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Solutions of Tetrahedron Equation from Quantum Cluster Algebra Associated with Symmetric Butterfly Quiver

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Abstract. We construct a new solution to the tetrahedron equation by further pursuing the quantum cluster algebra approach in our previous works. The key ingredients include a symmetric butterfly quiver attached to the wiring diagrams for the longest element of type A Weyl groups and the implementation of quantum Y-variables through the q-Weyl algebra. The solution consists of four products of quantum dilogarithms. By exploring both the coordinate and momentum representations, along with their modular double counterparts, our solution encompasses various known three-dimensional (3D) R-matrices. These include those obtained by Kapranov–Voevodsky (1994) utilizing the quantized coordinate ring, Bazhanov–Mangazeev–Sergeev (2010) from a quantum geometry perspective, Kuniba–Matsuike–Yoneyama (2023) linked with the quantized six-vertex model, and Inoue–Kuniba–Terashima (2023) associated with the Fock–Goncharov quiver. The 3D R-matrix presented in this paper offers a unified perspective on these existing solutions, coalescing them within the framework of quantum cluster algebra.

Key words: tetrahedron equation; quantum cluster algebra; q-Weyl algebra

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1 Introduction

The tetrahedron equation [24] is a generalization of the Yang–Baxter equation [1] to three-dimensional systems. A fundamental form of the equation in the so-called vertex formulation reads $R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}$, where R is a linear operator on $V^{\otimes 3}$ for some vector space V, and the indices specify the tensor components in $V^{\otimes 6}$ on which it acts non-trivially.

In this paper, we construct a new solution to the tetrahedron equation by an approach based on quantum cluster algebras [6, 7]. This method, initiated in [23] and further developed in [11, 12], commences with the Weyl group of type A and employing wiring diagrams to represent reduced expressions of the longest element in a standard format. One introduces a specific *quiver* and the corresponding quantum cluster algebra linked to these wiring diagrams. The pivotal element of the approach is the *cluster transformation* $\hat{R}$ serving as a counterpart of the cubic Coxeter relation. It acts on quantum Y-variables through a sequence of mutations and permutations. From the consideration about the embedding $A_2 \hookrightarrow A_3$, $\hat{R}$ is shown to satisfy the tetrahedron equation. Apart from the monomial part, $\hat{R}$ is described as an adjoint action of quantum dilogarithms. The next key step is to devise a realization of the quantum Y-variables in terms of a direct product of q-Weyl algebras which is an exponential version of the algebra of canonical coordinates u_i and momenta w_i with relations $\mathrm{e}^{\mathsf{u}_i}\mathrm{e}^{\mathsf{w}_j} = q^{\delta_{ij}}\mathrm{e}^{\mathsf{w}_j}\mathrm{e}^{\mathsf{u}_i}$, $\mathrm{e}^{\mathsf{u}_i}\mathrm{e}^{\mathsf{u}_j} = \mathrm{e}^{\mathsf{u}_j}\mathrm{e}^{\mathsf{u}_i}$ and $\mathrm{e}^{\mathsf{w}_i}\mathrm{e}^{\mathsf{w}_j} = \mathrm{e}^{\mathsf{w}_j}\mathrm{e}^{\mathsf{w}_i}$. It allows for the cluster transformation, including its monomial part, to be fully expressed in the adjoint form $\hat{R} = \mathrm{Ad}(R)$. It is this R which has many interesting features connected to existing solutions. The operator R can be endowed with several "spectral parameters" and satisfies the tetrahedron equation on its own including these parameters.

We execute the above program for the *symmetric butterfly* (SB) quiver, which is a symmetrized version of the butterfly quiver introduced in [23]. The vertices of an SB quiver are placed *both* on the vertices of the wiring diagram and within its domains. This contrasts with the Fock–Goncharov (FG) and the square quivers studied in [11] and [12, 23], respectively. In the former, vertices are assigned to the domains of the wiring diagrams, while in the latter, they are assigned to the edges.

Apart from $q = \mathrm{e}^{\hbar}$, our R-matrix $R = R_{123}$ involves parameters $C_1, \dots, C_8$ subject to $C_5 + C_6 = C_7 + C_8$. (See Remark 5.1.) Up to normalization, it is given by

$$
\begin{aligned}
R &= \Psi_q\big(\mathrm{e}^{2C_7+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)^{-1}\Psi_q\big(\mathrm{e}^{2C_5+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)^{-1} \\
&\quad\times P\Psi_q\big(\mathrm{e}^{2C_2+2C_3-2C_6+2C_8+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)\Psi_q\big(\mathrm{e}^{2C_2+2C_3+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big), \\
P &= \mathrm{e}^{\frac{1}{\hbar}(\mathsf{u}_3-\mathsf{u}_2)\mathsf{w}_1}\mathrm{e}^{\frac{1}{\hbar}\lambda_0(-\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3)}\mathrm{e}^{\frac{1}{\hbar}(\lambda_1\mathsf{u}_1+\lambda_2\mathsf{u}_2+\lambda_3\mathsf{u}_3)}\rho_{23},
\end{aligned}
\tag{1.1}
$$

where Ψ_q is the quantum dilogarithm (2.5), λ_i's are linear combinations of $C_1, \dots, C_8$ in (5.3), and ρ_{23} is the permutation $(\mathsf{u}_2, \mathsf{w}_2) \leftrightarrow (\mathsf{u}_3, \mathsf{w}_3)$.

The result (1.1) is universal within the current approach based on the SB quiver. In fact, one can project it onto various representations of the canonical variables. Our final result for the matrix elements $\langle \mathbf{n}|R|\mathbf{n}'\rangle$ (up to normalization) with bases labeled by $\mathbf{n} = (n_1, n_2, n_3)$ and $\mathbf{n}' = (n'_1, n'_2, n'_3) \in \mathbb{Z}^3$ reads

$$
\begin{aligned}
\langle \mathbf{n}|R|\mathbf{n}'\rangle &= \delta^{n_1+n_2}_{n'_1+n'_2}\delta^{n_2+n_3}_{n'_2+n'_3}\mathrm{e}^{\lambda_1 n'_1+\lambda_2 n'_3+\lambda_3 n'_2}\big(\mathrm{e}^{2C_5}q^{n_1+g_3}\big)^{n_3+g_3}q^{n'_2+g_2} \\
&\quad\times \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}z^{n'_2+g_2+1}}\frac{\big({-z}\mathrm{e}^{-2C_8}q^{2+n'_1+n'_3};q^2\big)_\infty\big({-z}\mathrm{e}^{-2C_7}q^{-n_1-n_3};q^2\big)_\infty}{\big({-z}\mathrm{e}^{-2C_6}q^{n'_1-n'_3};q^2\big)_\infty\big({-z}\mathrm{e}^{-2C_5}q^{n'_3-n'_1};q^2\big)_\infty}
\end{aligned}
\tag{1.2}
$$

in the coordinate representation of the q-Weyl algebras where u_i is diagonal (see Theorem 5.2), and

$$
\begin{aligned}
\langle \mathbf{n}|R|\mathbf{n}'\rangle &= q^{\psi_0}\big({-\mathrm{e}^{-2C_7}}\big)^{\frac{m_1}{2}}\big(\mathrm{e}^{2C_8-2C_3}\big)^{\frac{m_2}{2}}\big(\mathrm{e}^{-C_1-C_2-2C_3-C_4}\big)^{\frac{m_3}{2}}\big(\mathrm{e}^{C_1-C_2-2C_3-C_4}\big)^{\frac{m_4}{2}} \\
&\quad\times\frac{\big(\mathrm{e}^{2C_3};q^2\big)_{\frac{m_1}{2}}\big(\mathrm{e}^{-2C_2-2C_8};q^2\big)_{\frac{m_2}{2}}\big(\mathrm{e}^{2C_1-2C_3+2C_5};q^2\big)_{\frac{m_3}{2}}\big(\mathrm{e}^{-2C_1-2C_3+2C_6};q^2\big)_{\frac{m_4}{2}}}{\big(\mathrm{e}^{-4C_3+2C_5+2C_6};q^2\big)_{\frac{m_3+m_4}{2}}}
\end{aligned}
\tag{1.3}
$$

in the momentum representation where w_i is diagonal (see Theorem 5.5). Here

$$(g_1, g_2, g_3) = \frac{1}{\hbar}(C_7 - C_6, -C_4, C_7 - C_5),$$

m_i and ψ_0 are linear and quadratic forms of $\mathbf{n}$ and $\mathbf{n}'$ as given in (5.25) and (5.26).

Let us write (1.2) and (1.3) as $R^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$ and $S^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$, respectively. We have also evaluated the matrix elements in the modular double setting with the corresponding results $\mathcal{R}^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$ (see Theorem 6.1) and $\mathcal{S}^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$ (see Theorem 6.3). They are expressed in terms of the non-compact quantum dilogarithm (6.2). When the parameters are specialized appropriately, our R-matrices yield those obtained in [14] as the intertwiner of the quantized coordinate ring of SL_3 (see also [3, 17]), in [2] from a quantum geometry consideration, in [18] from a quantized six-vertex model, and in [11] from the quantum cluster algebra associated with the FG quiver. These results are summarized in Table 1.

	relevant quantum dilogarithm	coordinate rep.	momentum rep.	specialization adapted to the FG quiver
q-dilog R (5.4)	Ψ_q (2.5)	$R^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$, Theorem 5.2 [3, 14], Remark 5.4	$S^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$, Theorem 5.5 [18], Remark 5.6	[11], Theorem 8.2
modular $\mathcal{R}$ (6.9)	$\Phi_{\mathfrak{b}}$ (6.2)	$\mathcal{R}^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$, Theorem 6.1 [2], Remark 7.3	$\mathcal{S}^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$, Theorem 6.3 [18], Remark 6.4	[11], Proposition 8.4

Table 1. R-matrices in this paper. Relations to those in the literature and the relevant remarks or statements are given in the second line within each box.

In [12], the R-matrix in [22] was reproduced in a parallel story based on the *square* quiver. This solution also involves four quantum dilogarithms, but it differs from the one in this paper. In fact, even the special case of our solution mentioned in Remark 7.3 is related to [22] only through a highly non-trivial transformation called vertex-IRC (interaction round cube) duality [21]. Along with the current results obtained from the SB quiver, the quantum cluster algebra approach has successfully captured most of the significant solutions of the tetrahedron equation known to date for a generic q. Additionally, this approach has been extended to the 3D reflection equations [13, 17], as previously demonstrated with the FG quiver in [11]. In this paper we assume that q is generic throughout. We hope to explore the q root-of-unity case elsewhere.

The layout of the paper is as follows. In Section 2, we recall basic facts about quantum cluster algebras necessary in this paper. In Section 3, we introduce the SB quiver and study the cluster transformation $\hat{R}$. In Section 4, we realize the quantum Y-variables by q-Weyl algebras and extract R such that $\hat{R} = \mathrm{Ad}(R)$. The contents of Sections 3 and 4 are parallel with [12]. The matrix elements of R are calculated in Sections 5 and 6. In Section 7, we explain that the R-matrix in this paper satisfies the so-called $RLLL = LLLR$ relation for the L-operator which can be regarded as a quantized six-vertex model [2, 18]. It implies that the matrix elements obey linear recursion relations. In Section 8, we explain that the R-matrix for the FG quiver previously obtained in [11] arises as a special limit of the R-matrix in this paper. Appendix A is a supplement to Section 3.4. Appendix B provides another formula for R corresponding to a different choice of signs labeling the decomposition of mutations into monomial and automorphism parts. Appendix C contains integral formulas for non-compact quantum dilogarithm. Appendix D is a list of explicit forms of the $RLLL = LLLR$ relations.

2 Quantum cluster algebra

2.1 Mutation

Let us recall the definition of quantum cluster mutation following [5]. For a finite set I, set $B = (b_{ij})_{i,j\in I}$ with $b_{ij} = -b_{ji} \in \mathbb{Z}/2$. We call B the *exchange matrix*. In this article we will only encounter skew-symmetric exchange matrices with $b_{ij} \in \{\pm 1, \pm\frac{1}{2}, 0\}$. An exchange matrix will be depicted as a *quiver*. It is an oriented graph with vertices labeled with the elements of I and a solid arrow (resp. dotted arrow) from i to j when $b_{ij} = 1$ $\left(\text{resp. } b_{ij} = \frac{1}{2}\right)$.

Let $\mathcal{Y}(B)$ be a skew field generated by q-commuting variables $Y = (Y_i)_{i\in I}$ under the relations

$$Y_i Y_j = q^{2b_{ij}} Y_j Y_i. \tag{2.1}$$

The data (B, Y) will be called a quantum y-seed and Y_i a (quantum) Y-variable. We assume that the parameter q is generic throughout. For (B, Y) and for $k \in I$ such that $b_{ki} \neq \pm\frac{1}{2}$, the mutation μ_k transforms (B, Y) to $(B', Y') := \mu_k(B, Y)$, where

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise}, \end{cases} \tag{2.2}$$

$$Y'_i = \begin{cases} Y_k^{-1}, & i = k, \\ Y_i \displaystyle\prod_{j=1}^{|b_{ik}|} \left(1 + q^{2j-1} Y_k^{-\operatorname{sgn}(b_{ik})}\right)^{-\operatorname{sgn}(b_{ik})}, & i \neq k. \end{cases} \tag{2.3}$$

The mutations are involutive, $\mu_k\mu_k = \text{id.}$, and commutative, $\mu_k\mu_j = \mu_j\mu_k$ if $b_{jk} = b_{kj} = 0$. The mutation μ_k induces an isomorphism of skew fields $\mu_k^*\colon \mathcal{Y}(B') \to \mathcal{Y}(B)$, where $\mathcal{Y}(B')$ is a skew field generated by the variables $Y' = (Y'_i)_{i\in I}$ under the relations $Y'_i Y'_j = q^{2b'_{ij}} Y'_j Y'_i$.

The map μ_k^* is decomposed into two parts, a monomial part and an automorphism part [6], in two ways [16]. To explain it, let us introduce an isomorphism $\tau_{k,\varepsilon}$ of the skew fields for $\varepsilon \in \{+, -\}$ by

$$\tau_{k,\varepsilon}\colon\ \mathcal{Y}(B') \to \mathcal{Y}(B); \qquad Y'_i \mapsto \begin{cases} Y_k^{-1}, & i = k, \\ q^{-b_{ik}[\varepsilon b_{ik}]_+} Y_i Y_k^{[\varepsilon b_{ik}]_+}, & i \neq k, \end{cases} \tag{2.4}$$

where $[a]_+ := \max[0, a]$. The adjoint action $\mathrm{Ad}_{k,\varepsilon}$ on $\mathcal{Y}(B)$ is defined by $\mathrm{Ad}_{k,+} := \mathrm{Ad}(\Psi_q(Y_k))$, $\mathrm{Ad}_{k,-} := \mathrm{Ad}\left(\Psi_q\left(Y_k^{-1}\right)^{-1}\right)$, where $\mathrm{Ad}(Y)(X) = YXY^{-1}$. The symbol $\Psi_q(Y)$ appearing here denotes the quantum dilogarithm

$$\Psi_q(Y) = \frac{1}{\left(-qY; q^2\right)_\infty}, \qquad (z;q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n). \tag{2.5}$$

One has the expansions

$$\Psi_q(Y) = \sum_{n=0}^{\infty} \frac{(-qY)^n}{\left(q^2;q^2\right)_n}, \qquad \Psi_q(Y)^{-1} = \sum_{n=0}^{\infty} \frac{q^{n^2} Y^n}{\left(q^2;q^2\right)_n}, \tag{2.6}$$

where $\left(z;q^2\right)_n = \left(z;q^2\right)_\infty / \left(zq^{2n};q^2\right)_\infty$ for any n. Basic properties of the quantum dilogarithm are

$$\begin{aligned} &\Psi_q\left(q^2U\right)\Psi_q(U)^{-1} = 1 + qU, \\ &\Psi_q(U)\Psi_q(W) = \Psi_q(W)\Psi_q\left(q^{-1}UW\right)\Psi_q(U) \qquad \text{if} \quad UW = q^2WU, \end{aligned} \tag{2.7}$$

where the second one is called the pentagon identity.

Now the decomposition of μ_k^* in two ways mentioned in the above is given as

$$\mu_k^* = \mathrm{Ad}_{k,+} \circ \tau_{k,+} = \mathrm{Ad}_{k,-} \circ \tau_{k,-}. \tag{2.8}$$

Namely, one has the following diagram for both choices $\varepsilon = +, -$:

$$\begin{tikzcd}
\mathcal{Y}(B') \arrow[r, "\mu_k^*"] \arrow[dr, "\tau_{k,\varepsilon}"'] & \mathcal{Y}(B) \\
& \mathcal{Y}(B). \arrow[u, "\mathrm{Ad}_{k,\varepsilon}"']
\end{tikzcd}$$

Example 2.1. Let $I = \{1, 2\}$ and the 2-by-2 exchange matrix be given by $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, which implies $Y_1 Y_2 = q^2 Y_2 Y_1$. Consider the mutation $\mu_2(B, Y) = (B', Y')$, where $Y = (Y_1, Y_2)$ and $Y' = (Y_1', Y_2')$. Then $B' = -B$ from (2.2) and $Y_1' = Y_1\big(1 + qY_2^{-1}\big)^{-1}$ from (2.3). On the other hand, the same result is obtained also in the form $Y_1' \to Y_1\big(1 + qY_2^{-1}\big)^{-1}$ in two ways according to (2.8) as follows:

$$\begin{aligned}
Y_1' &\xrightarrow{\tau_{2,+}} q^{-1} Y_1 Y_2 \xrightarrow{\mathrm{Ad}_{2,+}} q^{-1} \Psi_q(Y_2) Y_1 Y_2 \Psi_q(Y_2)^{-1} \\
&\qquad = q^{-1} Y_1 \Psi_q\big(q^{-2} Y_2\big) \Psi_q(Y_2)^{-1} Y_2 = q^{-1} Y_1 \big(1 + q^{-1} Y_2\big)^{-1} Y_2, \\
Y_1' &\xrightarrow{\tau_{2,-}} Y_1 \xrightarrow{\mathrm{Ad}_{2,-}} \Psi_q\big(Y_2^{-1}\big)^{-1} Y_1 \Psi_q\big(Y_2^{-1}\big) = Y_1 \Psi_q\big(q^2 Y_2^{-1}\big)^{-1} \Psi_q\big(Y_2^{-1}\big) = Y_1\big(1 + qY_2^{-1}\big)^{-1}.
\end{aligned}$$

For later use, we introduce the quantum torus algebra $\mathcal{T}(B)$ associated to B. It is the $\mathbb{Q}(q)$-algebra generated by non-commutative variables Y^α $\big(\alpha \in \mathbb{Z}^I\big)$ satisfying the relations

$$q^{\langle \alpha, \beta \rangle} \mathsf{Y}^\alpha \mathsf{Y}^\beta = \mathsf{Y}^{\alpha+\beta}, \tag{2.9}$$

where $\langle\ ,\ \rangle$ is a skew-symmetric form defined by $\langle \alpha, \beta \rangle = -\langle \beta, \alpha \rangle = -\alpha \cdot B\beta$. Let e_i be the standard unit vector of $\mathbb{Z}^I$. We write Y^{e_i} simply as Y_i. Then $\mathsf{Y}_i \mathsf{Y}_j = q^{2b_{ij}} \mathsf{Y}_j \mathsf{Y}_i$ holds. We identify Y_i with Y_i, which is consistent with (2.1).

Let $\mathcal{FT}(B)$ be the fractional field of $\mathcal{T}(B)$. The mutations μ_k^* and their decompositions induce the morphisms for the fractional fields of the quantum torus algebras naturally. In particular, the monomial part (2.4) of μ_k^* is written as

$$\tau_{k,\varepsilon}\colon\ \mathcal{FT}\big(B'\big) \to \mathcal{FT}(B); \qquad \mathsf{Y}_i' \mapsto \begin{cases} \mathsf{Y}_k^{-1}, & i = k, \\ \mathsf{Y}^{e_i + e_k [\varepsilon b_{ik}]_+}, & i \neq k \end{cases} \tag{2.10}$$

under the identification $\mathsf{Y}_i = Y_i$, $\mathsf{Y}_i' = Y_i'$. Hence $\mathcal{FT}(B)$ (resp. $\mathcal{FT}\big(B'\big)$) is identified with $\mathcal{Y}(B)$ (resp. $\mathcal{Y}\big(B'\big)$).

2.2 Tropical y-variables and tropical sign

Let $\mathbb{P}(u) = \mathbb{P}_{\mathrm{trop}}(u_1, u_2, \ldots, u_p) := \big\{\prod_{i=1}^p u_i^{a_i};\ a_i \in \mathbb{Z}\big\}$ be the tropical semifield of rank p, endowed with the addition $\oplus$ and multiplication $\cdot$ defined by

$$\prod_{i=1}^p u_i^{a_i} \oplus \prod_{i=1}^p u_i^{b_i} = \prod_{i=1}^p u_i^{\min(a_i, b_i)}, \qquad \prod_{i=1}^p u_i^{a_i} \cdot \prod_{i=1}^p u_i^{b_i} = \prod_{i=1}^p u_i^{a_i + b_i}.$$

For $s = \prod_{i \in I} u_i^{a_i} \in \mathbb{P}(u)$, we write $s = u^\alpha$ with $\alpha = (a_i)_{i \in I} \in \mathbb{Z}^I$. We say that s is positive if $\alpha \in (\mathbb{Z}_{\geq 0})^I$ and negative if $\alpha \in (\mathbb{Z}_{\leq 0})^I$.

