by b and $1/\mathsf{b}$. We show that the corresponding Stokes automorphism is expressed in terms of a generalization of the non-compact quantum dilogarithm, and we conjecture that the Stokes constants are determined by the refined Donaldson–Thomas invariants counting spin-j BPS states. This jump in the refined topological string partition function is a special case (unit five-brane charge) of a more general transformation property of wave functions on quantum twisted tori introduced in earlier work by two of the authors. We show that this property follows from the transformation of a suitable refined dual partition function across BPS rays, defined by extending the Moyal star product to the realm of contact geometry.

Key words: resurgence; topological string theory; Borel resummation; Stokes automorphism

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1 Introduction and discussion

As a mathematically well-defined subsector of type II superstring theories, topological string theory provides a prime arena for exploring the non-perturbative completion of the asymptotic series predicted by the perturbative genus expansion in string theory. In cases where the target Calabi–Yau threefold X admits a large N dual, a non-perturbative formulation is available, which resums the perturbative series to all orders in the topological string coupling $g_s = 1/N$, for fixed values of the moduli $\boldsymbol{t}$ (corresponding to 't Hooft couplings in the dual gauge theory) at least for integer values of N. Non-perturbative corrections of order e^{-1/g_s} typically arise from tunneling (or instanton) effects in the dual matrix model. In general however, one has only access to the genus-g free energy $\mathcal{F}_g(\boldsymbol{t})$ occurring at order g_s^{2g-2} in the perturbative expansion, and non-perturbative corrections are ambiguous, without further assumptions.

Assuming that the putative, non-perturbative topological string partition function $\mathcal{F}(\boldsymbol{t}; g_s)$ belongs to a suitable class of resurgent functions, one can however extract much information about non-perturbative corrections from the growth of the coefficients $\mathcal{F}_g(\boldsymbol{t})$ at large genus. This method was proposed early on in [92] for general string theory models, and first applied to the topological string in [76, 77, 80, 91]. In particular, the instanton action $\mathcal{A}$ controlling the size of the leading instanton effects $\mathrm{e}^{-\mathcal{A}/g_s}$ can be read off from the location of the singularity closest to the origin in the Borel plane.

While the explicit computation of the free energies $\mathcal{F}_g(\boldsymbol{t})$ is usually impractical at large genus, they are strongly constrained by the holomorphic anomaly equations (HAE) [22]. Following [33, 34], much progress has been made recently in constructing the trans-series solution to the HAE [57, 59, 69, 81]. In particular, it was shown that the instanton actions $\mathcal{A}$ entering in the trans-series are equal (up to overall normalization) to the central charge $Z(\gamma)$ of vectors γ in the charge lattice (the Grothendieck lattice $K_0(X)$ in the A-model, or the homology lattice $H_3(X,\mathbb{Z})$ in the B-model). Moreover, the Stokes automorphism $\mathfrak{S}$ controlling the jump in the Borel resummation when a radial integration contour crosses a sequence of singularities was determined in [69]: the jump induces the following transformation in the partition function $\mathcal{Z}=\mathrm{e}^{\mathcal{F}}$,

$$\mathcal{Z} \;\mapsto\; \mathfrak{S}\mathcal{Z} = \exp\left[\frac{\omega(\gamma)}{2\pi\mathrm{i}}\left(\mathrm{Li}_2\left(\mathrm{e}^{2\pi x}\right) + 2\pi x\log\left(1-\mathrm{e}^{2\pi x}\right)\right)\right]\mathcal{Z}, \tag{1.1}$$

where $\omega(\gamma)$ is (up to normalization) the so-called Stokes constant,

$$\mathrm{Li}_2(z) = \sum_{k\geq 1} z^k/k^2$$

is the dilogarithm function and x is suitable operator $\big($equal to $\frac{\mathrm{i}g_s}{2\pi}c^a\partial_{t^a}$ in the simplest case considered in [69], which we also restrict to here for simplicity; here c^a are the components of γ in $H_4(X,\mathbb{Z})$, interpreted as D4-brane charges in type IIA set-up, or D3-brane charges in type IIB$\big)$. As observed in [69], the transformation (1.1) is exactly such that the so-called dual partition function

$$\tau(\boldsymbol{t},\boldsymbol{u};g_s) = \sum_{\boldsymbol{n}\in\mathbb{Z}^n} \mathrm{e}^{2\pi\boldsymbol{n}\cdot\boldsymbol{u}/g_s}\mathcal{Z}(\boldsymbol{t}-\mathrm{i}g_s\boldsymbol{n};g_s) \tag{1.2}$$

transforms by a simple factor, up to a $\boldsymbol{u}$-dependent shift in $\boldsymbol{t}$,

$$\mathfrak{S}\tau(\boldsymbol{t},\boldsymbol{u};g_s) = \mathrm{e}^{\frac{\omega(\gamma)}{2\pi\mathrm{i}}\mathrm{Li}_2(\mathrm{e}^{2\pi\boldsymbol{c}\cdot\boldsymbol{u}/g_s})}\,\tau\left(\boldsymbol{t}+\frac{\omega(\gamma)}{2\pi}g_s\boldsymbol{c}\log\left(1-\mathrm{e}^{2\pi\boldsymbol{c}\cdot\boldsymbol{u}/g_s}\right),\boldsymbol{u};g_s\right). \tag{1.3}$$

We note that dual partition functions (which are obtained as so-called Zak transforms of the conventional partition functions) have appeared in many different contexts, including supersymmetric gauge theories [87], topological string theory [2, 37], topological recursion [40, 41], the spectral theory/topological string correspondence [54] and isomonodromic tau functions [31, 32, 48, 49].

Quite remarkably, the transformations (1.1) and (1.3) have appeared before in yet a different context, namely the study of five-brane instanton corrections to the hypermultiplet moduli space in type II strings compactified on a Calabi–Yau threefold [9, 12]. The key idea of [9, 12] is that S-duality relates NS5-brane instanton corrections in type IIB string theory to D5-brane instanton corrections, which (for unit D5-brane charge) are governed by the topological string partition function by virtue of [82, 83]. More specifically, for fixed unit D5-brane charge $k=1$, the sum over D3-D1-D(-1) brane charges leads to a non-Gaussian theta series of the form (1.2) (with $\boldsymbol{n}$ being the D3-brane charge). Due to wall-crossing behavior of the Donaldson–Thomas (DT) invariants $\Omega(\gamma)$ counting D5-D3-D1-D(-1) instantons in type IIB (or equivalently D6-D4-D2-D0 black holes in type IIA, for in either case, these invariants count semi-stable objects in the derived category of coherent sheaves on X), the theta series becomes a section of a certain line bundle $\mathcal{L}$ over the (twistor space of) the hypermultiplet moduli space, with gluing conditions given by (1.3), which in turn imply the transformation (1.1) for the kernel of the theta series. This suggests in particular that the Stokes constant $\omega(\gamma)$ should be equated with the DT invariant $\Omega(\gamma)$ counting BPS states associated to the same ray, as proposed in different contexts in [18, 56, 57, 58, 59, 79]. These considerations were extended in [12] in two directions: first, to $k > 1$ units of five-brane charge, where the theta series becomes a section of the k-th power of the line bundle $\mathcal{L}$, and second, to include the effect of the refinement parameter y in Donaldson–Thomas theory, conjugate to angular momentum, which induces a non-commutative deformation of the (twistor space of the) hypermultiplet moduli space.

This brings us to the main goal of the present paper, which is to extend the results (1.1)–(1.3) to the refined topological string partition function. The latter can be viewed as a one-parameter deformation $\mathcal{F}(\boldsymbol{t}; g_s, \mathsf{b})$ of the usual topological free energy $\mathcal{F}(\boldsymbol{t}; g_s)$, which exists on non-compact Calabi–Yau threefolds with a $\mathbb{C}^\times$ action.[1] While the worldsheet definition of refined topological strings for $\mathsf{b} \neq 1$ remains obscure (see [20] for an early attempt), in space-time it is interpreted as the partition function of the five-dimensional gauge theory engineered by M-theory compactified on X times a five-dimensional Ω-background [86, 90] with parameters

$$\epsilon_1 = g_s \mathsf{b}, \qquad \epsilon_2 = -g_s \mathsf{b}^{-1}. \tag{1.4}$$

The resulting partition function leads to integer BPS invariants $N_{\boldsymbol{d}}^{j_L, j_R}$ which refine the usual Gopakumar–Vafa and Donaldson–Thomas invariants [28, 63, 88]. When X is a toric CY singularity, $\mathcal{F}(\boldsymbol{t}; g_s, \mathsf{b})$ can be computed by a refined version [68] of the topological vertex formalism [3]. Otherwise, the main computational tool is the refined version [67] of the standard holomorphic anomaly equations (HAE) [22], which can be integrated inductively with sufficient control on the behavior at boundaries of moduli space.

In this work, we extend the analysis of [59, 69] and construct the general trans-series solution to the refined HAE.[2] In particular, we find that there are now two instanton actions $\mathsf{b}Z(\gamma)$ and $\mathsf{b}^{-1}Z(\gamma)$ contributing to the trans-series for each γ in the charge lattice.[3] Moreover, in the simplest case where the singularity corresponds to a BPS state with unit Stokes constant $\omega(\gamma) = 1$ (as befits states which become massless at a conifold singularity), we find that the Stokes automorphism (1.1) is deformed to

$$\mathfrak{S}_{\mathsf{b}} \mathcal{Z} = \Phi_{\mathsf{b}}^{-1}(x) \mathcal{Z}, \tag{1.5}$$

where $\Phi_{\mathsf{b}}(x)$ is the Faddeev quantum dilogarithm defined in (A.5), reducing to (1.1) at $\mathsf{b} = 1$ due to (A.8). More generally, the Stokes automorphism associated to a singularity supported at $Z(\gamma)$ is given by

$$\mathfrak{S}_{\mathsf{b}} \mathcal{Z} = \prod_j \left[\prod_{m=-j}^{j} \Phi_{\mathsf{b}}\big(x + \mathrm{i}m(\mathsf{b} - \mathsf{b}^{-1})\big) \right]^{-\omega_{[j]}} \mathcal{Z}, \tag{1.6}$$

where the product runs over half integer m such that $m + j$ is integer, and $\omega_{[j]}$ are Stokes constants. This has a natural interpretation in terms of the motivic DT invariants $\Omega(\gamma, y)$

[1] It was suggested in [65] that the refined topological string could also be defined on compact CY threefolds, but the resulting free energies are no longer independent of complex structure moduli, and it is unclear if the HAE still holds.

[2] The trans-series solution in the NS limit of the refined theory [89] requires a particular treatment and was studied in [60].

[3] The occurrence of two instanton actions $\mathcal{A}/\mathsf{b}$ and $\mathsf{b}\mathcal{A}$ was already observed in an earlier study of the resurgent structure of the refined topological string on the resolved conifold [53]. Our main claim is that this feature arises for refined topological strings on arbitrary non-compact CY threefolds.

introduced in [73] (see also [38]). The latter can be decomposed as

$$\Omega(\gamma, y) = \sum_j \Omega_{[j]}(\gamma)\chi_j(y),$$

where

$$\chi_j(y) := \sum_{m=-j}^{j} y^{2m} = \frac{y^{2j+1} - y^{-2j-1}}{y - y^{-1}} \tag{1.7}$$

is the character of the spin j representation of SU(2). We conjecture that the Stokes constants $\omega_{[j]}$ associated to the singularity at $Z(\gamma)$ are equal to the coefficients $\Omega_{[j]}(\gamma)$ appearing in this decomposition.[4] This conjecture is motivated by the comparison of (1.6) with a structurally identical formula obtained in [12] in a different but related context, as we will review and extend in Section 4. The conjecture can be also verified directly in special cases, e.g., when γ corresponds to a D2-D0 BPS state, as shown below in equation (3.61).

It is then natural to ask whether there exists a deformation of the dual partition function (1.2), with kernel given by the refined topological string partition function $\mathcal{Z}$, whose transformation rules across rays would be guaranteed by the Stokes automorphism (1.5), or more generally (1.6). In fact, the Stokes automorphism (1.5) is a special case $k = 1$ of an operator $\mathbf{A}^{(k)}_{\mathsf{b}^2}$ acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^k$ which was introduced in [12, Section 4] in a rather *ad hoc* fashion, for the purpose of producing a doubly-quantized version of the standard 5-term relation for the quantum dilogarithm. In this work, we provide a rationale for the construction of [12, Section 4], by explicitly constructing a generalized theta series valued in a degree k line bundle $\mathcal{L}_{\mathsf{b}}$ over the quantum torus, which is well defined across BPS rays provided the kernel of the theta series transforms according to $\mathbf{A}^{(k)}_{\mathsf{b}^2}$, or more generally by a j-dependent extension of $\mathbf{A}^{(k)}_{\mathsf{b}^2}$. For $k = 1$, this provides a refined version of the dual partition function (1.2). Both the theta series and its transformation under wall-crossing are defined using a certain Moyal-type non-commutative product on twistor space. In fact, the non-commutativity makes it possible to introduce two different refined dual partition functions which are both related to the refined topological string partition function, but with one of the twistor space coordinates identified with (the inverse of) either of the two deformation parameters $\epsilon_{1,2}$ from (1.4).

These results raise several obvious questions: first, can one prove the equality between Stokes constants and Donaldson–Thomas invariants, perhaps in the framework of the Riemann–Hilbert problem proposed in [13, 21, 24, 25, 26]? Second, what is the physical interpretation of the refined dual partition function, for example in terms of free fermions, spectral determinants, tau functions, partition functions of line defects, or otherwise? Third, does there exists a rank-k version of the topological string (refined or unrefined) whose large order behavior would be governed by the Stokes automorphism $\mathbf{A}^{(k)}_{\mathsf{b}^2}$, and which would be related to the partition function of k five-branes constructed in [9, Section 5] by a rank-k version of the GV/DT correspondence [82]? We hope to return to some of these challenging problems in the near future.

The remainder of this work is organized as follows. In Section 2, we review basic properties of the refined topological string free energy. In Section 3, we construct the trans-series solution of the holomorphic anomaly equations, determine the boundary conditions at the conifold locus and in the large volume limit, deduce the Stokes automorphism associated to the singularities in the Borel plane, and perform numerical checks in the case of $X = K_{\mathbb{P}^2}$. In Section 4, we recover the same automorphism and generalize it to any integer $k \geq 1$ by constructing a class of dual partition functions on quantum twisted tori. Definitions and properties of several variants of the quantum dilogarithm function, which plays a central role in this work, are collected in Appendix A.

[4] The relation between the refinement parameters b and y will become apparent in (3.54).

2 The refined topological string

In this section, we review some basic facts about the refined topological string and how to calculate its free energy.

As mentioned above, the refined topological string is parametrized by two complex parameters $\epsilon_{1,2}$. Setting

$$\epsilon_1 = g_s \mathsf{b}, \qquad \epsilon_2 = -g_s \mathsf{b}^{-1}, \tag{2.1}$$

the parameter g_s can be identified as the topological string coupling constant, while b can be regarded as a deformation parameter. When $\mathsf{b} = 1$ one recovers the standard topological string. We note that many references (like [74]) use a parameter β, which is related to ours simply by $\mathsf{b} = \beta^{1/2}$. The refined topological string free energy is a formal power series [67]

$$\mathcal{F}(\boldsymbol{t}; \epsilon_1, \epsilon_2) = \sum_{i,j \geq 0} (\epsilon_1 + \epsilon_2)^{2i} (-\epsilon_1 \epsilon_2)^{j-1} \mathcal{F}^{(i,j)}(\boldsymbol{t}), \tag{2.2}$$

where the coefficients $\mathcal{F}^{(i,j)}(\boldsymbol{t})$ are sections of an appropriate line bundle on the moduli space of X (of Kähler structures in the A-model, or complex structures in the B-model), parametrized by the flat coordinates $\boldsymbol{t}$. (These are complexified Kähler parameters in the A-model, and periods of the holomorphic 3-form in the B-model. We recall that the coordinates are flat with respect to the Gauss–Manin connection.) For the purposes of this paper, we regard the free energy as a formal power series in g_s whose coefficients depend on the moduli and on the deformation parameter b,

$$\mathcal{F}(\boldsymbol{t}; g_s, \mathsf{b}) = \sum_{g \geq 0} g_s^{2g-2} \mathcal{F}_g(\boldsymbol{t}; \mathsf{b}), \tag{2.3}$$

where

$$\mathcal{F}_g(\boldsymbol{t}; \mathsf{b}) = \sum_{k=0}^{g} \mathcal{F}^{(k,g-k)}(\boldsymbol{t}) \left(\mathsf{b} - \mathsf{b}^{-1}\right)^{2k}. \tag{2.4}$$

The deformed free energies $\mathcal{F}_g(\boldsymbol{t}; \mathsf{b})$ are Laurent polynomials in b, of degree $2g$, and invariant under the exchange $\mathsf{b} \leftrightarrow \mathsf{b}^{-1}$. When $\mathsf{b} = 1$,

$$\mathcal{F}_g(\boldsymbol{t}; 1) = \mathcal{F}^{(0,g)}(\boldsymbol{t}) = \mathcal{F}_g(\boldsymbol{t}) \tag{2.5}$$

is the conventional, unrefined topological string free energy at genus g. The amplitudes $\mathcal{F}^{(k,g-k)}(\boldsymbol{t})$ are related to the ones defined in [67] by an overall sign $(-1)^k$.

