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Proof of Two Multivariate q-Binomial Sums Arising in Gromov–Witten Theory

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Abstract. We prove two multivariate q-binomial identities conjectured by Bousseau, Brini and van Garrel [*Geom. Topol.* **28** (2024), 393–496, arXiv:2011.08830] which give generating series for Gromov–Witten invariants of two specific log Calabi–Yau surfaces. The key identity in all the proofs is Jackson's q-analogue of the Pfaff–Saalschütz summation formula from the theory of basic hypergeometric series.

Key words: Looijenga pairs; log Calabi–Yau surfaces; Gromov–Witten invariants; q-binomial coefficients; basic hypergeometric series; Pfaff–Saalschütz summation formula

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Dedicated to Stephen Milne,
the pioneer of multidimensional
basic hypergeometric series

The purpose of this note is to prove two conjectured multivariate q-binomial summation identities from [1]. There, Bousseau, Brini and van Garrel deal with the computation of Gromov–Witten invariants of log Calabi–Yau surfaces (Looijenga pairs). For the two non-tame (but quasi-tame) surfaces $\mathrm{dP}_1(0,4)$ and $\mathbb{F}_0(0,4)$, conjectured closed-form expressions are given in [1] for the corresponding generating series (I refer to [1] for background and notation), namely (cf. [1, Conjecture B.3, equation (B-2)])

$$\mathsf{N}^{\log}_{(d_0,d_1)}(\mathrm{dP}_1(0,4))(\hbar) = \frac{[2d_0]_q}{[d_0]_q}\begin{bmatrix} d_0 \\ d_1 \end{bmatrix}_q \begin{bmatrix} d_0+d_1-1 \\ d_1-1 \end{bmatrix}_q \tag{1}$$

and (cf. [1, Conjecture B.3, equation (B-3)])

$$\mathsf{N}^{\log}_{(d_1,d_2)}(\mathbb{F}_0(0,4))(\hbar) = \frac{[2d_1+d_2]_q}{[d_2]_q}\begin{bmatrix} d_1+d_2-1 \\ d_2-1 \end{bmatrix}_q^2, \tag{2}$$

where $q=\mathrm{e}^{i\hbar}$. Here, the q-integers $[\alpha]_q$ are defined symmetrically according to physics convention, $[\alpha]_q := q^{\alpha/2} - q^{-\alpha/2}$, and the (corresponding) q-binomial coefficients $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ are defined by[1]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \dfrac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q[k-1]_q\cdots[1]_q} & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

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[1]Since this definition is based on the q-integers according to physics convention, these "physics q-binomial coefficients" differ from the "combinatorics q-binomial coefficients" (cf. [2, Exercise 1.2]) by a multiplicative factor of $q^{-k(n-k)/2}$.

In relation to the above two conjectures, we prove the following two multivariate summation identities.

Theorem 1. *For integers d_0 and d_1 with $d_0 > d_1 \ge 1$, we have*

$$\sum_{m\ge1}\sum_{\substack{k_1+\dots+k_m=d_1-k_0,\\ n_1k_1+\dots+n_mk_m=d_0-d_1,\\ k_1,\dots,k_m>0,\,k_0\ge0,\\ n_1>\dots>n_m>0}} \begin{bmatrix}2d_0\\ k_1\end{bmatrix}_q \begin{bmatrix}2d_0-2(n_1-n_2)k_1\\ k_2\end{bmatrix}_q \cdots \\ \times \begin{bmatrix}2d_0-2\sum_{j=1}^{m-1}(n_j-n_m)k_j\\ k_m\end{bmatrix}_q \begin{bmatrix}2d_1\\ k_0\end{bmatrix}_q = \frac{[2d_0]_q}{[d_0]_q}\begin{bmatrix}d_0\\ d_1\end{bmatrix}_q \begin{bmatrix}d_0+d_1-1\\ d_0\end{bmatrix}_q. \tag{3}$$

Theorem 2. *For positive integers d_1 and arbitrary d_2, we have*

$$\sum_{m\ge1}\sum_{\substack{n_1k_1+\dots+n_mk_m=d_1,\\ k_1,\dots,k_m>0,\\ n_1>\dots>n_m>0}} \begin{bmatrix}d_2+2d_1\\ k_1\end{bmatrix}_q \cdots \begin{bmatrix}d_2+2d_1+2\sum_{j=1}^{i}(n_i-n_j)k_j\\ k_i\end{bmatrix}_q \cdots \\ \times \begin{bmatrix}d_2+2d_1+2\sum_{j=1}^{m}(n_m-n_j)k_j\\ k_m\end{bmatrix}_q \begin{bmatrix}d_2\\ \sum_{j=1}^{m}k_j\end{bmatrix}_q = \frac{[2d_1+d_2]_q}{[d_2]_q}\begin{bmatrix}d_1+d_2-1\\ d_1\end{bmatrix}_q^2. \tag{4}$$

Remarks.

