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Asymptotics of the Humbert Function Ψ_1 for Two Large Arguments

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Abstract. Recently, Wald and Henkel (2018) derived the leading-order estimate of the Humbert functions Φ_2, Φ_3 and Ξ_2 for two large arguments, but their technique cannot handle the Humbert function Ψ_1. In this paper, we establish the leading asymptotic behavior of the Humbert function Ψ_1 for two large arguments. Our proof is based on a connection formula of the Gauss hypergeometric function and Nagel's approach (2004). This approach is also applied to deduce asymptotic expansions of the generalized hypergeometric function ${}_pF_q$ ($p \leqslant q$) for large parameters, which are not contained in NIST handbook.

Key words: Humbert function; asymptotics; generalized hypergeometric function

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1 Introduction

Humbert [14] introduced seven confluent hypergeometric functions of two variables which are denoted by Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1, Ξ_2. In this paper, we mainly focus on the Humbert function Ψ_1, which is defined by

$$\Psi_1[a,b;c,c';x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!}, \qquad |x|<1, \quad |y|<\infty,$$

where $a,b\in\mathbb{C}$ and $c,c'\notin\mathbb{Z}_{\leqslant 0}$. This function has a Kummer-type transformation [10, equation (2.54)]

$$\Psi_1[a,b;c,c';x,y] = (1-x)^{-a}\Psi_1\left[a,c-b;c,c';\frac{x}{x-1},\frac{y}{1-x}\right]. \tag{1.1}$$

Using the series manipulation technique, we can obtain [5, equation (83)]

$$\Psi_1[a,b;c,c';x,y] = \sum_{n=0}^{\infty}\frac{(a)_n}{(c')_n}{}_2F_1\left[\begin{matrix}a+n,b\\ c\end{matrix};x\right]\frac{y^n}{n!}, \tag{1.2}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function defined below in (1.3). Similar analysis of [20, equation (14)] gives

$$\left|{}_2F_1\left[\begin{matrix}a+n,b\\ c\end{matrix};x\right]\right| = \mathcal{O}\big(n^{-\omega}\rho_x^n\big), \qquad n\to\infty, \quad n\in\mathbb{Z}_{>0},$$

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$$\omega = \min\{\mathrm{Re}(b), \mathrm{Re}(c-b)\}, \qquad \rho_x = \max\{1, |1-x|^{-1}\}.$$

Then the summand in (1.2) has the order of magnitude

$$\mathcal{O}\left(n^{\mathrm{Re}(a-c')-\omega}\frac{(\rho_x|y|)^n}{n!}\right), \qquad n\to\infty,$$

which implies that the series (1.2) converges absolutely in the region

$$\mathbb{D}_{\Psi_1} := \{(x,y)\in\mathbb{C}^2 \colon x\neq 1,\ |\arg(1-x)|<\pi,\ |y|<\infty\}.$$

So the series in (1.2) provides an analytic continuation of Ψ_1 to $\mathbb{D}_{\Psi_1}$.

There are some useful identities about Ψ_1 in the literature (see [5, 10, 11, 14]), as well as many applications in physics (see [2, equation (5.2)] and [3, 4]). But we still know very little about the asymptotics of Ψ_1. Recently, in order to study the asymptotics of Saran's hypergeometric function F_K when two of its variables become simultaneously large, Hang and Luo [13] established asymptotic expansions of Ψ_1 for one large variable. By using a Tauberian theorem for Laplace transform, Wald and Henkel [26] derived the leading-order estimate of the Humbert functions Φ_2, Φ_3 and Ξ_2 when the absolute values of the two independent variables become simultaneously large. They also considered Ψ_1 and pointed out that their technique fails for Ψ_1 (see [26, p. 99]). In this paper, we give an incomplete answer to their problem by establishing the leading asymptotic behavior of Ψ_1 when $|x|\to\infty$ and $y\to+\infty$.

This paper is organised as follows. In Section 2, we demonstrate three lemmas which will be used later. Section 3 devotes to the asymptotics of Ψ_1 for two large arguments. In Section 4, we present asymptotic expansions of the generalized hypergeometric function ${}_pF_q$ ($p\leqslant q$) for large parameters, which are not contained in the NIST handbook [22]. The proofs in Sections 3 and 4 are based on Nagel's approach [21]. The main results are Theorems 3.6, 4.1, 4.3, 4.5 and 4.7.

Notation. In this paper, the number C generically denotes a positive constant independent of the parameter n, the index of summation ℓ and the variable z. Moreover, the generalized hypergeometric function ${}_pF_q$ is defined by (see, for example, [22, p. 404])

$${}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right] \equiv {}_pF_q[a_1,\dots,a_p;b_1,\dots,b_q;z] := \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \tag{1.3}$$

where $a_1,\dots,a_p\in\mathbb{C}$ and $b_1,\dots,b_q\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$.

2 Preliminary lemmas

In this section, we deduce three lemmas which will be used in the sequel. The first is a sharp bound for the ratio of Pochhammer symbols.

Lemma 2.1. *If $a\in\mathbb{C}$ and $b\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$, then*

$$\left|\frac{(a)_n}{(b)_n}\right| \leqslant Cn^{\mathrm{Re}(a-b)}, \qquad n\in\mathbb{Z}_{>0}.$$

Proof. The proof for $a\in\mathbb{Z}_{\leqslant 0}$ is trivial. If $a\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$, Stirling's formula implies that

$$\lim_{n\to\infty} n^{b-a}\frac{\Gamma(a+n)}{\Gamma(b+n)} = 1,$$

which concludes that $n^{b-a}\frac{(a)_n}{(b)_n}$ is bounded uniformly for $n\in\mathbb{Z}_{>0}$. ∎

The second is a simple estimate of the function $\Phi_a(x)$, which is the "horizontal" generating function of Stirling numbers of real order (see [8, Section 8]). More properties of $\Phi_a(x)$, containing the asymptotic behavior for fixed $x \in \mathbb{R}$ and $a \to \pm\infty$, are studied in [25, Section 3.2].