The refined free energies $\mathcal{F}_g(\boldsymbol{t}; \mathsf{b})$ can be calculated on certain local CY geometries using instanton calculus [86], and more generally with the refined topological vertex of [68]. These A-model-like calculations are intrinsically attached to the large radius of the geometry. In the refined case, the B-model approach to the free energies is based on the HAE of [22].[5] An extension of the HAE to the refined case was proposed in [67] (see also [74]), and tested in detail in, e.g., [28, 64]. In the framework of the HAE, one extends the free energies $\mathcal{F}_g(\boldsymbol{t}; \mathsf{b})$ to non-holomorphic functions, and the HAE control their non-holomorphic dependence. We will use Roman capital letters for the non-holomorphic free energies obtained from the HAE, and curly capital letters for their holomorphic limit, as in [57, 59].

[5] In the unrefined case, and in local CY geometries, the B-model is described by the topological recursion of [42], as proposed in [23, 77]. To our knowledge a refined version of the formalism for generic local CY geometries is not available yet, see however [70] for recent progress.

To write the HAE, we need some ingredients from special geometry (see, e.g., [71] for more details). In the local case, the mirror CY is encoded in an algebraic curve usually called the mirror curve. We recall that the moduli space $\mathcal{M}$ of complex structures of the mirror CY is a special Kähler manifold of complex dimension n. We will denote by z^a, with $a = 1, \dots, n$ a set of algebraic complex coordinates on this moduli space. The Kähler metric (here $\partial_a = \partial_{z^a}$)

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K \tag{2.6}$$

derives from a Kähler potential K, which is determined from the prepotential $\mathcal{F}_0$, equal to the genus zero free energy (the latter does not depend on b, as it clear from (2.4)). We introduce the covariant derivative D_a for the Levi–Civita connection associated to the metric,

$$\Gamma^a_{bc} = G^{a\bar{k}} \partial_b G_{c\bar{k}}. \tag{2.7}$$

In the case of compact CY threefolds, the covariant derivative involves as well a connection on the Hodge bundle $\mathcal{L}$ over $\mathcal{M}$. However, in the local case, this connection vanishes in the holomorphic limit. Therefore, we can solve the HAE by setting formally $K_a = 0$ [72]. An additional ingredient we will need is the Yukawa coupling, which is a tensor C_{abc}. In the coordinate system given by the flat coordinates t^i, the Yukawa coupling is given by the third derivatives of the genus 0 free energy, so we have in general

$$C_{abc} = \frac{\partial t^i}{\partial z^a} \frac{\partial t^j}{\partial z^b} \frac{\partial t^k}{\partial z^c} \frac{\partial^3 \mathcal{F}_0}{\partial t^i \partial t^j \partial t^k}. \tag{2.8}$$

Finally, we have to introduce an anti-holomorphic version of the Yukawa coupling, defined by

$$\overline{C}^{ab}_{\bar{c}} = \mathrm{e}^{2K} G^{a\bar{d}} G^{b\bar{e}} \overline{C}_{\bar{a}\bar{d}\bar{e}}. \tag{2.9}$$

We can now write the refined HAE, following [67]. In terms of the free energies $F^{(k,g-k)}$, they read

$$\partial_{\bar{c}} F^{(k,g-k)} = \frac{1}{2} \overline{C}^{ab}_{\bar{c}} \Big(D_a D_b F^{(k,g-k-1)} + \sum_{m,h} D_a F^{(m,h)} D_b F^{(k-m,g-k-h)} \Big), \tag{2.10}$$

where $g \geq 2$ and the sum over m, h is such that $m + h \geq 1$. However, it is easy to check that, in terms of the combinations $F_g(\mathsf{b})$ defined in (2.4), the HAE take the same form as in the unrefined case of [22]:

$$\partial_{\bar{c}} F_g(\mathsf{b}) = \frac{1}{2} \overline{C}^{ab}_{\bar{c}} \left(D_a D_b F_{g-1}(\mathsf{b}) + \sum_{h=1}^{g-1} D_a F_h(\mathsf{b}) D_b F_{g-h}(\mathsf{b}) \right), \qquad g \geq 2. \tag{2.11}$$

(Here, we only indicate the dependence of the refined free energies on the deformation parameter, but they of course also depend on z^a.) Therefore, the refined free energies satisfy the same HAE as the unrefined ones, but the starting point of the recursion, given by the free energies at $g = 1$, is different and given by

$$F_1(\mathsf{b}) = F_1(\mathsf{b} = 1) + \big(\mathsf{b} - \mathsf{b}^{-1}\big)^2 F^{(1,0)}. \tag{2.12}$$

Here, $F_1(\mathsf{b} = 1) = F^{(0,1)}$ is the conventional, unrefined free energy at genus one. The free energy $F^{(1,0)}$ turns out to be given by a holomorphic function of the moduli. It has the form [64, 67]

$$F^{(1,0)} = -\frac{1}{24} \log(f(\boldsymbol{z}) \Delta(\boldsymbol{z})), \tag{2.13}$$

where $\Delta(\boldsymbol{z})$ is the discriminant of the mirror curve, and $f(\boldsymbol{z})$ is a rational function of the moduli, explicitly known in many cases. The fact that $F^{(1,0)}$ is holomorphic will be important later on.

As in the unrefined local case, it is useful to introduce propagators S^{ab}, defined by the condition that [22]

$$\partial_{\bar{c}} S^{ab} = \overline{C}_{\bar{c}}^{ab}. \tag{2.14}$$

As a result, the HAE (2.11) can be rewritten in the form

$$\frac{\partial F_g(\mathsf{b})}{\partial S^{ab}} = \frac{1}{2}\left(D_a D_b F_{g-1}(\mathsf{b}) + \sum_{h=1}^{g-1} D_a F_h(\mathsf{b}) D_b F_{g-h}(\mathsf{b})\right), \qquad g \geq 2. \tag{2.15}$$

The initial condition (2.12) has to be also re-expressed in terms of propagators. Since $F^{(1,0)}$ is purely holomorphic, it does not depend on the propagators, whereas for $F^{(0,1)}$ one has [22]

$$\partial_a F^{(0,1)} = \frac{1}{2} C_{abc} S^{bc} + f_a(\boldsymbol{z}). \tag{2.16}$$

In this equation, $f_a(\boldsymbol{z})$ is a function of the moduli only. The equation (2.15), together with the initial conditions (2.12), (2.16) and (2.13), provides a recursive procedure to obtain the free energies $F_g(\mathsf{b})$ as polynomials in the propagators, which is sometimes called "direct integration" [17, 55, 66, 94]. At each genus g one has to fix an integration constant, independent of the propagators, but dependent on the moduli, and called the holomorphic ambiguity (the functions $f_a(\boldsymbol{z})$ in (2.16) can be regarded as examples thereof).

We would like to recall that the holomorphic free energies $\mathcal{F}_g(\boldsymbol{t};\mathsf{b})$ depend on what is called a choice of *symplectic frame*. This choice is determined, in the B-model, by a choice of a symplectic basis of periods, which determines in turn a choice of flat coordinates $\boldsymbol{t}$ and of "dual" coordinates $\partial\mathcal{F}_0/\partial\boldsymbol{t}$. A symplectic transformation of the periods leads to a change of frame, which is implemented at the level of the partition function $\mathcal{Z}$ by a generalized Fourier transform [1]. There are canonical choices of frame associated to special points in moduli space. In particular, the large radius frame is adapted to the large radius point in moduli space, and in this frame the holomorphic free energies have an expansion in terms of Gromov–Witten invariants. There is also a conifold frame adapted to the conifold point of the moduli space. We also note that the expression of the free energies in terms of the propagators S^{ab} is independent of the frame. However, the holomorphic limit of the propagators S^{ab}, which we will denote by $\mathcal{S}^{ab}$, does depend on a choice of frame, and different choices of this holomorphic limit in the solution of the HAE give the different, frame-dependent holomorphic free energies.

In the case of local CY manifolds the holomorphic ambiguity can be fixed in many cases by using the behavior of the free energies at special points in moduli space [61]. Let us review this behavior in the case of the refined topological string. We first consider the behavior at the conifold locus. Let t_c be an appropriately normalized flat coordinate vanishing at this locus, and parametrizing a normal direction to it. In the conifold frame, the free energy has the following behavior as $t_c \to 0$ [67, 74]

$$\mathcal{F}_g(\boldsymbol{t};\mathsf{b}) = \frac{c_g(\mathsf{b})}{t_c^{2g-2}} + \mathcal{O}(1), \qquad g \geq 2, \tag{2.17}$$

where the coefficient $c_g(\mathsf{b})$ is given by

$$c_g(\mathsf{b}) = -(2g-3)! \sum_{m=0}^{g} \widehat{B}_{2m} \widehat{B}_{2g-2m} \mathsf{b}^{2(2m-g)}. \tag{2.18}$$

In this formula

$$\widehat{B}_m = \left(2^{1-m} - 1\right)\frac{B_m}{m!} \tag{2.19}$$

and B_m is the Bernoulli number. In the unrefined limit $\mathsf{b} = 1$ one has

$$c_g(1) = \frac{B_{2g}}{2g(2g-2)}, \qquad g \geq 2, \tag{2.20}$$

and recovers the well-known universal conifold behavior of the standard topological string [51].

Another universal result concerns the behavior of the refined free energies at large radius and in the large radius frame. It is possible to generalize the Gopakumar–Vafa integrality structure [52] to a refined version [63, 67]. Let us introduce

$$\epsilon_{L,R} = \frac{\epsilon_1 \mp \epsilon_2}{2}. \tag{2.21}$$

In terms of the characters $\chi_j(y)$ defined in (1.7), the total free energy is then given, up to a cubic polynomial in $\boldsymbol{t}$, by a sum of the form

$$\mathcal{F}(\boldsymbol{t}; g_s, \mathsf{b}) = \sum_{w,\boldsymbol{d}} \sum_{j_L, j_R} \frac{\chi_{j_L}\left(\mathrm{e}^{\mathrm{i}w\epsilon_L}\right)\chi_{j_R}\left(\mathrm{e}^{\mathrm{i}w\epsilon_R}\right)}{4\sin\left(\frac{g_s w \mathsf{b}}{2}\right)\sin\left(\frac{g_s w}{2\mathsf{b}}\right)} N_{\boldsymbol{d}}^{j_L, j_R} Q_{\boldsymbol{d}}^{w}, \tag{2.22}$$

where $Q_{\boldsymbol{d}} = \mathrm{e}^{-\boldsymbol{d}\cdot\boldsymbol{t}}$ and $N_{\boldsymbol{d}}^{j_L,j_R}$ are integers (here, $j_L, j_R \in \mathbb{Z}^+/2$). The integers $N_{\boldsymbol{d}}^{j_L,j_R}$ count BPS states with charge $\boldsymbol{d}$ transforming with spin (j_L, j_R) under the little group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ in 5 dimensions, and are sometimes called BPS invariants. When $\mathsf{b} = 1$, we recover the integrality structure of the standard topological string free energy. In particular, we have the following relation between the genus zero Gopakumar–Vafa invariants $n_{\boldsymbol{d}}^{(0)}$ and the BPS invariants

$$n_{\boldsymbol{d}}^{(0)} = \sum_{j_L, j_R} d_L d_R N_{\boldsymbol{d}}^{j_L, j_R}, \qquad d_{L,R} = 2j_{L,R} + 1. \tag{2.23}$$

The refined (or motivic) Donaldson–Thomas invariants $\Omega(\boldsymbol{d}, y)$ in the large volume limit are characters for the diagonal $\mathrm{SU}(2)$ action:

$$\Omega(\boldsymbol{d}, y) = \sum_{j_L, j_R} \chi_{j_L}(y)\chi_{j_R}(y) N_{\boldsymbol{d}}^{j_L, j_R}. \tag{2.24}$$

In particular, they reduce to (2.23) in the unrefined limit $y \to 1$. Unlike the BPS invariants $N_{\boldsymbol{d}}^{j_L,j_R}$, which are only defined when $\boldsymbol{d}$ is a curve class, the refined DT invariants $\Omega(\gamma, y)$ are defined for any vector $\gamma \in K_0(X)$, i.e., for arbitrary D6-D4-D2-D0 brane charge in type IIA, or D5-D3-D1-D(-1) charge in type IIB. For later purposes, it will be convenient to decompose $\Omega(\gamma, y)$ as a sum of SU(2) characters,

$$\Omega(\gamma, y) := \sum_j \chi_j(y)\Omega_{[j]}(\gamma), \tag{2.25}$$

where the integers $\Omega_{[j]}(\gamma)$ count BPS multiplets of angular momentum $j \in \mathbb{Z}/2$, and y is the corresponding fugacity parameter. We note that the representation (2.25) implies that $\Omega(\gamma, y)$ is a Laurent polynomial in y, invariant under Poincaré duality $y \mapsto 1/y$. As emphasized in [84], for the standard definition of DT invariants using cohomology with compact supports, Poincaré duality is broken when X is non-compact, ultimately due to the non-invariance of the attractor indices counting D0-branes. Here we assume that the DT invariants which are pertinent for the refined topological string are in fact invariant under Poincaré duality.

3 Trans-series solutions and resurgent structure

In order to understand the resurgent structure of the refined topological string, we follow the strategy of [33, 34, 35, 57, 59] and look for trans-series solutions to the HAE. These define non-perturbative sectors of the theory, but they require boundary conditions, as in the perturbative case. As first pointed out in [33, 34], we can obtain these boundary conditions by looking at the large order behavior of the genus expansions near special points in moduli space. It turns out that the first aspect of this procedure, namely the construction of formal trans-series solutions to the HAE, is essentially identical to the unrefined case. Therefore, we will be rather succinct and refer the reader to [57, 59]. The second aspect of the procedure, namely the analysis of the boundary conditions, has some new ingredients, and we will analyze it in more detail.

3.1 Trans-series solutions

As in [57, 59], we want to construct multi-instanton trans-series solutions for the free energy and partition function. Let us denote by Z^{NP} a non-perturbative correction to the perturbative partition function Z (i.e., involving an exponentially small dependence on the string coupling constant). We will also introduce the "reduced" partition function

$$Z_{\mathrm{r}} = \frac{Z^{\mathrm{NP}}}{Z}, \tag{3.1}$$

where Z denotes the perturbative partition function. Following [33, 34, 57, 59], we will write down an equation satisfied by the reduced partition function Z_{r} as a consequence of the HAE. To do that, we introduce a pair of $\mathcal{A}$-dependent operators, W and D. In the local case these operators are defined as follows. Let us introduce

$$T^a = g_s \partial_b \mathcal{A}\big(S^{ab} - \mathcal{S}^{ab}_{\mathcal{A}}\big). \tag{3.2}$$

Here, $\mathcal{S}^{ab}_{\mathcal{A}}$ is the holomorphic limit of the propagator in a so-called $\mathcal{A}$-frame, i.e., a symplectic frame in which the action $\mathcal{A}$ is a linear combination of the flat coordinates t^a. Then

$$\mathsf{D} = T^a \frac{\partial}{\partial z^a}. \tag{3.3}$$

To define the operator W, we first introduce the derivation ω_S defined as

$$\omega_S = \frac{1}{g_s^2} T^a T^b \frac{\partial}{\partial S^{ab}}. \tag{3.4}$$

Then W is given by

$$\mathsf{W} = \omega_S - \sum_{g \geq 1} \left[\mathsf{D}\big(g_s^{2g-2} F_g\big)\right] \mathsf{D}, \tag{3.5}$$

where in the refined case the free energies are b-dependent. As shown in [57, 59], as a consequence of the HAE, the reduced partition function satisfies the linear equation

$$\mathsf{W} Z_{\mathrm{r}} = \frac{1}{2} \mathsf{D}^2 Z_{\mathrm{r}}. \tag{3.6}$$

We will consider a special class of solutions of the form

$$Z_{\mathrm{r}} = \exp(\Sigma_\lambda), \tag{3.7}$$

where $g_s\Sigma_\lambda$ is a formal power series in g_s. More precisely, we will write

$$\Sigma_\lambda = -\frac{\lambda}{g_s}\mathcal{A} + \mathcal{O}\big(g_s^0\big), \tag{3.8}$$

so that (3.7) is a non-perturbative, exponentially small quantity. In this equation, λ is an arbitrary constant, and following [33, 34, 39, 57] we will assume that $\mathcal{A}$ is an integer period of the mirror CY manifold. By analogy with instanton physics, we will sometimes call $\mathcal{A}$ an instanton action. We note that, since (3.6) is linear, arbitrary linear combinations of the basic solutions (3.7) are also solutions. As we will see, these solutions will be enough to construct the relevant trans-series for the refined topological string.