For a quiver Q whose vertex set is I, let $\mathbb{P}(u)$ be a tropical semifield of rank $|I|$. The data of the form (B, y) with B being the exchange matrix of Q and $y = (y_i)_{i\in I} \in \mathbb{P}(u)^I$ is called a tropical y-seed. For $k \in I$, the mutation[1] $\mu_k(B, y) =: \big(B', y'\big)$ is defined by (2.2) and

$$y'_i = \begin{cases} y_k^{-1}, & i = k, \\ y_i\big(1 \oplus y_k^{-\operatorname{sgn}(b_{ik})}\big)^{-b_{ik}}, & i \neq k. \end{cases} \tag{2.11}$$

For a tropical y-variable $y'_i = u^{\alpha'}$, the vector $\alpha' \in \mathbb{Z}^I$ is called the *c-vector* of y'_i. The following theorem states the *sign coherence* of the c-vectors.

Theorem 2.2 ([8, 10])**.** *Let $\big(B', y'\big) = \mu_{i_L} \cdots \mu_{i_2}\mu_{i_1}(B, u)$ be a tropical y-seed with $y' = (y'_i)_{i\in I}$. For any sequence $(i_1, \dots, i_L) \in I^L$, each $y'_i \in \mathbb{P}(u)$ is either positive or negative.*

Based on Theorem 2.2, for any tropical y-seed $\big(B', y'\big)$ with $y' = (y'_i)_{i\in I}$ obtained from (B, u) by applying mutations, we define the *tropical sign* ε'_i of y'_i to be $+1$ (resp. -1) if y'_i is positive (resp. y_i is negative). We also write $\varepsilon'_i = \pm$ for $\varepsilon'_i = \pm 1$ for simplicity.

Remark 2.3. For the mutation $\mu_k(B, y) = \big(B', y'\big)$ of a tropical y-seed, let c_i, c'_i, c_k be the c-vectors of y_i, y'_i, y_k, respectively, and let ε_k be the tropical sign of y_k. Then the tropical mutation (2.11) is expressed in terms of c-vectors as

$$c'_i = \begin{cases} -c_k, & i = k, \\ c_i + c_k[\varepsilon_k b_{ik}]_+, & i \neq k. \end{cases}$$

This coincides with the transformation of quantum torus (2.10) on $\mathbb{Z}^I$ (i.e., the power of (2.10)) when $\varepsilon = \varepsilon_k$.

2.3 Sequence of mutations

Let us describe the quantum Y-variables associated with the sequence of mutations $\mu_{i_l}\mu_{i_{l-1}} \cdots \mu_{i_2}\mu_{i_1}$:

$$\big(B^{(1)}, Y^{(1)}\big) \overset{\mu_{i_1}}{\longleftrightarrow} \big(B^{(2)}, Y^{(2)}\big) \overset{\mu_{i_2}}{\longleftrightarrow} \cdots \overset{\mu_{i_l}}{\longleftrightarrow} \big(B^{(l+1)}, Y^{(l+1)}\big). \tag{2.12}$$

For $t = 1, \dots, l+1$, let $\mathsf{Y}^{\alpha}(t)$ $\big(\alpha \in \mathbb{Z}^I\big)$ be the generators of the quantum torus $\mathcal{T}\big(B^{(t)}\big)$ in the sense explained around (2.9). We set $\mathsf{Y}_i(t) = \mathsf{Y}^{e_i}(t)$. Especially for $t = 1$, we use the simpler notations $\mathsf{Y}^{\alpha} = \mathsf{Y}^{\alpha}(1)$ and $\mathsf{Y}_i = \mathsf{Y}_i(1)$. As in (2.10), we identify Y_i with $Y_i = Y_i^{(1)}$, hence $\mathcal{Y}\big(B^{(1)}\big)$ with $\mathcal{FT}\big(B^{(1)}\big)$. Then the quantum Y-variables $Y^{(t+1)} = \big(Y_i^{(t+1)}\big)_{i\in I}$ $(t = 0, \dots, l)$ appearing in (2.12) are expressed as

$$\begin{aligned} Y_i^{(t+1)} &= \operatorname{Ad}\big(\Psi_q\big(\mathsf{Y}_{i_1}(1)^{\delta_1}\big)^{\delta_1}\big)\tau_{i_1,\delta_1} \cdots \operatorname{Ad}\big(\Psi_q\big(\mathsf{Y}_{i_t}(t)^{\delta_t}\big)^{\delta_t}\big)\tau_{i_t,\delta_t}(\mathsf{Y}_i(t+1)) \\ &= \operatorname{Ad}\big(\Psi_q\big(\mathsf{Y}^{\delta_1\beta_1}\big)^{\delta_1} \cdots \Psi_q\big(\mathsf{Y}^{\delta_t\beta_t}\big)^{\delta_t}\big)\tau_{i_1,\delta_1} \cdots \tau_{i_t,\delta_t}(\mathsf{Y}_i(t+1)). \end{aligned} \tag{2.13}$$

This formula is valid for any choice of the signs $\delta_1, \dots, \delta_l \in \{+, -\}$, on which the left-hand side is independent. Note that $Y_i^{(t+1)}$ is in general a "complicated" element in $\mathcal{Y}\big(B^{(1)}\big)$ generated from $\big(B^{(1)}, Y^{(1)}\big)$ by applying $\mu_{i_t} \cdots \mu_{i_2}\mu_{i_1}$ according to (2.3). On the other hand, $\mathsf{Y}_i(t+1)$ is just a basis of $\mathcal{T}\big(B^{(t+1)}\big)$. The first line of (2.13) says that $Y_i^{(t+1)}$ is also obtained as the image of $\mathsf{Y}_i(t+1)$ under the composition $\mu^*_{i_1} \cdots \mu^*_{i_{t-1}}\mu^*_{i_t}$ which is an isomorphism $\mathcal{FT}\big(B^{(t+1)}\big) \to \mathcal{FT}\big(B^{(1)}\big) = \mathcal{Y}\big(B^{(1)}\big)$. The second line is derived from the first line by pushing $\tau_{i,\delta}$'s to the right. Thus we have $\beta_1 = e_{i_1}$, and in general $\beta_r \in \mathbb{Z}^I$ is determined by $\mathsf{Y}^{\beta_r} = \tau_{i_1,\delta_1} \cdots \tau_{i_{r-1},\delta_{r-1}}(\mathsf{Y}_{i_r}(r))$.

[1]For simplicity, we use the same symbol μ_k to denote a mutation for quantum y-seeds (B, Y) and tropical y-seeds (B, y).

2.4 A useful theorem

Let $\sigma_{r,s} \in \mathfrak{S}_I$ $(r, s \in I)$ be a transposition. We let it act on either classical y-seeds (B, y) or quantum y-seeds (B, Y) as the exchange of the indices r and s. For quantum y-seeds, it is given by

$$((b_{ij})_{i,j\in I}, (Y_i)_{i\in I}) \mapsto ((b_{\sigma_{r,s}(i),\sigma_{r,s}(j)})_{i,j\in I}, (Y_{\sigma_{r,s}(i)})_{i\in I}), \tag{2.14}$$

where $\sigma_{r,s}(r) = s$, $\sigma_{r,s}(s) = r$ and $\sigma_{r,s}(i) = i$ for $i \neq r, s$. For classical y-seeds, the rule is similar.

Let

$$\nu = \nu_L \cdots \nu_1 := \sigma_{r_m,s_m} \cdots \mu_{i_l} \cdots \sigma_{r_1,s_1} \cdots \mu_{i_1}, \qquad L = l + m, \tag{2.15}$$

be a composition of l mutations $\mu_{i_1}, \ldots, \mu_{i_l}$ and m transpositions $\sigma_{r_1,s_1}, \ldots, \sigma_{r_m,s_m}$ in an arbitrary order. (So ν_L may actually be a mutation for example.) For simplicity, we also call ν a mutation sequence even though a part of it may involve transpositions.

Consider the tropical y-seeds starting from (B, y) and the quantum y-seeds starting from (B, Y) which are generated along the mutation sequences $\nu = \nu_L \cdots \nu_1$ and $\nu' = \nu'_{L'} \cdots \nu'_1$ as follows:

$$(B, y) =: \left(B^{(1)}, y^{(1)}\right) \overset{\nu_1}{\longleftrightarrow} \left(B^{(2)}, y^{(2)}\right) \overset{\nu_2}{\longleftrightarrow} \cdots \overset{\nu_L}{\longleftrightarrow} \left(B^{(L+1)}, y^{(L+1)}\right) = \nu(B, y), \tag{2.16}$$

$$(B, Y) =: \left(B^{(1)}, Y^{(1)}\right) \overset{\nu_1}{\longleftrightarrow} \left(B^{(2)}, Y^{(2)}\right) \overset{\nu_2}{\longleftrightarrow} \cdots \overset{\nu_L}{\longleftrightarrow} \left(B^{(L+1)}, Y^{(L+1)}\right) = \nu(B, Y), \tag{2.17}$$

$$(B, y) =: \left(B^{(1)\prime}, y^{(1)\prime}\right) \overset{\nu'_1}{\longleftrightarrow} \left(B^{(2)\prime}, y^{(2)\prime}\right) \overset{\nu'_2}{\longleftrightarrow} \cdots \overset{\nu'_{L'}}{\longleftrightarrow} \left(B^{(L+1)\prime}, y^{(L+1)\prime}\right) = \nu'(B, y), \tag{2.18}$$

$$(B, Y) =: \left(B^{(1)\prime}, Y^{(1)\prime}\right) \overset{\nu'_1}{\longleftrightarrow} \left(B^{(2)\prime}, Y^{(2)\prime}\right) \overset{\nu'_2}{\longleftrightarrow} \cdots \overset{\nu'_{L'}}{\longleftrightarrow} \left(B^{(L+1)\prime}, Y^{(L+1)\prime}\right) = \nu'(B, Y). \tag{2.19}$$

The following theorem is established by combining the synchronicity [20] among x-seeds, y-seeds and tropical y-seeds, and the synchronicity between classical and quantum seeds [7, Lemma 2.22], [15, Proposition 3.4].

Theorem 2.4. *In the situation in (2.16)–(2.19), the following two statements are equivalent:*

(1) *The tropical y-seeds satisfy $\nu(B, y) = \nu'(B, y)$.*

(2) *The quantum y-seeds satisfy $\nu(B, Y) = \nu'(B, Y)$.*

It is remarkable that (2) follows from (1) which is much simpler to check. We will utilize this fact efficiently in the subsequent arguments.

3 Cluster transformation $\widehat{R}$

3.1 Wiring diagram and symmetric butterfly quiver

Let us fix our convention of the wiring diagrams and associated square quivers using examples. See also [23, Section 3]. Let $W(A_n)$ be the Weyl group of A_n generated by the simple reflections $s_1, \ldots, s_n$ obeying the Coxeter relations $s_i^2 = 1$, $s_i s_j s_i = s_j s_i s_j$ $(|i - j| = 1)$ and $s_i s_j = s_j s_i$ $(|i - j| \geq 2)$. A reduced expression $s_{i_1} \cdots s_{i_l}$ of an element in $W(A_n)$ is identified with the (reduced) word $i_1 \ldots i_l \in [1, n]^l$. A wiring diagram is a collection of n wires which are horizontal except the vicinity of crossings. In the aforementioned context, i_k indicates that the k-th crossing from the left takes place at the i_k-th level, measured from the top. Crossings are required to occur at distinct horizontal positions, although this restriction can be relaxed due to the identification of topologically equivalent diagrams which are transformable by $s_i s_j = s_j s_i$ $(|i - j| \geq 2)$.

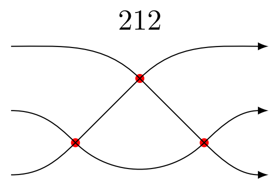


Figure 1. Wiring diagrams for the reduced words 212 and 121 of the longest element $s_2s_1s_2 = s_1s_2s_1$ of $W(A_2)$.

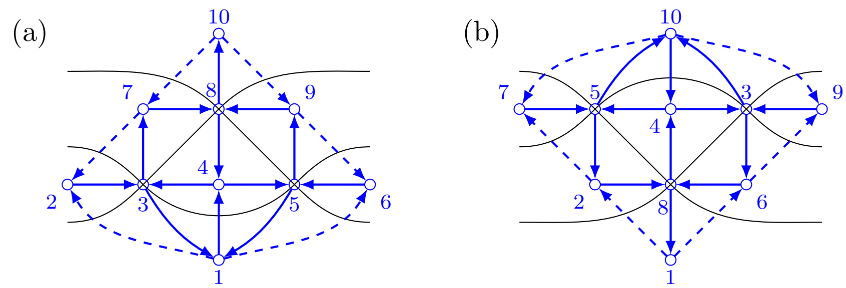


Figure 2. Symmetric butterfly quivers (depicted in blue) associated with the wiring diagrams. Given the labels $1, \dots, 10$ of the quiver vertices in (a), those in (b) are determined following the mutation sequence in Figure 3.

Given a wiring diagram, the associated symmetric butterfly quiver has the vertices in the domains and on the crossings of it. The vertices are interconnected by elementary triangles which are oriented with dotted arrows. A pair of dotted arrows pointing in the same (resp. opposite) direction are regarded as a solid arrow (resp. none). We choose the convention that quiver vertices on the crossings of the wiring diagram become sources vertically and sinks horizontally.

Remark 3.1. Let B be the exchange matrix corresponding to the symmetric butterfly quiver in Figure 2 (a). Then the skew filed $\mathcal{Y}(B)$ generated by $Y_1, \dots, Y_{10}$ has the center generated by

$$Y_1^{-1}Y_7Y_8Y_9Y_{10}^2, \qquad Y_2Y_4^{-1}Y_6Y_{10}, \qquad Y_3Y_4^2Y_6^{-2}Y_7^2Y_8, \qquad Y_5Y_6^2Y_8Y_9^2Y_{10}^2.$$

3.2 Cluster transformation $\widehat{R}$

Let $\big(B^{(1)}, Y^{(1)}\big) = (B, Y)$ and $\big(B^{(6)}, Y^{(6)}\big) = \big(B', Y'\big)$ be the quantum y-seeds corresponding to Figure 2 (a) and (b), respectively. We connect them by the following mutation sequence

$$\begin{aligned} &\big(B^{(1)}, Y^{(1)}\big) \overset{\mu_4}{\underset{\varepsilon_1}{\longleftrightarrow}} \big(B^{(2)}, Y^{(2)}\big) \overset{\mu_3}{\underset{\varepsilon_2}{\longleftrightarrow}} \big(B^{(3)}, Y^{(3)}\big) \\ &\qquad \overset{\mu_5}{\underset{\varepsilon_3}{\longleftrightarrow}} \big(B^{(4)}, Y^{(4)}\big) \overset{\mu_8}{\underset{\varepsilon_4}{\longleftrightarrow}} \big(B^{(5)}, Y^{(5)}\big) \overset{\sigma_{3,5}\sigma_{4,8}}{\longleftrightarrow} \big(B^{(6)}, Y^{(6)}\big), \end{aligned} \tag{3.1}$$

where $Y^{(t)} = \big(Y_1^{(t)}, \dots, Y_{10}^{(t)}\big)$. The symbol σ_{ij} denotes the exchange of the indices i and j in the exchange matrix and Y-variables. See (2.14). We have also attached the signs $\varepsilon_i = \pm 1$ along which the decomposition (2.8) into the automorphism part and the monomial part will be considered. See Figure 3.

For simplicity, we identify $Y_i^{(t)}$ and $\mathsf{Y}_i(t)$ in the description from now on. We introduce the cluster transformation $\widehat{R}\colon \mathcal{Y}\big(B'\big) \to \mathcal{Y}(B)$ corresponding to the mutation sequence (3.1) by applying (2.13) as

$$\begin{aligned} \widehat{R} = {}& \mathrm{Ad}\big(\Psi_q\big(\big(Y_4^{(1)}\big)^{\varepsilon_1}\big)^{\varepsilon_1}\big)\tau_{4,\varepsilon_1}\mathrm{Ad}\big(\Psi_q\big(\big(Y_3^{(2)}\big)^{\varepsilon_2}\big)^{\varepsilon_2}\big)\tau_{3,\varepsilon_2} \\ & \times \mathrm{Ad}\big(\Psi_q\big(\big(Y_5^{(3)}\big)^{\varepsilon_3}\big)^{\varepsilon_3}\big)\tau_{5,\varepsilon_3}\mathrm{Ad}\big(\Psi_q\big(\big(Y_8^{(4)}\big)^{\varepsilon_4}\big)^{\varepsilon_4}\big)\tau_{8,\varepsilon_4}\sigma_{3,5}\sigma_{4,8}. \end{aligned} \tag{3.2}$$

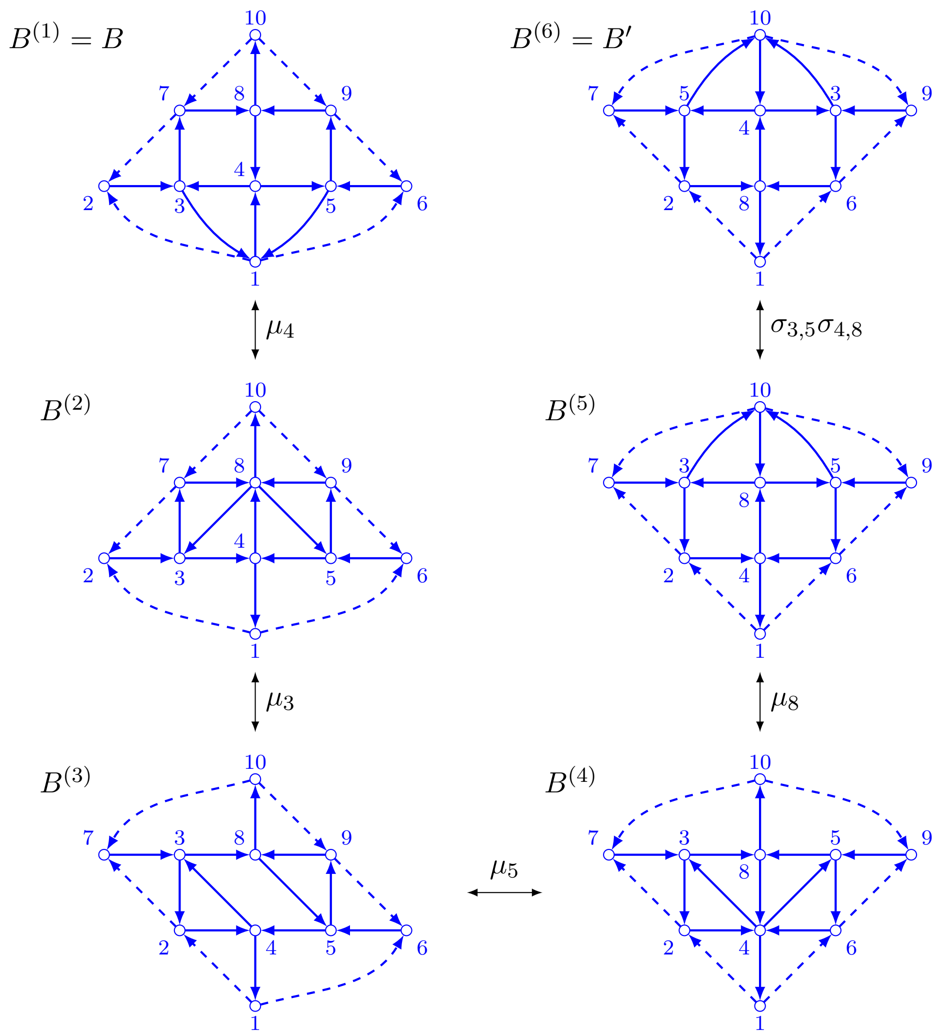


Figure 3. The quivers $B = B^{(1)}, \dots, B^{(6)} = B'$ and the mutations connecting them. We do not consider the wiring diagrams corresponding to the intermediate ones $B^{(2)}, \dots, B^{(5)}$.

The selection of $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{1, -1\}^4$ influences the expressions, but $\widehat{R}$ itself remains independent of it. We set

$$\tau_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} = \tau_{4,\varepsilon_1}\tau_{3,\varepsilon_2}\tau_{5,\varepsilon_3}\tau_{8,\varepsilon_4}\sigma_{3,5}\sigma_{4,8} \colon \ \mathcal{Y}(B') \to \mathcal{Y}(B), \tag{3.3}$$

and call it the monomial part of $\widehat{R}$.