Lemma 2.2. *Define*

$$\Phi_a(x) := \sum_{k=1}^{\infty} \frac{k^a}{k!} x^k, \qquad a, x \in \mathbb{R}.$$

Then $\Phi_a(x) \sim x^a \mathrm{e}^x$, $x \to +\infty$. *Thus*

$$\Phi_a(x) \leqslant K x^a \mathrm{e}^x, \qquad x \geqslant 1,$$

where $K > 0$ *is a constant independent of* x.

Proof. Take $a_n = \frac{n^a}{n!}$ and $b_n = \frac{1}{n!}$ in [24, p. 12, Problem 72]. ■

The third is a global estimate for the confluent hypergeometric function ${}_1F_1$.

Lemma 2.3. *Let* $a, c \in \mathbb{C}$. *Choose* $N \in \mathbb{Z}_{>0}$ *such that* $\mathrm{Re}(c + N + 1) > 0$ *and define*

$$G_\ell(z) := {}_1F_1\left[\begin{matrix} a+\ell \\ c+\ell \end{matrix}; z\right], \qquad \ell \in \mathbb{Z}_{\geqslant 0}. \tag{2.1}$$

Then for $\ell \geqslant N + 1$,

$$|G_\ell(z)| \leqslant C \mathrm{e}^{\gamma|z|}, \tag{2.2}$$

where $\gamma > 0$ *is a constant independent of* ℓ, N *and* z.

Proof. Since $G_\ell(0) = 1$, we can assume that $z \neq 0$. Recall the inequality [15, equation (2.3)]

$$\left|{}_1F_1\left[\begin{matrix} a_0 \\ b_0 \end{matrix}; z\right]\right| \leqslant \cos\frac{\theta}{2} \cdot {}_1F_1\left[\begin{matrix} |a_0| \\ |b_0| \end{matrix}; |z| \sec\frac{\theta}{2}\right] \tag{2.3}$$

with $\theta = \arg(b_0) \in (-\pi, \pi)$ and the inequality [9, p. 37, equation (3.5)]

$$\mathrm{e}^{\frac{a_0}{b_0}z} < {}_1F_1\left[\begin{matrix} a_0 \\ b_0 \end{matrix}; z\right] < 1 - \frac{a_0}{b_0} + \frac{a_0}{b_0}\mathrm{e}^z, \qquad b_0 > a_0 > 0, \quad z \neq 0. \tag{2.4}$$

But when $a_0 \geqslant b_0 > 0$ and $z > 0$, we have $\frac{(a_0)_k}{(b_0)_k} \leqslant \left(\frac{a_0}{b_0}\right)^k$ since $\frac{a_0+j}{b_0+j}$ decreases with respect to $j \geqslant 0$. Thus

$${}_1F_1\left[\begin{matrix} a_0 \\ b_0 \end{matrix}; z\right] \leqslant \mathrm{e}^{\frac{a_0}{b_0}z}, \qquad a_0 \geqslant b_0 > 0, \quad z > 0. \tag{2.5}$$

Recall $\mathrm{Re}(c+N+1) > 0$ and note that $\gamma(\ell) := \max\left\{1, \frac{|a+\ell|}{|c+\ell|}\right\}$ is bounded uniformly for $\ell \in \mathbb{Z}_{\geqslant 0}$. Therefore, a combination of the inequalities (2.3)–(2.5) claims that for $\ell \geqslant N + 1$,

$$|G_\ell(z)| \leqslant 2\gamma(\ell)\mathrm{e}^{\sqrt{2}\gamma(\ell)|z|} \leqslant C\mathrm{e}^{\gamma|z|},$$

where

$$\gamma := \sup_{\ell \in \mathbb{Z}_{\geqslant 0}} \sqrt{2}\gamma(\ell) \geqslant \sqrt{2}.$$

This completes the proof. ■

Remark 2.4. Lemma 2.3 can be easily generalized to the following form. Let $a_1, \dots, a_p, b_1, \dots, b_p \in \mathbb{C}$, and let $N \in \mathbb{Z}_{>0}$ such that $\mathrm{Re}(b_j + N + 1) > 0$, $1 \leqslant j \leqslant p$. Then for $\ell \geqslant N + 1$,

$$\left|{}_pF_p\left[\begin{matrix} a_1+\ell, \dots, a_p+\ell \\ b_1+\ell, \dots, b_p+\ell \end{matrix}; z\right]\right| \leqslant C\mathrm{e}^{\gamma|z|}, \tag{2.6}$$

where $\gamma > 0$ is a constant independent of ℓ, N and z.

3 Asymptotics of Ψ_1 for large arguments

In this section, we establish the leading asymptotic behavior of Ψ_1 under the condition

$$x \to \infty, \qquad |\arg(1-x)| < \pi, \qquad y \to +\infty, \qquad \left|\frac{y}{1-x}\right| = \gamma \tag{3.1}$$

satisfying $0 < \gamma_1 \leqslant \gamma \leqslant \gamma_2 < \infty$.

First of all, we derive a new series representation for Ψ_1. Our starting point is the behavior near unit argument of the Gauss hypergeometric function, which is given by the well-known connection formula [6, equation (1.2)]

$$\begin{aligned}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};z\right] &= \frac{\Gamma(a)\Gamma(b)\Gamma(s)}{\Gamma(a+s)\Gamma(b+s)}{}_2F_1\left[\begin{matrix}a,b\\1-s\end{matrix};1-z\right]\\&\quad+\Gamma(-s)(1-z)^s{}_2F_1\left[\begin{matrix}a+s,b+s\\1+s\end{matrix};1-z\right]\end{aligned} \tag{3.2}$$

with $|\arg z| < \pi$, $|\arg(1-z)| < \pi$ and $s = c-a-b$. Furthermore, (3.2) is valid if $s \notin \mathbb{Z}$.