Since D is a derivation, the linear equation (3.6) leads to the following operator equation for Σ_λ:

$$\mathsf{W}\Sigma_\lambda = \frac{1}{2}\big(\mathsf{D}^2\Sigma_\lambda + (\mathsf{D}\Sigma_\lambda)^2\big). \tag{3.9}$$

In [57, 59], this equation was solved as follows. Consider the formal series

$$G = \frac{\mathcal{A}}{g_s} + \sum_{g\geq 1}\mathsf{D}\big(g_s^{2g-2}F_g\big), \tag{3.10}$$

where $\mathcal{A}$ is the action appearing in (3.8). Let us now assume that G satisfies the equation

$$\mathsf{W}G = \frac{1}{2}\mathsf{D}^2 G. \tag{3.11}$$

Then

$$\Sigma_\lambda = \mathsf{O}^{(\lambda)}G, \tag{3.12}$$

where

$$\mathsf{O}^{(\lambda)} = \sum_{k\geq 1}\frac{(-\lambda)^k}{k!}\mathsf{D}^{k-1}, \tag{3.13}$$

satisfies (3.9). Therefore, it suffices to check that (3.11) is still true in the refined case. After using the HAE, one finds that (3.11) holds if and only if

$$\omega_S\left(\mathsf{D}F_1\right) - \mathsf{D}\left(\frac{\mathcal{A}}{g_s}\right)\mathsf{D}F_1 - \frac{1}{2}\mathsf{D}^2\left(\frac{\mathcal{A}}{g_s}\right) = 0. \tag{3.14}$$

This equation is true in the conventional topological string, when $F_1 = F_1(\mathsf{b}=1)$, as one can check by using (2.16). In particular, it holds for any choice of holomorphic ambiguity $f_i^{(1)}(\boldsymbol{z})$. It is easily checked to be true in the refined case as well: since $F_1(\mathsf{b})$ differs from $F_1(\mathsf{b}=1)$ in a purely holomorphic function of the moduli, as noted after (2.12), it follows immediately that (3.14) must also be true for $F_1(\mathsf{b})$. This can also be checked by a direct calculation.

Boundary conditions for trans-series solutions to the HAE are obtained by considering their holomorphic limit in an $\mathcal{A}$-frame. In this limit, the propagator S^{ab} has to be set to $\mathcal{S}^{ab}_{\mathcal{A}}$, and the T^a in (3.2), as well as the operator D, vanish. It is easy to see from the formulae above that Σ_λ, when evaluated in the $\mathcal{A}$-frame, is given by

$$\Sigma_{\lambda,\mathcal{A}} = -\lambda\frac{\mathcal{A}}{g_s}. \tag{3.15}$$

The solution (3.12) involves the full, non-holomorphic propagators, but in practice one wants to understand its holomorphic limit. This goes as follows. First, we write the action $\mathcal{A}$ as a linear combination of periods[6]

$$\mathcal{A} = -\mathrm{i}c^a \frac{\partial \mathcal{F}_0}{\partial t^a} + 2\pi d_a t^a + 4\pi^2 \mathrm{i} d_0. \tag{3.16}$$

For example, in the large radius frame, we have that $t^a = -\log(z^a) + \cdots$ are the complexified Kähler parameters, and the genus zero free energy $\mathcal{F}_0$ behaves as[7]

$$\mathcal{F}_0 = \frac{\kappa_{abc}}{3!} t^a t^b t^c + \mathcal{O}\left(\mathrm{e}^{-t^a}\right), \tag{3.17}$$

as $t^a \to +\infty$. In (3.16), (c^a, d_a, d_0) are integers, and in the large radius frame they correspond to D3-D1-D(-1) charges in type IIB set-up, or D4-D2-D0 charges in type IIA. The action $\mathcal{A}$ is then identified with the central charge $Z(\gamma)$, up to an overall factor $4\pi^2\mathrm{i}$.

When not all c^a vanish at once, one defines a new prepotential by

$$\mathcal{A} = -\mathrm{i}c^a \frac{\partial \widetilde{\mathcal{F}}_0}{\partial t^a}. \tag{3.18}$$

It differs from the conventional prepotential at most in a quadratic polynomial in the flat coordinates t^a. Then, in the holomorphic limit, one has

$$\mathsf{D} \to -\mathrm{i} g_s c^a \frac{\partial}{\partial t^a}, \tag{3.19}$$

and Σ_λ becomes

$$\Sigma_\lambda \to \tilde{\mathcal{F}}(\boldsymbol{t} + \mathrm{i}\lambda g_s \boldsymbol{c}; g_s, \mathsf{b}) - \tilde{\mathcal{F}}(\boldsymbol{t}; g_s, \mathsf{b}), \tag{3.20}$$

where

$$\tilde{\mathcal{F}}(\boldsymbol{t}; g_s, \mathsf{b}) = g_s^{-2} \tilde{\mathcal{F}}_0(\boldsymbol{t}) + \sum_{g\geq 1} g_s^{2g-2} \mathcal{F}_g(\boldsymbol{t}; \mathsf{b}), \tag{3.21}$$

i.e., it differs from the conventional free energy only in the genus zero part.

3.2 Boundary conditions from the conifold

Let us now discuss the boundary conditions for the trans-series. These follow from the behavior of the free energies, in appropriate frames, at special points in the moduli space. This behavior determines the large order asymptotics of the free energies, and this leads in turn to multi-instanton trans-series.

Let us first review how the conifold boundary condition for the trans-series is determined in the case of the standard topological string with $\mathsf{b} = 1$ [33, 34]. The free energies in the conifold frame have the following behavior near the conifold locus,

$$\mathcal{F}_g(\boldsymbol{t}; \mathsf{b} = 1) = \frac{B_{2g}}{2g(2g-2)} t_c^{2-2g} + \mathcal{O}(1), \qquad g \geq 2, \tag{3.22}$$

[6]For compact CY, there is an additional contribution $-\frac{c^0}{2\pi}(\mathcal{F}_0 - 2t^a \partial_a \mathcal{F}_0)$, corresponding at large radius to the D5-brane charge in IIB, or D6-brane charge in IIA, but this term is absent when X is non-compact.

[7]Here κ_{abc} is a rational number, which plays the role of the triple intersection number for a non-compact CY threefold. Note that in our conventions, monodromies around the large radius point induce shifts $t^a \mapsto t^a + 2\pi\mathrm{i}\epsilon^a$ with $\epsilon^a \in \mathbb{Z}$.

which follows from (2.17), (2.20). This singular behavior also determines the growth of the free energies at large genus close to the conifold locus where $t_c = 0$. To compute the large order behavior, one can use the following representation of the Bernoulli numbers

$$B_{2g} = (-1)^{g-1} \frac{2(2g)!}{(2\pi)^{2g}} \sum_{\ell \geq 1} \ell^{-2g} \tag{3.23}$$

to obtain the formula

$$\frac{B_{2g}}{2g(2g-2)} t_c^{2-2g} = \frac{1}{2\pi^2} \Gamma(2g-1) \sum_{\ell \geq 1} (\ell \mathcal{A})^{1-2g} \frac{\mathcal{A}}{\ell} \left(1 + \frac{1}{2g-2}\right), \qquad g \geq 2, \tag{3.24}$$

where

$$\mathcal{A} = 2\pi \mathrm{i} t_c. \tag{3.25}$$

Let us note that, in local CY manifolds, conifold flat coordinates are linear combinations of large radius periods $\partial F_0 / \partial t^i$ and of constant periods (see, e.g., [30]), therefore (3.25) is the central charge of a D4-D0 BPS state.

The formula (3.24) makes manifest the asymptotic factorial growth of the left-hand side for $g \gg 1$, and it includes all corrections to the large g behavior. Standard arguments in the theory of resurgence relate the large order behavior of asymptotic series to exponentially small corrections (see, e.g., [78, Section 3.3]). The basic idea is that factorial growth leads to singularities in the Borel transform of the asymptotic series, and these in turn lead to exponentially small discontinuities in lateral Borel resummations. By using these arguments, one finds that the Borel transform of the perturbative series given by the left-hand side of (3.24) has singularities at the point $\ell \mathcal{A}$, where $\ell \in \mathbb{Z} \backslash \{0\}$, and $\mathcal{A}$ is given by (3.25). In addition, one finds that each of these singularities leads to an ℓ-th instanton amplitude of the Pasquetti–Schiappa form [91],

$$\mathcal{F}_{\mathcal{A}}^{(\ell)} = \frac{1}{2\pi} \left[\frac{1}{\ell} \left(\frac{\mathcal{A}}{g_s} \right) + \frac{1}{\ell^2} \right] \mathrm{e}^{-\ell \mathcal{A}/g_s}, \qquad \ell \in \mathbb{Z} \backslash \{0\}. \tag{3.26}$$

This gives the ℓ-th order trans-series which will provide boundary conditions in the conifold frame.

Another way of finding the trans-series corresponding to the large genus asymptotics of (3.24) is to write an integral formula for the coefficient, of the form

$$\begin{aligned} (-1)^{g-1} \frac{B_{2g}}{2g(2g-2)} &= \frac{1}{2\pi^2} \int_0^\infty \frac{\mathrm{d}z}{z^{2g-1}} \left\{ \mathrm{Li}_2\left(\mathrm{e}^{-2\pi/z}\right) - \frac{2\pi}{z} \log\left(1 - \mathrm{e}^{-2\pi/z}\right) \right\} \\ &= -\frac{\mathrm{i}}{\pi} \int_0^\infty \frac{\mathrm{d}z}{z^{2g-1}} \log \Phi_1 \left(-\frac{1}{z} \right), \qquad g \geq 2, \end{aligned} \tag{3.27}$$

where $\Phi_{\mathsf{b}}(z)$ is Faddeev's quantum dilogarithm (see Appendix A). Up to the overall factor of $1/\pi$, the integrand in the last line of (3.27) gives the sum over all the multi-instantons trans-series with positive ℓ:

$$\sum_{\ell \geq 1} \mathcal{F}_{\mathcal{A}}^{(\ell)} = -\mathrm{i} \log \Phi_1 \left(-\frac{\mathcal{A}}{2\pi g_s} \right). \tag{3.28}$$

This function determines the structure of the Stokes automorphism, as explained in [69], and in line with what was obtained in [12].

Let us now consider the generalization of the above result to the refined topological string.[8] We need a formula for the coefficient $c_g(\mathsf{b})$ appearing in (2.17) which generalizes (3.24). This formula is

$$c_g(\mathsf{b}) = \frac{1}{2\pi}\Gamma(2g-2)\sum_{\ell\geq 1}\left[\frac{(-1)^{\ell-1}}{\ell}\frac{1}{\sin\left(\frac{\pi\ell}{\mathsf{b}^2}\right)}\left(\frac{2\pi\mathrm{i}\ell}{\mathsf{b}}\right)^{2-2g} + \left(\mathsf{b}\leftrightarrow\mathsf{b}^{-1}\right)\right], \qquad g\geq 2, \quad (3.29)$$

and it can be derived by using Borel transform techniques, as follows. Let us define the more convenient set of coefficients:

$$\tilde{c}_0(\mathsf{b}) = 1, \qquad \tilde{c}_1(\mathsf{b}) = \frac{1}{12}\left(\mathsf{b}^2+\mathsf{b}^{-2}\right), \qquad \tilde{c}_g(\mathsf{b}) = (-1)^{g-1}\frac{(2g)!}{(2g-3)!}c_g(\mathsf{b}), \qquad g\geq 2, (3.30)$$

and its generating function

$$\varphi(x) = \sum_{g\geq 0}\tilde{c}_g(\mathsf{b})x^{2g}. \quad (3.31)$$

The Borel transform of $\varphi(x)$ can be obtained in closed form, as

$$\widehat{\varphi}(\zeta) = \frac{\zeta^2}{4\sin\left(\frac{\zeta\mathsf{b}}{2}\right)\sin\left(\frac{\zeta\mathsf{b}^{-1}}{2}\right)}. \quad (3.32)$$

When b^2 is not a rational number, this function has simple pole singularities at

$$\zeta = \frac{2\pi\ell}{\mathsf{b}} \qquad \text{and} \qquad 2\pi\ell\mathsf{b}, \qquad \ell\in\mathbb{Z}\backslash\{0\}, \quad (3.33)$$

with residues

$$\frac{(2\pi\ell)^2(-1)^\ell}{\mathsf{b}^3}\frac{2}{\sin\left(\frac{\pi\ell}{\mathsf{b}^2}\right)} \quad (3.34)$$

at the first set of poles. The residues at the second set are obtained by exchanging $\mathsf{b}\leftrightarrow\mathsf{b}^{-1}$. The standard connection between singularities of the Borel transform and large order asymptotics gives then the analogue of the formula (3.24) for the coefficients $\tilde{c}_g(\mathsf{b})$, and by going back to the original coefficients $c_g(\mathsf{b})$ we obtain (3.29). In our derivation, we have assumed that b^2 is not rational. When $\mathsf{b}^2\in\mathbb{Q}$, we have singularities in the denominators in (3.29). One can verify though that the singularities cancel between the two summands related by $\mathsf{b}\leftrightarrow\mathsf{b}^{-1}$, as noted in [62]. The resulting expression for Faddeev's quantum dilogarithm when b^2 is rational can be found in [50].

We can now use the expression (A.10) for Faddeev's quantum dilogarithm to obtain the following generalization of (3.27)

$$(-1)^{g-1}c_g(\mathsf{b}) = -\frac{\mathrm{i}}{\pi}\int_0^\infty \frac{\mathrm{d}z}{z^{2g-1}}\log\Phi_{\mathsf{b}}\left(-\frac{1}{z}\right). \quad (3.35)$$

Using (A.10), we conclude that the relevant trans-series are given by

$$\mathcal{F}^{(\ell)}_{\mathcal{A},\mathsf{b}} = \frac{(-1)^{\ell-1}}{\ell}\frac{\mathrm{e}^{-\frac{\ell\mathcal{A}}{\mathsf{b}g_s}}}{2\sin\left(\frac{\pi\ell}{\mathsf{b}^2}\right)}, \qquad \mathcal{F}^{(\ell)}_{\mathcal{A},\mathsf{b}^{-1}} = \frac{(-1)^{\ell-1}}{\ell}\frac{\mathrm{e}^{-\frac{\ell\mathcal{A}\mathsf{b}}{g_s}}}{2\sin\left(\pi\ell\mathsf{b}^2\right)}, \quad (3.36)$$

[8]See [16, 53] for earlier studies of the resurgent structure of the refined topological string on the resolved conifold.

and

$$\sum_{\ell\geq 1}\left(\mathcal{F}^{(\ell)}_{\mathcal{A},\mathsf{b}}+\mathcal{F}^{(\ell)}_{\mathcal{A},\mathsf{b}^{-1}}\right)=-\mathrm{i}\log\Phi_{\mathsf{b}}\left(-\frac{\mathcal{A}}{2\pi g_s}\right). \tag{3.37}$$

In particular, the trans-series involves two different actions $\mathcal{A}/\mathsf{b}$ and $\mathsf{b}\mathcal{A}$, as first observed in [53].