Example 3.2. τ_{--++} and τ^{-1}_{--++} are given as follows:

$$\tau_{--++}\colon \ \begin{cases} Y_1' \mapsto Y_1, \qquad Y_2' \mapsto Y_2, \qquad Y_3' \mapsto Y_8, \qquad Y_4' \mapsto Y_4^{-1}Y_5^{-1}Y_8^{-1}, \\ Y_5' \mapsto Y_3^{-1}Y_5Y_8, \qquad Y_6' \mapsto Y_4Y_5Y_6, \qquad Y_7' \mapsto Y_3Y_4Y_7, \qquad Y_8' \mapsto Y_3, \\ Y_9' \mapsto Y_9, \qquad Y_{10}' \mapsto Y_{10}, \end{cases}$$

$$\tau^{-1}_{--++}\colon \ \begin{cases} Y_1 \mapsto Y_1', \qquad Y_2 \mapsto Y_2', \qquad Y_3 \mapsto Y_8', \qquad Y_4 \mapsto Y_4'^{-1}Y_5'^{-1}Y_8'^{-1}, \\ Y_5 \mapsto Y_3'^{-1}Y_5'Y_8', \qquad Y_6 \mapsto Y_3'Y_4'Y_6', \qquad Y_7 \mapsto Y_4'Y_5'Y_7', \qquad Y_8 \mapsto Y_3', \\ Y_9 \mapsto Y_9', \qquad Y_{10} \mapsto Y_{10}'. \end{cases}$$

By using them, $\widehat{R}$ in (3.2) for the choice $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-,-,+,+)$ is expressed as

$$\widehat{R} = \mathrm{Ad}\big(\Psi_q\big(\big(Y_4^{(1)}\big)^{-1}\big)^{-1}\big)\tau_{4,-}\mathrm{Ad}\big(\Psi_q\big(\big(Y_3^{(2)}\big)^{-1}\big)^{-1}\big)\tau_{3,-}$$

$$
\begin{aligned}
&\times \operatorname{Ad}\big(\Psi_q\big(Y_5^{(3)}\big)\big)\tau_{5,+}\operatorname{Ad}\big(\Psi_q\big(Y_8^{(4)}\big)\big)\tau_{8,+}\sigma_{3,5}\sigma_{4,8}\\
&= \operatorname{Ad}\big(\Psi_q\big(Y_4^{-1}\big)^{-1}\Psi_q\big(qY_3^{-1}Y_4^{-1}\big)^{-1}\Psi_q\big(q^{-1}Y_4Y_5\big)\Psi_q\big(Y_4Y_5Y_8\big)\big)\tau_{--++} \qquad (3.4)\\
&= \operatorname{Ad}\big(\Psi_q\big(Y_4^{-1}\big)^{-1}\Psi_q\big(qY_3^{-1}Y_4^{-1}\big)^{-1}\big)\tau_{--++}\operatorname{Ad}\big(\Psi_q\big(qY_3'^{-1}Y_4'^{-1}\big)\Psi_q\big(Y_4'^{-1}\big)\big). \qquad (3.5)
\end{aligned}
$$

The formula (3.5) is derived from (3.4) by moving τ_{--++} to the left by using τ^{-1}_{--++}.

Example 3.3. τ_{-+-+} and τ^{-1}_{-+-+} are given as follows:

$$
\tau_{-+-+}\colon\ \begin{cases}
Y_1' \mapsto Y_1, \quad Y_2' \mapsto Y_2Y_3Y_4, \quad Y_3' \mapsto Y_3Y_5^{-1}Y_8, \quad Y_4' \mapsto q^2Y_3^{-1}Y_4^{-1}Y_8^{-1},\\
Y_5' \mapsto Y_8, \quad Y_6' \mapsto Y_6, \quad Y_7' \mapsto Y_7, \quad Y_8' \mapsto Y_5,\\
Y_9' \mapsto q^{-2}Y_4Y_5Y_9, \quad Y_{10}' \mapsto Y_{10},
\end{cases}
$$

$$
\tau^{-1}_{-+-+}\colon\ \begin{cases}
Y_1 \mapsto Y_1', \quad Y_2 \mapsto Y_2'Y_4'Y_5', \quad Y_3 \mapsto Y_3'Y_5'^{-1}Y_8', \quad Y_4 \mapsto q^2Y_3'^{-1}Y_4'^{-1}Y_8'^{-1},\\
Y_5 \mapsto Y_8', \quad Y_6 \mapsto Y_6', \quad Y_7 \mapsto Y_7', \quad Y_8 \mapsto Y_5',\\
Y_9 \mapsto q^2Y_3'Y_4'Y_9', \quad Y_{10} \mapsto Y_{10}'.
\end{cases}
$$

By using them, $\widehat{R}$ in (3.2) for the choice $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) = (-,+,-,+)$ is expressed as

$$
\begin{aligned}
\widehat{R} &= \operatorname{Ad}\big(\Psi_q\big(\big(Y_4^{(1)}\big)^{-1}\big)^{-1}\big)\tau_{4,-}\operatorname{Ad}\big(\Psi_q\big(Y_3^{(2)}\big)\big)\tau_{3,+}\\
&\quad\times \operatorname{Ad}\big(\Psi_q\big(\big(Y_5^{(3)}\big)^{-1}\big)^{-1}\big)\tau_{5,-}\operatorname{Ad}\big(\Psi_q\big(Y_8^{(4)}\big)\big)\tau_{8,+}\sigma_{3,5}\sigma_{4,8}\\
&= \operatorname{Ad}\big(\Psi_q\big(Y_4^{-1}\big)^{-1}\Psi_q(qY_3Y_4)\Psi_q\big(qY_5^{-1}Y_4^{-1}\big)^{-1}\Psi_q\big(q^2Y_3Y_4Y_8\big)\big)\tau_{-+-+} \qquad (3.6)\\
&= \operatorname{Ad}\big(\Psi_q\big(Y_4^{-1}\big)^{-1}\Psi_q(qY_3Y_4)^{-1}\big)\tau_{-+-+}\operatorname{Ad}\big(\Psi_q(qY_3'Y_4')^{-1}\Psi_q\big(Y_4'^{-1}\big)\big). \qquad (3.7)
\end{aligned}
$$

Performing a straightforward calculation using any one of the formulas for $\widehat{R}$ in Examples 3.2 and 3.3, we get the following.

Proposition 3.4. *The cluster transformation $\widehat{R}\colon \mathcal{Y}(B') \to \mathcal{Y}(B)$ is given by*

$$
\begin{aligned}
&Y_1' \mapsto q\Lambda_4^{-1}Y_4Y_1, \quad Y_2' \mapsto qY_3\Lambda_4\Lambda_3^{-1}Y_2, \quad Y_3' \mapsto q^2\Lambda_0^{-1}Y_3Y_4Y_8,\\
&Y_4' \mapsto q\Lambda_4^{-1}Y_3^{-1}Y_4^{-1}Y_5^{-1}Y_8^{-1}\Lambda_3\Lambda_5, \quad Y_5' \mapsto \Lambda_0^{-1}Y_4Y_5Y_8, \quad Y_6' \mapsto qY_5\Lambda_4\Lambda_5^{-1}Y_6,\\
&Y_7' \mapsto Y_7\Lambda_3, \quad Y_8' \mapsto Y_4^{-1}\Lambda_0, \quad Y_9' \mapsto Y_9\Lambda_5, \quad Y_{10}' \mapsto Y_{10}\Lambda_5^{-1}\Lambda_3^{-1}\Lambda_0,
\end{aligned}
$$

where Λ_0, Λ_3, Λ_4 and Λ_5 are given as follows:

$$
\begin{aligned}
&\Lambda_0 = \Lambda_3\Lambda_5 + Y_4Y_3Y_5Y_8\Lambda_4, \quad \Lambda_3 = 1 + qY_3 + Y_4Y_3, \quad \Lambda_4 = 1 + qY_4,\\
&\Lambda_5 = 1 + qY_5 + Y_4Y_5.
\end{aligned}
$$

Remark 3.5. $\widehat{R}$ preserves the following combinations:

$$
\begin{aligned}
&\widehat{R}\big(Y_3'Y_8'\big) = Y_3Y_8, \quad \widehat{R}\big(Y_5'Y_8'\big) = Y_5Y_8,\\
&\widehat{R}\big(Y_1'Y_2'Y_4'Y_6'Y_7'Y_8'^{-1}Y_9'Y_{10}'\big) = Y_1Y_2Y_4Y_6Y_7Y_8^{-1}Y_9Y_{10}.
\end{aligned}
$$

3.3 $\widehat{R}$ satisfies the tetrahedron equation

In the situation in Figure 4, $\widehat{R}$ is a transformation of the 10 variables $\{Y_1',\dots,Y_{10}'\}$ into $\{Y_1,\dots,Y_{10}\}$ as in Proposition 3.4. We denote it simply by $\widehat{R}_{123}$, where the indices 1, 2, 3 are the vertices 1, 2, 3 of the wiring diagram (highlighted in red).

The following result is essentially due to [23].

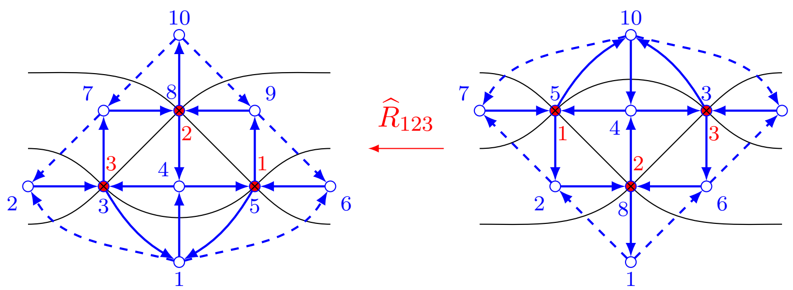


Figure 4. Cluster transformation $\widehat{R}_{123}$, which acts on the q-Weyl variables attached to the vertices 1, 2, 3 of the wiring diagram colored in red.

Proposition 3.6. *$\widehat{R}$ satisfies the tetrahedron equation twisted by permutations of Y-variables*

$$\widehat{R}_{124}\widehat{R}_{135}\widehat{R}_{236}\widehat{R}_{456}\sigma_{7,12} = \widehat{R}_{456}\widehat{R}_{236}\widehat{R}_{135}\widehat{R}_{124}\sigma_{7,14}. \tag{3.8}$$

Proof. For each reduced word for the longest element of the Weyl group $W(A_3)$, draw a wiring diagram and a symmetric butterfly quiver extending Figure 4 naturally. The quivers and the crossings of the wiring diagrams (red vertices $1,\dots,6$) are connected by the cluster transformations $\widehat{R}_{ijk}$ as in Figure 5. In Figure 5, let ν and ν' be the mutation sequences corresponding to the left path

$$\begin{aligned}&\big(B^{(1)},Y^{(1)}\big) \to \big(B^{(6)},Y^{(6)}\big) \to \big(B^{(11)},Y^{(11)}\big) \to \big(B^{(16)},Y^{(16)}\big) \to \big(B^{(22)},Y^{(22)}\big)\\ &\quad = \nu\big(B^{(1)},Y^{(1)}\big)\end{aligned}$$

and the right path

$$\begin{aligned}&\big(B^{(1)\prime},Y^{(1)\prime}\big) \to \big(B^{(6)\prime},Y^{(6)\prime}\big) \to \big(B^{(11)\prime},Y^{(11)\prime}\big) \to \big(B^{(16)\prime},Y^{(16)\prime}\big) \to \big(B^{(22)\prime},Y^{(22)\prime}\big)\\ &\quad = \nu'\big(B^{(1)},Y^{(1)}\big),\end{aligned}$$

respectively. Let $\nu\big(B^{(1)},y^{(1)}\big)$ and $\nu'\big(B^{(1)},y^{(1)}\big)$ be the tropical y-seeds generated by the same mutation sequences. It has been checked [23, Section A.2] that they satisfy the equality $\nu\big(B^{(1)},y^{(1)}\big) = \nu'\big(B^{(1)},y^{(1)}\big)$. Thus Theorem 2.4 enforces the equality of quantum y-seeds $\nu\big(B^{(1)},Y^{(1)}\big) = \nu'\big(B^{(1)\prime},Y^{(1)\prime}\big)$. In terms of cluster transformations, it means that the twisted tetrahedron equation $\widehat{R}_{124}\widehat{R}_{135}\widehat{R}_{236}\widehat{R}_{456}\sigma_{7,12} = \widehat{R}_{456}\widehat{R}_{236}\widehat{R}_{135}\widehat{R}_{124}\sigma_{7,14}$ is valid. ■

3.4 Monomial solutions to the tetrahedron equation

In this subsection, we provide additional details regarding Figure 5 and Proposition 3.6. Let $\big(B^{(1)},Y^{(1)}\big) = \big(B^{(1)\prime},Y^{(1)\prime}\big)$ be the initial quantum y-seed corresponding to the quiver at the bottom of Figure 5. The quantum y-seeds $\big(B^{(t)},Y^{(t)}\big)$ and $\big(B^{(t)\prime},Y^{(t)\prime}\big)$ $(t = 2,\dots,21)$, which pertain to the left and the right paths are determined from it by the mutation sequences, and we have just shown that the final results coincide, i.e., $\big(B^{(22)},Y^{(22)}\big) = \big(B^{(22)\prime},Y^{(22)\prime}\big)$. Set $Y^{(t)} = \big(Y^{(t)}_1,\dots,Y^{(t)}_{17}\big)$ and $Y^{(t)\prime} = \big(Y^{(t)\prime}_1,\dots,Y^{(t)\prime}_{17}\big)$.

The quantum y-seeds $\big(B^{(t)},Y^{(t)}\big)$ $(t = 2,\dots,21)$ are determined from the initial one $\big(B^{(1)}, Y^{(1)}\big)$ by

$$\begin{aligned}&\big(B^{(1)},Y^{(1)}\big) \underset{\varepsilon_1}{\overset{\mu_{14}}{\longleftrightarrow}} \big(B^{(2)},Y^{(2)}\big) \underset{\varepsilon_2}{\overset{\mu_{13}}{\longleftrightarrow}} \big(B^{(3)},Y^{(3)}\big) \underset{\varepsilon_3}{\overset{\mu_{15}}{\longleftrightarrow}} \big(B^{(4)},Y^{(4)}\big)\\ &\qquad \underset{\varepsilon_4}{\overset{\mu_{8}}{\longleftrightarrow}} \big(B^{(5)},Y^{(5)}\big) \overset{\sigma_{13,15}\sigma_{8,14}}{\longleftrightarrow} \big(B^{(6)},Y^{(6)}\big),\end{aligned}$$

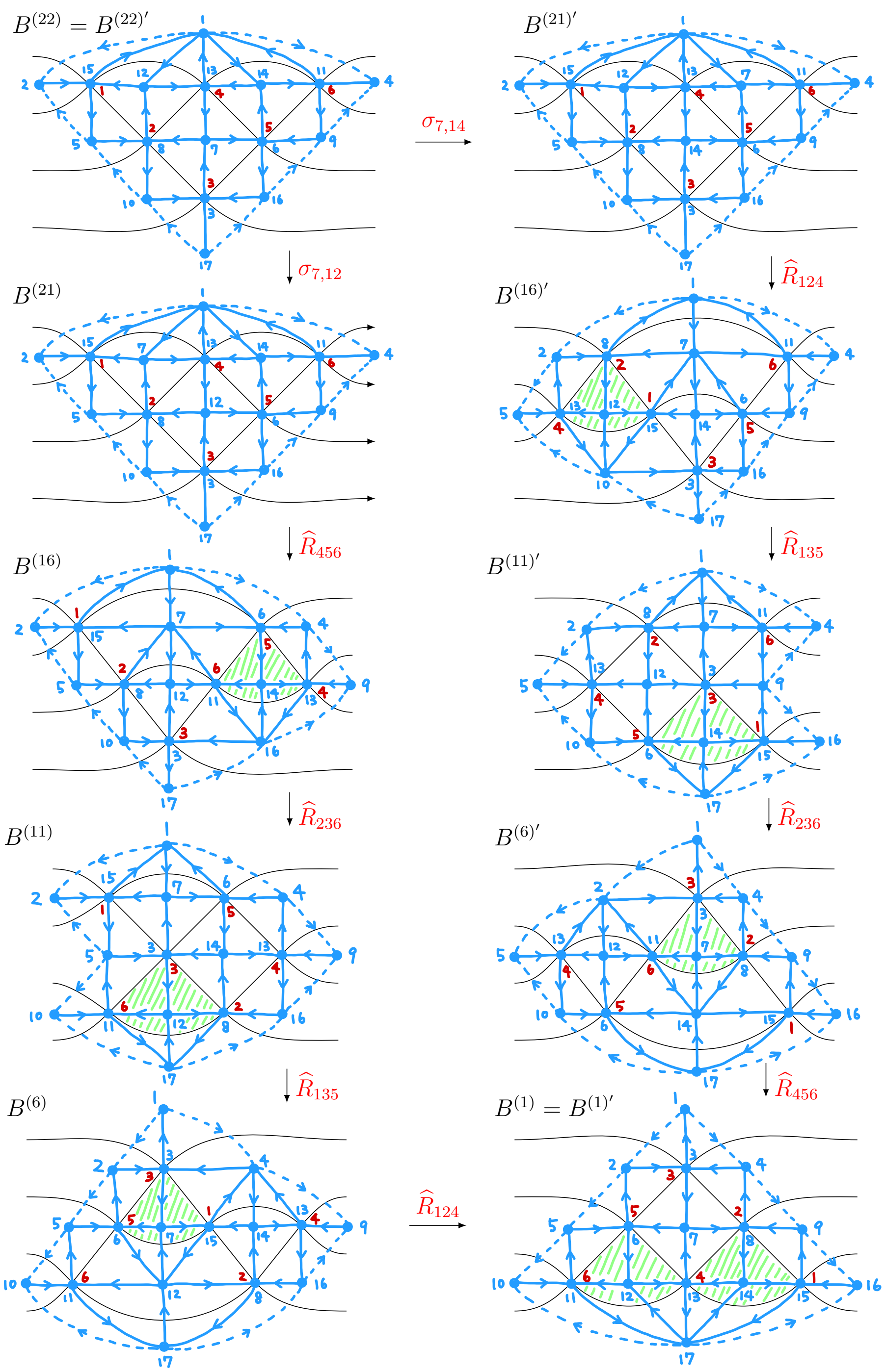


Figure 5. Cluster transformations $\widehat{R}_{ijk}$. The wiring diagrams have 6 crossings (red). The quivers (blue) have 17 vertices. Triangles relevant to the image of $\hat{R}_{ijk}$ are hatched in green. The seeds $\big(B^{(t)}, Y^{(t)}\big)$ and $\big(B^{(t)\prime}, Y^{(t)\prime}\big)$ will be explained in detail in Section 3.4.

$$
\begin{aligned}
&\big(B^{(6)}, Y^{(6)}\big) \underset{\varepsilon_1}{\overset{\mu_7}{\longleftrightarrow}} \big(B^{(7)}, Y^{(7)}\big) \underset{\varepsilon_2}{\overset{\mu_6}{\longleftrightarrow}} \big(B^{(8)}, Y^{(8)}\big) \underset{\varepsilon_3}{\overset{\mu_{15}}{\longleftrightarrow}} \big(B^{(9)}, Y^{(9)}\big) \\
&\qquad \underset{\varepsilon_4}{\overset{\mu_3}{\longleftrightarrow}} \big(B^{(10)}, Y^{(10)}\big) \overset{\sigma_{6,15}\sigma_{3,7}}{\longleftrightarrow} \big(B^{(11)}, Y^{(11)}\big), \\
&\big(B^{(11)}, Y^{(11)}\big) \underset{\varepsilon_1}{\overset{\mu_{12}}{\longleftrightarrow}} \big(B^{(12)}, Y^{(12)}\big) \underset{\varepsilon_2}{\overset{\mu_{11}}{\longleftrightarrow}} \big(B^{(13)}, Y^{(13)}\big) \underset{\varepsilon_3}{\overset{\mu_8}{\longleftrightarrow}} \big(B^{(14)}, Y^{(14)}\big) \\
&\qquad \underset{\varepsilon_4}{\overset{\mu_3}{\longleftrightarrow}} \big(B^{(15)}, Y^{(15)}\big) \overset{\sigma_{8,11}\sigma_{3,12}}{\longleftrightarrow} \big(B^{(16)}, Y^{(16)}\big),
\end{aligned}
$$

$$\begin{aligned}
&\big(B^{(16)},Y^{(16)}\big)\underset{\varepsilon_1}{\overset{\mu_{14}}{\longleftrightarrow}}\big(B^{(17)},Y^{(17)}\big)\underset{\varepsilon_2}{\overset{\mu_{11}}{\longleftrightarrow}}\big(B^{(18)},Y^{(18)}\big)\underset{\varepsilon_3}{\overset{\mu_{13}}{\longleftrightarrow}}\big(B^{(19)},Y^{(19)}\big)\\
&\qquad\underset{\varepsilon_4}{\overset{\mu_{6}}{\longleftrightarrow}}\big(B^{(20)},Y^{(20)}\big)\overset{\sigma_{11,13}\sigma_{6,14}}{\longleftrightarrow}\big(B^{(21)},Y^{(21)}\big),\\
&\big(B^{(21)},Y^{(21)}\big)\overset{\sigma_{7,12}}{\longleftrightarrow}\big(B^{(22)},Y^{(22)}\big),
\end{aligned}\tag{3.9}$$

where the notation is parallel with (3.1). In particular the choice of signs $\varepsilon_1,\dots,\varepsilon_4$ does not influence the mutations themselves. According to (3.2), each line in the above corresponds to a cluster transformation appearing in the left path of Figure 5 as follows:

$$\begin{aligned}
\widehat{R}_{124}&=\mathrm{Ad}\big(\Psi_q\big(\big(Y_{14}^{(1)}\big)^{\varepsilon_1}\big)^{\varepsilon_1}\big)\tau_{14,\varepsilon_1}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{13}^{(2)}\big)^{\varepsilon_2}\big)^{\varepsilon_2}\big)\tau_{13,\varepsilon_2}\\
&\quad\times\mathrm{Ad}\big(\Psi_q\big(\big(Y_{15}^{(3)}\big)^{\varepsilon_3}\big)^{\varepsilon_3}\big)\tau_{15,\varepsilon_3}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{8}^{(4)}\big)^{\varepsilon_4}\big)^{\varepsilon_4}\big)\tau_{8,\varepsilon_4}\sigma_{13,15}\sigma_{8,14},\\
\widehat{R}_{135}&=\mathrm{Ad}\big(\Psi_q\big(\big(Y_{7}^{(6)}\big)^{\varepsilon_1}\big)^{\varepsilon_1}\big)\tau_{7,\varepsilon_1}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{6}^{(7)}\big)^{\varepsilon_2}\big)^{\varepsilon_2}\big)\tau_{6,\varepsilon_2}\\
&\quad\times\mathrm{Ad}\big(\Psi_q\big(\big(Y_{15}^{(8)}\big)^{\varepsilon_3}\big)^{\varepsilon_3}\big)\tau_{15,\varepsilon_3}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{3}^{(9)}\big)^{\varepsilon_4}\big)^{\varepsilon_4}\big)\tau_{3,\varepsilon_4}\sigma_{6,15}\sigma_{3,7},\\
\widehat{R}_{236}&=\mathrm{Ad}\big(\Psi_q\big(\big(Y_{12}^{(11)}\big)^{\varepsilon_1}\big)^{\varepsilon_1}\big)\tau_{12,\varepsilon_1}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{11}^{(12)}\big)^{\varepsilon_2}\big)^{\varepsilon_2}\big)\tau_{11,\varepsilon_2}\\
&\quad\times\mathrm{Ad}\big(\Psi_q\big(\big(Y_{8}^{(13)}\big)^{\varepsilon_3}\big)^{\varepsilon_3}\big)\tau_{8,\varepsilon_3}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{3}^{(14)}\big)^{\varepsilon_4}\big)^{\varepsilon_4}\big)\tau_{3,\varepsilon_4}\sigma_{8,11}\sigma_{3,12},\\
\widehat{R}_{456}&=\mathrm{Ad}\big(\Psi_q\big(\big(Y_{14}^{(16)}\big)^{\varepsilon_1}\big)^{\varepsilon_1}\big)\tau_{14,\varepsilon_1}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{11}^{(17)}\big)^{\varepsilon_2}\big)^{\varepsilon_2}\big)\tau_{11,\varepsilon_2}\\
&\quad\times\mathrm{Ad}\big(\Psi_q\big(\big(Y_{13}^{(18)}\big)^{\varepsilon_3}\big)^{\varepsilon_3}\big)\tau_{13,\varepsilon_3}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{6}^{(19)}\big)^{\varepsilon_4}\big)^{\varepsilon_4}\big)\tau_{6,\varepsilon_4}\sigma_{11,13}\sigma_{6,14}.
\end{aligned}\tag{3.10}$$