Expanding the right-hand side of (3.2) as follows (see [7, equation (1.1)]):

$$\begin{aligned}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};z\right] &= \sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(s-n)}{\Gamma(a+s)\Gamma(b+s)n!}(1-z)^n\\&\quad+\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(a+s+n)\Gamma(b+s+n)\Gamma(-s-n)}{\Gamma(a+s)\Gamma(b+s)n!}(1-z)^{n+s}\end{aligned} \tag{3.3}$$

and then applying (3.3) to (1.2), we get

$$\begin{aligned}\Psi_1[a,b;c,c';x,y] &= C_1\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b-c+1)_n}{}_2F_2\left[\begin{matrix}a-c+1,a+n\\c',a+b-c+1+n\end{matrix};y\right]\frac{(1-x)^n}{n!}\\&\quad+C_2(1-x)^{c-a-b}\sum_{n=0}^{\infty}\frac{(c-a)_n(c-b)_n}{(c-a-b+1)_n}{}_2F_2\left[\begin{matrix}a-c+1,a+b-c-n\\c',a-c+1-n\end{matrix};\frac{y}{1-x}\right]\frac{(1-x)^n}{n!},\end{aligned} \tag{3.4}$$

where $|\arg(1-x)| < \pi$, $a+b-c \notin \mathbb{Z}$,

$$C_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad C_2 = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

and ${}_2F_2$ is defined in (1.3). Both series in (3.4) converge absolutely for $|x-1| < 1$ and $|y| < \infty$.

Combining (1.1) with (3.4) gives the following series representation.

Theorem 3.1. *Assume that $c, c' \notin \mathbb{Z}_{\leqslant 0}$ and $a-c \notin \mathbb{Z}$. Then when $a-b \notin \mathbb{Z}$,*

$$\Psi_1[a,b;c,c';x,y] = \mathfrak{f}_c(b,a)(1-x)^{-a}V_1(x,y) + \mathfrak{f}_c(a,b)(1-x)^{-b}V_2(x,y) \tag{3.5}$$

holds for $|\arg(1-x)| < \pi$, $|x-1| > 1$ and $|y| < \infty$, where

$$V_1(x,y) := \sum_{n=0}^{\infty}\frac{(a)_n(c-b)_n}{(a-b+1)_n}{}_2F_2\left[\begin{matrix}a-c+1,a+n\\c',a-b+1+n\end{matrix};\frac{y}{1-x}\right]\frac{(1-x)^{-n}}{n!}, \tag{3.6}$$

$$V_2(x,y) := \sum_{n=0}^{\infty}\frac{(b)_n(c-a)_n}{(b-a+1)_n}{}_2F_2\left[\begin{matrix}a-c+1,a-b-n\\c',a-c+1-n\end{matrix};y\right]\frac{(1-x)^{-n}}{n!}, \tag{3.7}$$

and

$$\mathfrak{f}_\gamma(a,b) := \frac{\Gamma(\gamma)\Gamma(a-b)}{\Gamma(a)\Gamma(\gamma-b)}.$$

Remark 3.2.

(1) The series (3.6) and (3.7) converge absolutely for $|x-1|>1$ and $|y|<\infty$.

(2) When $s=c-a-b\in\mathbb{Z}$ in (3.2), the corresponding connection formulas of the Gauss hypergeometric function appear as [1, equations (15.3.10)–(15.3.12)]. Thus if $s=c-a-b\in\mathbb{Z}$, one may derive the corresponding series representations of Ψ_1.

Next we derive a uniform estimate of ${}_2F_2$ for large parameters by using Nagel's approach [21, equations (A16)–(A19)].

Define for $\varepsilon=\pm 1$ that

$$f_n^\varepsilon(z) := {}_2F_2\left[\begin{matrix}a, b+\varepsilon n\\ c, d+\varepsilon n\end{matrix}; z\right], \qquad g_n^\varepsilon(z) := \frac{\Gamma(b)\Gamma(d+\varepsilon n)}{\Gamma(d)\Gamma(b+\varepsilon n)}\,{}_2F_2\left[\begin{matrix}a, b\\ c, d\end{matrix}; z\right], \tag{3.8}$$

where ${}_2F_2$ is given by (1.3).

Lemma 3.3. *Let $c\notin\mathbb{Z}_{\leqslant 0}$ and $d\notin\mathbb{Z}$. Then*

$$\left|f_n^\varepsilon(z)-g_n^\varepsilon(z)\right| \leqslant CB^n\cdot\max\left\{1, n^{\operatorname{Re}(d-b)}\right\}z^p\mathrm{e}^z, \qquad n\geqslant 1, \quad z\geqslant z_0, \tag{3.9}$$

where

$$B := \sup_{m\in\mathbb{Z}}\left|\frac{b+m}{d+m}\right| \geqslant 1, \qquad p := \operatorname{Re}(a-c)+\max\{0, \operatorname{Re}(b-d)\},$$

and $z_0\geqslant 1$ chosen such that $z^p\mathrm{e}^z\geqslant 1$ holds for $z\geqslant z_0$.

Proof. The constant B exists and satisfies $B\geqslant 1$, since $\lim_{|m|\to\infty}\frac{b+m}{d+m}=1$. Next, we just prove the inequality for $\varepsilon=-1$ since the proof for $\varepsilon=1$ is similar.