(1) Both identities actually hold when q is considered as a formal variable. Furthermore, in the statement of Theorem 2, the phrase "arbitrary d_2" means that the identity holds when d_2 is considered as a formal variable.

(2) For any fixed d_0 and d_1, the sums over m in (3) and (4) are *finite* sums since all of the k_i's and the n_i's are at least 1, which implies an obvious bound on m.

By [1, Theorem B.1], Theorem 1 implies (1). Similarly, by [1, Theorem B.2], Theorem 2 implies (2).[2]

The identities (1) and (2) are actually special cases of a more general conjecture, namely [1, Conjecture 4.7], which predicts a closed-form formula for $\mathsf{N}^{\log}_{(d_0,d_1,d_2,d_3)}(\mathrm{dP}_3(0,2))(\hbar)$. It is conceivable that the ideas of this note, or similar ones, may lead to a proof of this more general conjecture. However, as is explained in [1, Section 4.2], in order to obtain an expression for $\mathsf{N}^{\log}_{(d_0,d_1,d_2,d_3)}(\mathrm{dP}_3(0,2))(\hbar)$ to start with, one would have to perform certain scattering diagram calculations. This is deemed "daunting" by the authors of [1] (see the paragraph below Conjecture 4.7), and they do not carry out these calculations.

As is the case frequently, the identities in Theorems 1 and 2 are difficult (impossible?) to prove directly since the parameters in these identities do not allow for enough flexibility, in particular if one has an inductive approach in mind (which we do). The key in proving (3) and (4) is to *generalise*, or, in this case, to *refine*. By experimenting with the sums in (3) and (4), I noticed that one can still get closed forms if we fix the sum of the k_i's, $i = 1, 2, \dots, m$. This leads us to the following key result.

Proposition 3. *Let k_0 and d_1 be integers with $1 \le k_0 \le d_1$. Furthermore, for arbitrary d_0 set*

$$f(d_0,d_1,k_0)=\sum_{m\ge1}\sum_{\substack{k_1+\dots+k_m=k_0,\\ n_1k_1+\dots+n_mk_m=d_1,\\ k_1,\dots,k_m>0,\\ n_1>\dots>n_m>0}} \prod_{i=1}^{m}\begin{bmatrix}2d_0-2\sum_{j=1}^{i-1}(n_j-n_i)k_j\\ k_i\end{bmatrix}_q. \tag{5}$$

[2] For the sake of consistency, in comparison to Theorem B.2 in [1], here we have reversed the indexing of the k_i's and the n_i's, that is, we have replaced k_i by k_{m-i+1} and n_i by n_{m-i+1}, $i = 1, 2, \dots, m$.

Then

$$f(d_0,d_1,k_0)=\frac{[2d_0]_q}{[k_0]_q}\begin{bmatrix}2d_0-d_1+k_0-1\\k_0-1\end{bmatrix}_q\begin{bmatrix}d_1-1\\k_0-1\end{bmatrix}_q.$$

Remarks.

(1) Again, the identity actually holds when q is considered as a formal variable.

(2) For the meaning of “arbitrary d_0”, see Remarks (1) below Theorem 2.