Let us now use these boundary conditions to obtain appropriate trans-series in an arbitrary frame. After exponentiating, we find

$$Z_{\mathrm{r},\mathcal{A}}=\exp\left[\sum_{\ell\geq 1}(\mathcal{F}^{(\ell)}_{\mathcal{A},\mathsf{b}}+\mathcal{F}^{(\ell)}_{\mathcal{A},\mathsf{b}^{-1}})\right]=\sum_{n,m\geq 0}C_{n,m}\exp\left(-\frac{n\mathcal{A}}{\mathsf{b}g_s}-\frac{m\mathsf{b}\mathcal{A}}{g_s}\right), \tag{3.38}$$

where $C_{n,m}$ are constants (depending on b) which can be read from the representation (A.10) of $\log\Phi_{\mathsf{b}}\left(-\frac{1}{z}\right)$. We have, for example,

$$C_{1,0}=\frac{1}{2\sin\left(\frac{\pi\ell}{\mathsf{b}^2}\right)},\qquad C_{0,1}=\frac{1}{2\sin\left(\pi\ell\mathsf{b}^2\right)}. \tag{3.39}$$

We note that each of the terms in the sum of the right-hand side in (3.38) is of the form $\exp(\Sigma_\lambda)$, with

$$\lambda=\frac{n}{\mathsf{b}}+m\mathsf{b}. \tag{3.40}$$

Therefore, we find by linearity

$$Z_{\mathrm{r}}=\sum_{n,m\geq 0}C_{n,m}\exp\left(\Sigma_{\frac{n}{\mathsf{b}}+m\mathsf{b}}\right), \tag{3.41}$$

with holomorphic limit

$$\mathcal{Z}_{\mathrm{r}}=\sum_{n,m\geq 0}C_{n,m}\exp\left[\tilde{\mathcal{F}}(\boldsymbol{t}+\mathrm{i}g_s\boldsymbol{c}(n/\mathsf{b}+m\mathsf{b}))-\tilde{\mathcal{F}}(\boldsymbol{t})\right]. \tag{3.42}$$

For example, the one-instanton amplitude is

$$\mathcal{F}^{(1)}=\frac{1}{2\sin\left(\frac{\pi}{\mathsf{b}^2}\right)}\exp\left[\tilde{\mathcal{F}}(\boldsymbol{t}+\mathrm{i}g_s\boldsymbol{c}/\mathsf{b})-\tilde{\mathcal{F}}(\boldsymbol{t})\right]+\left(\mathsf{b}\leftrightarrow\mathsf{b}^{-1}\right). \tag{3.43}$$

When expanded in g_s, we find

$$\begin{aligned}\mathcal{F}^{(1)}=&\frac{\mathrm{e}^{-\frac{\mathcal{A}}{\mathsf{b}g_s}}}{2\sin\left(\frac{\pi}{\mathsf{b}^2}\right)}\exp\left(-\frac{c^ac^b}{2\mathsf{b}^2}\tau_{ab}\right)\left[1-\mathrm{i}\left(\frac{c^a}{\mathsf{b}}\frac{\partial\mathcal{F}_1(\mathsf{b})}{\partial t^a}-\frac{1}{6\mathsf{b}^3}c^ac^bc^eC^t_{abe}\right)g_s+\cdots\right]\\&+\left(\mathsf{b}\leftrightarrow\mathsf{b}^{-1}\right),\end{aligned} \tag{3.44}$$

where

$$\tau_{ab}=\frac{\partial^2\widetilde{\mathcal{F}}_0}{\partial t^a\partial t^b},\qquad C^t_{abe}=\frac{\partial^3\mathcal{F}_0}{\partial t^a\partial t^b\partial t^e}. \tag{3.45}$$

Note that $\mathcal{F}_1(\mathsf{b})$ is the holomorphic limit of the full, b-dependent genus one amplitude (2.12). We expect that the trans-series (3.44) will control the asymptotic behavior of $\mathcal{F}_g(\boldsymbol{t};\mathsf{b})$ at large genus, not far from the conifold locus. We will test this numerically in Section 3.5.

3.3 Boundary conditions from large radius

As we have seen in the last section, the behavior of the refined topological string predicts the existence of a Borel singularity given (up to a normalization) by the conifold flat coordinate, and leads to an explicit form for the corresponding multi-instanton trans-series. This can be transformed to an arbitrary frame, leading to (3.42). In [57], it was shown that, by using the Gopakumar–Vafa formula [52], one can obtain information on the Borel singularities and their trans-series near the large radius point. We will now generalize the argument in [57] to the refined case.

The starting point is the representation (2.22) of the topological free energy in terms of the BPS invariants. Let us introduce

$$\mathsf{b}_\pm = \mathsf{b} \pm \mathsf{b}^{-1}, \tag{3.46}$$

and the coefficients $s_g^{j_L,j_R}(\mathsf{b})$ defined in terms of the generating function

$$\frac{\chi_{j_L}\left(\mathrm{e}^{\frac{\mathrm{i}x\mathsf{b}_+}{2}}\right)\chi_{j_R}\left(\mathrm{e}^{\frac{\mathrm{i}x\mathsf{b}_-}{2}}\right)}{4\sin\left(\frac{x\mathsf{b}}{2}\right)\sin\left(\frac{x}{2\mathsf{b}}\right)} = \sum_{g\geq 0} s_g^{j_L,j_R}(\mathsf{b}) x^{2g-2}. \tag{3.47}$$

We have, for example,

$$s_0^{j_L,j_R}(\mathsf{b}) = d_L d_R, \qquad s_1^{j_L,j_R}(\mathsf{b}) = \frac{d_L d_R}{24}\left[\mathsf{b}^2 + \mathsf{b}^{-2} + \left(1-d_L^2\right)\mathsf{b}_+^2 + \left(1-d_R^2\right)\mathsf{b}_-^2\right]. \tag{3.48}$$

By expanding both sides of (2.22) in powers of g_s, we obtain the refined multicovering formula

$$\mathcal{F}_g(\boldsymbol{t};\mathsf{b}) = \sum_{\boldsymbol{d}}\sum_{j_L,j_R} N_{\boldsymbol{d}}^{j_L,j_R} s_g^{j_L,j_R}(\mathsf{b}) \mathrm{Li}_{3-2g}(Q_{\boldsymbol{d}}). \tag{3.49}$$

Now, as in [57], we use

$$\sum_{n\in\mathbb{Z}} \frac{1}{(2\pi n - \mathrm{i}t)^m} = \frac{\mathrm{i}^m}{(m-1)!}\mathrm{Li}_{-m+1}\left(\mathrm{e}^{-t}\right), \qquad m \geq 2, \tag{3.50}$$

to write the free energy as

$$\mathcal{F}_g(\boldsymbol{t};\mathsf{b}) = \sum_{\boldsymbol{d}}\sum_{n\in\mathbb{Z}}\sum_{j_L,j_R} N_{\boldsymbol{d}}^{j_L,j_R} r_g^{j_L,j_R}(\mathsf{b}) \mathcal{A}_{\boldsymbol{d},n}^{2-2g}, \qquad g \geq 2, \tag{3.51}$$

where

$$r_g^{j_L,j_R}(\mathsf{b}) = (-1)^{g-1}(2\pi\mathrm{i})^{2g-2}(2g-3)! s_g^{j_L,j_R}(\mathsf{b}) \tag{3.52}$$

and

$$\mathcal{A}_{\boldsymbol{d},n} = 2\pi\boldsymbol{d}\cdot\boldsymbol{t} + 4\pi^2\mathrm{i}n. \tag{3.53}$$

We want to obtain the large genus behavior of (3.51), and extract the corresponding trans-series.[9] To do this, we need to know the large order behavior of the coefficients $r_g^{j_L,j_R}(\mathsf{b})$. This can be obtained by using an argument similar to the one employed in equations (3.32)–(3.34) to derive (3.29). One notes first that the series (3.47) can be regarded as the Borel transform

[9] In the case of the resolved conifold, the Borel transform of the total free energy was calculated in [53].

of the divergent series with coefficients $(2g)!s_g^{j_L,j_R}$. The singularities of this Borel transform are also at (3.33), and the calculation of the residues is straightforward. Introducing the variables

$$y_{\mathsf{b}} = -\mathrm{e}^{\pi \mathrm{i} \mathsf{b}^2}, \qquad \tilde{y}_{\mathsf{b}} = -\mathrm{e}^{-\pi \mathrm{i}/\mathsf{b}^2}, \tag{3.54}$$

one finds the following asymptotic expansion for $g \gg 1$

$$r_g^{j_L,j_R}(\mathsf{b}) \sim \frac{1}{2\pi}\Gamma(2g-2)\sum_{\ell\geq 1}\left[\frac{(-1)^{\ell-1}}{\ell}\frac{\chi_{j_L}\left(\tilde{y}_{\mathsf{b}}^{\ell}\right)\chi_{j_R}\left(\tilde{y}_{\mathsf{b}}^{\ell}\right)}{\sin\left(\frac{\pi\ell}{\mathsf{b}^2}\right)}\left(\frac{\ell}{\mathsf{b}}\right)^{-2g+2} + \left(\mathsf{b}\to\mathsf{b}^{-1}\right)\right]. \tag{3.55}$$

We conclude that there is a sequence of Borel singularities at

$$\frac{\ell}{\mathsf{b}}\mathcal{A}_{\boldsymbol{d},n} \qquad \text{and} \qquad \ell\mathsf{b}\mathcal{A}_{\boldsymbol{d},n}, \qquad \ell\in\mathbb{Z}_{>0}, \qquad n\in\mathbb{Z}, \tag{3.56}$$

and the corresponding trans-series are

$$\begin{aligned}
\mathcal{F}^{(\ell)}_{\mathcal{A},\mathsf{b},j_L,j_R} &= \frac{(-1)^{\ell-1}}{\ell}\frac{\chi_{j_L}\left(\tilde{y}_{\mathsf{b}}^{\ell}\right)\chi_{j_R}\left(\tilde{y}_{\mathsf{b}}^{\ell}\right)}{2\sin\left(\frac{\pi\ell}{\mathsf{b}^2}\right)}\mathrm{e}^{-\frac{\ell\mathcal{A}}{\mathsf{b}g_s}},\\
\mathcal{F}^{(\ell)}_{\mathcal{A},\mathsf{b}^{-1},j_L,j_R} &= \frac{(-1)^{\ell-1}}{\ell}\frac{\chi_{j_L}\left(y_{\mathsf{b}}^{\ell}\right)\chi_{j_R}\left(y_{\mathsf{b}}^{\ell}\right)}{2\sin\left(\pi\ell\mathsf{b}^2\right)}\mathrm{e}^{-\frac{\ell\mathsf{b}\mathcal{A}}{g_s}},
\end{aligned} \tag{3.57}$$

where $\mathcal{A} = \mathcal{A}_{\boldsymbol{d},n}$. When $j_L = j_R = 0$, we recover (3.36). The sum over all multi-instanton sectors gives

$$\sum_{\ell\geq 1}\left(\mathcal{F}^{(\ell)}_{\mathcal{A},\mathsf{b},j_L,j_R} + \mathcal{F}^{(\ell)}_{\mathcal{A},\mathsf{b}^{-1},j_L,j_R}\right) = -\mathrm{i}\log\Phi_{\mathsf{b}}^{[j_L,j_R]}\left(-\frac{\mathcal{A}}{2\pi g_s}\right), \tag{3.58}$$

where the series on the right-hand side is the following generalization of Faddeev's quantum dilogarithm

$$\begin{aligned}
&\log\Phi_{\mathsf{b}}^{[j_L,j_R]}\left(-\frac{\mathcal{A}}{2\pi g_s}\right)\\
&\quad = \frac{1}{\mathrm{i}}\sum_{\ell\geq 1}\frac{(-1)^{\ell}}{\ell}\left[\frac{\chi_{j_L}\left(\tilde{y}_{\mathsf{b}}^{\ell}\right)\chi_{j_R}\left(\tilde{y}_{\mathsf{b}}^{\ell}\right)}{2\sin\left(\frac{\pi\ell}{\mathsf{b}^2}\right)}\mathrm{e}^{-\frac{\ell\mathcal{A}}{\mathsf{b}g_s}} + \frac{\chi_{j_L}\left(y_{\mathsf{b}}^{\ell}\right)\chi_{j_R}\left(y_{\mathsf{b}}^{\ell}\right)}{2\sin\left(\pi\ell\mathsf{b}^2\right)}\mathrm{e}^{-\frac{\ell\mathcal{A}\mathsf{b}}{g_s}}\right].
\end{aligned} \tag{3.59}$$

When $j_L = j_R = 0$, we recover the conventional quantum dilogarithm. In fact, (3.59) can be expressed in terms of a more elementary function $\Phi_{\mathsf{b}}^{[j]}(z)$ defined in (A.14). Indeed, from the representation of $\Phi_{\mathsf{b}}^{[j]}(z)$ given in the first line of (A.15), one finds the following identity

$$\log\Phi_{\mathsf{b}}^{[j_L,j_R]}(z) = \sum_{j=|j_L-j_R|}^{j_L+j_R}\log\Phi_{\mathsf{b}}^{[j]}(z), \tag{3.60}$$

where the sum runs over half-integer j such that $j - j_L - j_R$ is integer. The third line in (A.15) also shows that $\Phi_{\mathsf{b}}^{[j]}(z)$ can be expressed as a product of the ordinary quantum dilogarithms with shifted arguments.

The full trans-series corresponding to all the singularities near large radius is a sum of trans-series of the form (3.59),

$$\begin{aligned}
&-\mathrm{i}\sum_{\boldsymbol{d}}\sum_{n\in\mathbb{Z}}\sum_{j_L,j_R} N_{\boldsymbol{d}}^{j_L,j_R}\log\Phi_{\mathsf{b}}^{[j_L,j_R]}\left(-\frac{\mathcal{A}_{\boldsymbol{d},n}}{2\pi g_s}\right)\\
&\quad = -\mathrm{i}\sum_{\boldsymbol{d}}\sum_{n\in\mathbb{Z}}\sum_{j}\Omega_{[j]}(\boldsymbol{d})\log\Phi_{\mathsf{b}}^{[j]}\left(-\frac{\mathcal{A}_{\boldsymbol{d},n}}{2\pi g_s}\right)
\end{aligned} \tag{3.61}$$

where

$$\Omega_{[j]}(\boldsymbol{d}) = \sum_{\substack{j_L, j_R \\ |j_L - j_R| \le j \le j_L + j_R}} N_{\boldsymbol{d}}^{j_L, j_R} \,. \tag{3.62}$$

This supports the identification of the Stokes constant near the large radius point with the refined DT invariants $\Omega_{[j]}(\boldsymbol{d})$ advocated below (2.25). Note that the Stokes constants are common to all singularities (3.56) with different values of ℓ and n. In the limit $\mathsf{b} \to 1$, upon using (2.23), we recover the result of [57] identifying the Stokes constants near large radius with the genus 0 GV invariants,

$$-\mathrm{i} \sum_{\boldsymbol{d}} \sum_{n \in \mathbb{Z}} \left(\sum_{j_L, j_R} d_L d_R N_{\boldsymbol{d}}^{j_L, j_R} \right) \log \Phi_1 \left(-\frac{\mathcal{A}_{\boldsymbol{d},n}}{2\pi g_s} \right) = -\mathrm{i} \sum_{\boldsymbol{d}} \sum_{n \in \mathbb{Z}} n_{\boldsymbol{d}}^{(0)} \log \Phi_1 \left(-\frac{\mathcal{A}_{\boldsymbol{d},n}}{2\pi g_s} \right). \tag{3.63}$$

3.4 Stokes automorphism in the refined case

From the above results, and using the methods of [69], one can easily obtain the Stokes automorphism corresponding to the trans-series (3.37), (3.58). We recall that, in the theory of resurgence, the Stokes automorphism can be obtained from the discontinuity of Borel resummations as we cross a ray of singularities. This discontinuity is expressed as a formal linear combination of resummed trans-series associated to the singularities, whose coefficients are the Stokes constants. The Stokes automorphism is given by this formal linear combination of trans-series. We refer, e.g., to [69] for additional background and references on the Stokes automorphism and related aspects of the theory of resurgence.