Note that

$$g_n^\varepsilon(z) = \frac{(1-b)_n}{(1-d)_n}\sum_{\ell=0}^{\infty}\frac{(a)_\ell(b)_\ell}{(c)_\ell(d)_\ell}\frac{z^\ell}{\ell!}.$$

Since $q:=\operatorname{Re}(a+b-c-d)\leqslant p$, we get from Lemmas 2.1 and 2.2 that for $z\geqslant z_0$,

$$\begin{aligned}\left|g_n^\varepsilon(z)\right| &\leqslant Cn^{\operatorname{Re}(d-b)}+Cn^{\operatorname{Re}(d-b)}\sum_{\ell=1}^{\infty}\ell^q\frac{z^\ell}{\ell!}\leqslant Cn^{\operatorname{Re}(d-b)}(1+\Phi_q(z))\\ &\leqslant Cn^{\operatorname{Re}(d-b)}z^q\mathrm{e}^z\leqslant Cn^{\operatorname{Re}(d-b)}z^p\mathrm{e}^z.\end{aligned} \tag{3.10}$$

Assume that $z\geqslant z_0$ and write

$$f_n^\varepsilon(z)-1 = \left(\sum_{\ell=1}^{n}+\sum_{\ell=n+1}^{\infty}\right)\frac{(a)_\ell(b-n)_\ell}{(c)_\ell(d-n)_\ell}\frac{z^\ell}{\ell!} =: S_1^*+S_2^*.$$

If $1\leqslant\ell\leqslant n$, then

$$\left|\frac{(b-n)_\ell}{(d-n)_\ell}\right| = \prod_{j=0}^{\ell-1}\left|\frac{b-n+j}{d-n+j}\right| \leqslant B^\ell,$$

and using Lemma 2.1 can get

$$|S_1^*| \leqslant C\sum_{\ell=1}^{n}B^\ell\ell^{\operatorname{Re}(a-c)}\frac{z^\ell}{\ell!} \leqslant CB^n\sum_{\ell=1}^{n}\ell^{\operatorname{Re}(a-c)}\frac{z^\ell}{\ell!}. \tag{3.11}$$

If $\ell \geqslant n+1$, adopt Lemma 2.1 to obtain

$$\left|\frac{(b-n)_\ell}{(d-n)_\ell}\right| = \left|\frac{(1-b)_n(b)_{\ell-n}}{(1-d)_n(d)_{\ell-n}}\right| \leqslant Cn^{\mathrm{Re}(d-b)}(\ell-n)^{\mathrm{Re}(b-d)}.$$

Thus

$$|S_2^*| \leqslant Cn^{\mathrm{Re}(d-b)} \sum_{\ell=n+1}^{\infty} \ell^{\mathrm{Re}(a-c)}(\ell-n)^{\mathrm{Re}(b-d)}\frac{z^\ell}{\ell!}.$$

Since $1 \leqslant \ell - n < \ell$, we have $(\ell-n)^{\mathrm{Re}(b-d)} \leqslant \ell^{\max\{0,\mathrm{Re}(b-d)\}}$. Therefore,

$$|S_2^*| \leqslant Cn^{\mathrm{Re}(d-b)} \sum_{\ell=n+1}^{\infty} \ell^{\mathrm{Re}(a-c)+\max\{0,\mathrm{Re}(b-d)\}}\frac{z^\ell}{\ell!}. \tag{3.12}$$

Using Lemma 2.2, the inequality (3.9) follows from (3.10)–(3.12). ■

A direct application of Lemma 3.3 gives the following theorem.

Theorem 3.4. *Let $\{v_n\}$ be a sequence satisfying $v_0 = 1$ and $v_n = \mathcal{O}(\mathrm{e}^{rn})$, where $r > 1$. Define*

$$V(z) := \sum_{n=0}^{\infty} v_n f_n^{\varepsilon}(z) z^{-n}, \qquad V^*(z) := \sum_{n=0}^{\infty} v_n g_n^{\varepsilon}(z) z^{-n}, \qquad z \neq 0,$$

where $f_n^{\varepsilon}(z)$ and $g_n^{\varepsilon}(z)$ are given by (3.8), $c \notin \mathbb{Z}_{\leqslant 0}$ and $d \notin \mathbb{Z}$. Then

$$V(z) - V^*(z) = \mathcal{O}\big(z^{p-1}\mathrm{e}^z\big), \qquad z \to +\infty,$$

where $p = \mathrm{Re}(a-c) + \max\{0, \mathrm{Re}(b-d)\}$.

Remark 3.5. By substituting the asymptotic expansion of ${}_2F_2$ (see [23, p. 380])

$${}_2F_2\left[\begin{matrix} a, b+\nu \\ c, d+\nu \end{matrix}; z\right] \sim \frac{\Gamma(c)\Gamma(d+\nu)}{\Gamma(a)\Gamma(b+\nu)} z^{a+b-c-d}\mathrm{e}^z, \qquad z \to \infty, \quad |\arg z| < \pi/2 \tag{3.13}$$

into the series

$$A(z) := \sum_{\nu=0}^{\infty} \frac{z^{-\nu}\Gamma(\alpha+\gamma+\nu)}{\nu!\Gamma(\beta+1+\nu)} {}_2F_2\left[\begin{matrix} \alpha+1, \beta+\nu \\ \delta, \beta+1+\nu \end{matrix}; -\frac{1}{z}\right],$$

Juršėnas [16, Section 4.2] claimed that

$$A(z) \sim \frac{\Gamma(\alpha+\gamma)\Gamma(\delta)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha-\gamma)} \mathrm{e}^{-1/z}(-z)^{\gamma+\delta}, \qquad |z| \to 0, \quad \mathrm{Re}(z) < 0. \tag{3.14}$$

The reminder term in the expansion (3.13) is given in [18], but it is not valid uniformly for large ν and large z. Thus, Juršėnas' expansion is not rigorous. Furthermore, Lemma 3.3 fails for the asymptotics of $A(z)$, so it is of interest to give a rigorous proof of (3.14) in our future work.

We now state and prove the main result.