Before we can embark on the proof of the proposition, we need to introduce the standard notation for basic hypergeometric series,

$$ {}_{r+1}\phi_r\left[\begin{matrix}a_1,\dots,a_{r+1}\\b_1,\dots,b_r\end{matrix};q,z\right]=\sum_{\ell=0}^{\infty}\frac{(a_1;q)_\ell\cdots(a_{r+1};q)_\ell}{(q;q)_\ell(b_1;q)_\ell\cdots(b_r;q)_\ell}z^\ell, \tag{6}$$

where $(a;q)_0=1$ and $(a;q)_m=\prod_{k=0}^{m-1}(1-aq^k)$. The “bible” [2] of the theory of basic hypergeometric series contains many summation and transformation formulae for such series. The formula that we need here is Jackson’s q-analogue of the Pfaff–Saalschütz summation (see [2, equation (1.7.2); Appendix II.12])

$$ {}_3\phi_2\left[\begin{matrix}a,b,q^{-N}\\c,abq^{1-N}/c\end{matrix};q,q\right]=\frac{(c/a;q)_N\,(c/b;q)_N}{(c;q)_N\,(c/ab;q)_N}, \tag{7}$$

where N is a nonnegative integer.

Proof of Proposition 3. We prove the claim by induction on k_0.

First we consider the start of the induction, $k_0=1$. In this case, the summation on the right-hand side of (5) reduces to $m=1$, $k_1=1$, $n_1=d_1$, and hence

$$f(d_0,d_1,1)=\begin{bmatrix}2d_0\\1\end{bmatrix}_q,$$

in agreement with our assertion.

For the induction step, we rewrite the definition of $f(d_0,d_1,k_0)$ in (5) in the form

$$\begin{aligned}f(d_0,d_1,k_0)&=\chi(k_0\mid d_1)\begin{bmatrix}2d_0\\k_0\end{bmatrix}_q\\&\quad+\sum_{m\ge2}\sum_{k_m=1}^{k_0-1}\sum_{n_m=1}^{\lfloor(d_1-k_0+k_m)/k_0\rfloor}\sum_{\substack{k_1+\dots+k_{m-1}=k_0-k_m,\\ \bar n_1k_1+\dots+\bar n_{m-1}k_{m-1}=d_1-n_mk_0,\\ k_1,\dots,k_{m-1}>0,\\ \bar n_1>\dots>\bar n_{m-1}>0}}\begin{bmatrix}2d_0\\k_1\end{bmatrix}_q\begin{bmatrix}2d_0-2(\bar n_1-\bar n_2)k_1\\k_2\end{bmatrix}_q\cdots\\&\quad\times\begin{bmatrix}2d_0-2\sum_{j=1}^{m-2}(\bar n_j-\bar n_{m-1})k_j\\k_{m-1}\end{bmatrix}_q\begin{bmatrix}2d_0-2d_1+2n_mk_0\\k_m\end{bmatrix}_q,\end{aligned} \tag{8}$$

where $\bar n_i=n_i-n_m$, $i=1,2,\dots,m-1$, and where $\chi(\cdot)$ denotes the usual truth function, that is, $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise. There are two details in this expression which require further explanation. First of all, the first term on the right-hand side of (8) gives the contribution of the sum in (5) for $m=1$. Indeed, for $m=1$ the summation conditions in (5) require $k_1=k_0$ and $n_1k_1=d_1$. Consequently, a corresponding summand occurs only if $k_0=k_1\mid d_1$; if this divisibility condition is satisfied, the summand equals $\left[\begin{smallmatrix}2d_0\\k_0\end{smallmatrix}\right]_q$. Second, by the conditions imposed on $k_1,\dots,k_{m-1}$ and $\bar n_1,\dots,\bar n_{m-1}$, the sum $\bar n_1k_1+\dots+\bar n_{m-1}k_{m-1}$ must be at least as large as the sum $k_1+\dots+k_{m-1}$. This explains the upper bound on n_m in the inner sum.

Now, again with (5) in mind, the equation (8) may be written as

$$f(d_0,d_1,k_0)=\chi(k_0\mid d_1)\begin{bmatrix}2d_0\\k_0\end{bmatrix}_q+\sum_{k=1}^{k_0-1}\sum_{n=1}^{\lfloor (d_1-k_0+k)/k_0\rfloor} f(d_0,d_1-nk_0,k_0-k)\begin{bmatrix}2d_0-2d_1+2nk_0\\k\end{bmatrix}_q.$$