We found that, for the topological free energies in the large radius frame, the leading Borel singularities near the large radius point are at $\ell \mathsf{b}^{\pm 1} \mathcal{A}_{\boldsymbol{d},n}$, with $\ell \in \mathbb{Z}\backslash 0$, corresponding to D2-D0 branes in type IIA, or D1-D(-1) instantons in IIB. We note that, for general b and $\ell > 0$, we have two rays of singularities. For convenience, we consider the Stokes automorphism associated to the discontinuity as we cross both rays.[10] In this case this automorphism is purely multiplicative, and is obtained by exponentiating the sum of multi-instantons (3.58) (with an additional factor of $-\mathrm{i}$ due to conventions in the definition of the Stokes automorphism). We then obtain

$$\mathfrak{S}(\mathcal{Z}) = \prod_{j_L, j_R} \left[\Phi_{\mathsf{b}}^{[j_L, j_R]} \left(-\frac{\mathcal{A}}{2\pi g_s} \right) \right]^{-N_{\boldsymbol{d}}^{j_L, j_R}} \mathcal{Z} = \prod_j \left[\Phi_{\mathsf{b}}^{[j]} \left(-\frac{\mathcal{A}}{2\pi g_s} \right) \right]^{-\Omega_{[j]}(\boldsymbol{d})} \mathcal{Z}. \tag{3.64}$$

The case (3.37) was obtained by analyzing the behavior of the topological string free energies in the conifold frame, near the conifold point where a D4-brane becomes massless (or a D3-instanton becomes of vanishing action). The resulting Stokes automorphism is again purely multiplicative, and is in fact a particular case of (3.64) in which only $j = 0$ contributes with $\Omega_{[j]} = 1$. In general, when we consider the partition function in an $\mathcal{A}$-frame, we expect the Stokes automorphism to be given by the general expression (3.64), extending what is found in the unrefined case [57, 59].

Let us now consider what happens at an arbitrary frame, i.e., not necessarily an $\mathcal{A}$-frame. In that case, not all c^a appearing in (3.16) vanish. As shown in [69], the Stokes automorphism can be obtained from the multiplicative one, after promoting the exponential of the action $\mathcal{A}$ to a shift operator. One then finds

$$\mathfrak{S}(\tilde{\mathcal{Z}}) = \prod_j \left[\Phi_{\mathsf{b}}^{[j]} \left(\mathrm{i} g_s c^a \frac{\partial}{\partial t^a} \right) \right]^{-\Omega_{[j]}(\gamma)} \tilde{\mathcal{Z}}, \tag{3.65}$$

[10] One could in principle consider separately the Stokes automorphism for the ray containing the singularities $\ell \mathsf{b} \mathcal{A}_{\boldsymbol{d},n}$, $\ell > 0$, and the Stokes automorphism for the ray containing $\ell \mathsf{b}^{-1} \mathcal{A}_{\boldsymbol{d},n}$, $\ell > 0$. However, when b is rational, they are both singular separately, and it is only by adding both that one obtains a finite answer.

where we have denoted $\tilde{\mathcal{Z}} = \mathrm{e}^{\tilde{\mathcal{F}}}$ the partition function associated to the modified free energy (3.21). When $\mathsf{b} = 1$, (3.65) reduces to the result of [69] for the standard topological string. Furthermore, when $\mathsf{b}^2 = 2$, using (A.9), we recover the Stokes automorphism which appears in the real topological string at the conifold point [81]. The relevance of this special value for topological strings on orientifolds was anticipated in [74].

We expect (3.64), (3.65) to give the Stokes automorphism of the refined topological string due to arbitrary singularities at $\ell\mathsf{b}^{\pm 1}\mathcal{A}$, $\ell \in \mathbb{Z}_{>0}$. The Stokes constants $\Omega_{[j]}$ appearing in these formulae should be identified with the coefficients of the character expansion of the refined DT invariant (2.25) associated to the corresponding BPS state, as conjectured around (1.6). The formulae (3.64), (3.65) were in fact already proposed by [12], in a different but related context, and we will rederive them in Section 4.

3.5 Numerical checks for $X = K_{\mathbb{P}^2}$

As is well-known, the trans-series obtained in resurgent analysis, if correct, should control the large order behavior of the original perturbative series. In particular, the one-instanton series (3.44) should give the leading asymptotic behavior in the region of moduli space in which the leading singularity in the Borel plane is given by either $\mathsf{b}\mathcal{A}$ or $\mathcal{A}/\mathsf{b}$ (i.e., the one which is closer to the origin). We will now verify that (3.44) indeed gives the correct answer in the simplest non-trivial local CY, namely, $X = K_{\mathbb{P}^2}$, the total space of the canonical bundle over $\mathbb{P}^2$, also known as local $\mathbb{P}^2$. We follow the notations of [59] and denote by z the standard algebraic coordinate on Kähler moduli space, such that $z = 0$, $z = -\frac{1}{27}$ and $z = \infty$ correspond to the large volume, conifold and orbifold points, respectively. For simplicity, we shall restrict ourselves to negative values of z in the large radius region of the moduli space,

$$-\frac{1}{27} < z < 0, \tag{3.66}$$

since with our conventions the free energies in the large radius frame are real in this interval.

From previous studies [33, 35, 59], it is known that, in most of the range in (3.66), the leading Borel singularity corresponds to the central charge of the D4-brane which becomes massless at the conifold point $z = -1/27$, and is given by the conifold result (3.25). In particular, the Borel singularity $\mathsf{b}^{\pm 1}\mathcal{A}_{1,0}$ in (3.53) only becomes relevant very close to the large radius limit, when $-10^{-6} < z < 0$. We will denote

$$\mathcal{A}_c = 2\pi \mathrm{i} t_c, \tag{3.67}$$

where t_c is the appropriately normalized conifold flat coordinate.[11] Therefore, in this region the large genus behavior of the free energies $\mathcal{F}_g(\mathsf{b})$ in the large radius frame is determined by (3.44). Let us note that, in our conventions, the free energy $F^{(1,0)}$ is given by

$$\mathcal{F}^{(1,0)} = -\frac{1}{24}\log\left(-\frac{\Delta}{z}\right), \qquad \Delta = 1 + 27z. \tag{3.68}$$

Since $b_2(X) = 1$, there is only one coefficient c^a in (3.44), which is equal to $c = -3$ in the present conventions [59]. Writing $\mathcal{F}^{(1)}$ as

$$\mathcal{F}^{(1)} = \mathrm{e}^{-\widetilde{\mathcal{A}}/g_s}\left(\mu_0 + \mu_1 g_s + \cdots\right), \tag{3.69}$$

standard resurgent analysis predicts

$$\mathcal{F}_g(\mathsf{b}) \sim \frac{1}{\pi}\widetilde{\mathcal{A}}^{-2g+2}\Gamma(2g-2)\left(\mu_0 + \frac{\mu_1\widetilde{\mathcal{A}}}{2g-3} + \cdots\right), \qquad g \gg 1. \tag{3.70}$$

[11] We follow the conventions of [81] for the normalization of t_c, which differs from the one used in [59] by a factor of $\sqrt{3}$.

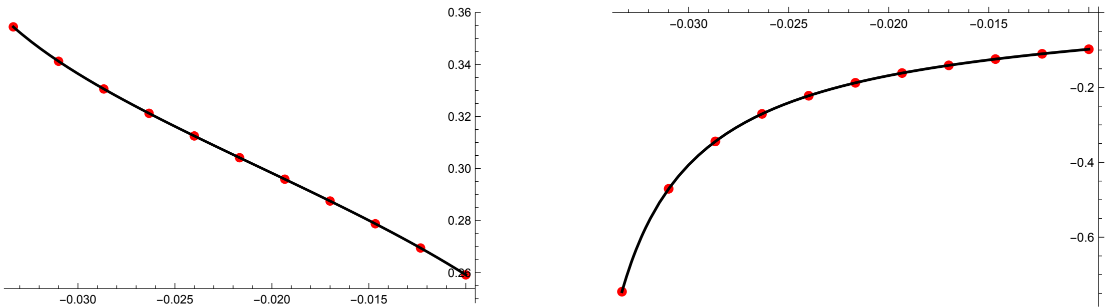

Figure 1. Values of the coefficients μ_0 (left) and μ_1 (right) in (3.70), for different values of z in the range $-\frac{1}{27} < z < 0$ and for a fixed value $\mathsf{b} = \pi$, extracted from the asymptotic behaviour of $\mathcal{F}_g(\mathsf{b})$ up to $g = 35$ (after using three Richardson transforms to improve convergence). The numerical values are very close to the theoretical predictions shown in black lines, with 10^{-10} and 10^{-9} accuracy for μ_0 and μ_1, respectively.

We can extract numerically the value of the action $\widetilde{\mathcal{A}}$ and of the coefficients $\mu_{0,1}$ from the perturbative free energies at sufficiently large order, for different values of the modulus z and the parameter b, and compare those to the theoretical prediction in (3.44). This prediction implies in particular that, if $|\mathsf{b}| > 1$, we will have

$$\widetilde{\mathcal{A}} = \frac{\mathcal{A}_c}{\mathsf{b}}. \tag{3.71}$$

If $|\mathsf{b}| < 1$, the action $\widetilde{\mathcal{A}}$ is obtained by exchanging $\mathsf{b} \leftrightarrow \mathsf{b}^{-1}$ in (3.71) (note that the sequence $\mathcal{F}_g$ is invariant under this exchange). Computing the free energies up to $g = 35$ for $\mathsf{b} = \pi$ (a convenient irrational number) and several values of z, we already find excellent agreement, see Figure 1. A similar agreement is obtained for other values of b, including complex ones.

4 Refined dual partition functions

In this section, we propose an extension of the notion of dual partition function, realized as certain generalized theta series encoding five-brane instantons, that incorporates the refinement. The construction relies on the use of a non-commutative Moyal star product and its extension to the realm of contact geometry. Our main result is that the same Stokes automorphism (3.64) that governs the refined topological string also arises as the transformation of the kernel of the refined dual partition function induced by a non-commutative wall-crossing transformation. Furthermore, by considering higher five-brane charge $k > 1$, we obtain a vector-valued generalization of (3.64), which realizes the double quantization proposed in [12, Section 4].

4.1 Five-brane instanton corrections and dual partition functions

Let us first revisit the construction of dual partition functions and their behavior under wall-crossing from [12]. This construction can be motivated by considering five-brane instanton corrections to the vector multiplet moduli space $\mathcal{M}$ arising by compactifying type IIA strings on $X \times S_1$ down to 3 dimensions.[12] When X is a compact CY threefold, $\mathcal{M}$ is a quaternion-Kähler (QK) manifold of real dimension $4b_2(X) + 4$, of the form

$$\mathcal{M} \simeq \mathbb{R}^+_R \times \mathcal{SK}_{z^a} \times \mathcal{T}_{\zeta^\Lambda, \tilde{\zeta}_\Lambda} \times S^1_\sigma, \tag{4.1}$$

[12] Equivalently, one can consider the hypermultiplet moduli space of type IIB string theory compactified on X, or type IIA on the mirror threefold $\widehat{X}$.

where $\mathbb{R}^+$ parametrizes the radius of the circle, $\mathcal{SK}$ the complexified Kähler structure on X, $\mathcal{T}$ the holonomies of the Ramond gauge fields around $H_{\rm even}(X,\mathbb{Z})$, and S^1 the scalar Poincaré-dual to the Kaluza–Klein gauge field in 3 dimensions. In (4.1), we have indicated in subscript the coordinates used to parametrize the various factors, with a running over $1,\dots,b_2 := b_2(X) = b_4(X)$, $\Lambda = 0,\dots,b_2$, with ζ^Λ and $\tilde\zeta_\Lambda$ associated to $H_0\oplus H_2$ and $H_4\oplus H_6$, respectively. Topologically, the level sets of R are principal bundles over $\mathcal{SK}$, whose fibers are twisted tori $\widetilde{\mathcal{T}} = \mathbb{R}^{2b_2+3}_{\zeta^\Lambda,\tilde\zeta_\Lambda,\sigma}/\mathbb{H} \simeq \mathcal{T}\times S^1$ where $\mathbb{H}$ is the non-commutative Heisenberg group of large gauge transformations parametrized by $\Theta = \left(\eta^\Lambda,\tilde\eta_\Lambda,\kappa\right)\in\mathbb{Z}^{2b_2+3}$ and acting by

$$\mathscr{T}_\Theta\colon\ \left(\zeta^\Lambda,\tilde\zeta_\Lambda,\sigma\right) \ \mapsto\ \left(\zeta^\Lambda+\eta^\Lambda,\tilde\zeta_\Lambda+\tilde\eta_\Lambda,\sigma+2\kappa-\tilde\eta_\Lambda\zeta^\Lambda+\eta^\Lambda\tilde\zeta_\Lambda-\eta^\Lambda\tilde\eta_\Lambda\right). \tag{4.2}$$

The coordinate σ can be viewed as parametrizing the phase of a section of the theta line bundle $\mathcal{L}$ with connection $\mathcal{A}$ over the torus parametrized by $\left(\zeta^\Lambda,\tilde\zeta_\Lambda\right)$, with curvature $\mathrm{d}\mathcal{A} = \mathrm{d}\tilde\zeta_\Lambda\wedge\mathrm{d}\zeta^\Lambda$ [10]. When X is non-compact, three-dimensional gravity decouples and $\mathcal{M}$ becomes a family of hyperkähler manifolds of dimension $4b_2$ parametrized by the (non-dynamical) radius R, of the form $\mathcal{SK}_{z^a}\times\mathcal{T}_{\zeta^a,\tilde\zeta_a}$, obtained as a rigid limit of (4.1).

As $R\to\infty$, the QK metric on $\mathcal{M}$ is simply obtained from the special Kähler metric on $\mathcal{SK}$ by the c-map construction [45], but for finite R there are $\mathcal{O}\left(\mathrm{e}^{-R}\right)$ corrections from Euclidean D-branes wrapped on even cycles in X times S^1, and $\mathcal{O}\left(\mathrm{e}^{-R^2}\right)$ corrections from Euclidean five-branes.[13] Both types of corrections must preserve the QK property of the metric, which is equivalent [75] to the existence of a complex contact structure on the twistor space $\mathcal{Z}\simeq\mathbb{P}^1_{\boldsymbol{z}}\times\mathcal{M}$, where the first factor parametrizes the sphere-worth of almost complex structures on $\mathcal{M}$. Such a structure is guaranteed by the existence of coordinate patches parametrized by complex Darboux coordinates ξ^Λ, $\tilde\xi_\Lambda$, $\tilde\alpha$ such that the contact one-form

$$\mathcal{X} = -\frac{1}{2}\left(\mathrm{d}\tilde\alpha-\xi^\Lambda\mathrm{d}\tilde\xi_\Lambda+\tilde\xi_\Lambda\mathrm{d}\xi^\Lambda\right) \tag{4.3}$$

is globally well defined up to rescaling by a non-vanishing holomorphic function. The Darboux coordinates are functions of t, R, z^a, ζ^Λ, $\tilde\zeta_\Lambda$, σ, holomorphic in $\boldsymbol{z}$ in the respective patches, and can be chosen such that the Heisenberg group acts in the same way as in (4.2),

$$\mathscr{T}_\Theta\colon\ \left(\xi^\Lambda,\tilde\xi_\Lambda,\tilde\alpha\right) \ \mapsto\ \left(\xi^\Lambda+\eta^\Lambda,\tilde\xi_\Lambda+\tilde\eta_\Lambda,\tilde\alpha+2\kappa-\tilde\eta_\Lambda\xi^\Lambda+\eta^\Lambda\tilde\xi_\Lambda-\eta^\Lambda\tilde\eta_\Lambda\right). \tag{4.4}$$

This is an example of contact transformation, i.e., preserving the contact one-form (4.3). As a result, the twistor space $\mathcal{Z}$ can be obtained by gluing together algebraic twisted tori

$$\widetilde{\mathcal{T}}_{\mathbb{C}} = (\mathbb{C}^\times)^{2b_2+3}_{\xi^\Lambda,\tilde\xi_\Lambda,\tilde\alpha}/\mathbb{H}.$$

By omitting the coordinate $\tilde\alpha$, the latter project to algebraic tori $\mathcal{T}_{\mathbb{C}} = (\mathbb{C}^\times)^{2b_2+2}_{\xi^\Lambda,\tilde\xi_\Lambda}$, with $\mathbb{C}^\times$ fiber parametrized by $\mathrm{e}^{-\pi\mathrm{i}\tilde\alpha}$.