Theorem 3.6. *Assume that*

$$c, c' \notin \mathbb{Z}_{\leqslant 0}, \qquad a-b, a-c \notin \mathbb{Z}, \qquad \mathrm{Re}(c-b) > 0.$$

Then under the condition (3.1),

$$\Psi_1[a, b; c, c'; x, y] \sim \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(c-b)} \left(\frac{y}{1-x}\right)^b y^{a-2b-c'}\mathrm{e}^y. \tag{3.15}$$

Proof. Recall (3.5) and note that $V_1(x,y)$ converges absolutely. Thus, the main contribution of Ψ_1 comes from $V_2(x,y)$. Using (3.13) and Theorem 3.4, we can obtain

$$\begin{aligned} V_2(x,y) &= {}_1F_1\left[\begin{matrix} a-b \\ c' \end{matrix}; y\right] {}_1F_0\left[\begin{matrix} b \\ - \end{matrix}; \frac{1}{1-x}\right] + \mathcal{O}\big(y^{p-1}\mathrm{e}^y\big) \\ &\sim \frac{\Gamma(c')}{\Gamma(a-b)} y^{a-b-c'}\mathrm{e}^y + \mathcal{O}\big(y^{p-1}\mathrm{e}^y\big), \end{aligned} \tag{3.16}$$

where $p = \mathrm{Re}(a-c-c'+1) + \max\{0, \mathrm{Re}(c-b-1)\}$. Since

$$\mathrm{Re}(a-b-c') > p-1 \ \Leftrightarrow\ \mathrm{Re}(c-b) > 0,$$

the result follows from (3.5) and (3.16). ■

Numerical verification of Theorem 3.6 is given in Appendix A.

4 Asymptotics of the generalized hypergeometric function

Our derivation in Section 3 depends on a rough estimate of ${}_2F_2$ for large $-n$. In this section, Nagel's approach is also used to explicitly establish the asymptotic behavior of ${}_2F_2$ for large parameters. We also present the asymptotic behavior of ${}_pF_q$ for large $-n$.

4.1 Asymptotics of ${}_2F_2$

Let us examine the complete asymptotic expansions of ${}_2F_2$ for large parameters.

Theorem 4.1. *Let $\delta \in (0,\pi)$, $c \notin \mathbb{Z}_{\leqslant 0}$ and $z \neq 0$. Then for any positive integer N,*

$${}_2F_2\left[\begin{matrix} a, b+\lambda \\ c, d+\lambda \end{matrix}; z\right] = \sum_{k=0}^{N-1} \frac{(a)_k (d-b)_k}{(c)_k (d+\lambda)_k} \frac{(-z)^k}{k!} {}_1F_1\left[\begin{matrix} a+k \\ c+k \end{matrix}; z\right] + \mathcal{O}\big(\lambda^{-N}\big), \tag{4.1}$$

where $\lambda \to \infty$ in the sector $|\arg(\lambda+d)| \leqslant \pi-\delta$.

Proof. If denoting

$$v(z) := {}_1F_1\left[\begin{matrix} a \\ c \end{matrix}; z\right], \qquad d_k := \frac{(a)_k}{(c)_k k!},$$

we can obtain

$${}_2F_2\left[\begin{matrix} a, b+\lambda \\ c, d+\lambda \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(\lambda+b)_k}{(\lambda+d)_k} d_k z^k.$$

Note that the k-th derivative of $v(z)$ is given by [22, p. 405, equation (16.3.1)]

$$v^{(k)}(z) = \frac{(a)_k}{(c)_k} {}_1F_1\left[\begin{matrix} a+k \\ c+k \end{matrix}; z\right].$$

Applying Fields' result [12, Theorem 3] yields

$${}_2F_2\left[\begin{matrix} a, b+\lambda \\ c, d+\lambda \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(d-b)_k}{(d+\lambda)_k} \frac{(-z)^k}{k!} v^{(k)}(z) = \sum_{k=0}^{\infty} \frac{(a)_k (d-b)_k}{(c)_k (d+\lambda)_k} \frac{(-z)^k}{k!} {}_1F_1\left[\begin{matrix} a+k \\ c+k \end{matrix}; z\right] \tag{4.2}$$

with $\lambda + d \notin \mathbb{Z}_{\leqslant 0}$, and also gives the asymptotic expansion (4.1). ■

Remark 4.2.

(1) The series (4.2) can be reformulated as follows:

$$ {}_2F_2\left[\begin{matrix}a,b\\c,d\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(a)_k(d-b)_k}{(c)_k(d)_k}\frac{(-z)^k}{k!}{}_1F_1\left[\begin{matrix}a+k\\c+k\end{matrix};z\right], \tag{4.3} $$

which is a specialization of Luke's result [19, Section 9.1, equation (34)]

$$ \begin{aligned} &{}_{p+1}F_{q+1}\left[\begin{matrix}a_1,\dots,a_p,b\\c_1,\dots,c_q,d\end{matrix};z\right]\\ &\quad=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_q)_k}\frac{(d-b)_k}{(d)_k}\frac{(-z)^k}{k!}{}_pF_q\left[\begin{matrix}a_1+k,\dots,a_p+k\\c_1+k,\dots,c_q+k\end{matrix};z\right], \end{aligned} \tag{4.4} $$

where $p\leqslant q$ with $z\in\mathbb{C}$, or $p=q+1$ with $\mathrm{Re}(z)<\frac{1}{2}$.

(2) Juršėnas [16, Section 4.1, p. 69] mentioned an asymptotic formula

$$ {}_2F_2\left[\begin{matrix}\alpha+1,\beta+\nu\\\delta,\beta+1+\nu\end{matrix};-\frac{1}{z}\right]\sim{}_1F_1\left[\begin{matrix}\alpha+1\\\delta\end{matrix};-\frac{1}{z}\right],\qquad \nu\in\mathbb{Z}_{\geqslant0},\quad \nu\to\infty. \tag{4.5} $$

without proof, whereas our Theorem 4.1 gives a full asymptotic expansion of (4.5).

The asymptotics of ${}_2F_2$ for $\lambda=-n$ is given below.