We may now use the induction hypothesis, and obtain

$$\begin{aligned}f(d_0,d_1,k_0)&=\chi(k_0\mid d_1)\begin{bmatrix}2d_0\\k_0\end{bmatrix}_q+\sum_{k=1}^{k_0-1}\sum_{n=1}^{\lfloor (d_1-k_0+k)/k_0\rfloor}\frac{[2d_0]_q}{[k_0-k]_q}\\&\quad\times\begin{bmatrix}2d_0-d_1+nk_0+k_0-k-1\\k_0-k-1\end{bmatrix}_q\begin{bmatrix}d_1-nk_0-1\\k_0-k-1\end{bmatrix}_q\begin{bmatrix}2d_0-2d_1+2nk_0\\k\end{bmatrix}_q\\&=\chi(k_0\mid d_1)\begin{bmatrix}2d_0\\k_0\end{bmatrix}_q+\sum_{n=1}^{\lfloor (d_1-1)/k_0\rfloor}\sum_{k=0}^{k_0-1}\frac{[2d_0]_q}{[k_0-k]_q}\\&\quad\times\begin{bmatrix}2d_0-d_1+nk_0+k_0-k-1\\k_0-k-1\end{bmatrix}_q\begin{bmatrix}d_1-nk_0-1\\k_0-k-1\end{bmatrix}_q\begin{bmatrix}2d_0-2d_1+2nk_0\\k\end{bmatrix}_q\\&\quad-\sum_{n=1}^{\lfloor (d_1-1)/k_0\rfloor}\frac{[2d_0]_q}{[k_0]_q}\begin{bmatrix}2d_0-d_1+nk_0+k_0-1\\k_0-1\end{bmatrix}_q\begin{bmatrix}d_1-nk_0-1\\k_0-1\end{bmatrix}_q.\end{aligned}\tag{9}$$

Now we write the sum over k in terms of the standard basic hypergeometric notation (6). Thus, this sum over k becomes

$$\frac{[2d_0]_q}{[k_0]_q}\begin{bmatrix}2d_0-d_1+nk_0+k_0-1\\k_0-1\end{bmatrix}_q\begin{bmatrix}d_1-nk_0-1\\k_0-1\end{bmatrix}_q\times{}_3\phi_2\left[\begin{matrix}q^{-2d_0+2d_1-2k_0n},q^{-k_0},q^{1-k_0}\\q^{1+d_1-k_0-k_0n},q^{1-2d_0+d_1-k_0-k_0n}\end{matrix};q,q\right].$$

The ${}_3\phi_2$-series can be evaluated by means of the q-Pfaff–Saalschütz summation (7). After simplification, we arrive at the expression

$$\frac{[2d_0]_q}{[k_0]_q}\begin{bmatrix}2d_0-d_1+k_0n-1\\k_0-1\end{bmatrix}_q\begin{bmatrix}d_1-k_0n+k_0-1\\k_0-1\end{bmatrix}_q.$$

If we substitute this in (9), then we get

$$\begin{aligned}f(d_0,d_1,k_0)&=\chi(k_0\mid d_1)\begin{bmatrix}2d_0\\k_0\end{bmatrix}_q\\&\quad+\sum_{n=1}^{\lfloor (d_1-1)/k_0\rfloor}\frac{[2d_0]_q}{[k_0]_q}\begin{bmatrix}2d_0-d_1+k_0n-1\\k_0-1\end{bmatrix}_q\begin{bmatrix}d_1-k_0n+k_0-1\\k_0-1\end{bmatrix}_q\\&\quad-\sum_{n=1}^{\lfloor (d_1-1)/k_0\rfloor}\frac{[2d_0]_q}{[k_0]_q}\begin{bmatrix}2d_0-d_1+nk_0+k_0-1\\k_0-1\end{bmatrix}_q\begin{bmatrix}d_1-nk_0-1\\k_0-1\end{bmatrix}_q.\end{aligned}$$

As is straightforward to verify, the first term in this expression can be integrated into the first sum, so that we obtain

$$f(d_0,d_1,k_0)=\sum_{n=1}^{\lfloor d_1/k_0\rfloor}\frac{[2d_0]_q}{[k_0]_q}\begin{bmatrix}2d_0-d_1+k_0n-1\\k_0-1\end{bmatrix}_q\begin{bmatrix}d_1-k_0n+k_0-1\\k_0-1\end{bmatrix}_q\\-\sum_{n=1}^{\lfloor (d_1-1)/k_0\rfloor}\frac{[2d_0]_q}{[k_0]_q}\begin{bmatrix}2d_0-d_1+nk_0+k_0-1\\k_0-1\end{bmatrix}_q\begin{bmatrix}d_1-nk_0-1\\k_0-1\end{bmatrix}_q. \tag{10}$$