The quantum y-seeds $\big(B^{(t)\prime},Y^{(t)\prime}\big)$ $(t=2,\dots,21)$ are determined from the initial one $\big(B^{(1)\prime},Y^{(1)\prime}\big)$ by

$$\begin{aligned}
&\big(B^{(1)\prime},Y^{(1)\prime}\big)\underset{\varepsilon_1}{\overset{\mu_{12}}{\longleftrightarrow}}\big(B^{(2)\prime},Y^{(2)\prime}\big)\underset{\varepsilon_2}{\overset{\mu_{11}}{\longleftrightarrow}}\big(B^{(3)\prime},Y^{(3)\prime}\big)\underset{\varepsilon_3}{\overset{\mu_{13}}{\longleftrightarrow}}\big(B^{(4)\prime},Y^{(4)\prime}\big)\\
&\qquad\underset{\varepsilon_4}{\overset{\mu_{6}}{\longleftrightarrow}}\big(B^{(5)\prime},Y^{(5)\prime}\big)\overset{\sigma_{11,13}\sigma_{6,12}}{\longleftrightarrow}\big(B^{(6)\prime},Y^{(6)\prime}\big),\\
&\big(B^{(6)\prime},Y^{(6)\prime}\big)\underset{\varepsilon_1}{\overset{\mu_{7}}{\longleftrightarrow}}\big(B^{(7)\prime},Y^{(7)\prime}\big)\underset{\varepsilon_2}{\overset{\mu_{11}}{\longleftrightarrow}}\big(B^{(8)\prime},Y^{(8)\prime}\big)\underset{\varepsilon_3}{\overset{\mu_{8}}{\longleftrightarrow}}\big(B^{(9)\prime},Y^{(9)\prime}\big)\\
&\qquad\underset{\varepsilon_4}{\overset{\mu_{3}}{\longleftrightarrow}}\big(B^{(10)\prime},Y^{(10)\prime}\big)\overset{\sigma_{8,11}\sigma_{3,7}}{\longleftrightarrow}\big(B^{(11)\prime},Y^{(11)\prime}\big),\\
&\big(B^{(11)\prime},Y^{(11)\prime}\big)\underset{\varepsilon_1}{\overset{\mu_{14}}{\longleftrightarrow}}\big(B^{(12)\prime},Y^{(12)\prime}\big)\underset{\varepsilon_2}{\overset{\mu_{6}}{\longleftrightarrow}}\big(B^{(13)\prime},Y^{(13)\prime}\big)\underset{\varepsilon_3}{\overset{\mu_{15}}{\longleftrightarrow}}\big(B^{(14)\prime},Y^{(14)\prime}\big)\\
&\qquad\underset{\varepsilon_4}{\overset{\mu_{3}}{\longleftrightarrow}}\big(B^{(15)\prime},Y^{(15)\prime}\big)\overset{\sigma_{6,15}\sigma_{3,14}}{\longleftrightarrow}\big(B^{(16)\prime},Y^{(16)\prime}\big)\\
&\big(B^{(16)\prime},Y^{(16)\prime}\big)\underset{\varepsilon_1}{\overset{\mu_{12}}{\longleftrightarrow}}\big(B^{(17)\prime},Y^{(17)\prime}\big)\underset{\varepsilon_2}{\overset{\mu_{13}}{\longleftrightarrow}}\big(B^{(18)\prime},Y^{(18)\prime}\big)\underset{\varepsilon_3}{\overset{\mu_{15}}{\longleftrightarrow}}\big(B^{(19)\prime},Y^{(19)\prime}\big)\\
&\qquad\underset{\varepsilon_4}{\overset{\mu_{8}}{\longleftrightarrow}}\big(B^{(20)\prime},Y^{(20)\prime}\big)\overset{\sigma_{13,15}\sigma_{8,12}}{\longleftrightarrow}\big(B^{(21)\prime},Y^{(21)\prime}\big),\\
&\big(B^{(21)\prime},Y^{(21)\prime}\big)\overset{\sigma_{7,14}}{\longleftrightarrow}\big(B^{(22)\prime},Y^{(22)\prime}\big).
\end{aligned}\tag{3.11}$$

They correspond to the cluster transformations in the right path of Figure 5 as follows:

$$\begin{aligned}
\widehat{R}_{456}&=\mathrm{Ad}\big(\Psi_q\big(\big(Y_{12}^{(1)\prime}\big)^{\varepsilon_1}\big)^{\varepsilon_1}\big)\tau_{12,\varepsilon_1}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{11}^{(2)\prime}\big)^{\varepsilon_2}\big)^{\varepsilon_2}\big)\tau_{11,\varepsilon_2}\\
&\quad\times\mathrm{Ad}\big(\Psi_q\big(\big(Y_{13}^{(3)\prime}\big)^{\varepsilon_3}\big)^{\varepsilon_z3}\big)\tau_{13,\varepsilon_3}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{6}^{(4)\prime}\big)^{\varepsilon_4}\big)^{\varepsilon_4}\big)\tau_{6,\varepsilon_4}\sigma_{11,13}\sigma_{6,12},\\
\widehat{R}_{236}&=\mathrm{Ad}\big(\Psi_q\big(\big(Y_{7}^{(6)\prime}\big)^{\varepsilon_1}\big)^{\varepsilon_1}\big)\tau_{7,\varepsilon_1}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{11}^{(7)\prime}\big)^{\varepsilon_2}\big)^{\varepsilon_2}\big)\tau_{11,\varepsilon_2}\\
&\quad\times\mathrm{Ad}\big(\Psi_q\big(\big(Y_{8}^{(8)\prime}\big)^{\varepsilon_3}\big)^{\varepsilon_3}\big)\tau_{8,\varepsilon_3}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{3}^{(9)\prime}\big)^{\varepsilon_4}\big)^{\varepsilon_4}\big)\tau_{3,\varepsilon_4}\sigma_{8,11}\sigma_{3,7},\\
\widehat{R}_{135}&=\mathrm{Ad}\big(\Psi_q\big(\big(Y_{14}^{(11)\prime}\big)^{\varepsilon_1}\big)^{\varepsilon_1}\big)\tau_{14,\varepsilon_1}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{6}^{(12)\prime}\big)^{\varepsilon_2}\big)^{\varepsilon_2}\big)\tau_{6,\varepsilon_2}\\
&\quad\times\mathrm{Ad}\big(\Psi_q\big(\big(Y_{15}^{(13)\prime}\big)^{\varepsilon_3}\big)^{\varepsilon_3}\big)\tau_{15,\varepsilon_3}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{3}^{(14)\prime}\big)^{\varepsilon_4}\big)^{\varepsilon_4}\big)\tau_{3,\varepsilon_4}\sigma_{6,15}\sigma_{3,14},
\end{aligned}$$

$$
\begin{aligned}
\widehat{R}_{124} = {} & \mathrm{Ad}\big(\Psi_q\big(\big(Y_{12}^{(16)\prime}\big)^{\varepsilon_1}\big)^{\varepsilon_1}\big)\tau_{12,\varepsilon_1}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{13}^{(17)\prime}\big)^{\varepsilon_2}\big)^{\varepsilon_2}\big)\tau_{13,\varepsilon_2} \\
& \times \mathrm{Ad}\big(\Psi_q\big(\big(Y_{15}^{(18)\prime}\big)^{\varepsilon_3}\big)^{\varepsilon_3}\big)\tau_{15,\varepsilon_3}\mathrm{Ad}\big(\Psi_q\big(\big(Y_{8}^{(19)\prime}\big)^{\varepsilon_4}\big)^{\varepsilon_4}\big)\tau_{8,\varepsilon_4}\sigma_{13,15}\sigma_{8,12}.
\end{aligned} \tag{3.12}
$$

Although the formulas (3.10) and (3.12) may appear distinct, they all signify the same transformation described in Proposition 3.4 for the corresponding subsets of Y-variables. This fact justifies denoting them by the common symbol $\widehat{R}$.

Remark 3.7. Consider the tropical y-seeds generated by the same mutation sequences from the initial one $\big(B^{(1)}, y^{(1)}\big) = \big(B^{(1)\prime}, y^{(1)\prime}\big)$. Suppose $y_i^{(1)} = y_i^{(1)\prime}$ is positive for all $i = 1, \dots, 17$. Then the four mutations highlighted in red in (3.9) and (3.11) are associated to a negative tropical sign of the y-variable at the mutation point (the y-seed in the left), while the remaining ones are positive.

Let us introduce the monomial parts of the cluster transformations (3.10) and (3.12)

$$
\begin{aligned}
&\tau_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} := \tau_{14,\varepsilon_1}\tau_{13,\varepsilon_2}\tau_{15,\varepsilon_3}\tau_{8,\varepsilon_4}\sigma_{13,15}\sigma_{8,14}\colon\ \mathcal{Y}\big(B^{(6)}\big) \to \mathcal{Y}\big(B^{(1)}\big), \\
&\tau_{135|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} := \tau_{7,\varepsilon_1}\tau_{6,\varepsilon_2}\tau_{15,\varepsilon_3}\tau_{3,\varepsilon_4}\sigma_{6,15}\sigma_{3,7}\colon\ \mathcal{Y}\big(B^{(11)}\big) \to \mathcal{Y}\big(B^{(6)}\big), \\
&\tau_{236|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} := \tau_{12,\varepsilon_1}\tau_{11,\varepsilon_2}\tau_{8,\varepsilon_3}\tau_{3,\varepsilon_4}\sigma_{8,11}\sigma_{3,12}\colon\ \mathcal{Y}\big(B^{(16)}\big) \to \mathcal{Y}\big(B^{(11)}\big), \\
&\tau_{456|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} := \tau_{14,\varepsilon_1}\tau_{11,\varepsilon_2}\tau_{13,\varepsilon_3}\tau_{6,\varepsilon_4}\sigma_{11,13}\sigma_{6,14}\colon\ \mathcal{Y}\big(B^{(21)}\big) \to \mathcal{Y}\big(B^{(16)}\big), \\
&\tau'_{456|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} := \tau_{12,\varepsilon_1}\tau_{11,\varepsilon_2}\tau_{13,\varepsilon_3}\tau_{6,\varepsilon_4}\sigma_{11,13}\sigma_{6,12}\colon\ \mathcal{Y}\big(B^{(6)\prime}\big) \to \mathcal{Y}\big(B^{(1)\prime}\big), \\
&\tau'_{236|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} := \tau_{7,\varepsilon_1}\tau_{11,\varepsilon_2}\tau_{8,\varepsilon_3}\tau_{3,\varepsilon_4}\sigma_{8,11}\sigma_{3,7}\colon\ \mathcal{Y}\big(B^{(11)\prime}\big) \to \mathcal{Y}\big(B^{(6)\prime}\big), \\
&\tau'_{135|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} := \tau_{14,\varepsilon_1}\tau_{6,\varepsilon_2}\tau_{15,\varepsilon_3}\tau_{3,\varepsilon_4}\sigma_{6,15}\sigma_{3,14}\colon\ \mathcal{Y}\big(B^{(16)\prime}\big) \to \mathcal{Y}\big(B^{(11)\prime}\big), \\
&\tau'_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} := \tau_{12,\varepsilon_1}\tau_{13,\varepsilon_2}\tau_{15,\varepsilon_3}\tau_{8,\varepsilon_4}\sigma_{13,15}\sigma_{8,12}\colon\ \mathcal{Y}\big(B^{(21)\prime}\big) \to \mathcal{Y}\big(B^{(16)\prime}\big).
\end{aligned}
$$

The primes in $\tau'_{ijk|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ are added just for distinction. The maps $\tau_{ijk|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ and $\tau'_{ijk|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ consistently adhere to $\tau_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ in (3.3) with respect to the subset of Y-variables.

Now we are ready to explain monomial solutions to the twisted tetrahedron equation. Proposition 3.6, Figure 5 and Remark 2.3 indicate the equality of the tropical y-variables $y^{(22)} = y^{(22)\prime}$ *provided* that all the signs associated with the monomial part of the mutation are chosen to be the tropical signs. Considering Remark 3.7 alongside, we find that

$$
\begin{aligned}
&\tau_{124|++++}\tau_{135|++++}\tau_{236|++-+}\tau_{456|-+++}\sigma_{7,12} \\
&\quad = \tau'_{456|++++}\tau'_{236|++++}\tau'_{135|+-++}\tau'_{124|-+++}\sigma_{7,14}
\end{aligned} \tag{3.13}
$$

is valid instead of the naive choice of $\tau_{ijk|++++}$ and $\tau'_{ijk|++++}$ everywhere. This is an inhomogeneous version of the twisted tetrahedron equation, as the maps involved are not always uniform in their sign indices. The coincident image of $Y_1^{(22)}, \dots, Y_{17}^{(22)}$ by the two sides are sign coherent monomials in the initial Y-variables $Y^{(1)} = (Y_1, \dots, Y_{17})$. Their explicit form is available in (A.1).

A natural question is whether there are monomial solutions to the twisted tetrahedron equation with the signs homogeneously chosen as $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$

$$
\begin{aligned}
&\tau_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau_{135|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau_{236|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau_{456|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\sigma_{7,12} \\
&\quad = \tau'_{456|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau'_{236|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau'_{135|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau'_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\sigma_{7,14}.
\end{aligned} \tag{3.14}
$$

The answer is given by a direct calculation as follows.

Proposition 3.8. *The monomial part satisfies the tetrahedron equation* (3.14) *if and only if* $\varepsilon_1 = -\varepsilon_4 = -$, *i.e.,* $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{(-,+,+,+), (-,+,-,+), (-,-,+,+), (-,-,-,+)\}$.

Examples 3.2 and 3.3 describe the monomial parts τ_{--++} and τ_{-+-+} explicitly. Analogous information is supplied for the remaining two cases in Appendix A.

3.5 Dilogarithm identities

Now we turn to the dilogarithm identities that will be utilized later. Substitute (3.10) and (3.12) into the left-hand side and the right-hand side of (3.8) respectively. The result takes the form

$$\begin{aligned}
&\mathrm{Ad}\big(\Psi_q(U_1)^{\varepsilon_1}\cdots\Psi_q(U_4)^{\varepsilon_4}\big)\tau_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\mathrm{Ad}\big(\Psi_q(U_5)^{\varepsilon_1}\cdots\Psi_q(U_8)^{\varepsilon_4}\big)\tau_{135|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\\
&\qquad\times\mathrm{Ad}\big(\Psi_q(U_9)^{\varepsilon_1}\cdots\Psi_q(U_{12})^{\varepsilon_4}\big)\tau_{236|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\\
&\qquad\times\mathrm{Ad}\big(\Psi_q(U_{13})^{\varepsilon_1}\cdots\Psi_q(U_{16})^{\varepsilon_4}\big)\tau_{456|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\sigma_{7,12}\\
&\quad=\mathrm{Ad}\big(\Psi_q\big(U'_1\big)^{\varepsilon_1}\cdots\Psi_q\big(U'_4\big)^{\varepsilon_4}\big)\tau'_{456|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\mathrm{Ad}\big(\Psi_q\big(U'_5\big)^{\varepsilon_1}\cdots\Psi_q\big(U'_8\big)^{\varepsilon_4}\big)\tau'_{236|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\\
&\qquad\times\mathrm{Ad}\big(\Psi_q\big(U'_9\big)^{\varepsilon_1}\cdots\Psi_q\big(U'_{12}\big)^{\varepsilon_4}\big)\tau'_{135|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\\
&\qquad\times\mathrm{Ad}\big(\Psi_q\big(U'_{13}\big)^{\varepsilon_1}\cdots\Psi_q\big(U'_{16}\big)^{\varepsilon_4}\big)\tau'_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\sigma_{7,14},
\end{aligned}\tag{3.15}$$

where U_t and U'_t $(t=1,\dots,16)$ denote the Y-variables depending on $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$. Pushing the monomial parts to the right brings (3.15) into the form

$$\begin{aligned}
&\mathrm{Ad}\big(\Psi_q(Z_1)^{\varepsilon_1}\cdots\Psi_q(Z_{16})^{\varepsilon_4}\big)\tau_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau_{135|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau_{236|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau_{456|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\sigma_{7,12}\\
&\quad=\mathrm{Ad}\big(\Psi_q\big(Z'_1\big)^{\varepsilon_1}\cdots\Psi_q\big(Z'_{16}\big)^{\varepsilon_4}\big)\tau'_{456|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau'_{236|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\\
&\quad\times\tau'_{135|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau'_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\sigma_{7,14},
\end{aligned}\tag{3.16}$$

where Z_i and Z'_i are monomials of $Y_1,\dots,Y_{16}$ determined by

$$Z_i=\begin{cases}U_i, & i=1,\dots,4,\\ \tau_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(U_i), & i=5,\dots,8,\\ \tau_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau_{135|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(U_i), & i=9,\dots,12,\\ \tau_{124|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau_{135|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\tau_{236|\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(U_i), & i=13,\dots,16.\end{cases}\tag{3.17}$$

The elements Z'_i are similarly determined from the right-hand side of (3.15). From (3.16) and Proposition 3.8, we deduce

$$\mathrm{Ad}\big(\Psi_q(Z_1)^{\varepsilon_1}\cdots\Psi_q(Z_{16})^{\varepsilon_4}\big)=\mathrm{Ad}\big(\Psi_q\big(Z'_1\big)^{\varepsilon_1}\cdots\Psi_q\big(Z'_{16}\big)^{\varepsilon_4}\big)\tag{3.18}$$

for $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\in\{(-,+,+,+),(-,+,-,+),(-,-,+,+),(-,-,-,+)\}$. Actually a stronger equality holds.