At large but finite radius R, D-brane instantons generate corrections of order $\mathrm{e}^{-2\pi R|Z(\gamma)|}$ to the QK metric on $\mathcal{M}$, where $\gamma = \left(p^\Lambda,q_\Lambda\right)\in H_{\rm even}(X,\mathbb{Z})$ is the instanton charge and $Z(\gamma)$ the corresponding central charge. We denote by $\mathcal{M}_D$ the QK metric on $\mathcal{M}$ incorporating all D-instanton corrections. The twistor space $\mathcal{Z}_D$ associated to $\mathcal{M}_D$ can be constructed by postulating discontinuities of the Darboux coordinates across the so-called BPS rays ℓ_γ on $\mathbb{P}^1_{\boldsymbol{z}}$ (the latter being the loci where the phase of $\boldsymbol{z}$ coincides with the central charge $Z(\gamma)$). Namely,

[13] In the present context, these are Kaluza–Klein five-branes of the form $\mathrm{TN}_k\times X$, where the first factor is a Taub-NUT gravitational instanton of charge k which asymptotes to $\mathbb{R}^3\times S_1(R)$. Under T-duality along the circle, this becomes a Neveu–Schwarz five-brane instanton correcting the hypermultiplet moduli space, which is the equivalent set-up used in [9, 12].

one requires that across ℓ_γ they change as [4, 11, 14]

$$\mathscr{V}_\gamma\colon\ \begin{aligned} &\xi^\Lambda \mapsto \xi'^\Lambda = \xi^\Lambda + \frac{p^\Lambda}{2\pi\mathrm{i}}\Omega(\gamma)\log(1-\mathcal{X}_\gamma),\\ &\tilde\xi_\Lambda \mapsto \tilde\xi'_\Lambda = \tilde\xi_\Lambda + \frac{q_\Lambda}{2\pi\mathrm{i}}\Omega(\gamma)\log(1-\mathcal{X}_\gamma),\\ &\tilde\alpha \mapsto \tilde\alpha' = \tilde\alpha + \frac{1}{2\pi^2}\Omega(\gamma)L_{\sigma(\gamma)}(\mathcal{X}_\gamma). \end{aligned} \tag{4.5}$$

Here $\Omega(\gamma)$ is the generalized DT invariant counting BPS states of charge γ in four dimensions,

$$\mathcal{X}_\gamma = \sigma(\gamma)\mathrm{e}^{-2\pi\mathrm{i}(q_\Lambda\xi^\Lambda - p^\Lambda\tilde\xi_\Lambda)}$$

are the twisted Fourier modes, with $\sigma(\gamma)$ a quadratic refinement of the symplectic intersection pairing $\langle\gamma,\gamma'\rangle$ on the lattice of charges such that

$$\mathcal{X}_\gamma\mathcal{X}_{\gamma'} = (-1)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}, \tag{4.6}$$

and $L_\sigma(z)$ is the twisted Rogers dilogarithm

$$L_\epsilon(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log\left(\epsilon^{-1}z\right)\log(1-z). \tag{4.7}$$

Forgetting the action on $\tilde\alpha$, the transformation (4.5) can be written in terms of the Fourier modes $\mathcal{X}_\gamma$ as

$$\mathscr{U}_\gamma\colon\ \mathcal{X}_{\gamma'} \ \mapsto\ \mathcal{X}'_{\gamma'} = \mathcal{X}_{\gamma'}(1-\mathcal{X}_\gamma)^{\langle\gamma,\gamma'\rangle\Omega(\gamma)}. \tag{4.8}$$

This is recognized as the Kontsevich–Soibelman (KS) symplectomorphism encoding wall-crossing transformations [46, 73], or the Delabaere–Dillinger–Pham (DDP) formula controlling the Stokes automorphisms for quantum periods [36]. The action on the additional coordinate $\tilde\alpha$ lifts $\mathscr{U}_\gamma$ to a contact transformation $\mathscr{V}_\gamma$. The gluing conditions for $\tilde\alpha$ further imply that $\mathrm{e}^{-\pi\mathrm{i}\tilde\alpha}$ is a section of the theta line bundle $\mathcal{L}$ over the algebraic tori $\mathcal{T}_\mathbb{C}$ [9, 85], which descends to a hyperholomorphic line bundle over the hyperkähler space obtained from the D-instanton corrected QK space $\mathcal{M}_D$ by the QK/HK correspondence [11].

By contrast, Neveu–Schwarz five-brane instantons induce corrections of order $\mathrm{e}^{-4\pi kR^2}$ and are poorly understood, beyond linear order around the D-instanton corrected twistor space $\mathcal{Z}_D$ [9] (see also [5, 6, 7]). At linear order, instanton corrections from charge k five-branes are described by sections of $\mathcal{L}^k$, the k-th power of the theta line bundle $\mathcal{L}$. In practice, this means that on local coordinate patches they are described by functions $H_k\big(\xi,\tilde\xi,\tilde\alpha\big) := \mathrm{e}^{-\mathrm{i}\pi k\tilde\alpha}H_k\big(\xi,\tilde\xi\big)$ which stay invariant under the contact transformations $\mathscr{T}_\Theta$ and are mapped to each other under $\mathscr{V}_\gamma$. The functions $H_k\big(\xi,\tilde\xi,\tilde\alpha\big)$ give rise to additional discontinuities of the Darboux coordinates obtained by applying to them the operator $\mathrm{e}^{\{H_k,\cdot\}_{1,0}}$ where $\{\cdot,\cdot\}_{1,0}$ denotes the so called contact bracket [5, 6, 14], in the same way as the discontinuities (4.5) are obtained by applying $\mathrm{e}^{\{h_\gamma,\cdot\}_{1,0}}$ with $h_\gamma = \frac{\Omega(\gamma)}{4\pi^2}\mathrm{Li}_2(\mathcal{X}_\gamma)$. The instanton corrected metric $\mathcal{M}_D$ can then be derived following the procedure explained in detail in [15].

The invariance under the Heisenberg action (4.4) requires that

$$H_k\big(\xi+\eta,\tilde\xi+\tilde\eta\big) = (-1)^{k\eta^\Lambda\tilde\eta_\Lambda}\mathrm{e}^{\pi\mathrm{i}k(\eta^\Lambda\tilde\xi_\Lambda - \tilde\eta_\Lambda\xi^\Lambda)}H_k\big(\xi,\tilde\xi\big). \tag{4.9}$$

This implies the following 'non-Abelian' Fourier expansion

$$H_k\big(\xi,\tilde\xi,\tilde\alpha\big) = \sum_{\ell^\Lambda\in\frac{\mathbb{Z}^{b_2}}{|k|\mathbb{Z}^{b_2}}}\ \sum_{n^\Lambda\in\mathbb{Z}^{b_2}+\frac{\ell^\Lambda}{k}} \mathrm{e}^{-\pi\mathrm{i}k(\tilde\alpha+\xi^\Lambda\tilde\xi_\Lambda)+2\pi\mathrm{i}kn^\Lambda\tilde\xi_\Lambda}\mathcal{H}_{k,\ell^\Lambda}\big(\xi^\Lambda-n^\Lambda\big), \tag{4.10}$$

where $\mathcal{H}_{k,\ell^\Lambda}(\xi^\Lambda)$ is referred to as the wave-function. The reason for this terminology is that it transforms in the metaplectic representation under a change of symplectic basis. For example, upon exchanging the 'position' coordinates ξ^Λ with the conjugate 'momenta' $\tilde\xi_\Lambda$, one gets

$$H_k(\xi,\tilde\xi,\tilde\alpha) = \sum_{l_\Lambda\in\frac{\mathbb{Z}}{|k|\mathbb{Z}}}\ \sum_{m_\Lambda\in\mathbb{Z}+\frac{\ell^\Lambda}{k}} \mathrm{e}^{-\pi \mathrm{i}k(\tilde\alpha-\xi^\Lambda\tilde\xi_\Lambda)-2\pi\mathrm{i}k m_\Lambda\xi^\Lambda}\mathcal{H}'_{k,l_\Lambda}(\tilde\xi_\Lambda-m_\Lambda), \tag{4.11}$$

where the wave-functions are related by Fourier transform

$$\mathcal{H}_{k,\ell^\Lambda}(\xi) = \sum_{m_\Lambda\in\frac{\mathbb{Z}}{|k|\mathbb{Z}}} \mathrm{e}^{-2\pi\mathrm{i}m_\Lambda\ell^\Lambda/k}\int \mathrm{d}\tilde\xi_\Lambda \mathrm{e}^{2\pi\mathrm{i}k\xi^\Lambda\tilde\xi_\Lambda}\mathcal{H}'_{k,m_\Lambda}(\tilde\xi). \tag{4.12}$$

For $k=1$, it was argued in [9] using S-duality in type IIB string theory that the wave function $\mathcal{H}_1(\xi)$ should be identified with the topological string partition function in real polarization $\mathcal{Z}_{\mathbb{R}}(\xi)$, analytically continued to the complex domain and evaluated on the Darboux coordinates ξ^Λ. As a result, (4.10) becomes proportional to the 'dual partition function':

$$H_1(\xi,\tilde\xi,\tilde\alpha) = \mathrm{e}^{2\pi\mathrm{i}\alpha}\sum_{n^\Lambda\in\mathbb{Z}}\mathrm{e}^{2\pi\mathrm{i}n^\Lambda\tilde\xi_\Lambda}\mathcal{Z}_{\mathbb{R}}(\xi^\Lambda-n^\Lambda), \tag{4.13}$$

where we introduced

$$\alpha = -\frac{1}{2}(\tilde\alpha+\xi^\Lambda\tilde\xi_\Lambda) \tag{4.14}$$

such that

$$\mathcal{X} = \mathrm{d}\alpha + \xi^\Lambda\mathrm{d}\tilde\xi_\Lambda. \tag{4.15}$$

In fact, $\mathcal{Z}_{\mathbb{R}}(\xi)$ can be shown to be proportional to the usual holomorphic topological string partition function $\mathcal{Z}(\boldsymbol{t};g_s)$ provided one identifies[14]

$$t^a = 2\pi\mathrm{i}\frac{\xi^a}{\xi^0}, \qquad g_s = \frac{2\pi}{\xi^0}. \tag{4.16}$$

Thus, the five-brane instantons turn out to be described by the unrefined version of the partition function studied in the previous sections.

One of the main results of [12] was to determine the transformation property of the wave functions $\mathcal{H}_{k,\ell^\Lambda}$ under the contact transformations (4.5). Taking into account the transformation of $\tilde\alpha$, it is immediate to see that

$$\mathscr{V}_\gamma:\ H_k(\xi,\tilde\xi) \mapsto\ H'_k(\xi,\tilde\xi) = \mathrm{e}^{\frac{k}{2\pi\mathrm{i}}\Omega(\gamma)L_{\sigma(\gamma)}(\mathcal{X}_\gamma)}H_k(\xi',\tilde\xi'). \tag{4.17}$$

In the pure electric case $\gamma=(0,q^\Lambda)$, it was shown that this implies

$$\mathscr{V}_\gamma:\ \mathcal{H}_{k,\ell^\Lambda}(\xi)\ \mapsto\ \left[\mathbf{A}^{(k)}(-q_\Lambda\xi^\Lambda,-q_\Lambda\ell^\Lambda)\right]^{-\Omega(\gamma)}\mathcal{H}_{k,\ell^\Lambda}(\xi), \tag{4.18}$$

where[15]

$$\mathbf{A}^{(k)}(x,\ell) = \left(1-\mathrm{e}^{2\pi\mathrm{i}(x+\ell/k)}\right)^\ell\left[\Phi_1\left(\mathrm{i}(x+\ell/k)\right)\right]^k. \tag{4.19}$$

[14] The easiest way to establish these relations is to compare the Fourier modes $\mathcal{X}_\gamma$ with $\mathrm{e}^{-\mathcal{A}/g_s}$, where the instanton action $\mathcal{A}$ is given in (3.16), by identifying the integer valued charges q_Λ and (d_a,d_0). Another consistency check is that in (1.2) the argument of $\mathcal{Z}$ becomes $t^a-\mathrm{i}g_sn^a = 2\pi\mathrm{i}(\xi^a-n^a)/\xi^0$, which is consistent with (4.13) if one sets $n^0=0$. Since n^0 plays the role of D6-brane charge, this restriction is indeed necessary in the non-compact case. Finally, one can also check that the relations (4.16) ensure that in (1.3) the shift of t^a in the argument of τ is consistent with the KS transformation of ξ^a given in (4.5).

[15] The function defined in (4.19) is the inverse of $\mathbf{A}^{(k)}(x,\ell)$ in [12].

The transformation property across generic BPS rays can be obtained by conjugating (4.18) by the metaplectic representation, or by extending by a rank 2 hyperbolic lattice, see [12, equation (2.11)]. For $k = 1$, (4.18) and (4.16) imply that the topological string amplitude $\mathcal{Z}(\boldsymbol{t}; g_s)$ gets multiplied by $\Phi_1(-(q_a t^a + 2\pi \mathrm{i} q_0)/g_s)$, which coincides with the Stokes automorphism for the topological string partition function, as noticed in [69].

4.2 Refined contact structure

Our goal is to incorporate the refinement parameter into the above construction. In fact, a refined version of the function $\mathbf{A}^{(k)}(x, \ell)$ was already put forward in [12, Section 4]. However, it was not derived nor justified by any invariance or transformation property. To fill this gap, we have to find a proper generalization of the dual partition function (4.13) (or more generally (4.10)) and compute the action of a refined version of the transformation $\mathscr{V}_\gamma$ (4.5) on its kernel. However, as will be discussed shortly, while the refinement of the symplectomorphism $\mathscr{U}_\gamma$ is well understood, this is not so for the contact transformation $\mathscr{V}_\gamma$. Therefore, the first step is to find how to lift refined symplectomorphisms to refined contact transformations. Since the defining property of the latter in the absence of refinement was that they preserve the contact one-form, this can be seen as constructing a refined version of the contact structure. This is the problem that we address in this subsection.

Physically, the standard way to introduce a refinement is to switch on an Ω-background. As was observed in [27, 47] in the gauge theory context, its effect is to deform the Riemann–Hilbert problem defining the instanton-corrected metric on the Coulomb branch into a non-commutative one [21]. This is achieved by replacing the KS symplectomorphism (4.8) by its quantum version

$$\hat{\mathscr{U}}_\gamma\colon\ \mathcal{X}_{\gamma'} \ \mapsto\ \mathcal{X}'_{\gamma'} = \mathcal{U}_\gamma \star \mathcal{X}_{\gamma'} \star \mathcal{U}_\gamma^{-1}. \tag{4.20}$$

Here $\star$ denotes the non-commutative Moyal product

$$f \star g = f \exp\left[\frac{\mathrm{i}\epsilon}{2\pi} \sum_\Lambda \left(\overleftarrow{\partial}_{\xi^\Lambda} \overrightarrow{\partial}_{\tilde{\xi}_\Lambda} - \overleftarrow{\partial}_{\tilde{\xi}_\Lambda} \overrightarrow{\partial}_{\xi^\Lambda}\right)\right] g, \tag{4.21}$$

where ϵ is the refinement parameter to be related to b in the next subsection. It is easy to check that with respect to the Moyal product the relation (4.6) gets deformed to

$$\mathcal{X}_\gamma \star \mathcal{X}_{\gamma'} = (-y)^{\langle \gamma, \gamma' \rangle} \mathcal{X}_{\gamma+\gamma'}, \qquad y = \mathrm{e}^{2\pi \mathrm{i}\epsilon}. \tag{4.22}$$

The function $\mathcal{U}_\gamma$ in (4.20), which generates the quantum KS transformation, is defined in terms the compact quantum dilogarithm $E_y(x)$, described in Appendix A, as

$$\mathcal{U}_\gamma = \prod_{n\in\mathbb{Z}} E_y\left(y^n \mathcal{X}_\gamma\right)^{\Omega_n(\gamma)}, \tag{4.23}$$

where $\Omega_n(\gamma)$ are the Laurent coefficients of the refined BPS indices (to be distinguished from the multiplicities $\Omega_{[j]}(\gamma)$ defined in (2.25))

$$\Omega(\gamma, y) = \sum_{n\in\mathbb{Z}} \Omega_n(\gamma) y^n. \tag{4.24}$$

Note that the product in (4.23) is finite, since $\Omega_n(\gamma)$ vanishes for $|n|$ large enough. Substituting (4.23) into (4.20) and evaluating the star product explicitly, one finds

$$\begin{aligned} \mathcal{X}'_{\gamma'} &= \mathcal{X}_{\gamma'} \prod_{n\in\mathbb{Z}} \left[\frac{E_y\left(y^{n+\langle\gamma,\gamma'\rangle} \mathcal{X}_\gamma\right)}{E_y\left(y^{n-\langle\gamma,\gamma'\rangle} \mathcal{X}_\gamma\right)}\right]^{\Omega_n(\gamma)} \\ &= \mathcal{X}_{\gamma'} \prod_{n\in\mathbb{Z}} \prod_{k=0}^{|\langle\gamma,\gamma'\rangle|-1} \left(1 - y^{n+2k-|\langle\gamma,\gamma'\rangle|+1} \mathcal{X}_\gamma\right)^{\operatorname{sgn}\langle\gamma,\gamma'\rangle \Omega_n(\gamma)}, \end{aligned} \tag{4.25}$$

where in the second step we used the property (A.4) of the quantum dilogarithm. In the unrefined limit $y \to 1$, this transformation reduces to the classical symplectomorphism (4.8), as it should.