Theorem 4.3. *Let $c\notin\mathbb{Z}_{\leqslant0}$, $d\notin\mathbb{Z}$, $b-d\notin\mathbb{Z}_{\geqslant0}$ and $z\neq0$. Then for any positive integer N,*

$$ {}_2F_2\left[\begin{matrix}a,b-n\\c,d-n\end{matrix};z\right]=\sum_{k=0}^{N-1}\frac{(a)_k(d-b)_k}{(c)_k(d-n)_k}\frac{(-z)^k}{k!}{}_1F_1\left[\begin{matrix}a+k\\c+k\end{matrix};z\right]+\mathcal{O}\big(n^{-N}\big), \tag{4.6} $$

where $n\to+\infty$ through integer values.

Proof. We shall follow Nagel's approach. For nonnegative integers n and ℓ, write

$$ F(z):={}_2F_2\left[\begin{matrix}a,b-n\\c,d-n\end{matrix};z\right],\qquad a_\ell(n):=\frac{(-1)^\ell}{(d-n)_\ell},\qquad g_\ell:=\frac{(a)_\ell(d-b)_\ell}{(c)_\ell\ell!}. $$

Clearly, $|g_\ell|\leqslant C\ell^{\mathrm{Re}(\alpha)}$, where $\alpha:=a+d-b-c-1$. Moreover, (4.3) suggests that

$$ R(z):=F(z)-G_0(z)=\sum_{\ell=1}^{\infty}a_\ell(n)g_\ell G_\ell(z)z^\ell, \tag{4.7} $$

where $G_\ell(z)$ is given by (2.1).

Take n large, let $m=\lceil\log n\rceil$ and divide the series (4.7) into five parts:

$$ R(z)=\sum_{1}^{N}+\sum_{N+1}^{m}+\sum_{m+1}^{n/2}+\sum_{n/2+1}^{n}+\sum_{n+1}^{\infty}=:S_0+S_1+S_2+S_3+S_4, \tag{4.8} $$

where $N\in\mathbb{Z}_{>0}$ is chosen so that $\mathrm{Re}(c+N+1)>0$. Therefore, the inequality (2.2) holds.

Let us derive estimates for the sums in (4.8).

Case 1. $1\leqslant\ell\leqslant N$. Now $a_\ell(n)\sim n^{-\ell}$. Thus

$$ S_0=\sum_{\ell=1}^{N-1}a_\ell(n)g_\ell G_\ell(z)z^\ell+\mathcal{O}\big(n^{-N}\big). $$

Case 2. $N+1 \leqslant \ell \leqslant m$. Now both n and $n-\ell$ are large. The use of the identity $(z)_\ell = (-1)^\ell (1-z-\ell)_\ell$ gives

$$a_\ell(n) = \frac{1}{(1-d+n-\ell)_\ell} = \frac{\Gamma(1-d+n-\ell)}{\Gamma(1-d+n)}. \tag{4.9}$$

By Stirling's formula, we get $a_\ell(n) \sim n^{-\ell}$ and thus

$$|S_1| \leqslant C\mathrm{e}^{\gamma|z|} \sum_{\ell=N+1}^{m} \ell^{\mathrm{Re}(\alpha)} \left(\frac{|z|}{n}\right)^\ell.$$

For n large, $\ell^{\mathrm{Re}(\alpha)}\left(\frac{|z|}{n}\right)^\ell$ decreases with respect to $\ell \geqslant N+1$ and then

$$|S_1| \leqslant C\mathrm{e}^{\gamma|z|} m(N+1)^{\mathrm{Re}(\alpha)} \left(\frac{|z|}{n}\right)^{N+1} = \mathcal{O}\left(\frac{\log n}{n^{N+1}}\right).$$

Case 3. $m+1 \leqslant \ell \leqslant \frac{n}{2}$. Stirling's formula shows that

$$a_\ell(n) = \frac{\Gamma(1-d+n-\ell)}{\Gamma(1-d+n)} \sim \left(1-\frac{\ell}{n}\right)^{-d} \frac{(n-\ell)!}{n!}.$$

It follows from $\frac{1}{2} \leqslant 1-\frac{\ell}{n} < 1$ that

$$|a_\ell(n)| \leqslant C\frac{(n-\ell)!}{n!} = \frac{C}{(n-\ell+1)\cdots(n-1)n} \leqslant C\left(\frac{2}{n}\right)^\ell.$$

As in Case 2, the monotonicity gives

$$|S_2| \leqslant C\mathrm{e}^{\gamma|z|} \sum_{\ell=m+1}^{n/2} \ell^{\mathrm{Re}(\alpha)} \left(\frac{2|z|}{n}\right)^\ell \leqslant C\mathrm{e}^{\gamma|z|} n m^{\mathrm{Re}(\alpha)} \left(\frac{2|z|}{n}\right)^{m+1} = \mathcal{O}\big(n^{-\frac{1}{2}\log n}\big).$$

Case 4. $\frac{n}{2}+1 \leqslant \ell \leqslant n$. Choose the least number $r \in \mathbb{Z}_{\geqslant 0}$ so that

$$|(1-d+n-\ell)+r| \geqslant 4|z| > 0.$$

Note that $\frac{1}{|z+n|}$ is bounded uniformly for $n \geqslant 1$. Using (4.9) gives

$$|a_\ell(n)| = \prod_{j=0}^{\ell-1} \big|(1-d+n-\ell)+j\big|^{-1} \leqslant C(4|z|)^{r-\ell}.$$

Thus

$$|S_3| \leqslant C\mathrm{e}^{\gamma|z|}(4|z|)^r \sum_{\ell=n/2+1}^{n} \ell^{\mathrm{Re}(\alpha)} 4^{-\ell} = \mathcal{O}\big(n^{\mathrm{Re}(\alpha)+1} 2^{-n}\big).$$