On the right-hand side, these are, up to a shift of the index n and the slightly deviating lower and upper bounds on the summation index, essentially the same sums. After cancellation, only the term corresponding to $n=1$ of the first sum and the term corresponding to $n=\lfloor(d_1-1)/k_0\rfloor$ of the second sum survives, however the latter only if $k_0\nmid d_1$. (If $k_0\mid d_1$, then no term of the second sum survives the cancellation.) Since, for $n=\lfloor(d_1-1)/k_0\rfloor$, we have

$$0\le d_1-nk_0-1=d_1-\left\lfloor\frac{d_1-1}{k_0}\right\rfloor k_0-1<d_1-\left(\frac{d_1}{k_0}-1\right)k_0-1=k_0-1,$$

the q-binomial coefficient $\left[\begin{smallmatrix}d_1-nk_0-1\\k_0-1\end{smallmatrix}\right]_q$ vanishes for this choice of parameters. In other words, after cancellation of terms in (10), only the term corresponding to $n=1$ of the first sum survives and gives a non-zero contribution. These arguments yield

$$f(d_0,d_1,k_0)=\frac{[2d_0]_q}{[k_0]_q}\begin{bmatrix}2d_0-d_1+k_0-1\\k_0-1\end{bmatrix}_q\begin{bmatrix}d_1-1\\k_0-1\end{bmatrix}_q.$$

This completes the induction step and the proof of the proposition. ■

Now we are in the position to prove Theorems 1 and 2.

Proof of Theorem 1. The left-hand side of (3) can be rewritten as

$$\sum_{k_0=0}^{d_1}f(d_0,d_0-d_1,d_1-k_0)\begin{bmatrix}2d_1\\k_0\end{bmatrix}_q.$$

If we now use Proposition 3 for the evaluation of $f(d_0,d_0-d_1,d_1-k_0)$ and write the result in basic hypergeometric notation (6), the above sum becomes

$$\frac{[2d_0]_q}{[d_1]_q}\begin{bmatrix}d_0+2d_1-1\\d_1-1\end{bmatrix}_q\begin{bmatrix}d_0-d_1-1\\d_1-1\end{bmatrix}_q{}_3\phi_2\left[\begin{matrix}q^{-2d_1},q^{-d_1},q^{1-d_1}\\q^{1+d_0-2d_1},q^{1-d_0-2d_1}\end{matrix};q,q\right].$$

Also this ${}_3\phi_2$-series can be evaluated by means of the q-Pfaff–Saalschütz summation (7). After simplification, one obtains

$$\frac{[2d_0]_q}{[d_0]_q}\begin{bmatrix}d_0\\d_1\end{bmatrix}_q\begin{bmatrix}d_0+d_1-1\\d_0\end{bmatrix}_q,$$

as desired. ■

Proof of Theorem 2. With $k_0=\sum_{j=1}^m k_j$, the sum on the left-hand side of (4) can be rewritten as

$$\sum_{k_0\ge1}f\left(d_1+\frac{d_2}{2},d_1,k_0\right)\begin{bmatrix}d_2\\k_0\end{bmatrix}_q.$$

If we now use Proposition 3 for the evaluation of $f(d_1 + \frac{d_2}{2}, d_1, k_0)$ and write the result in basic hypergeometric notation (6), the above sum becomes

$$\frac{[2d_1+d_2]_q\,[d_2]_q}{[1]_q^2}\,{}_3\phi_2\left[\begin{matrix} q^{1+d_1+d_2}, q^{1-d_2}, q^{1-d_1} \\ q^2, q^2 \end{matrix}; q, q\right].$$

Again, the ${}_3\phi_2$-series can be evaluated by means of the q-Pfaff–Saalschütz summation (7). As a result, we obtain

$$\frac{[2d_1+d_2]_q}{[d_2]_q}\begin{bmatrix} d_1+d_2-1 \\ d_1 \end{bmatrix}_q^2,$$

as desired. ■

We close with the remark that it is highly surprising that in all three proofs the identity from the theory of basic hypergeometric series that is required is the q-Pfaff–Saalschütz summation (7). Usually, one needs the q-Chu–Vandermonde summation here, a transformation formula there, and maybe the q-Pfaff–Saalschütz summation somewhere. However, remarkably, here it is *exclusively* the q-Pfaff–Saalschütz summation.

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