The analysis of [8] suggests that a similar non-commutative deformation is induced by the refinement in full string theory as well. But in this case we also need to understand how to extend $\hat{\mathscr{U}}_\gamma$ to act on the full twistor space, including the coordinate $\tilde{\alpha}$. In other words, we need to extend the construction to the case of twisted quantum tori. To this end, we observe that the contact one-form (4.15) on $\widetilde{\mathcal{T}}_{\mathbb{C}}$ arises by projectivizing the symplectic form ω on a $\mathbb{C}^\star$ bundle over $\widetilde{\mathcal{T}}_{\mathbb{C}}$ (which coincides at least locally with the Swann bundle [93] of the QK space $\mathcal{M}$), with Darboux coordinates $\left(\eta^I, \mu_I\right)$, $I = \flat, 0, \dots, b_2$, such that [15]

$$\omega = \sum_I \mathrm{d}\eta^I \wedge \mathrm{d}\mu_I. \tag{4.26}$$

Indeed, under the identification

$$\xi^\Lambda = \frac{\eta^\Lambda}{\eta^\flat}, \qquad \tilde{\xi}_\Lambda = \mu_\Lambda, \qquad \alpha = \mu_\flat, \tag{4.27}$$

we have

$$\omega = \mathrm{d}\eta^\flat \wedge \left(\mathrm{d}\mu_\flat + \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda\right) + \eta^\flat \mathrm{d}\xi^\Lambda \wedge \mathrm{d}\tilde{\xi}_\Lambda = \mathrm{d}\eta^\flat \wedge \mathcal{X} + \eta^\flat \mathrm{d}\mathcal{X}. \tag{4.28}$$

Thus, we can define a star product on functions of ξ^Λ, $\tilde{\xi}_\Lambda$, α $\left(\text{or equivalently functions of } \xi^\Lambda,\ \mu_I\right)$ by viewing them as functions of $\left(\eta^I, \mu_I\right)$ which are invariant under the $\mathbb{C}^\times$ action $\eta^I \to \lambda \eta^I$, and using the Moyal product on $\mathbb{C}^{2b_2+4}$, with a deformation parameter that we denote by $\epsilon^\flat$,

$$f \star g = f \exp\left[\frac{\mathrm{i}\epsilon^\flat}{2\pi} \sum_I \left(\overleftarrow{\partial}_{\eta^I} \overrightarrow{\partial}_{\mu_I} - \overleftarrow{\partial}_{\mu_I} \overrightarrow{\partial}_{\eta^I}\right)\right] g. \tag{4.29}$$

In general, the resulting Moyal product is not invariant under the $\mathbb{C}^\times$ action $\eta^I \to \lambda \eta^I$, unless this action also affects the deformation parameter via $\epsilon^\flat \to \lambda \epsilon^\flat$. In other words, the result is not only a function of $\xi^\Lambda \equiv \eta^\Lambda/\eta^\flat$ and μ_I but also depends on $\epsilon \equiv \epsilon^\flat/\eta^\flat$. Thus, the Moyal product (4.29) defines a non-commutative deformation of the product of functions $f\left(\xi^\Lambda, \tilde{\xi}_\Lambda, \tilde{\alpha}, \epsilon\right)$ on $\widetilde{\mathcal{T}}_{\mathbb{C}} \times \mathbb{C}_\epsilon$, which by construction preserves associativity.

It is easy to see that the corresponding Moyal bracket

$$\{f, g\}_\star = \frac{\pi}{\mathrm{i}\epsilon}\left(f \star g - g \star f\right) \tag{4.30}$$

in the unrefined limit $\epsilon \to 0$ reproduces the contact bracket $\{\cdot,\cdot\}_{0,0}$ introduced in [14].[16] Since the latter essentially encodes the contact structure, e.g., it generates classical contact transformations via exponentiation $\exp\{h,\cdot\}_{1,0}$ [6], the star product (4.29) can be thought as providing a definition of the refined contact structure.

[16] In general, the contact bracket is defined on sections of $\mathcal{O}(2m)$ and $\mathcal{O}(2n)$ bundles by $\{f, g\}_{m,n} = \partial_{\xi^\Lambda} f \partial_{\tilde{\xi}_\Lambda} g + \left(m - \xi^\Lambda \partial_{\xi^\Lambda}\right) f \partial_\alpha g - \partial_{\xi^\Lambda} g \partial_{\tilde{\xi}_\Lambda} f - \left(n - \xi^\Lambda \partial_{\xi^\Lambda}\right) g \partial_\alpha f$. Arbitrary values of m and n can easily be incorporated into the above construction since a section of $\mathcal{O}(2m)$ bundle is described by a homogeneous function of degree m on $\widetilde{\mathcal{T}}_{\mathbb{C}} \times \mathbb{C}_\epsilon$, i.e., it is sufficient to postulate $f = \left(\eta^\flat\right)^m f\left(\eta^\Lambda/\eta^\flat, \mu_\Lambda, \mu_\flat, \epsilon^\flat/\eta^\flat\right)$ and evaluate the same star product (4.29). However, in this work for our purposes it is sufficient to restrict to $m = 0$.

4.3 Wave functions and non-commutative wall-crossing

Equipped with the star product, we can now construct a refined analogue of the dual partition function (4.10). To this end, we simply replace the usual product by the non-commutative one,[17]

$$H_k^{(\mathrm{ref})}(\xi, \tilde{\xi}, \alpha) = \sum_{\ell^\Lambda \in \frac{\mathbb{Z}}{|k|\mathbb{Z}}} \sum_{n^\Lambda \in \mathbb{Z}+\ell^\Lambda/k} \mathrm{e}^{2\pi\mathrm{i}k(\alpha+n^\Lambda\tilde{\xi}_\Lambda)} \star \mathcal{H}^{(\mathrm{ref})}_{k,\ell^\Lambda}(\xi^\Lambda - n^\Lambda). \tag{4.31}$$

In fact, the star product can be evaluated explicitly using the property

$$\mathrm{e}^{2\pi\mathrm{i}(k\alpha+p^\Lambda\tilde{\xi}_\Lambda)} \star f(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha, \epsilon) = \mathrm{e}^{2\pi\mathrm{i}(k\alpha+p^\Lambda\tilde{\xi}_\Lambda)} f\left(\frac{\xi^\Lambda + \epsilon p^\Lambda}{1+\epsilon k}, \tilde{\xi}_\Lambda, \alpha, \frac{\epsilon}{1+\epsilon k}\right). \tag{4.32}$$

In particular, it allows to see that the invariance under the Heisenberg group still holds. Indeed, the star product changes the argument of $\mathcal{H}^{(\mathrm{ref})}_{k,\ell^\Lambda}$ to

$$\frac{\xi^\Lambda + k\epsilon n^\Lambda}{1+k\epsilon} - n^\Lambda = \frac{\xi^\Lambda - n^\Lambda}{1+k\epsilon}$$

and thus the refined wave function is still a function of the difference $\xi^\Lambda - n^\Lambda$, which ensures the invariance.

Our goal now is to determine the transformation property of the refined wave-function $\mathcal{H}^{(\mathrm{ref})}_{k,\ell}(\xi)$ under the quantum KS transformations $\hat{\mathscr{U}}_\gamma$ lifted to the twisted torus by means of (4.29). Denoting by $\hat{\mathscr{V}}_\gamma$ the corresponding lift, we arrive at the following condition

$$\begin{aligned} &\sum_{\ell^\Lambda, n^\Lambda} \mathrm{e}^{2\pi\mathrm{i}k(\alpha+n^\Lambda\tilde{\xi}_\Lambda)} \star \hat{\mathscr{V}}_\gamma\left[\mathcal{H}^{(\mathrm{ref})}_{k,\ell^\Lambda}(\xi - n)\right] \\ &\qquad = \mathcal{U}_\gamma \star \sum_{\ell^\Lambda, n^\Lambda} \mathrm{e}^{2\pi\mathrm{i}k(\alpha+n^\Lambda\tilde{\xi}_\Lambda)} \star \mathcal{H}^{(\mathrm{ref})}_{k,\ell^\Lambda}(\xi - n) \star \mathcal{U}_\gamma^{-1}. \end{aligned} \tag{4.33}$$

Restricting to the electric case $\gamma = (0, q_\Lambda)$, such that $\mathcal{U}_\gamma$ becomes a function of ξ^Λ and ϵ only and therefore commutes with the wave function, and using (4.32) to evaluate the star product, we obtain that the right-hand side of (4.33) is given by

$$\sum_{\ell^\Lambda, n^\Lambda} \mathrm{e}^{2\pi\mathrm{i}k(\alpha+n^\Lambda\tilde{\xi}_\Lambda)} \star \frac{\mathcal{U}_\gamma\left(\frac{\xi - 2k\epsilon n}{1-2k\epsilon}; \frac{\epsilon}{1-2k\epsilon}\right)}{\mathcal{U}_\gamma(\xi;\epsilon)} \mathcal{H}^{(\mathrm{ref})}_{k,\ell^\Lambda}(\xi - n). \tag{4.34}$$

Furthermore, using that

$$\mathrm{e}^{-2\pi\mathrm{i}q_\Lambda\xi^\Lambda} = \mathrm{e}^{-2\pi\mathrm{i}q_\Lambda(\xi^\Lambda - n^\Lambda + \ell^\Lambda/k)}, \qquad \mathrm{e}^{-2\pi\mathrm{i}q_\Lambda\frac{\xi^\Lambda - 2k\epsilon n^\Lambda}{1-2k\epsilon}} = \mathrm{e}^{-2\pi\mathrm{i}q_\Lambda\left(\frac{\xi^\Lambda - n^\Lambda}{1-2k\epsilon} + \frac{\ell^\Lambda}{k}\right)}, \tag{4.35}$$

it is easy to see that the factor generated by the transformation depends on ξ^Λ and n^Λ only through their difference. Thus, the condition (4.33) requires that the wave function should transform as

$$\hat{\mathscr{V}}_\gamma\left[\mathcal{H}^{(\mathrm{ref})}_{k,\ell^\Lambda}(\xi)\right] = \Upsilon_{k,\ell^\Lambda}(\xi)\mathcal{H}^{(\mathrm{ref})}_{k,\ell^\Lambda}(\xi), \qquad \Upsilon_{k,\ell^\Lambda}(\xi) = \frac{\mathcal{U}_\gamma\left(\frac{\xi}{1-2k\epsilon} + \frac{\ell}{k}; \frac{\epsilon}{1-2k\epsilon}\right)}{\mathcal{U}_\gamma(\xi + \ell/k;\epsilon)}. \tag{4.36}$$

[17] In the refined case, we define $H_k^{(\mathrm{ref})}(\xi, \tilde{\xi}, \alpha)$ to be a function of α rather than $\tilde{\alpha}$ because it is α that coincides with one of the Darboux coordinates for the symplectic form (4.26) used to define the star product, but one can always use (4.14) to translate between the two variables.

It turns out that the function $\Upsilon_{k,\ell^\Lambda}$ can be expressed through the Faddeev quantum dilogarithm or its appropriate generalization, which are all described in Appendix A. For simplicity, let us first consider the unit multiplicity case, $\Omega_n(\gamma) = \delta_n$, such that

$$\mathcal{U}_\gamma(\xi;\epsilon) = E_y\big(\mathrm{e}^{-2\pi\mathrm{i}q_\Lambda\xi^\Lambda}\big) \tag{4.37}$$

with $y = \mathrm{e}^{2\pi\mathrm{i}\epsilon}$. We will distinguish between cases of positive and negative five-brane charge k because they lead to different relations between the refinement parameters ϵ and b. For $k > 0$, we identify

$$\epsilon = \frac{1}{2k}\big(1-\mathsf{b}^{-2}\big), \tag{4.38}$$

such that $1 - 2k\epsilon = \mathsf{b}^{-2}$ and

$$y = \mathrm{e}^{\pi\mathrm{i}/k}\mathrm{e}^{-\pi\mathrm{i}/(\mathsf{b}^2k)} := \tilde{y}_{\mathsf{b},k}, \qquad \mathrm{e}^{\frac{2\pi\mathrm{i}\epsilon}{1-2k\epsilon}} = \mathrm{e}^{-\pi\mathrm{i}/k}\mathrm{e}^{\pi\mathrm{i}\mathsf{b}^2/k} := y_{\mathsf{b},k}. \tag{4.39}$$

which generalize the variables (3.54) to generic k and satisfy $\tilde{y}_{\mathsf{b},k} = \bar{y}_{1/\bar{\mathsf{b}},k}$. Thus, the function $\Upsilon_{k,\ell^\Lambda}$ in (4.36) takes the form

$$\Upsilon_{k,\ell^\Lambda}(\xi) = \frac{E_{y_{\mathsf{b},k}}\big(\mathrm{e}^{-2\pi\mathrm{i}(\mathsf{b}^2q_\Lambda\xi^\Lambda+\ell/k)}\big)}{E_{\tilde{y}_{\mathsf{b},k}}\big(\mathrm{e}^{-2\pi\mathrm{i}(q_\Lambda\xi^\Lambda+\ell/k)}\big)}, \tag{4.40}$$

where we denoted $\ell = q_\Lambda\ell^\Lambda$. Setting also $s_\ell = \mathrm{sgn}(\ell)$ and using (A.4), the numerator can be rewritten as

$$\begin{aligned} &E_{y_{\mathsf{b},k}}\big(y_{\mathsf{b},k}^{2\ell}\mathrm{e}^{-2\pi\mathrm{i}\mathsf{b}^2(q_\Lambda\xi^\Lambda+\ell/k)}\big) \\ &\quad = E_{y_{\mathsf{b},k}}\big(\mathrm{e}^{-2\pi\mathrm{i}\mathsf{b}^2(q_\Lambda\xi^\Lambda+\ell/k)}\big)\prod_{j=0}^{|\ell|-1}\big(1-y_{\mathsf{b},k}^{2\ell-s_\ell(2j+1)}\mathrm{e}^{-2\pi\mathrm{i}\mathsf{b}^2(q_\Lambda\xi^\Lambda+\ell/k)}\big)^{s_\ell}. \end{aligned} \tag{4.41}$$

We can then use the property (A.3) to get

$$\begin{aligned} \Upsilon_{k,\ell^\Lambda}(\xi) &= \prod_{j=0}^{k-1}\frac{E_{\mathrm{e}^{\pi\mathrm{i}\mathsf{b}^2}}\big(-\mathrm{e}^{-2\pi\mathrm{i}\mathsf{b}\Xi^+_{k,j}}\big)}{E_{\mathrm{e}^{-\pi\mathrm{i}/\mathsf{b}^2}}\big(-\mathrm{e}^{-2\pi\mathrm{i}\mathsf{b}^{-1}\Xi^+_{k,j}}\big)}\prod_{j=0}^{|\ell|-1}\big(1-\mathrm{e}^{-2\pi\mathrm{i}(\mathsf{b}^2q_\Lambda\xi^\Lambda+\frac{\ell}{k}+\frac{s_\ell}{k}(\mathsf{b}^2-1)(j+\frac12))}\big)^{s_\ell} \\ &= \prod_{j=0}^{k-1}\Phi_{\mathsf{b}}^{-1}\big(-\mathrm{i}\Xi^+_{k,j}\big)\prod_{j=0}^{|\ell|-1}\big(1-\mathrm{e}^{-2\pi\mathrm{i}(\mathsf{b}^2q_\Lambda\xi^\Lambda+\frac{\ell}{k}+\frac{s_\ell}{k}(\mathsf{b}^2-1)(j+\frac12))}\big)^{s_\ell}, \end{aligned} \tag{4.42}$$

where

$$\Xi^+_{k,j} = \mathsf{b}q_\Lambda\xi^\Lambda + \frac{\mathsf{b}\ell}{k} - \frac{1}{k}\big(\mathsf{b}-\mathsf{b}^{-1}\big)\left(j-\frac{k-1}{2}\right). \tag{4.43}$$

For $\ell > 0$ (hence $s = 1$), (4.42) is recognized as the function $\mathbf{A}^{(k)}_\hbar(t,\ell)$ defined in [12, equation (4.1)] evaluated at $\hbar = \mathsf{b}^2$ and $t = -\mathrm{i}\mathsf{b}^2q_\Lambda\xi^\Lambda$.