Proposition 3.9. *For $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\in\{(-,+,-,+),(-,-,+,+)\}$, the products of quantum dilogarithms within* Ad *in (3.18) are well defined formal Laurent series in the nine Y-variables Y_3, Y_6, Y_7, Y_8, Y_{11}, Y_{12}, Y_{13}, Y_{14} and Y_{15}. Moreover, they are equal, i.e.,*

$$\Psi_q(Z_1)^{\varepsilon_1}\cdots\Psi_q(Z_{16})^{\varepsilon_4}=\Psi_q\big(Z'_1\big)^{\varepsilon_1}\cdots\Psi_q\big(Z'_{16}\big)^{\varepsilon_4}.\tag{3.19}$$

Proof. We show the claim for $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=(-,-,+,+)$. The case $(-,+,-,+)$ is similar. The data $Z_1,\dots,Z_{16}$ for $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=(-,-,+,+)$ is given by

$$\begin{pmatrix}
Y_{14}^{-1} & qY_{13}^{-1}Y_{14}^{-1} & q^{-1}Y_{14}Y_{15} & q^{-2}Y_8Y_{14}Y_{15}\\
q^2Y_7^{-1}Y_{13}^{-1}Y_{14}^{-1} & q^3Y_6^{-1}Y_7^{-1}Y_{13}^{-1}Y_{14}^{-1} & q^{-3}Y_7Y_8Y_{14}Y_{15} & q^{-4}Y_3Y_7Y_8Y_{14}Y_{15}\\
Y_{12}^{-1} & qY_{11}^{-1}Y_{12}^{-1} & q^{-1}Y_{12}Y_{13} & q^{-2}Y_6Y_{12}Y_{13}\\
Y_7^{-1} & qY_6^{-1}Y_7^{-1} & q^{-1}Y_7Y_8 & q^{-2}Y_3Y_7Y_8
\end{pmatrix},\tag{3.20}$$

where the element at ith row and the jth column from the top left signifies Z_{4i+j-4}. Similarly, the data $Z'_1,\dots,Z'_{16}$ is given as follows:

$$\begin{pmatrix}
Y_{12}^{-1} & qY_{11}^{-1}Y_{12}^{-1} & q^{-1}Y_{12}Y_{13} & q^{-2}Y_6Y_{12}Y_{13}\\
Y_7^{-1} & qY_6^{-1}Y_7^{-1} & q^{-1}Y_7Y_8 & q^{-2}Y_3Y_7Y_8\\
Y_{12}^{-1}Y_{13}^{-1}Y_{14}^{-1} & qY_{11}^{-1}Y_{12}^{-1}Y_{13}^{-1}Y_{14}^{-1} & q^{-1}Y_{12}Y_{13}Y_{14}Y_{15} & q^{-2}Y_6Y_{12}Y_{13}Y_{14}Y_{15}\\
Y_6Y_{11}^{-1}Y_{14}^{-1} & qY_{13}^{-1}Y_{14}^{-1} & q^{-1}Y_{14}Y_{15} & q^{-2}Y_3Y_6^{-1}Y_8Y_{14}Y_{15}
\end{pmatrix}.\tag{3.21}$$

Note that Z'_{13} and Z'_{16} in (3.21) are not sign coherent. In order to show the well-definedness of the left-hand side of (3.19), expand the 16 Ψ_q's via (2.6) with the summation variables $n_1, \dots, n_{16} \in \mathbb{Z}_{\geq 0}$. By using the q-commutativity of Y-variables, one can arrange each term of the expansion uniquely as

$$\begin{aligned}
&\Psi_q(Z_1)^{-1}\Psi_q(Z_2)^{-1}\Psi_q(Z_3)\Psi_q(Z_4)\cdots\Psi_q(Z_{13})^{-1}\Psi_q(Z_{14})^{-1}\Psi_q(Z_{15})\Psi_q(Z_{16})\\
&\quad = \sum_{\mathbf{n}\in(\mathbb{Z}_{\geq 0})^{16}} C(\mathbf{n}) Y_3^{p_1}Y_6^{p_2}Y_7^{p_3}Y_8^{p_4}Y_{11}^{p_5}Y_{12}^{p_6}Y_{13}^{p_7}Y_{14}^{p_8}Y_{15}^{p_9},
\end{aligned} \tag{3.22}$$

where $C(\mathbf{n})$ is a rational function of q depending on $\mathbf{n} = (n_1, \dots, n_{16})$. The powers p_i's are given by

$$\begin{aligned}
&p_1 = n_8 + n_{16}, \qquad p_2 = -n_6 + n_{12} - n_{14}, \qquad p_3 = -n_5 - n_6 - n_7 + n_{12} - n_{14} - n_{16},\\
&p_4 = n_4 - n_7 + n_{12} + n_{13} - n_{16}, \qquad p_5 = -n_{10}, \qquad p_6 = -n_9 - n_{10} - n_{11} - n_{13} - n_{15},\\
&p_7 = -n_2 - n_5 - n_6 - n_{11} - n_{13} - n_{15}, \qquad p_8 = -n_1 - n_2 - n_3 - n_5 - n_6 - n_{11} - n_{13},\\
&p_9 = -n_3 - n_{11} - n_{13}.
\end{aligned} \tag{3.23}$$

The series (3.22) is well defined if the coefficient of the monomial

$$Y_3^{p_1}Y_6^{p_2}Y_7^{p_3}Y_8^{p_4}Y_{11}^{p_5}Y_{12}^{p_6}Y_{13}^{p_7}Y_{14}^{p_8}Y_{15}^{p_9}$$

for any given $(p_1, \dots, p_9) \in \mathbb{Z}^9$ is finite. This is shown by checking that there are none or finitely many $\mathbf{n} \in (\mathbb{Z}_{\geq 0})^{16}$ satisfying the nine equations (3.23). This is straightforward. The well-definedness of the right-hand side is verified in the same manner.

Next we prove (3.19). Write Φ for (LHS of (3.19))(RHS of (3.19))$^{-1}$. From an argument similar to the proof of [15, Theorem 3.5], we prove that $\Phi = c$ where c only depends on q as follows. We can extend the degenerate exchange matrix $B^{(1)}$ to a non-degenerate one $\tilde{B}$ which has a twice size as $B^{(1)}$ (see [15, Example 2.5]). Then, due to the extension theorem [19, Theorem 4.3] the periodicity of the seed $\big(B^{(1)}, Y^{(1)}\big)$ is also that of the seed $\big(\tilde{B}, \tilde{Y}\big)$. Hence Φ commutes with any element of the quantum torus algebra $\mathcal{T}\big(\tilde{B}\big)$. This means that $\Phi = c$, since $\tilde{B}$ is nondegenerate. To determine c, we compare the constant terms contained in the left-hand side and the right-hand side of (3.19). For the left-hand side, one looks for $\mathbf{n} \in (\mathbb{Z}_{\geq 0})^{16}$ such that $p_1 = \cdots = p_9 = 0$. It is easy to see that $\mathbf{n} = (0, \dots, 0)$ is the only solution indicating that the constant term of the left-hand side is 1. Similarly, the constant term of the right-hand side is found to be 1. Therefore, $c = 1$. ■

Remark 3.10. For the two cases $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-,+,+,+), (-,-,-,+)$ in Proposition 3.8, there are infinitely many choices of $\mathbf{n} \in (\mathbb{Z}_{\geq 0})^{16}$ satisfying (3.23). Therefore, the simple argument in the above proof is not valid.

4 Realization in terms of q-Weyl algebras

4.1 Y-variables and q-Weyl algebras

Hereafter we also use $\hbar$ which is related to q by $q = e^{\hbar}$. By a q-Weyl algebra we mean an associative algebra generated by $U^{\pm 1}$ and $W^{\pm 1}$ obeying the relations $UU^{-1} = U^{-1}U = WW^{-1} = W^{-1}W = 1$ and $UW = qWU$. To each crossing i of the wiring diagram we associate parameters $\mathcal{P}_i = (a_i, b_i, c_i, d_i, e_i)$ and canonical variables u_i, w_i satisfying

$$[\mathsf{u}_i, \mathsf{w}_j] = \hbar\delta_{ij}, \qquad [\mathsf{u}_i, \mathsf{u}_j] = [\mathsf{w}_i, \mathsf{w}_j] = 0. \tag{4.1}$$

$$a_i + b_i + c_i + d_i + e_i = 0. \tag{4.2}$$

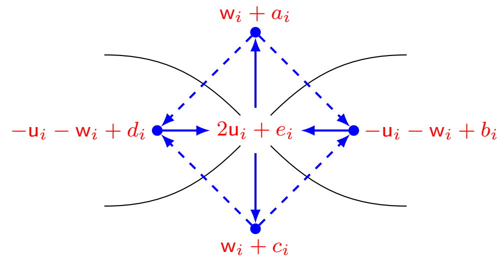


Figure 6. Graphical rule to parametrize the Y-variables in terms of q-Weyl algebra generators in the vicinity of the crossing i (center) of the wiring diagram (black). A Y-variable situated at a vertex (blue circle) of the symmetric butterfly quiver acquires factors $\mathrm{e}^{a_i+\mathsf{w}_i}$, $\mathrm{e}^{b_i-\mathsf{u}_i-\mathsf{w}_i}$, $\mathrm{e}^{c_i+\mathsf{w}_i}$, $\mathrm{e}^{d_i-\mathsf{u}_i-\mathsf{w}_i}$ from the neighboring crossing i of the wiring diagram if the vertex is located at the north, east, south, west of i, respectively. A Y-variable on the vertex i is $\mathrm{e}^{e_i+2\mathsf{u}_i}$. The ordering of these factors from different i's (if any) is inconsequential due to the commutativity of the associated canonical variables.

The exponentials of the canonical variables obey the relations in a direct product of q-Weyl algebras, e.g., $\mathrm{e}^{\mathsf{u}_i}\mathrm{e}^{\mathsf{w}_j} = q^{\delta_{ij}}\mathrm{e}^{\mathsf{w}_j}\mathrm{e}^{\mathsf{u}_i}$. Given a wiring diagram and the associated symmetric butterfly quiver, we "parametrize" the Y-variables by the graphical rule explained in Figure 6.

The claim is that the relation $Y_rY_s = q^{2b_{rs}}Y_sY_r$ (2.1) is satisfied under this parametrization. To state it formally, let $\mathcal{W}_n$ be the direct product of the q-Weyl algebras generated by $\mathrm{e}^{\pm\mathsf{u}_i}$, $\mathrm{e}^{\pm\mathsf{w}_i}$ for $i = 1, \dots, n$. Let further $\mathcal{A}_n$ be the non-commuting fractional field of $\mathcal{W}_n$. Then for B corresponding to the left diagram in Figure 4, we have a morphism $\phi_{\mathrm{SB}}\colon \mathcal{Y}(B) \to \mathcal{A}_3$ defined by

$$\phi_{\mathrm{SB}}\colon \begin{cases} Y_1 \mapsto \mathrm{e}^{c_1+c_3+\mathsf{w}_1+\mathsf{w}_3}, & Y_6 \mapsto \mathrm{e}^{b_1-\mathsf{u}_1-\mathsf{w}_1}, \\ Y_2 \mapsto \mathrm{e}^{d_3-\mathsf{u}_3-\mathsf{w}_3}, & Y_7 \mapsto \mathrm{e}^{d_2+a_3-\mathsf{u}_2-\mathsf{w}_2+\mathsf{w}_3}, \\ Y_3 \mapsto \mathrm{e}^{e_3+2\mathsf{u}_3}, & Y_8 \mapsto \mathrm{e}^{e_2+2\mathsf{u}_2}, \\ Y_4 \mapsto \mathrm{e}^{d_1+c_2+b_3-\mathsf{u}_1-\mathsf{w}_1+\mathsf{w}_2-\mathsf{u}_3-\mathsf{w}_3}, & Y_9 \mapsto \mathrm{e}^{a_1+b_2+\mathsf{w}_1-\mathsf{u}_2-\mathsf{w}_2}, \\ Y_5 \mapsto \mathrm{e}^{e_1+2\mathsf{u}_1}, & Y_{10} \mapsto \mathrm{e}^{a_2+\mathsf{w}_2}, \end{cases} \tag{4.3}$$

where SB signifies "symmetric butterfly". Similarly, for the right diagram of Figure 4, we have a morphism $\phi'_{\mathrm{SB}}\colon \mathcal{Y}(B') \to \mathcal{A}_3$ as

$$\phi'_{\mathrm{SB}}\colon \begin{cases} Y'_1 \mapsto \mathrm{e}^{c_2+\mathsf{w}_2}, & Y'_6 \mapsto \mathrm{e}^{b_2+c_3-\mathsf{u}_2-\mathsf{w}_2+\mathsf{w}_3}, \\ Y'_2 \mapsto \mathrm{e}^{c_1+d_2+\mathsf{w}_1-\mathsf{u}_2-\mathsf{w}_2}, & Y'_7 \mapsto \mathrm{e}^{d_1-\mathsf{u}_1-\mathsf{w}_1}, \\ Y'_3 \mapsto \mathrm{e}^{e_3+2\mathsf{u}_3}, & Y'_8 \mapsto \mathrm{e}^{e_2+2\mathsf{u}_2}, \\ Y'_4 \mapsto \mathrm{e}^{b_1+a_2+d_3-\mathsf{u}_1-\mathsf{w}_1+\mathsf{w}_2-\mathsf{u}_3-\mathsf{w}_3}, & Y'_9 \mapsto \mathrm{e}^{b_3-\mathsf{u}_3-\mathsf{w}_3}, \\ Y'_5 \mapsto \mathrm{e}^{e_1+2\mathsf{u}_1}, & Y'_{10} \mapsto \mathrm{e}^{a_1+a_3+\mathsf{w}_1+\mathsf{w}_3}. \end{cases} \tag{4.4}$$

Remark 4.1. In the parametrization (4.3), the centers in Remark 3.1 take the following values:

$$\begin{aligned} &\phi_{\mathrm{SB}}\big(Y_1^{-1}Y_7Y_8Y_9Y_{10}^2\big) = \mathrm{e}^{a_1-c_1+a_2-c_2+a_3-c_3}, \\ &\phi_{\mathrm{SB}}\big(Y_2Y_4^{-1}Y_6Y_{10}\big) = \mathrm{e}^{b_1-d_1+a_2-c_2+d_3-b_3}, \\ &\phi_{\mathrm{SB}}\big(Y_3Y_4^2Y_6^{-2}Y_7^2Y_8\big) = \mathrm{e}^{2d_1-2b_1+2c_2+2d_2+e_2+2a_3+2b_3+e_3}, \\ &\phi_{\mathrm{SB}}\big(Y_5Y_6^2Y_8Y_9^2Y_{10}^2\big) = q^{-4}\mathrm{e}^{2a_1+2b_1+e_1+2a_2+2b_2+e_2}. \end{aligned}$$

In both parametrization (4.3) and (4.4), the invariants in Remark 3.5 take the following forms:

$$\phi_{\mathrm{SB}}(Y_3Y_8) = \phi'_{\mathrm{SB}}\big(Y'_3Y'_8\big) = \mathrm{e}^{e_2+e_3+\mathsf{u}_2+\mathsf{u}_3}, \tag{4.5}$$

$$\phi_{\rm SB}(Y_5Y_8) = \phi'_{\rm SB}\big(Y'_5Y'_8\big) = \mathrm{e}^{e_1+e_2+\mathsf{u}_1+\mathsf{u}_2}, \tag{4.6}$$
$$\phi_{\rm SB}\big(Y_1Y_2Y_4Y_6Y_7Y_8^{-1}Y_9Y_{10}\big) = \phi'_{\rm SB}\big(Y'_1Y'_2Y'_4Y'_6Y'_7Y'^{-1}_8Y'_9Y'_{10}\big) = \mathrm{e}^{-e_1-2e_2-e_3-2\mathsf{u}_1-4\mathsf{u}_2-2\mathsf{u}_3}.$$

The combinations $\mathsf{u}_1+\mathsf{u}_2$ and $\mathsf{u}_2+\mathsf{u}_3$ will reemerge as conserved quantities within the delta functions in the matrix elements of the R-matrix in coordinate representations. See (5.10) and (6.12).

4.2 Extracting R_{123} from $\widehat{R}_{123}$

Let us illustrate the action of the monomial part $\tau_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ (3.3) of $\widehat{R}_{123}$ on the canonical variables for the case $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=(-,-,+,+)$. From Example 3.2 and (4.3)–(4.4), we find that τ_{--++} is translated into a transformation $\tau^{\mathsf{uw}}_{--++}$ of the canonical variables[2] as

$$\tau^{\mathsf{uw}}_{--++}\colon\ \begin{cases} \mathsf{u}_1\mapsto \mathsf{u}_1+\mathsf{u}_2-\mathsf{u}_3+\lambda_0, & \mathsf{w}_1\mapsto \mathsf{w}_1+\lambda_1,\\ \mathsf{u}_2\mapsto \mathsf{u}_3-\lambda_0, & \mathsf{w}_2\mapsto \mathsf{w}_1+\mathsf{w}_3+\lambda_3,\\ \mathsf{u}_3\mapsto \mathsf{u}_2+\lambda_0, & \mathsf{w}_3\mapsto -\mathsf{w}_1+\mathsf{w}_2+\lambda_2,\end{cases} \tag{4.7}$$

where $\lambda_r=\lambda_r(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$ for $r=0,1,2,3$ is defined, under the condition (4.2), by

$$\lambda_0=\frac{e_2-e_3}{2},\qquad \lambda_1=a_2-a_3+b_2-b_3+\lambda_0,\qquad \lambda_2=-a_1-b_2+b_3-\lambda_0,$$
$$\lambda_3=c_1-c_2+c_3. \tag{4.8}$$

In order to realize (4.7) as an adjoint action, we introduce the group N_n generated by

$$\mathrm{e}^{\pm\frac{1}{\hbar}\mathsf{u}_i\mathsf{w}_j}\quad (i\neq j),\qquad \mathrm{e}^{\frac{a}{\hbar}\mathsf{u}_i},\qquad \mathrm{e}^{\frac{a}{\hbar}\mathsf{w}_i}\quad (a\in\mathbb{C}),\qquad b\in\mathbb{C}^\times$$

with $i,j\in\{1,\dots,n\}$. The multiplication is defined by the (generalized) Baker–Campbell–Hausdorff (BCH) formula and (4.1), which is well defined due to the grading by $\hbar^{-1}$. Let $\mathfrak{S}_n$ be the symmetric group generated by the transpositions ρ_{ij} $(i,j\in\{1,\dots,n\})$. It acts on N_n via the adjoint action, inducing permutations of the indices of the canonical variables. Thus one can form the semi-direct product $N_n\rtimes\mathfrak{S}_n$, and let it act on $\mathcal{W}_n$ by the adjoint action.

Now the monomial part $\tau^{\mathsf{uw}}_{--++}$ is described as the adjoint action as follows:

$$\tau^{\mathsf{uw}}_{--++}=\mathrm{Ad}(P_{--++}), \tag{4.9}$$
$$P_{--++}=\mathrm{e}^{\frac{1}{\hbar}(\mathsf{u}_3-\mathsf{u}_2)\mathsf{w}_1}\mathrm{e}^{\frac{\lambda_0}{\hbar}(-\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3)}\mathrm{e}^{\frac{1}{\hbar}(\lambda_1\mathsf{u}_1+\lambda_2\mathsf{u}_2+\lambda_3\mathsf{u}_3)}\rho_{23}\in N_3\rtimes\mathfrak{S}_3, \tag{4.10}$$

where ρ_{23} acts trivially on the parameters λ_r. Extending the indices and suppressing the sign choice of $--++$ in (4.9) and (4.10), we introduce

$$\tau^{\mathsf{uw}}_{ijk}=\mathrm{Ad}(P_{ijk}), \tag{4.11}$$
$$P_{ijk}=\mathrm{e}^{\frac{1}{\hbar}(\mathsf{u}_k-\mathsf{u}_j)\mathsf{w}_i}\mathrm{e}^{\frac{\lambda_0}{\hbar}(-\mathsf{w}_i-\mathsf{w}_j+\mathsf{w}_k)}\mathrm{e}^{\frac{1}{\hbar}(\lambda_1\mathsf{u}_i+\lambda_2\mathsf{u}_j+\lambda_3\mathsf{u}_k)}\rho_{jk}\in N_6\rtimes\mathfrak{S}_6, \tag{4.12}$$

where $\lambda_r=\lambda_r(\mathcal{P}_i,\mathcal{P}_j,\mathcal{P}_k)$ is given by (4.8) by replacing the parameters as $(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)\to(\mathcal{P}_i,\mathcal{P}_j,\mathcal{P}_k)$. By a straightforward calculation using the BCH formula, one can prove the following.

Lemma 4.2. *P_{ijk} satisfies the tetrahedron equation in $N_6\rtimes\mathfrak{S}_6$ by itself*

$$P_{124}P_{135}P_{236}P_{456}=P_{456}P_{236}P_{135}P_{124}. \tag{4.13}$$

[2] $\tau^{\mathsf{uw}}_{--++}$ is naturally regarded also as a transformation in $\mathcal{W}_3$ via exponentials.

The fact that P_{ijk} acts on the canonical variables rather than Y-variables has led to the tetrahedron equation *without* a twist by permutations.

Define $R_{123} = R(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)_{123}$ by

$$\begin{aligned} R_{123} &= \Psi_q\bigl(\mathrm{e}^{-d_1-c_2-b_3+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\bigr)^{-1}\Psi_q\bigl(\mathrm{e}^{-d_1-c_2-b_3-e_3+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\bigr)^{-1} \\ &\quad\times \Psi_q\bigl(\mathrm{e}^{d_1+e_1+c_2+b_3+\mathsf{u}_1-\mathsf{u}_3-\mathsf{w}_1+\mathsf{w}_2-\mathsf{w}_3}\bigr) \\ &\quad\times \Psi_q\bigl(\mathrm{e}^{d_1+e_1+c_2+e_2+b_3+\mathsf{u}_1+2\mathsf{u}_2-\mathsf{u}_3-\mathsf{w}_1+\mathsf{w}_2-\mathsf{w}_3}\bigr)P_{123} \end{aligned} \tag{4.14}$$

$$\begin{aligned} &= \Psi_q\bigl(\mathrm{e}^{-d_1-c_2-b_3+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\bigr)^{-1}\Psi_q\bigl(\mathrm{e}^{-d_1-c_2-b_3-e_3+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\bigr)^{-1} \\ &\quad\times P_{123}\Psi_q\bigl(\mathrm{e}^{-b_1-a_2-d_3-e_3+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\bigr) \\ &\quad\times \Psi_q\bigl(\mathrm{e}^{-b_1-a_2-d_3+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\bigr). \end{aligned} \tag{4.15}$$

Let $\widehat{R}^{\mathsf{uw}}_{123}$ be the cluster transformation $\widehat{R}_{123}$ (3.5) viewed as the one for the canonical variables $\{\mathsf{u}_i, \mathsf{w}_i\}_{i=1,2,3}$. Then from (4.3), (4.4) and (4.9), we have

$$\widehat{R}^{\mathsf{uw}}_{123} = \mathrm{Ad}\bigl(R(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)_{123}\bigr). \tag{4.16}$$

Note that the right-hand side is invariant under $R \to cR$ for any scalar c. A proper normalization of R based on a symmetry consideration will be proposed in Section 4.3.