Case 5. $\ell \geqslant n+1$. Now we can obtain $|a_\ell(n)| \leqslant C\frac{n^{\mathrm{Re}(d)}(\ell-n)^{1-\mathrm{Re}(d)}}{n!(\ell-n)!}$ from

$$a_\ell(n) = \frac{(-1)^{\ell-n}}{(1-d)_n (d)_{\ell-n}} = \frac{(1)_n (1)_{\ell-n}}{(1-d)_n (d)_{\ell-n}} \frac{(-1)^{\ell-n}}{n!(\ell-n)!}.$$

It follows that

$$|S_4| \leqslant C\mathrm{e}^{\gamma|z|}\frac{n^{\mathrm{Re}(d)}}{n!}\sum_{\ell=n+1}^{\infty}\ell^{\mathrm{Re}(\alpha)}(\ell-n)^{1-\mathrm{Re}(d)}\frac{|z|^{\ell}}{(\ell-n)!}$$
$$= C\mathrm{e}^{\gamma|z|}n^{\mathrm{Re}(d)}\frac{|z|^n}{n!}\sum_{j=1}^{\infty}(j+n)^{\mathrm{Re}(\alpha)}j^{1-\mathrm{Re}(d)}\frac{|z|^j}{j!}.$$

Note that $\max\{n,j\} \leqslant j+n \leqslant 2\cdot\max\{n,j\}$. Then for $p,q\in\mathbb{R}$, we have

$$\sum_{j=1}^{\infty}(j+n)^p j^q\frac{|z|^j}{j!} \leqslant Cn^p\sum_{j=1}^{n}j^q\frac{|z|^j}{j!} + C\sum_{j=n+1}^{\infty}j^{p+q}\frac{|z|^j}{j!}$$
$$\leqslant Cn^{\max\{0,p\}}\sum_{j=1}^{\infty}j^{\max\{q,p+q\}}\frac{|z|^j}{j!},$$

which shows that

$$|S_4| \leqslant C\mathrm{e}^{\gamma|z|}n^{\max\{\mathrm{Re}(d),\mathrm{Re}(\alpha+d)\}}\frac{|z|^n}{n!}\sum_{j=1}^{\infty}j^{1-\mathrm{Re}(d)+\max\{0,\mathrm{Re}(\alpha)\}}\frac{|z|^j}{j!}.$$

Now the asymptotic expansion (4.6) follows from the estimates above. ■

Remark 4.4. When $b-d\in\mathbb{Z}_{>0}$, the series (4.3) terminates. Therefore, when $c\notin\mathbb{Z}_{\leqslant 0}$, $d\notin\mathbb{Z}$ and $b-d\in\mathbb{Z}_{>0}$, take $N=b-d+1$ in Theorem 4.3, and as a result, the asymptotic expansion of ${}_2F_2[a,b-n;c,d-n;z]$ is given by (4.6) with the error term vanishing.

4.2 Asymptotics of ${}_pF_q$ $(p\leqslant q)$

We have established the asymptotics of ${}_2F_2$ for large parameters. More generally, by using Nagel's approach, we can further obtain the following result about the generalized hypergeometric functions ${}_pF_q$ defined in (1.3).

Theorem 4.5.

(1) *Let p, q, r and s be nonnegative integers satisfying $q\geqslant p+1$ and $s\geqslant r-1$. Define*

$$\mathcal{F}_n^{(1)}(z) := {}_{p+r}F_{q+s}\left[\begin{matrix}a_1-n,\dots,a_p-n,b_1,\dots,b_r\\ c_1-n,\dots,c_q-n,d_1,\dots,d_s\end{matrix};z\right]$$

and assume that $c_1,\dots,c_q\notin\mathbb{Z}$ and $d_1,\dots,d_s\notin\mathbb{Z}_{\leqslant 0}$. Then for any positive integer N,

$$\mathcal{F}_n^{(1)}(z) = \sum_{k=0}^{N-1}\frac{(a_1-n)_k\cdots(a_p-n)_k}{(c_1-n)_k\cdots(c_q-n)_k}\frac{(b_1)_k\cdots(b_r)_k}{(d_1)_k\cdots(d_s)_k}\frac{z^k}{k!} + \mathcal{O}\big(n^{(p-q)N}\big)$$

as $n\to+\infty$ through integer values.

(2) *Let $p\in\mathbb{Z}_{\geqslant 0}$. Define*

$$\mathcal{F}_n^{(2)}(z) := {}_{p+1}F_{p+1}\left[\begin{matrix}a_1,\dots,a_p,b-n\\ c_1,\dots,c_p,d-n\end{matrix};z\right]$$

and assume that $c_1,\dots,c_q\notin\mathbb{Z}_{\geqslant 0}$ and $b,d\notin\mathbb{Z}$. Then for any positive integer N,

$$\mathcal{F}_n^{(2)}(z) = \sum_{k=0}^{N-1}\frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_p)_k}\frac{(d-b)_k}{(d-n)_k}\frac{(-z)^k}{k!}{}_pF_p\left[\begin{matrix}a_1+k,\dots,a_p+k\\ c_1+k,\dots,c_p+k\end{matrix};z\right] + \mathcal{O}\big(n^{-N}\big)$$

as $n\to+\infty$ through integer values.

Proof. The proof is much akin to that of Theorem 4.3, so it is sufficient to give some key estimates. To get (1), as in the proof of Theorem 4.3, establish the estimate

$$\frac{(a_1-n)_\ell\cdots(a_p-n)_\ell}{(c_1-n)_\ell\cdots(c_p-n)_\ell}=\begin{cases}1+\mathcal{O}(n^{-1}), & 1\leqslant \ell\leqslant N,\\ 1+\mathcal{O}\left(\dfrac{\log n}{n}\right), & N<\ell\leqslant m,\\ \mathcal{O}(1), & m<\ell\leqslant \frac{n}{2},\\ \mathcal{O}(\omega^n), & \frac{n}{2}<\ell\leqslant n,\\ \mathcal{O}\big(n^{\Delta}(\ell-n)^{-\Delta}\big), & \ell>n\end{cases}$$

as $n\to+\infty$ through integer values, where $m=\lceil\log n\rceil$, $\Delta=\sum_{j=1}^{p}\operatorname{Re}(b_j-a_j)$ and $\omega>1$ is a constant independent of ℓ and n. To get (2), use (2.6) and (4.4). And the rest is the same. ∎

Remark 4.6.