For $k < 0$, we replace the identification (4.38) by

$$\epsilon = \frac{1}{2k}\big(1-\mathsf{b}^2\big), \tag{4.44}$$

such that $1 - 2k\epsilon = \mathsf{b}^2$ and

$$y = y_{\mathsf{b},-k}, \qquad \mathrm{e}^{\frac{2\pi\mathrm{i}\epsilon}{1-2k\epsilon}} = \tilde{y}_{\mathsf{b},-k}. \tag{4.45}$$

Repeating the same steps as above, one arrives at the following representation for the function $\Upsilon_{k,\ell^\Lambda}$ (4.36):

$$\Upsilon_{k,\ell^\Lambda}(\xi)=\prod_{j=0}^{|k|-1}\Phi_{\mathsf{b}}(-\mathrm{i}\Xi^-_{k,j})\prod_{j=0}^{|\ell|-1}\left(1-\mathrm{e}^{-2\pi\mathrm{i}(\mathsf{b}^{-2}q_\Lambda\xi^\Lambda+\frac{\ell}{k}+\frac{s_\ell}{k}(\mathsf{b}^{-2}-1)(j+\frac12))}\right)^{s_\ell},\tag{4.46}$$

where now

$$\Xi^-_{k,j}=\mathsf{b}^{-1}q_\Lambda\xi^\Lambda+\frac{\ell}{\mathsf{b}k}-\frac{1}{|k|}\left(\mathsf{b}-\mathsf{b}^{-1}\right)\left(j-\frac{|k|-1}{2}\right).\tag{4.47}$$

For BPS rays carrying general refined BPS indices $\Omega_n(\gamma)$, the wall-crossing transformation can be expressed through a generalization $\Phi^{[j]}_{\mathsf{b},k}(z)$ of the Faddeev quantum dilogarithm defined in (A.17), which also has a representation as a product of the usual quantum dilogarithms with shifted arguments. Again, proceeding as above, it is straightforward to show that

$$\begin{aligned}\Upsilon_{k,\ell^\Lambda}(\xi)=\prod_j\Bigg[&\prod_{i=0}^{|k|-1}\left(\Phi^{[j]}_{\mathsf{b},k}(-\mathrm{i}\Xi^{s_k}_{k,i})\right)^{s_k}\\&\times\prod_{m=-j}^{j}\prod_{i=0}^{|\ell|-1}\left(1-\mathrm{e}^{-2\pi\mathrm{i}(\mathsf{b}^{2s_k}q_\Lambda\xi^\Lambda+\frac{\ell}{k}+\frac{s_\ell}{k}(\mathsf{b}^{2s_k}-1)(i-m+\frac12))}\right)^{-s_\ell}\Bigg]^{-\Omega_{[j]}},\end{aligned}\tag{4.48}$$

where $s_k=\mathrm{sgn}(k)$. For $k=1$, in which case $\ell=0$, this formula reduces to

$$\Upsilon_1(\xi)=\prod_j\left[\Phi^{[j]}_{\mathsf{b}}\left(-\mathrm{i}\mathsf{b}q_\Lambda\xi^\Lambda\right)\right]^{-\Omega_{[j]}}.\tag{4.49}$$

This result is to be compared with (3.64) where $\mathcal{A}=2\pi q_a t^a+4\pi^2\mathrm{i}q_0$. It is easy to see that the factor appearing in (3.64) coincides with $\Upsilon_1(\xi)$ provided one relates the variables as in (4.16) with ξ^Λ replaced by $\mathsf{b}\xi^\Lambda$, i.e.,

$$t^a=2\pi\mathrm{i}\frac{\xi^a}{\xi^0},\qquad g_s=\frac{2\pi}{\mathsf{b}\xi^0}.\tag{4.50}$$

Note that the last relation is equivalent to

$$\xi^0=\frac{2\pi}{\epsilon_1},\tag{4.51}$$

where ϵ_1 is one the deformation parameters of the Ω-background, see (2.1). The match of the Stokes factors suggests that the wave function $\mathcal{H}^{(\mathrm{ref})}_1$ is equal to the refined topological string up to a b-dependent constant,

$$\mathcal{H}^{(\mathrm{ref})}_1(\xi)\sim\mathcal{Z}(\boldsymbol{t};g_s,\mathsf{b}),\tag{4.52}$$

with the parameters identified as in (4.50). This generalizes a similar relation in the unrefined case [9].

Note that if we had chosen the opposite ordering in the definition of the refined dual partition function (4.31), the results for the cases of positive and negative k would effectively be swapped. Indeed, exchanging the ordering of factors in (4.32) leads to the flip of signs in front of k and p^Λ in the arguments of the function f. As a result, denoting by tilde the quantities corresponding to the opposite ordering, one has

$$\tilde{\Upsilon}_{k,\ell^\Lambda}(\xi)=\Upsilon^{-1}_{-k,-\ell^\Lambda}(\xi).\tag{4.53}$$

In particular, using (4.48), one finds that

$$\tilde{\Upsilon}_1(\xi) = \prod_j \left[\Phi_{\mathsf{b}}^{[j]}\left(-\mathrm{i}\mathsf{b}^{-1} q_\Lambda \xi^\Lambda\right)\right]^{-\Omega_{[j]}}, \tag{4.54}$$

which in turn implies the identification

$$\tilde{\mathcal{H}}_1^{(\mathrm{ref})}(\xi) \sim \mathcal{Z}\left(2\pi\mathrm{i}\frac{\xi^a}{\xi^0}; \frac{2\pi\mathsf{b}}{\xi^0}, \mathsf{b}\right). \tag{4.55}$$

Note that in this case the relation (4.51) is replaced by

$$\xi^0 = -\frac{2\pi}{\epsilon_2}. \tag{4.56}$$

We observe that the effect of the refinement is to introduce a factor of b in the relation between ξ^0 and the topological string coupling g_s. It spoils the symmetry $\mathsf{b} \leftrightarrow \mathsf{b}^{-1}$, unless one simultaneously changes the ordering in the definition of the refined dual partition function. Of course, this factor could be absorbed in the definition of the Darboux coordinates ξ^Λ, but the price to pay is a modification of the quasi-periodicity conditions (4.4) and (4.9), and it would reappear anyway in the quantization condition of charges.

A Quantum dilogarithms

In this Appendix, we introduce several versions of the quantum dilogarithm function which play a role in this work.

The standard (sometimes called compact) quantum dilogarithm $E_y(x)$ is defined for $x, y \in \mathbb{C}$, $|y| < 1$ as[18]

$$E_y(x) := \exp\left[\sum_{k=1}^{\infty} \frac{(xy)^k}{k\left(1-y^{2k}\right)}\right] = \prod_{n=0}^{\infty}\left(1 - xy^{2n+1}\right)^{-1} = \left(xy; y^2\right)_\infty^{-1}, \tag{A.1}$$

where $(a;q)_n := \prod_{k=0}^{n-1}\left(1-aq^k\right)$ is the q-Pochhammer symbol. The quantum dilogarithm satisfies the following properties:

$$E_y\left(xy^2\right) = (1-xy)E_y(x), \tag{A.2}$$

$$E_{y^{1/k}}(x) = \prod_{j=0}^{k-1} E_y\left(y^{\frac{2j+1}{k}-1}x\right), \qquad k \in \mathbb{N}. \tag{A.3}$$

The first property has the obvious generalization

$$E_y\left(y^{2\ell}x\right) = E_y(x)\prod_{j=0}^{\ell-1}\left(1 - xy^{2j+1}\right), \qquad \ell \in \mathbb{N}. \tag{A.4}$$

A different version of the quantum dilogarithm (sometimes called non-compact) was introduced by Faddeev [43], through the contour integral

$$\Phi_{\mathsf{b}}(z) := \exp\left(\int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{e}^{-2\mathrm{i}zv}}{4\sinh(v\mathsf{b})\sinh(v/\mathsf{b})}\frac{\mathrm{d}v}{v}\right) \tag{A.5}$$

[18]There are different conventions in the literature, e.g., [21, 29] define the quantum dilogarithm as $\mathbb{E}_q(x) = (x,q)_\infty$, which is related to our definition by $E_y(x) = \left[\mathbb{E}_{y^2}(xy)\right]^{-1}$.

over the real line, circumventing the pole at $v = 0$ by deviating into the upper half plane. This integral converges for $\operatorname{Re}\mathsf{b} \neq 0$, $|\operatorname{Im} z| < |\operatorname{Im} c_{\mathsf{b}}|$ with $c_{\mathsf{b}} := \frac{\mathrm{i}}{2}(\mathsf{b} + \mathsf{b}^{-1})$. It possesses several beautiful properties listed, for example, in [12, Section A.2]. Here we mention the quasi-periodicity

$$\Phi_{\mathsf{b}}\left(x - \mathrm{i}\mathsf{b}^{\pm 1}/2\right) = \left(1 + \mathrm{e}^{2\pi\mathsf{b}^{\pm 1}x}\right)\Phi_{\mathsf{b}}\left(x + \mathrm{i}\mathsf{b}^{\pm 1}/2\right), \tag{A.6}$$

the classical limit $\mathsf{b} \to 0$ [19, Section 13.4]

$$\Phi_{\mathsf{b}}(x) = \exp\left(\frac{\mathrm{Li}_2\left(-\mathrm{e}^{2\pi\mathsf{b}x}\right)}{2\pi\mathrm{i}\mathsf{b}^2}\right)\left(1 + \mathcal{O}\left(\mathsf{b}^2\right)\right), \tag{A.7}$$

the special value at $\mathsf{b} = 1$

$$\Phi_1(x) = \exp\left[\frac{\mathrm{i}}{2\pi}\left(\mathrm{Li}_2\left(\mathrm{e}^{2\pi x}\right) + 2\pi x \log\left(1 - \mathrm{e}^{2\pi x}\right)\right)\right], \tag{A.8}$$

and the special value at $\mathsf{b} = \sqrt{2}$

$$\Phi_{\sqrt{2}}\left(\frac{x}{\sqrt{2}}\right) = \exp\left[\frac{\mathrm{i}}{4\pi}\left(\mathrm{Li}_2\left(-\mathrm{e}^{2\pi x}\right) + 2\pi x \log\left(1 + \mathrm{e}^{2\pi x}\right) + 2\pi \tan^{-1}\left(\mathrm{e}^{\pi x}\right)\right)\right], \tag{A.9}$$

which follows from the more general results obtained in [50].

Evaluating the integral in (A.5) by residues, one finds that

$$\log \Phi_{\mathsf{b}}(z) = \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{2\mathrm{i}\ell}\left(\frac{\mathrm{e}^{2\pi\ell z\mathsf{b}}}{\sin\left(\ell\pi\mathsf{b}^2\right)} + \frac{\mathrm{e}^{2\pi\ell z/\mathsf{b}}}{\sin\left(\ell\pi/\mathsf{b}^2\right)}\right), \tag{A.10}$$

which allows to establish the following relation between the two versions of quantum dilogarithm [44, Section A], valid whenever $\operatorname{Im}\left(b^2\right) > 0$,

$$\Phi_{\mathsf{b}}(z) = \frac{E_{\mathrm{e}^{-\mathrm{i}\pi/\mathsf{b}^2}}\left(-\mathrm{e}^{2\pi z/\mathsf{b}}\right)}{E_{\mathrm{e}^{\mathrm{i}\pi\mathsf{b}^2}}\left(-\mathrm{e}^{2\pi z\mathsf{b}}\right)} = \frac{E_{\tilde{y}_{\mathsf{b}}}\left(\mathrm{e}^{2\pi z/\mathsf{b}}\right)}{E_{y_{\mathsf{b}}}\left(\mathrm{e}^{2\pi z\mathsf{b}}\right)}, \tag{A.11}$$

where in the second representation we used the variables defined in (3.54).

Let us now introduce a generalization of the compact quantum dilogarithm, which appeared in [38, equation (2.12)] and depends on an additional label $j \in \mathbb{Z}/2$,

$$E_y^{[j]}(x) = \exp\left[\sum_{k=1}^{\infty} \frac{(xy)^k \chi_j\left(y^k\right)}{k\left(1 - y^{2k}\right)}\right] = \prod_{m=-j}^{j}\left(xy^{2m+1}; y^2\right)_\infty^{-1}, \tag{A.12}$$

where $\chi_j(y)$ is the character (1.7) of the SU(2) representation of spin j. It is easy to see that

$$E_y^{[j]}(x) = \prod_{m=-j}^{j} E_y\left(xy^{2m}\right), \tag{A.13}$$

where the product runs over half-integers m such that $m - j$ is integer. Following (A.11), we then define the corresponding generalization of the Faddeev quantum dilogarithm

$$\Phi_{\mathsf{b}}^{[j]}(z) = \frac{E_{\tilde{y}_{\mathsf{b}}}^{[j]}\left(\mathrm{e}^{2\pi z/\mathsf{b}}\right)}{E_{y_{\mathsf{b}}}^{[j]}\left(\mathrm{e}^{2\pi z\mathsf{b}}\right)}. \tag{A.14}$$

From this definition and (A.12), it follows that this new function can be written in one of the following forms

$$\begin{aligned}\Phi_{\mathsf{b}}^{[j]}(z) &= \exp\left[\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell}}{2\mathrm{i}\ell}\left(\frac{\chi_j\left(-\mathrm{e}^{\pi\mathrm{i}\ell\mathsf{b}^2}\right)\mathrm{e}^{2\pi\ell z\mathsf{b}}}{\sin\left(\ell\pi\mathsf{b}^2\right)}+\frac{\chi_j\left(-\mathrm{e}^{-\pi\mathrm{i}\ell/\mathsf{b}^2}\right)\mathrm{e}^{2\pi\ell z/\mathsf{b}}}{\sin\left(\ell\pi/\mathsf{b}^2\right)}\right)\right]\\ &= \exp\left(\int_{\mathbb{R}+\mathrm{i}\epsilon}\frac{\chi_j\left(\mathrm{e}^{(\mathsf{b}-\mathsf{b}^{-1})v}\right)\mathrm{e}^{-2\mathrm{i}zv}}{4\sinh(v\mathsf{b})\sinh(v/\mathsf{b})}\frac{\mathrm{d}v}{v}\right)\\ &= \prod_{m=-j}^{j}\Phi_{\mathsf{b}}\left(z+\mathrm{i}m\left(\mathsf{b}-\mathsf{b}^{-1}\right)\right) \end{aligned}\tag{A.15}$$

in accordance with (A.13).

Finally, we incorporate one additional integer parameter k corresponding to the five-brane charge. To this end, we define

$$E_{y,k}^{[j]}(x) = \prod_{m=-j}^{j} E_y\left(xy^{2m/k}\right) \tag{A.16}$$

and

$$\Phi_{\mathsf{b},k}^{[j]}(z) := \frac{E_{\tilde{y}_{\mathsf{b}},k}^{[j]}\left(\mathrm{e}^{2\pi z/\mathsf{b}}\right)}{E_{y_{\mathsf{b}},k}^{[j]}\left(\mathrm{e}^{2\pi z\mathsf{b}}\right)} = \prod_{m=-j}^{j}\Phi_{\mathsf{b}}\left(z+\frac{\mathrm{i}m}{k}\left(\mathsf{b}-\mathsf{b}^{-1}\right)\right). \tag{A.17}$$

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