Formally the results (4.7) and (4.16) may be stated as the commutativity of the diagrams

$$\begin{tikzcd} \mathcal{Y}(B') \arrow[r, "\tau_{--++}"] \arrow[d, "\phi'_{\mathrm{SB}}"'] & \mathcal{Y}(B) \arrow[d, "\phi_{\mathrm{SB}}"] \\ \mathcal{A}_3 \arrow[r, "\tau^{\mathsf{uw}}_{--++}"] & \mathcal{A}_3, \end{tikzcd} \qquad \begin{tikzcd} \mathcal{Y}(B') \arrow[r, "\widehat{R}_{123}"] \arrow[d, "\phi'_{\mathrm{SB}}"'] & \mathcal{Y}(B) \arrow[d, "\phi_{\mathrm{SB}}"] \\ \mathcal{A}_3 \arrow[r, "\widehat{R}^{\mathsf{uw}}_{123}"] & \mathcal{A}_3. \end{tikzcd}$$

Extending (4.14) and (4.15), we introduce $R_{ijk} = R(\mathcal{P}_i, \mathcal{P}_j, \mathcal{P}_k)_{ijk}$ by

$$\begin{aligned} R_{ijk} &= \Psi_q\bigl(\mathrm{e}^{-d_i-c_j-b_k+\mathsf{u}_i+\mathsf{u}_k+\mathsf{w}_i-\mathsf{w}_j+\mathsf{w}_k}\bigr)^{-1}\Psi_q\bigl(\mathrm{e}^{-d_i-c_j-b_k-e_k+\mathsf{u}_i-\mathsf{u}_k+\mathsf{w}_i-\mathsf{w}_j+\mathsf{w}_k}\bigr)^{-1} \\ &\quad\times \Psi_q\bigl(\mathrm{e}^{d_i+e_i+c_j+b_k+\mathsf{u}_i-\mathsf{u}_k-\mathsf{w}_i+\mathsf{w}_j-\mathsf{w}_k}\bigr) \\ &\quad\times \Psi_q\bigl(\mathrm{e}^{d_i+e_i+c_j+e_j+b_k+\mathsf{u}_i+2\mathsf{u}_j-\mathsf{u}_k-\mathsf{w}_i+\mathsf{w}_j-\mathsf{w}_k}\bigr)P_{ijk} \end{aligned} \tag{4.17}$$

$$\begin{aligned} &= \Psi_q\bigl(\mathrm{e}^{-d_i-c_j-b_k+\mathsf{u}_i+\mathsf{u}_k+\mathsf{w}_i-\mathsf{w}_j+\mathsf{w}_k}\bigr)^{-1}\Psi_q\bigl(\mathrm{e}^{-d_i-c_j-b_k-e_k+\mathsf{u}_i-\mathsf{u}_k+\mathsf{w}_i-\mathsf{w}_j+\mathsf{w}_k}\bigr)^{-1} \\ &\quad\times P_{ijk}\Psi_q\bigl(\mathrm{e}^{-b_i-a_j-d_k-e_k+\mathsf{u}_i-\mathsf{u}_k+\mathsf{w}_i-\mathsf{w}_j+\mathsf{w}_k}\bigr)\Psi_q\bigl(\mathrm{e}^{-b_i-a_j-d_k+\mathsf{u}_i+\mathsf{u}_k+\mathsf{w}_i-\mathsf{w}_j+\mathsf{w}_k}\bigr), \end{aligned} \tag{4.18}$$

where P_{ijk} is given by (4.12). Now we state the main result of the paper.

Theorem 4.3. *$R(\mathcal{P}_i, \mathcal{P}_j, \mathcal{P}_k)_{ijk}$ satisfies the tetrahedron equation*

$$\begin{aligned} &R(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_4)_{124}R(\mathcal{P}_1, \mathcal{P}_3, \mathcal{P}_5)_{135}R(\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_6)_{236}R(\mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6)_{456} \\ &\quad= R(\mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6)_{456}R(\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_6)_{236}R(\mathcal{P}_1, \mathcal{P}_3, \mathcal{P}_5)_{135}R(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_4)_{124}. \end{aligned} \tag{4.19}$$

Proof. Consider the dilogarithm identity (3.19) with $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-, -, +, +)$ in terms of canonical variables[3]

$$\Psi_q\bigl(\tilde{Z}_1\bigr)^{\varepsilon_1}\cdots\Psi_q\bigl(\tilde{Z}_{16}\bigr)^{\varepsilon_4} = \Psi_q\bigl(\tilde{Z}'_1\bigr)^{\varepsilon_1}\cdots\Psi_q\bigl(\tilde{Z}'_{16}\bigr)^{\varepsilon_4}. \tag{4.20}$$

Here $\tilde{Z}_i = \tilde{\phi}_{\mathrm{SB}}(Z_i)$ and $\tilde{Z}'_i = \tilde{\phi}'_{\mathrm{SB}}(Z'_i)$ with Z_i and Z'_i given in (3.20) and (3.21). The morphisms $\tilde{\phi}_{\mathrm{SB}}$ and $\tilde{\phi}'_{\mathrm{SB}}$ are natural generalizations of (4.3) and (4.4). They send the Y-variables

[3]The signs ε_1, ε_2, ε_3, ε_4 are actually $-$, $-$, $+$, $+$, but for clarity, they are left as they are.

to $\mathcal{W}_6$ according to the rule in Figure 6 applied to the bottom right diagram in Figure 5.[4] Multiplication of (4.13) to (4.20) from the right leads to

$$\begin{aligned}&\Psi_q\big(\tilde{Z}_1\big)^{\varepsilon_1}\cdots\Psi_q\big(\tilde{Z}_{16}\big)^{\varepsilon_4}P_{124}P_{135}P_{236}P_{456}\\ &\qquad=\Psi_q\big(\tilde{Z'}_1\big)^{\varepsilon_1}\cdots\Psi_q\big(\tilde{Z'}_{16}\big)^{\varepsilon_4}P_{456}P_{236}P_{135}P_{124}.\end{aligned}\tag{4.21}$$

Let us consider the left-hand side. It is obviously equal to

$$\begin{aligned}&\Psi_q\big(\tilde{Z}_1\big)^{\varepsilon_1}\cdots\Psi_q\big(\tilde{Z}_4\big)^{\varepsilon_4}P_{124}P_{124}^{-1}\Psi_q\big(\tilde{Z}_5\big)^{\varepsilon_1}\cdots\Psi_q\big(\tilde{Z}_8\big)^{\varepsilon_4}P_{124}\cdot P_{135}\\ &\quad\times P_{135}^{-1}P_{124}^{-1}\Psi_q\big(\tilde{Z}_9\big)^{\varepsilon_1}\cdots\Psi_q\big(\tilde{Z}_{12}\big)^{\varepsilon_4}P_{124}P_{135}\cdot P_{236}\\ &\quad\times P_{236}^{-1}P_{135}^{-1}P_{124}^{-1}\Psi_q\big(\tilde{Z}_{13}\big)^{\varepsilon_1}\cdots\Psi_q\big(\tilde{Z}_{16}\big)^{\varepsilon_4}P_{124}P_{135}P_{236}\cdot P_{456}.\end{aligned}\tag{4.22}$$

On the other hand from (4.11) and the image of (3.17) by $\tilde{\phi}_{\mathrm{SB}}$, we know

$$\tilde{U}_i=\begin{cases}\tilde{Z}_i, & i=1,\dots,4,\\ P_{124}^{-1}\tilde{Z}_iP_{124}, & i=5,\dots,8,\\ P_{135}^{-1}P_{124}^{-1}\tilde{Z}_iP_{124}P_{135}, & i=9,\dots,12,\\ P_{236}^{-1}P_{135}^{-1}P_{124}^{-1}\tilde{Z}_iP_{124}P_{135}P_{236}, & i=13,\dots,16,\end{cases}$$

where $\tilde{U}_i=\tilde{\phi}_{\mathrm{SB}}(U_i)$. Thus (4.22) is cast into the form

$$\begin{aligned}&\Psi_q\big(\tilde{U}_1\big)^{\varepsilon_1}\cdots\Psi_q\big(\tilde{U}_4\big)^{\varepsilon_4}P_{124}\Psi_q\big(\tilde{U}_5\big)^{\varepsilon_1}\cdots\Psi_q\big(\tilde{U}_8\big)^{\varepsilon_4}P_{135}\\ &\quad\times\Psi_q\big(\tilde{U}_9\big)^{\varepsilon_1}\cdots\Psi_q\big(\tilde{U}_{12}\big)^{\varepsilon_4}P_{236}\Psi_q\big(\tilde{U}_{13}\big)^{\varepsilon_1}\cdots\Psi_q\big(\tilde{U}_{16}\big)^{\varepsilon_4}P_{456}.\end{aligned}$$

This is identified with the left-hand side of (4.19) for (4.17). The right-hand side of (4.19) is similarly derived from that in (4.21). ■

The monomial part $\tau_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ which admits the adjoint action description as (4.9) can be searched in the same manner as explained around [12, equation (4.27)]. We find that $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=(-,-,+,+)$ and $(-,+,-,+)$ are the only such cases. The formulas corresponding to $(-,+,-,+)$ are summarized in Appendix B. They are obtained from $(-,-,+,+)$ case by the interchange of the parameters $a_i\leftrightarrow a_{4-i}$, $c_i\leftrightarrow c_{4-i}$, $e_i\leftrightarrow e_{4-i}$ and $b_i\leftrightarrow d_{4-i}$ which is compatible with (4.2).

4.3 Symmetries of $\boldsymbol{R}$-matrices

Let (B,Y) and $\big(B',Y'\big)$ be the quantum y-seeds corresponding to the quivers in Figure 2 (a) and (b), respectively, and $\big(\widetilde{B},\widetilde{Y}\big)=\mu_8\mu_5\mu_3(B,Y)$ and $\big(\widetilde{B}',\widetilde{Y}'\big)=\mu_8\mu_5\mu_3\big(B',Y'\big)$. Note that $\widetilde{B}=-B$ and $\widetilde{B}'=-B'$. Define isomorphisms

$$\begin{aligned}&\alpha:=\sigma_{2,6}\sigma_{3,5}\sigma_{7,9}\colon\ \mathcal{Y}(B)\to\mathcal{Y}(B),\qquad \alpha':=\sigma_{2,6}\sigma_{3,5}\sigma_{7,9}\colon\ \mathcal{Y}\big(B'\big)\to\mathcal{Y}\big(B'\big),\\ &\beta:=\big(Y\to Y'\big)\sigma_{1,10}\sigma_{2,9}\sigma_{6,7}\colon\ \mathcal{Y}(B)\to\mathcal{Y}\big(B'\big),\\ &\beta':=\big(Y'\to Y\big)\sigma_{1,10}\sigma_{2,9}\sigma_{6,7}\colon\ \mathcal{Y}\big(B'\big)\to\mathcal{Y}(B),\\ &\gamma:=\big(q\to q^{-1}\big)\big(\widetilde{Y}\to Y^{-1}\big)\mu_3^*\mu_5^*\mu_8^*\colon\ \mathcal{Y}(B)\to\mathcal{Y}(B),\\ &\gamma':=\big(q\to q^{-1}\big)\big(\widetilde{Y}'\to Y'^{-1}\big)\mu_3^*\mu_5^*\mu_8^*\colon\ \mathcal{Y}\big(B'\big)\to\mathcal{Y}\big(B'\big),\end{aligned}$$

where $(x\to y)$ denotes the operation of replacing x with y.

[4]The map ϕ_{SB} does not spoil the well-definedness of the expansion like (3.22) with respect to $\mathrm{e}^{\mathsf{u}_i}$ and $\mathrm{e}^{\mathsf{w}_i}$ since it preserves the rank of the quantum torus generated by Y_i $(i=3,6,7,8,11,12,13,14,15)$.

Proposition 4.4. *The cluster transformation* $\widehat{R}_{123}\colon \mathcal{Y}(B') \to \mathcal{Y}(B)$ *satisfies*

$$\alpha\widehat{R}_{123} = \widehat{R}_{123}\alpha', \tag{4.23}$$

$$\beta\widehat{R}_{123} = \widehat{R}_{123}^{-1}\beta', \tag{4.24}$$

$$\gamma\widehat{R}_{123} = \widehat{R}_{123}\gamma'. \tag{4.25}$$

Proof. The first equation is a consequence of the reflection symmetries of the quivers in Figure 2 and the mutation sequence for $\widehat{R}$ about the vertical axis going through vertices 1 and 10. The mutation sequence is symmetric because vertices 3 and 5 are disconnected in the relevant quiver.

The second equation can be understood by turning Figure 3 upside down. Since

$$\sigma_{3,5}\sigma_{4,8}\mu_8\mu_5\mu_3\mu_4 = \mu_4\mu_3\mu_5\mu_8\sigma_{3,5}\sigma_{4,8},$$

the mutation sequence going from $B^{(1)}$ to $B^{(6)}$ in the upside down figure is the same as the reverse sequence going from $B^{(6)}$ to $B^{(1)}$ in the original figure, with labels 2 and 9, labels 6 and 7, and labels 1 and 10 swapped.

To show the third equation, we use the fact that

$$\sigma_{3,5}\sigma_{4,8}\mu_8\mu_5\mu_3\mu_4\mu_8\mu_5\mu_3(B,y) = \mu_8\mu_5\mu_3\sigma_{3,5}\sigma_{4,8}\mu_8\mu_5\mu_3\mu_4(B,y),$$

as one can check by direct calculation. By Theorem 2.4, this implies the following equality between maps from $\mathcal{Y}(B')$ to $\mathcal{Y}(\widetilde{B})$

$$\mu_3^*\mu_5^*\mu_8^*(\mu_4^*\mu_3^*\mu_5^*\mu_8^*\sigma_{3,5}\sigma_{4,8}) = (\mu_4^*\mu_3^*\mu_5^*\mu_8^*\sigma_{3,5}\sigma_{4,8})\mu_3^*\mu_5^*\mu_8^*.$$

Multiplying both sides with $(q \to q^{-1})(\widetilde{Y} \to Y^{-1})$, we get

$$\gamma\widehat{R} = (q \to q^{-1})(\widetilde{Y} \to Y^{-1})(\mu_4^*\mu_3^*\mu_5^*\mu_8^*\sigma_{3,5}\sigma_{4,8})\mu_3^*\mu_5^*\mu_8^*.$$

This is the desired equation because $\widetilde{Y}_i'$ transforms under mutations in the same way as $Y_i'^{-1}$, except that q appearing in the formula is replaced by q^{-1}. ■

Let α^{uw}, β^{uw}, γ^{uw} be α, β, γ expressed in terms of the canonical variables $\{\mathsf{u}_i, \mathsf{w}_i\}_{i=1,2,3}$. In other words, these are operators such that $\alpha^{\mathsf{uw}} \circ \phi_{\mathrm{SB}} = \phi_{\mathrm{SB}} \circ \alpha$ and $\alpha^{\mathsf{uw}} \circ \phi'_{\mathrm{SB}} = \phi'_{\mathrm{SB}} \circ \alpha$, etc. Explicitly, they act on the parameters and the canonical variables as follows:

$$\begin{aligned}
\alpha^{\mathsf{uw}}\colon\ & a_1 \leftrightarrow a_3, \quad b_1 \leftrightarrow d_3, \quad c_1 \leftrightarrow c_3, \quad d_1 \leftrightarrow b_3, \quad e_1 \leftrightarrow e_3, \quad b_2 \leftrightarrow d_2,\\
& \mathsf{u}_1 \leftrightarrow \mathsf{u}_3, \quad \mathsf{w}_1 \leftrightarrow \mathsf{w}_3,\\
\beta^{\mathsf{uw}}\colon\ & a_i \leftrightarrow c_i, \quad b_i \leftrightarrow d_i,\\
\gamma^{\mathsf{uw}}\colon\ & q \mapsto q^{-1}, \quad a_i \leftrightarrow c_i, \quad b_i \leftrightarrow d_i, \quad \mathrm{e}^{-\mathsf{w}_i} \mapsto \mathrm{e}^{\mathsf{w}_i + a_i + c_i}\big(1 + q\mathrm{e}^{2\mathsf{u}_i + e_i}\big).
\end{aligned} \tag{4.26}$$

Acting on (4.23), (4.24) and (4.25) with ϕ_{SB}, we obtain the following relations that hold in $\phi_{\mathrm{SB}}(\mathcal{Y}(B'))$:

$$\alpha^{\mathsf{uw}}\widehat{R}_{123}^{\mathsf{uw}} = \widehat{R}_{123}^{\mathsf{uw}}\alpha^{\mathsf{uw}}, \qquad \beta^{\mathsf{uw}}\widehat{R}_{123}^{\mathsf{uw}} = \widehat{R}_{123}^{\mathsf{uw}-1}\beta^{\mathsf{uw}}, \tag{4.27}$$

$$\gamma^{\mathsf{uw}}\widehat{R}_{123}^{\mathsf{uw}} = \widehat{R}_{123}^{\mathsf{uw}}\gamma^{\mathsf{uw}}. \tag{4.28}$$

The symmetry (4.27) can also be deduced from the formula (4.14) for R_{123} and its counterpart (B.2) for the sign choice $(-,+,-,+)$, which are mapped to each other by α^{uw}. In fact, not only the adjoint action of R_{123} but R_{123} itself enjoys the symmetries α^{uw} and β^{uw}.

Proposition 4.5. *Let f be a function of the parameters $(a_i, b_i, c_i, d_i)_{i=1,2,3}$ such that*

$$\alpha^{\mathsf{uw}}(f)f^{-1} = \exp\left(-\frac{1}{4\hbar}(e_1 - e_3)(a_3 - 2c_2 + c_3 - d_1 + d_3 + a_1 + c_1 - b_3 + b_1)\right),$$
$$\beta^{\mathsf{uw}}(f)f = \exp\left(\frac{1}{2\hbar}(e_2 - e_3)(a_1 - a_2 + a_3 + c_1 - c_2 + c_3)\right). \tag{4.29}$$

As an operator in either the u-diagonal representation or the w-diagonal representation introduced in Section 5, R_{123} satisfies

$$\alpha^{\mathsf{uw}}(fR_{123}) = fR_{123}, \tag{4.30}$$
$$\beta^{\mathsf{uw}}(fR_{123}) = (fR_{123})^{-1}. \tag{4.31}$$

Proof. The symmetry (4.30) under α^{uw} can be seen from the formulas for the matrix elements of R_{123} in the u-diagonal representation and the w-diagonal representation, obtained in Theorems 5.2 and 5.5, respectively. In both cases, the only part of the matrix elements that is not manifestly invariant under α^{uw} is the factor $\mathrm{e}^{-C_5^2}$. (See Remark 5.3.) The symmetry (4.31) under β^{uw} actually holds at the level of an element of $N_3 \rtimes \mathfrak{S}_3$, as one can check by calculating $\beta^{\mathsf{uw}}(R_{123})R_{123}$, say using the expression (4.15) for R_{123}. ∎

An example of a function that satisfies the above two conditions is

$$f = \exp\left(\frac{1}{4\hbar}(e_2 - e_3)(a_1 + a_3 + c_1 - 2c_2 + c_3 + b_1 - b_3 - d_1 + d_3)\right).$$

5 Matrix elements of R

In this section, we derive explicit formulas for the elements of the R-matrix given in (4.15) and (4.12) in some infinite dimensional representations of the q-Weyl algebra. When the overall normalization is not our primary concern, we will use notation such as $A^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} \equiv B^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$ to imply $A^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} = cB^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$ for some c that does not depend on the indices n_1, n_2, n_3, n'_1, n'_2, n'_3. When discussing the symmetry of the elements, it is important to consider whether c depends on the parameters $C_1, \dots, C_8$ in (5.1) or not. In such a circumstance, we will address the dependence case by case. For simplicity, $\left(z; q^2\right)_m$ will be denoted as $(z)_m$.

5.1 Parameters

Recall that we have the parameters $\mathcal{P}_i = (a_i, b_i, c_i, d_i, e_i)$ satisfying (4.2) attached to each vertex i of the quiver. In what follows, we will also use the following:

$$C_1 = \frac{1}{2}(b_1 - b_2 + c_1 - c_3 + d_2 - d_3), \qquad C_2 = -\frac{1}{2}(c_1 - c_2 + c_3 + b_1 + a_2 + d_3),$$
$$C_3 = \frac{1}{2}(c_1 - c_2 + c_3), \qquad C_4 = \frac{1}{2}(a_2 + b_2 + c_2 + d_2), \qquad C_5 = \frac{1}{2}(a_3 - c_2 + c_3 - d_1 + d_3),$$
$$C_6 = \frac{1}{2}(a_1 + b_1 - b_3 + c_1 - c_2), \qquad C_7 = \frac{1}{2}(-d_1 - c_2 - b_3),$$
$$C_8 = \frac{1}{2}(a_1 + a_3 + b_1 + c_1 - c_2 + c_3 + d_3). \tag{5.1}$$

They satisfy the relation

$$C_5 + C_6 - C_7 - C_8 = 0. \tag{5.2}$$

The parameters e_i in (4.2) and λ_i in (4.8) are expressed as

$$\begin{aligned}
&\lambda_0 = -C_4 + C_5 - C_7, \qquad \lambda_1 = -C_1 - C_2 - C_5 + C_7, \qquad e_1 = 2(C_7 - C_6), \qquad g_1 = \frac{e_1}{2\hbar},\\
&\lambda_2 = C_1 - C_2 - C_6 - C_8, \qquad e_2 = -2C_4, \qquad g_2 = \frac{e_2}{2\hbar},\\
&\lambda_3 = 2C_3, \quad e_3 = 2(C_7 - C_5), \qquad g_3 = \frac{e_3}{2\hbar},
\end{aligned} \tag{5.3}$$

where we have also introduced g_1, g_2 and g_3. Now the R-matrix $R = R_{123}$ (4.15) is expressed as

$$\begin{aligned}
R = {}& \Psi_q\big(\mathrm{e}^{2C_7+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)^{-1}\Psi_q\big(\mathrm{e}^{2C_5+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)^{-1}\\
&\times P\Psi_q\big(\mathrm{e}^{\alpha_6+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)\Psi_q\big(\mathrm{e}^{\alpha_8+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big),
\end{aligned} \tag{5.4}$$

where $\alpha_6 = -b_1 - a_2 - d_3 - e_3$ and $\alpha_8 = -b_1 - a_2 - d_3$. They are also expressed as

$$\begin{pmatrix}\alpha_6\\ \alpha_8\end{pmatrix} = \begin{pmatrix}-\lambda_1 - \lambda_2 + \lambda_3 - 2C_6\\ -\lambda_1 - \lambda_2 + \lambda_3 - 2C_8\end{pmatrix} = \begin{pmatrix}2C_2 + 2C_3 + 2C_5 - 2C_7\\ 2C_2 + 2C_3\end{pmatrix}. \tag{5.5}$$

The operator $P = P_{123}$ (4.12) reads

$$P = \mathrm{e}^{\frac{1}{\hbar}(\mathsf{u}_3-\mathsf{u}_2)\mathsf{w}_1}\mathrm{e}^{g_{23}(-\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3)}\mathrm{e}^{\frac{1}{\hbar}(\lambda_1\mathsf{u}_1+\lambda_2\mathsf{u}_2+\lambda_3\mathsf{u}_3)}\rho_{23}, \tag{5.6}$$

where we have set $g_{23} = g_2 - g_3$, which is equal to $\frac{\lambda_0}{\hbar} = \frac{-C_4+C_5-C_7}{\hbar}$. The formula (5.4) is the same with (1.1). We note that the transformation α^{uw} (4.26) is expressed as

$$\alpha^{\mathsf{uw}}\colon\ C_1 \to -C_1, \qquad C_5 \leftrightarrow C_6, \qquad \text{other } C_i\text{'s are invariant}, \tag{5.7}$$

as far as the parameters are concerned.