(1) Knottnerus [17] derived the asymptotic expansions of

$${}_pF_q\left[\begin{matrix}a_1+r,\dots,a_p+r\\ b_1+r,\dots,b_q+r\end{matrix};z\right],\qquad r\to+\infty,\quad r\in\mathbb{Z}$$

with $p\leqslant q+1$ and

$${}_{p+1}F_p\left[\begin{matrix}a_1+r,\dots,a_{k-1}+r,a_k,\dots,a_{p+1}\\ b_1+r,\dots,b_k+r,b_{k+1},\dots,b_p\end{matrix};z\right],\qquad r\to+\infty,\quad r\in\mathbb{Z}$$

with $1\leqslant k\leqslant p$. These results are also quoted and presented in [19, Section 7.3], [21, Appendix 1] and [22, Section 16.11 (iii)]. Our Theorem 4.5 gives the full asymptotic expansion of ${}_pF_q$ $(p\leqslant q)$ for large $-n$, which does not appear in NIST handbook [22] and Luke's book [19].

(2) Nagel's approach cannot be applied to the asymptotics of

$${}_{p+1}F_p\left[\begin{matrix}a_1-n,\dots,a_{k-1}-n,a_k,\dots,a_{p+1}\\ b_1-n,\dots,b_k-n,b_{k+1},\dots,b_p\end{matrix};z\right],\qquad n\to+\infty,\quad n\in\mathbb{Z},$$

where $1\leqslant k\leqslant p$, since the condition $\operatorname{Re}(b_1-n)>\operatorname{Re}(a_1-n)>0$ is needed for the Euler-type integral representation of ${}_{p+1}F_p$. We are interested in finding a more effective method than Nagel's approach.

We end this section with the other results of ${}_2F_2$ for large parameters.

Theorem 4.7.

(1) *Assume that $c,d\notin\mathbb{Z}$. Then for any positive integer N,*

$${}_2F_2\left[\begin{matrix}a-n,b\\ c-n,d-n\end{matrix};z\right]=\sum_{k=0}^{N-1}\frac{(a-n)_k(b)_k}{(c-n)_k(d-n)_k}\frac{z^k}{k!}+\mathcal{O}\big(n^{-N}\big)$$

as $n\to+\infty$ through integer values.

(2) *Assume that $a,b,c,d\notin\mathbb{Z}$. Then for any positive integer N,*

$${}_2F_2\left[\begin{matrix}a-n,b-n\\ c-n,d-n\end{matrix};z\right]=\mathrm{e}^z\sum_{k=0}^{N-1}\frac{(a-n)_k(d-b)_k}{(c-n)_k(d-n)_k}\frac{(-z)^k}{k!}\,{}_1F_1\left[\begin{matrix}c-a\\ c-n+k\end{matrix};z\right]+\mathcal{O}\big(n^{-N}\big)$$

as $n\to+\infty$ through integer values.

Proof. Assertion (1) follows immediately from Theorem 4.5 (1). In order to prove Assertion (2), we only need to note that combining (4.3) and the Kummer transformation [22, equation (13.2.39)] yields

$$ {}_2F_2\left[\begin{matrix}a-n, b-n\\ c-n, d-n\end{matrix}; z\right] = \mathrm{e}^z \sum_{k=0}^{\infty} \frac{(a-n)_k (d-b)_k}{(c-n)_k (d-n)_k} \frac{(-z)^k}{k!} {}_1F_1\left[\begin{matrix}c-a\\ c-n+k\end{matrix}; z\right]. $$

In addition, it is easy to verify that

$$ \left| {}_1F_1\left[\begin{matrix}c-a\\ c-n+k\end{matrix}; z\right] \right| \leqslant K, $$

where K is independent of n and k. The rest of the proof is similar to that of Theorem 4.5 and is omitted here. ∎

A Numerical verification of Theorem 3.6

By using Mathematica 12.1, we provide a numerical verification of Theorem 3.6. The value of Ψ_1 is evaluated by using the following integral representation

$$ \Psi_1[a, b; c, c'; x, y] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} {}_1F_1\left[\begin{matrix}a\\ c'\end{matrix}; \frac{y}{1-xt}\right] \mathrm{d}t, $$

where $a \in \mathbb{C}$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $c' \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, $y \in \mathbb{C}$ and $x \in \mathbb{C} \setminus [1, +\infty)$. The value of the right-hand side of (3.15) is denoted by AE_{Ψ_1}. Tables 1 and 2 below clearly illustrate that the ratio $\frac{\Psi_1}{\mathrm{AE}_{\Psi_1}}$ approaches to 1 as $x \to -\infty$ ($y = \gamma(1-x) \to +\infty$).

Table 1. Numerical comparison when $c' = a = 3$, $b = \frac{3}{2}$, $c = \frac{5}{2}$ and $\gamma = 1$.

	x	$\frac{\Psi_1}{\mathrm{AE}_{\Psi_1}}$
1	-10	1.06951
2	-100	1.00745
3	-1000	1.00075
4	-2000	1.00037
5	-3000	1.00025

Table 2. Numerical comparison when $a = 3$, $b = \frac{3}{2}$, $c = \frac{5}{2}$, $c' = 2$ and $\gamma = \frac{1}{5}$.

	x	$\frac{\Psi_1}{\mathrm{AE}_{\Psi_1}}$
1	-10	0.98215
2	-100	1.00223
3	-1000	1.00025
4	-2000	1.00012
5	-3000	1.00008

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