Remark 5.1. By shifting the canonical variables u_i and w_i, one can set $c_i = d_i = 0$ without loss of generality. See Figure 6. In this parametrization, the constraint $C_3 = C_1 + C_2 + C_4 = 0$ holds in addition to (5.2). Consequently, our solution (1.1) involves five parameters, in addition to the parameter q.

5.2 Elements of R in u-diagonal representation

Let $F = \bigoplus_{n_1,n_2,n_3\in\mathbb{Z}} \mathbb{C}|n_1, n_2, n_3\rangle$ and $F^* = \bigoplus_{n_1,n_2,n_3\in\mathbb{Z}} \mathbb{C}\langle n_1, n_2, n_3|$ be the infinite dimensional spaces. Define the representations of the direct product of the q-Weyl algebra (see the explanation around (4.1)) on them by

$$\begin{aligned}
&\mathrm{e}^{\mathsf{u}_k}|\mathbf{n}\rangle = iq^{n_k+\frac{1}{2}}|\mathbf{n}\rangle, \qquad \mathrm{e}^{\mathsf{w}_k}|\mathbf{n}\rangle = |\mathbf{n}+\mathbf{e}_k\rangle, \qquad \langle\mathbf{n}|\mathrm{e}^{\mathsf{u}_k} = \langle\mathbf{n}|iq^{n_k+\frac{1}{2}},\\
&\langle\mathbf{n}|\mathrm{e}^{\mathsf{w}_k} = \langle\mathbf{n}-\mathbf{e}_k|
\end{aligned} \tag{5.8}$$

for $k = 1, 2, 3$. Here $|n_1, n_2, n_3\rangle$ (resp. $\langle n_1, n_2, n_3|$) is simply denoted by $|\mathbf{n}\rangle$ (resp. $\langle\mathbf{n}|$) with $\mathbf{n} \in \mathbb{Z}^3$, and $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$. The dual pairing of F and F^* is defined by $\langle\mathbf{n}|\mathbf{n}'\rangle = \delta_{\mathbf{n},\mathbf{n}'}$, which satisfies $(\langle\mathbf{n}|X)|\mathbf{n}'\rangle = \langle\mathbf{n}|(X|\mathbf{n}'\rangle)$.

In the rest of this subsection, we assume that g_i's defined in (5.3) satisfy the condition

$$g_i \in \mathbb{Z} \quad (i = 1, 2, 3). \tag{5.9}$$

Theorem 5.2. *Under the assumption* (5.9)*, the element* $R^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} := \langle n_1, n_2, n_3|R\big|n'_1, n'_2, n'_3\rangle$ *of the* R*-matrix* (5.4) *is given by*

$$R^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} = \kappa\delta^{n_1+n_2}_{n'_1+n'_2}\delta^{n_2+n_3}_{n'_2+n'_3}\mathrm{e}^{\lambda_1 n'_1+\lambda_2 n'_3+\lambda_3 n'_2}\big(\mathrm{e}^{2C_5}q^{n_1+g_3}\big)^{n_3+g_3}q^{n'_2+g_2}$$

$$\times \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z^{n_2'+g_2+1}} \frac{\left(-z\mathrm{e}^{-2C_8} q^{2+n_1'+n_3'}\right)_\infty \left(-z\mathrm{e}^{-2C_7} q^{-n_1-n_3}\right)_\infty}{\left(-z\mathrm{e}^{-2C_6} q^{n_1'-n_3'}\right)_\infty \left(-z\mathrm{e}^{-2C_5} q^{n_3'-n_1'}\right)_\infty}, \tag{5.10}$$

where $\kappa = \mathrm{e}^{\frac{1}{2}(\frac{\mathrm{i}\pi}{\hbar}+1)(\lambda_1+\lambda_2+\lambda_3)}$. *The integral encircles* $z = 0$ *anti-clockwise picking the residue at the origin.*

Proof. From the expansions (2.5) and the commutation relations $\mathrm{e}^{\mathsf{u}_i}\mathrm{e}^{\mathsf{w}_j} = q^{\delta_{ij}}\mathrm{e}^{\mathsf{w}_j}\mathrm{e}^{\mathsf{u}_i}$, $\mathrm{e}^{\mathsf{u}_i}\mathrm{e}^{\mathsf{u}_j} = \mathrm{e}^{\mathsf{u}_j}\mathrm{e}^{\mathsf{u}_i}$ and $\mathrm{e}^{\mathsf{w}_i}\mathrm{e}^{\mathsf{w}_j} = \mathrm{e}^{\mathsf{w}_j}\mathrm{e}^{\mathsf{w}_i}$, one has

$$\Psi_q\left(\mathrm{e}^{2C_7+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right)^{-1} = \sum_{k\ge 0} \frac{1}{\left(q^2\right)_k} \left(\mathrm{e}^{2C_7+\mathsf{u}_1+\mathsf{u}_3}\right)^k \left(\mathrm{e}^{\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right)^k,$$

$$\Psi_q\left(\mathrm{e}^{2C_5+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right)^{-1} = \sum_{l\ge 0} \frac{q^{l^2}}{\left(q^2\right)_l} \left(\mathrm{e}^{2C_5+\mathsf{u}_1-\mathsf{u}_3}\right)^l \left(\mathrm{e}^{\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right)^l,$$

$$\Psi_q\left(\mathrm{e}^{\alpha_6+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right) = \sum_{r\ge 0} \frac{q^r}{\left(q^2\right)_r} \left(\mathrm{e}^{\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right)^r \left(-\mathrm{e}^{\alpha_6+\mathsf{u}_1-\mathsf{u}_3}\right)^r,$$

$$\Psi_q\left(\mathrm{e}^{\alpha_8+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right) = \sum_{s\ge 0} \frac{q^{s^2+s}}{\left(q^2\right)_s} \left(\mathrm{e}^{\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right)^s \left(-\mathrm{e}^{\alpha_8+\mathsf{u}_1+\mathsf{u}_3}\right)^s.$$

By utilizing them and (5.8), we get

$$\begin{aligned}
&\langle n_1, n_2, n_3| \Psi_q\left(\mathrm{e}^{2C_7+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right)^{-1} \Psi_q\left(\mathrm{e}^{2C_5+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right)^{-1} \\
&\quad = \sum_{k,l\ge 0} \frac{q^{l^2}}{\left(q^2\right)_k \left(q^2\right)_l} \left(-\mathrm{e}^{2C_7} q^{1+n_1+n_3}\right)^k \\
&\qquad \times \left(\mathrm{e}^{2C_5} q^{n_1-n_3}\right)^l \langle n_1-k-l, n_2+k+l, n_3-k-l|,
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
&\Psi_q\left(\mathrm{e}^{\alpha_6+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right) \Psi_q\left(\mathrm{e}^{\alpha_8+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\right) \left|n_1', n_2', n_3'\right\rangle \\
&\quad = \sum_{r,s\ge 0} \frac{q^{s^2}}{\left(q^2\right)_r \left(q^2\right)_s} \left(\mathrm{e}^{\alpha_8} q^{2+n_1'+n_3'}\right)^s \\
&\qquad \times \left(-\mathrm{e}^{\alpha_6} q^{1+n_1'-n_3'}\right)^r \left|n_1'+r+s, n_2'-r-s, n_3'+r+s\right\rangle.
\end{aligned} \tag{5.12}$$

Elements of P (5.6) are calculated as

$$\begin{aligned}
&\langle n_1, n_2, n_3| P \left|n_1', n_2', n_3'\right\rangle \\
&\quad = \langle n_1, n_2, n_3| \mathrm{e}^{(n_3-n_2)\mathsf{w}_1} \mathrm{e}^{g_{23}(-\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3)} \mathrm{e}^{\frac{1}{\hbar}(\lambda_1\mathsf{u}_1+\lambda_2\mathsf{u}_2+\lambda_3\mathsf{u}_3)} \left|n_1', n_3', n_2'\right\rangle \\
&\quad = \kappa \langle n_1+n_2-n_3, n_2, n_3| \mathrm{e}^{g_{23}(-\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3)} \left|n_1', n_3', n_2'\right\rangle \mathrm{e}^{\lambda_1 n_1'+\lambda_2 n_3'+\lambda_3 n_2'} \\
&\quad = \kappa \langle n_1+n_2-n_3, n_2, n_3| \left|n_1'-g_{23}, n_3'-g_{23}, n_2'+g_{23}\right\rangle \mathrm{e}^{\lambda_1 n_1'+\lambda_2 n_3'+\lambda_3 n_2'} \\
&\quad = \kappa \delta^{n_1+n_2}_{n_1'+n_2'} \delta^{n_2+n_3}_{n_2'+n_3'} \delta^{n_3}_{n_2'+g_{23}} \mathrm{e}^{\lambda_1 n_1'+\lambda_2 n_3'+\lambda_3 n_2'},
\end{aligned} \tag{5.13}$$

where κ is defined after (5.10). Combining (5.11), (5.12) and (5.13), we get

$$\begin{aligned}
&R^{n_1,n_2,n_3}_{n_1',n_2',n_3'} \\
&\quad = \kappa \delta^{n_1+n_2}_{n_1'+n_2'} \delta^{n_2+n_3}_{n_2'+n_3'} \sum_{k,l,r,s\ge 0} \delta^{n_3-k-l}_{n_2'-r-s+g_{23}} q^{l^2+s^2} \mathrm{e}^{\lambda_1(n_1'+r+s)+\lambda_2(n_3'+r+s)+\lambda_3(n_2'-r-s)} \\
&\qquad \times \frac{\left(-\mathrm{e}^{2C_7} q^{1+n_1+n_3}\right)^k \left(\mathrm{e}^{2C_5} q^{n_1-n_3}\right)^l \left(\mathrm{e}^{\alpha_8} q^{2+n_1'+n_3'}\right)^s \left(-\mathrm{e}^{\alpha_6} q^{1+n_1'-n_3'}\right)^r}{\left(q^2\right)_k \left(q^2\right)_l \left(q^2\right)_s \left(q^2\right)_r}
\end{aligned}$$

$$
\begin{aligned}
&= \kappa \delta^{n_1+n_2}_{n'_1+n'_2} \delta^{n_2+n_3}_{n'_2+n'_3} \mathrm{e}^{\lambda_1 n'_1+\lambda_2 n'_3+\lambda_3 n'_2} \sum_{k,l,r,s\geq 0} \delta^{n_3-k-l}_{n'_2-r-s+g_{23}} q^{l^2+s^2} \\
&\quad \times \frac{\left(-\mathrm{e}^{2C_7} q^{1+n_1+n_3}\right)^k \left(\mathrm{e}^{2C_5} q^{n'_1-n'_3}\right)^l \left(\mathrm{e}^{-2C_8} q^{2+n'_1+n'_3}\right)^s \left(-\mathrm{e}^{-2C_6} q^{1+n'_1-n'_3}\right)^r}{\left(q^2\right)_k \left(q^2\right)_l \left(q^2\right)_s \left(q^2\right)_r},
\end{aligned}
$$

where the last step uses (5.5) and $n_1 - n_3 = n'_1 - n'_3$ under the two Kronecker delta's. The last line is the coefficient of $z^{n'_2-n_3+g_{23}}$ of

$$
\begin{aligned}
&\sum_{k,l,r,s\geq 0} \frac{\left(-z^{-1}\mathrm{e}^{2C_7} q^{1+n_1+n_3}\right)^k}{\left(q^2\right)_k} \frac{q^{l^2}\left(z^{-1}\mathrm{e}^{2C_5} q^{n'_1-n'_3}\right)^l}{\left(q^2\right)_l} \frac{q^{s^2}\left(z\mathrm{e}^{-2C_8} q^{2+n'_1+n'_3}\right)^s}{\left(q^2\right)_s} \\
&\qquad \times \frac{\left(-z\mathrm{e}^{-2C_6} q^{1+n'_1-n'_3}\right)^r}{\left(q^2\right)_r} = \frac{\left(-z^{-1}\mathrm{e}^{2C_5} q^{1+n'_1-n'_3}\right)_\infty \left(-z\mathrm{e}^{-2C_8} q^{3+n'_1+n'_3}\right)_\infty}{\left(-z^{-1}\mathrm{e}^{2C_7} q^{1+n_1+n_3}\right)_\infty \left(-z\mathrm{e}^{-2C_6} q^{1+n'_1-n'_3}\right)_\infty} = \Gamma(z) H(z),
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma(z) &= \frac{\left(-z\mathrm{e}^{-2C_7} q^{1-n_1-n_3}\right)_\infty \left(-z\mathrm{e}^{-2C_8} q^{3+n'_1+n'_3}\right)_\infty}{\left(-z\mathrm{e}^{-2C_5} q^{1+n'_3-n'_1}\right)_\infty \left(-z\mathrm{e}^{-2C_6} q^{1+n'_1-n'_3}\right)_\infty}, \\
H(z) &= \frac{\left(-z\mathrm{e}^{-2C_5} q^{1+n'_3-n'_1}\right)_\infty \left(-z^{-1}\mathrm{e}^{2C_5} q^{1+n'_1-n'_3}\right)_\infty}{\left(-z\mathrm{e}^{-2C_7} q^{1-n_1-n_3}\right)_\infty \left(-z^{-1}\mathrm{e}^{2C_7} q^{1+n_1+n_3}\right)_\infty}.
\end{aligned}
$$

Thus far we have shown

$$
R^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} = \kappa \delta^{n_1+n_2}_{n'_1+n'_2} \delta^{n_2+n_3}_{n'_2+n'_3} \mathrm{e}^{\lambda_1 n'_1+\lambda_2 n'_3+\lambda_3 n'_2} \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} z^{-n'_2+n_3-g_{23}} \Gamma(z) H(z). \tag{5.14}
$$

Note that $f(\xi) := (-q\xi)_\infty \left(-q\xi^{-1}\right)_\infty$ satisfies

$$
f(\xi) = q\xi^{-1} f\left(\xi q^{-2}\right) = q^{m^2} \xi^{-m} f\left(\xi q^{-2m}\right)
$$

for any $m \in \mathbb{Z}$. Setting $\xi = z\mathrm{e}^{-2C_5} q^{n'_3-n'_1} = z\mathrm{e}^{-2C_5} q^{n_3-n_1}$, $m = n_3 + g_3$, and using $\mathrm{e}^{-2C_5} = \mathrm{e}^{-2C_7+e_3} = \mathrm{e}^{-2C_7} q^{2g_3}$, we find

$$
H(z) = f(\xi)/f\left(\xi q^{-2m}\right) = \left(\mathrm{e}^{2C_5} q^{n_1+g_3}\right)^{n_3+g_3} z^{-n_3-g_3}.
$$

Substituting this into (5.14) and replacing z by $q^{-1}z$, we obtain (5.10). ■

Remark 5.3. The factor $\mathrm{e}^{\lambda_1 n'_1+\lambda_2 n'_3+\lambda_3 n'_2} \left(\mathrm{e}^{2C_5} q^{n_1+g_3}\right)^{n_3+g_3} q^{n'_2+g_2}$ in the first line of (5.10) is equal to

$$
\mathrm{e}^{-C_4+\frac{1}{\hbar}(-C_5^2+C_7^2)+C_7(n_1+n_3)+(C_1+C_5-C_6)(n'_3-n'_1)-(C_2+C_8)(n'_1+n'_3)+2C_3 n'_2} q^{n_1 n_3+n'_2}
$$

under the condition implied by $\delta^{n_1+n_2}_{n'_1+n'_2} \delta^{n_2+n_3}_{n'_2+n'_3}$. Therefore, the result (5.10) fulfills the symmetry (4.30)[5] due to (5.7) and the fact that the right-hand side of (4.29) is equal to $\exp\left(\left(C_6^2 - C_5^2\right)/\hbar\right)$.

Remark 5.4. When $a_i = b_i = c_i = d_i = e_i = 0$ for $i = 1, 2, 3$, hence $\forall C_i = \forall \lambda_i = \forall g_i = 0$, the formula (5.10) coincides, including the overall normalization, with $R^{n'_1,n'_2,n'_3}_{n_1,n_2,n_3}$ from [17, equation (3.81)] for the elements of the R-matrix [14] originally discovered from the representation theory of quantized coordinate ring. A similar integral formula was recognized earlier in the footnote of [3, p. 5]. In this case, $\Gamma(z)$ reduces to a rational function of z (see the explanation after [17, equation (3.81)]), and the tetrahedron equation closes among elements with non-negative integer indices.

[5] α^{uw} should also be accompanied by the interchange $\left(n_i, n'_i\right) \leftrightarrow \left(n_{4-i}, n'_{4-i}\right)$ in view of (4.26).

5.3 Elements of R in w-diagonal representation

Let us turn to the representation of the canonical variables in which w_k's are diagonal,

$$\mathrm{e}^{\mathsf{u}_k}|\mathbf{n}\rangle = |\mathbf{n}-\mathbf{e}_k\rangle, \qquad \mathrm{e}^{\mathsf{w}_k}|\mathbf{n}\rangle = q^{n_k}|\mathbf{n}\rangle, \qquad \langle\mathbf{n}|\mathrm{e}^{\mathsf{u}_k} = \langle\mathbf{n}+\mathbf{e}_k|,$$
$$\langle\mathbf{n}|\mathrm{e}^{\mathsf{w}_k} = \langle\mathbf{n}|q^{n_k}, \tag{5.15}$$

where notations are similar to (5.8). We employ the same pairing $\langle\mathbf{n}|\mathbf{n}'\rangle = \delta_{\mathbf{n},\mathbf{n}'}$ and the notation $g_{23} = g_2 - g_3 = (-C_4 + C_5 - C_7)\hbar^{-1}$ introduced after (5.6). In this subsection, we assume

$$\ell_i := \frac{\lambda_i}{\hbar} \in \mathbb{Z}, \qquad i = 1,2,3, \tag{5.16}$$

where λ_i's are defined in (5.3).

Theorem 5.5. *Under the assumption (5.16), the element $S^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} := \langle n_1,n_2,n_3|R\big|n'_1,n'_2,n'_3\rangle$ of the R-matrix (5.4) is given by*

$$S^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} = \frac{(-1)^{\frac{\nu_1}{2}} q^{\psi+\omega}(q^{\nu_3+\nu_4})_\infty \big(q^2\big)^3_\infty}{(q^{\nu_1})_\infty(q^{\nu_2})_\infty(q^{\nu_3})_\infty(q^{\nu_4})_\infty}, \tag{5.17}$$
$$\nu_1 = 2C_3\hbar^{-1} + n_1 + n_3 - n'_2, \qquad \nu_2 = -2(C_2+C_8)\hbar^{-1} + n_2 - n'_1 - n'_3, \tag{5.18}$$
$$\nu_3 = 2(C_1 - C_3 + C_5)\hbar^{-1} - n_1 - n_2 + n_3 + n'_1 + n'_2 - n'_3, \tag{5.19}$$
$$\nu_4 = 2(-C_1 - C_3 + C_6)\hbar^{-1} + n_1 - n_2 - n_3 - n'_1 + n'_2 + n'_3, \tag{5.20}$$
$$\begin{aligned}\psi &= \tfrac14\big(-(\nu_1-\nu_2)(\nu_3+\nu_4) + \nu_3\nu_4 - \nu_1^2 + 2\nu_1\big)\\ &\quad + \tfrac{1}{2\hbar}\big((C_8 - C_7)(\nu_1+\nu_2) + (C_8 - C_6 - C_4)\nu_3 + (C_8 - C_5 - C_4)\nu_4\big),\end{aligned} \tag{5.21}$$

where ω is independent of n_i, n'_i, and given by $\omega = (C_5 + C_6)(C_4 - C_5 + C_7)/\hbar^2$.

Proof. The derivation is similar to Theorem 5.2. We have

$$\begin{aligned}&\langle n_1,n_2,n_3|\Psi_q\big(\mathrm{e}^{2C_7+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)^{-1}\Psi_q\big(\mathrm{e}^{2C_5+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)^{-1}\\ &\qquad = \sum_{k,l\ge 0}\frac{q^{2k^2+l^2}}{\big(q^2\big)_k\big(q^2\big)_l}\big(\mathrm{e}^{2C_7}q^{n_1-n_2+n_3}\big)^k A^l\langle n_1+k+l, n_2, n_3+k-l|,\\ &\Psi_q\big(\mathrm{e}^{\alpha_6+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)\Psi_q\big(\mathrm{e}^{\alpha_8+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)\big|n'_1,n'_2,n'_3\big\rangle\\ &\qquad = \sum_{r,s\ge 0}\frac{q^{-s^2+s+r}}{\big(q^2\big)_r\big(q^2\big)_s}\big(-\mathrm{e}^{\alpha_8}q^{n'_1-n'_2+n'_3}\big)^s B^r\big|n'_1-r-s, n'_2, n'_3+r-s\big\rangle,\\ &\langle n_1,n_2,n_3|P\big|n'_1,n'_2,n'_3\big\rangle = q^{g_{23}(n_1-n_2+n_3)}\delta^{n_1+\ell_1}_{n'_1}\delta^{n_1+n_3+\ell_3}_{n'_2}\delta^{n_2-n_1+\ell_2}_{n'_3},\end{aligned}$$

where $A = \mathrm{e}^{2C_5}q^{n_1-n_2+n_3+2k}$ and $B = -\mathrm{e}^{\alpha_6}q^{n'_1-n'_2+n'_3-2s}$. They lead to

$$\begin{aligned}\langle n_1,n_2,n_3|R\big|n'_1,n'_2,n'_3\big\rangle &= \frac{q^{2k^2-s^2+s}}{\big(q^2\big)_k\big(q^2\big)_s}q^{g_{23}(n_1-n_2+n_3)}\big(\mathrm{e}^{2C_7}q^{n_1-n_2+n_3}\big)^k\big(-\mathrm{e}^{\alpha_8}q^{n'_1-n'_2+n'_3}\big)^s\\ &\quad\times\sum_{l+r=M}\frac{q^{l^2+r}}{\big(q^2\big)_r\big(q^2\big)_l}A^l B^r\end{aligned} \tag{5.22}$$

with k, s and M fixed as[6]

$$k = \frac12\big(n'_2 - n_1 - n_3 - \ell_3\big) = -\frac{\nu_1}{2}, \qquad s = \frac12\big(n'_1 + n'_3 - n_2 - \ell_1 - \ell_2\big) = -\frac{\nu_2}{2},$$

[6]There is a parity condition on n_i, n'_i, ℓ_i in order to ensure $k,s\in\mathbb{Z}$ in (5.23). However the final formula (5.17) makes sense for generic C_i's.

$$\begin{aligned} M &= n'_1 - n_1 - k - s - \ell_1 \\ &= \frac{1}{2}\bigl(-n_1 + n_2 + n_3 + n'_1 - n'_2 - n'_3 - \ell_1 + \ell_2 + \ell_3\bigr) = -\frac{\nu_4}{2}. \end{aligned} \tag{5.23}$$

The last line of (5.22) is $(qB)^M\bigl(-AB^{-1}\bigr)_M/\bigl(q^2\bigr)_M$ with $AB^{-1} = -q^{\nu_3+\nu_4}$. Thus (5.22) is equal to

$$\frac{q^{\psi'}\bigl(q^{\nu_3+\nu_4}\bigr)_{-\nu_4/2}}{\bigl(q^2\bigr)_{-\nu_1/2}\bigl(q^2\bigr)_{-\nu_2/2}\bigl(q^2\bigr)_{-\nu_4/2}} = \frac{q^{\psi'}\bigl(q^{\nu_3+\nu_4}\bigr)_\infty}{\bigl(q^2\bigr)_{-\nu_1/2}\bigl(q^2\bigr)_{-\nu_2/2}\bigl(q^2\bigr)_{-\nu_4/2}(q^{\nu_3})_\infty}$$

for some power ψ'. Rewriting $\bigl(q^2\bigr)_{-\nu_i/2}$ in the denominator ($i = 1,2,4$) as $(-1)^{-\frac{\nu_i}{2}}q^{\frac{\nu_i}{2}(\frac{\nu_i}{2}-1)}$ $\times\,(q^{\nu_i})_\infty/\bigl(q^2\bigr)_\infty$, we obtain (5.17)–(5.21). ■

It is easily confirmed that the result (5.17) fulfills the symmetry (4.30).

A slightly more explicit form of (5.17) is

$$\begin{aligned} &S^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} \\ &\quad\equiv q^{\psi_0}\bigl(-\mathrm{e}^{-2C_7}\bigr)^{\frac{m_1}{2}}\bigl(\mathrm{e}^{2C_8-2C_3}\bigr)^{\frac{m_2}{2}}\bigl(\mathrm{e}^{-C_1-C_2-2C_3-C_4}\bigr)^{\frac{m_3}{2}}\bigl(\mathrm{e}^{C_1-C_2-2C_3-C_4}\bigr)^{\frac{m_4}{2}} \\ &\qquad\times\frac{\bigl(\mathrm{e}^{-4C_3+2C_5+2C_6}q^{m_3+m_4}\bigr)_\infty\bigl(q^2\bigr)^3_\infty}{\bigl(\mathrm{e}^{2C_3}q^{m_1}\bigr)_\infty\bigl(\mathrm{e}^{-2C_2-2C_8}q^{m_2}\bigr)_\infty\bigl(\mathrm{e}^{2C_1-2C_3+2C_5}q^{m_3}\bigr)_\infty\bigl(\mathrm{e}^{-2C_1-2C_3+2C_6}q^{m_4}\bigr)_\infty}, \end{aligned} \tag{5.24}$$

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} n_1 + n_3 - n'_2 \\ n_2 - n'_1 - n'_3 \end{pmatrix}, \qquad \begin{pmatrix} m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} -n_1 - n_2 + n_3 + n'_1 + n'_2 - n'_3 \\ n_1 - n_2 - n_3 - n'_1 + n'_2 + n'_3 \end{pmatrix}, \tag{5.25}$$

$$\psi_0 = \frac{1}{4}\bigl(-(m_1 - m_2)(m_3 + m_4) + m_3m_4 - m_1^2 + 2m_1\bigr), \tag{5.26}$$

up to an overall factor depending on $C_1,\dots,C_8$. The formula (1.3) is obtained, up to an overall factor, by replacing the infinite products $(zq^m)_\infty$ appearing here with $\bigl(z;q^2\bigr)_\infty/\bigl(z;q^2\bigr)_{\frac{m}{2}}$.[7]

Let us compare the above $S^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$ with the R-matrix R^{ZZZ} obtained in [18, equation (45)]. We write the element $R^{a,b,c}_{i,j,k}$ therein as $X^{a,b,c}_{i,j,k}$ here for distinction. It contains twelve parameters (r_j,s_j,t_j,w_j) ($j = 1,2,3$). Apply [18, equation (51)] to rewrite the first factor in its denominator and replace the parameters as $(t_j,w_j) \to \bigl(-iq^{-\frac{1}{2}}t_j, t_j^{-1}w_j\bigr)$. The result reads

$$\begin{aligned} X^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} &= q^{\varphi}\left(\frac{r_1r_3}{t_3w_1}\right)^{\frac{m_1}{2}}\left(-\frac{s_2}{t_1w_3}\right)^{\frac{m_2}{2}}\left(\frac{t_2}{s_1t_3}\right)^{\frac{m_3}{2}}\left(\frac{w_2}{s_3w_1}\right)^{\frac{m_4}{2}} \\ &\quad\times\frac{\Theta_{m_1}\bigl(\frac{r_2}{r_1r_3}\bigr)\Theta_{m_2}\bigl(\frac{s_1s_3}{s_2}\bigr)\Theta_{m_3}\bigl(\frac{r_3t_1w_2}{s_3t_2w_1}\bigr)\Theta_{m_4}\bigl(\frac{r_1t_2w_3}{s_1t_3w_2}\bigr)}{\Theta_{m_3+m_4}\bigl(\frac{r_1r_3t_1w_3}{s_1s_3t_3w_1}\bigr)}, \\ \varphi &= \frac{1}{4}\bigl((m_1 - m_2)(m_3 + m_4) + m_3m_4 - m_2^2 + 2m_2\bigr), \end{aligned} \tag{5.27}$$

where m_j's are those in (5.25). The function $\Theta_m(z)$ is defined up to normalization by $\Theta_{m+2}(z) = (1 - zq^m)\Theta_m(z)$.

Remark 5.6. With the choice $\Theta_m(z) = 1/\bigl(zq^m;q^2\bigr)_\infty$ and the identification of parameters as

$$\begin{aligned} &\mathrm{e}^{C_1} = \sqrt{\frac{r_1t_2w_3}{r_3t_1w_2}}, \qquad \mathrm{e}^{C_2} = \sqrt{\frac{r_2t_1w_3}{r_1r_3s_2}}, \qquad \mathrm{e}^{C_3} = \sqrt{\frac{r_1r_3}{r_2}}, \qquad \mathrm{e}^{C_4} = \sqrt{\frac{r_2s_2}{t_2w_2}}, \\ &\mathrm{e}^{C_5} = \sqrt{\frac{r_3s_3w_1}{r_2w_3}}, \qquad \mathrm{e}^{C_6} = \sqrt{\frac{r_1s_1t_3}{r_2t_1}}, \qquad \mathrm{e}^{C_7} = \sqrt{\frac{t_3w_1}{r_2}}, \qquad \mathrm{e}^{C_8} = \sqrt{\frac{r_1r_3s_1s_3}{r_2t_1w_3}}, \end{aligned} \tag{5.28}$$

[7] For a proper treatment of indices with both parities, see [18, equation (49)] and also [18, Proposition 2].

the elements (5.24) and (5.27) are related as

$$S^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} \equiv X^{-n'_3,-n'_2,-n'_1}_{-n_3,-n_2,-n_1}\big|_{(r_i,s_i,t_i,w_i)\to(s_{4-i},r_{4-i},t_{4-i},w_{4-i})}.$$

The replacement $\left(n_1,n_2,n_3,n'_1,n'_2,n'_3\right) \to \left(-n'_3,-n'_2,-n'_1,-n_3,-n_2,-n_1\right)$ in the right-hand side induces the exchange $m_1 \leftrightarrow m_2$ and $m_3 \leftrightarrow m_4$, converting φ into ψ_0. Thus we have elucidated a quantum cluster algebra theoretic origin of the R-matrix R^{ZZZ} [18].

Remark 5.7. Apart from trivial rescaling of generators, a q-Weyl algebra $\langle \mathrm{e}^{\pm\mathsf{u}}, \mathrm{e}^{\pm\mathsf{w}}\rangle$ with the commutation relation $\mathrm{e}^{\mathsf{u}}\mathrm{e}^{\mathsf{w}} = q\mathrm{e}^{\mathsf{w}}\mathrm{e}^{\mathsf{u}}$ has the automorphism labeled with $\mathrm{SL}(2,\mathbb{Z})$

$$\iota_f\colon\ \begin{cases}\mathrm{e}^{\mathsf{u}} \mapsto \mathrm{e}^{\alpha\mathsf{u}}\mathrm{e}^{\beta\mathsf{w}},\\ \mathrm{e}^{\mathsf{w}} \mapsto \mathrm{e}^{\gamma\mathsf{u}}\mathrm{e}^{\delta\mathsf{w}},\end{cases} \qquad f = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Recall the n-fold direct product of the q-Weyl algebra $\mathcal{W}_n$ introduced in Section 4. Given a representation $\rho_1\otimes\rho_2\otimes\rho_3\colon \mathcal{W}_3 \to \mathrm{End}(V_1\otimes V_2\otimes V_3)$ of $\mathcal{W}_3$, one generates an infinite family of representations via the pullback

$$\rho_1^{f_1}\otimes\rho_2^{f_2}\otimes\rho_3^{f_3}\colon\ \mathcal{W}_3 \xrightarrow{\iota_{f_1}\otimes\iota_{f_2}\otimes\iota_{f_3}} \mathcal{W}_3 \xrightarrow{\rho_1\otimes\rho_2\otimes\rho_3} \mathrm{End}(V_1\otimes V_2\otimes V_3).$$

The u-diagonal representation and the w-diagonal representation considered in this section are essentially transformed by the above automorphism. They are just two special "homogeneous" cases $\rho_1 = \rho_2 = \rho_3$, where the computation of the elements of $\mathrm{e}^{(\mathsf{u}_3-\mathsf{u}_2)\mathsf{w}_1/\hbar}$ in (5.6) is simple. A similar remark applies also to the treatment in the next section. In the context of the $RLLL$ relation (cf. Section 7), a study of the case $\rho^{f_1}\otimes\rho^{f_2}\otimes\rho^{f_3}$ with non-identical f_1, f_2, f_3 has been undertaken in [18].

6 Modular $\mathcal{R}$ and its elements

6.1 Modular $\mathcal{R}$

We use a parameter $\mathfrak{b}$ and set

$$\hbar = \mathrm{i}\pi\mathfrak{b}^2, \qquad q = \mathrm{e}^{\mathrm{i}\pi\mathfrak{b}^2}, \qquad q^\vee = \mathrm{e}^{\mathrm{i}\pi\mathfrak{b}^{-2}}, \qquad \bar{q} = \mathrm{e}^{-\mathrm{i}\pi\mathfrak{b}^{-2}}, \qquad \eta = \frac{\mathfrak{b}+\mathfrak{b}^{-1}}{2}. \tag{6.1}$$

The non-compact quantum dilogarithm is defined by

$$\Phi_{\mathfrak{b}}(z) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+\mathrm{i}0} \frac{\mathrm{e}^{-2\mathrm{i}zw}}{\sinh(w\mathfrak{b})\sinh(w/\mathfrak{b})}\frac{\mathrm{d}w}{w}\right) = \frac{\left(\mathrm{e}^{2\pi(z+\mathrm{i}\eta)\mathfrak{b}}; q^2\right)_\infty}{\left(\mathrm{e}^{2\pi(z-\mathrm{i}\eta)\mathfrak{b}^{-1}}; \bar{q}^2\right)_\infty}, \tag{6.2}$$

where the integral avoids the singularity at $w = 0$ from above. The infinite product formula is valid in the so-called strong coupling regime $0 < \eta < 1$ with $0 < \operatorname{Im}\mathfrak{b} < \frac{\pi}{2}$. It enjoys the symmetry $\Phi_{\mathfrak{b}}(z) = \Phi_{\mathfrak{b}^{-1}}(z)$, and has the following properties (see also [4])

$$\Phi_{\mathfrak{b}}(z)\Phi_{\mathfrak{b}}(-z) = \mathrm{e}^{\mathrm{i}\pi z^2 - \mathrm{i}\pi(1-2\eta^2)/6}, \tag{6.3}$$

$$\frac{\Phi_{\mathfrak{b}}\left(z - \mathrm{i}\mathfrak{b}^{\pm1}/2\right)}{\Phi_{\mathfrak{b}}\left(z + \mathrm{i}\mathfrak{b}^{\pm1}/2\right)} = 1 + \mathrm{e}^{2\pi z\mathfrak{b}^{\pm1}}, \tag{6.4}$$

$$\Phi_{\mathfrak{b}}(z) \to \begin{cases}1, & \operatorname{Re} z \to -\infty,\\ \mathrm{e}^{\mathrm{i}\pi z^2 - \mathrm{i}\pi(1-2\eta^2)/6}, & \operatorname{Re} z \to \infty,\end{cases} \tag{6.5}$$

$$\text{poles of } \Phi_{\mathfrak{b}}(z)^{\pm1} = \left\{\pm\left(\mathrm{i}\eta + \mathrm{i}m\mathfrak{b} + \mathrm{i}n\mathfrak{b}^{-1}\right) \mid m,n \in \mathbb{Z}_{\geq 0}\right\}. \tag{6.6}$$

Recall that u_k, w_k $(k = 1, 2, 3)$ are canonical variables obeying (4.1). In this section, we work with the $\pi\mathfrak{b}$-rescaled canonical variables $\hat{x}_k$, $\hat{p}_k$ and the parameters defined as follows:

$$\begin{aligned}
&\mathsf{u}_k = \pi\mathfrak{b}\hat{x}_k, \qquad \mathsf{w}_k = \pi\mathfrak{b}\hat{p}_k, \qquad [\hat{x}_j, \hat{p}_k] = \frac{\mathrm{i}}{\pi}\delta_{j,k},\\
&(a_k, b_k, c_k, d_k, e_k) = \pi\mathfrak{b}\big(\tilde{a}_k, \tilde{b}_k, \tilde{c}_k, \tilde{d}_k, \tilde{e}_k\big), \qquad \lambda_k = \pi\mathfrak{b}\tilde{\lambda}_k, \qquad C_k = \pi\mathfrak{b}\mathcal{C}_k.
\end{aligned} \tag{6.7}$$

From (2.7) and (6.4), we have

$$\frac{\Psi_q\big(\mathrm{e}^{2\pi\mathfrak{b}(z+\mathrm{i}\mathfrak{b}/2)}\big)}{\Psi_q\big(\mathrm{e}^{2\pi\mathfrak{b}(z-\mathrm{i}\mathfrak{b}/2)}\big)} = \frac{\Phi_\mathfrak{b}(z-\mathrm{i}\mathfrak{b}/2)}{\Phi_\mathfrak{b}(z+\mathrm{i}\mathfrak{b}/2)}.$$

It indicates the formal correspondence

$$\Psi_q\big(\mathrm{e}^{2\pi\mathfrak{b}z}\big) \leftrightarrow \Phi_\mathfrak{b}(z)^{-1}. \tag{6.8}$$

Applying it to (5.4) and (5.6), we define

$$\begin{aligned}
\mathcal{R} = {}& f\big(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\big)\Phi_\mathfrak{b}\left(\frac{1}{2}(\hat{x}_1 + \hat{x}_3 + \hat{p}_1 - \hat{p}_2 + \hat{p}_3 + 2\mathcal{C}_7)\right)\\
&\times \Phi_\mathfrak{b}\left(\frac{1}{2}(\hat{x}_1 - \hat{x}_3 + \hat{p}_1 - \hat{p}_2 + \hat{p}_3 + 2\mathcal{C}_5)\right)\\
&\times \mathcal{P}\Phi_\mathfrak{b}\left(\frac{1}{2}(\hat{x}_1 - \hat{x}_3 + \hat{p}_1 - \hat{p}_2 + \hat{p}_3 + \tilde{\alpha}_6)\right)^{-1}\\
&\times \Phi_\mathfrak{b}\left(\frac{1}{2}(\hat{x}_1 + \hat{x}_3 + \hat{p}_1 - \hat{p}_2 + \hat{p}_3 + \tilde{\alpha}_8)\right)^{-1},\\
\mathcal{P} = {}& \mathrm{e}^{\pi\mathrm{i}(\hat{x}_2 - \hat{x}_3)\hat{p}_1}\mathrm{e}^{\pi\mathrm{i}\tilde{\lambda}_0(\hat{p}_1 + \hat{p}_2 - \hat{p}_3)}\mathrm{e}^{-\pi\mathrm{i}(\tilde{\lambda}_1\hat{x}_1 + \tilde{\lambda}_2\hat{x}_2 + \tilde{\lambda}_3\hat{x}_3)}\rho_{23},
\end{aligned} \tag{6.9}$$

where $\tilde{\alpha}_6 = -\tilde{b}_1 - \tilde{a}_2 - \tilde{d}_3 - \tilde{e}_3$ and $\tilde{\alpha}_8 = -\tilde{b}_1 - \tilde{a}_2 - \tilde{d}_3$. They are also determined by $\tilde{\alpha}_6 + \tilde{\lambda}_1 + \tilde{\lambda}_2 - \tilde{\lambda}_3 = -2\mathcal{C}_6$ and $\tilde{\alpha}_8 + \tilde{\lambda}_1 + \tilde{\lambda}_2 - \tilde{\lambda}_3 = -2\mathcal{C}_8$ as in (5.5).

The normalization of $\mathcal{R}$ remains inherently undetermined from the postulate on $\mathrm{Ad}(\mathcal{R})$. Following the symmetry argument in Section 4.3 with the rescaling (6.7) of parameters, we choose the prefactor $f\big(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\big)$ as

$$\begin{aligned}
f\big(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\big) &= \exp\left(\mathrm{i}\pi(\mathcal{C}_4 - \mathcal{C}_5 + \mathcal{C}_7)(\mathcal{C}_5 + \mathcal{C}_6)\right)\\
&= \exp\left(-\frac{\mathrm{i}\pi}{4}(\tilde{e}_2 - \tilde{e}_3)\big(\tilde{a}_1 + \tilde{a}_3 + \tilde{c}_1 - 2\tilde{c}_2 + \tilde{c}_3 + \tilde{b}_1 - \tilde{b}_3 - \tilde{d}_1 + \tilde{d}_3\big)\right).
\end{aligned} \tag{6.10}$$

Then $\mathrm{Ad}(\mathcal{R})$ is invariant under the simultaneous exchange $1 \leftrightarrow 3$ and $\tilde{b}_i \leftrightarrow \tilde{d}_i$ of indices and parameters. Furthermore, $\mathrm{Ad}(\mathcal{R})$ becomes $\mathrm{Ad}(\mathcal{R})^{-1}$ under the exchange $\tilde{a}_i \leftrightarrow \tilde{c}_i$, $\tilde{b}_i \leftrightarrow \tilde{d}_i$ of parameters. We can multiply f by any function $h\big(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\big)$ such that

$$\begin{aligned}
&h\big(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\big)h\big(\tilde{c}, \tilde{d}, \tilde{a}, \tilde{b}\big) = 1,\\
&h\big(\tilde{a}_1, \tilde{b}_1, \tilde{c}_1, \tilde{d}_1, \tilde{a}_2, \tilde{b}_2, \tilde{c}_2, \tilde{d}_2, \tilde{a}_3, \tilde{b}_3, \tilde{c}_3, \tilde{d}_3\big) = h\big(\tilde{a}_3, \tilde{d}_3, \tilde{c}_3, \tilde{b}_3, \tilde{a}_2, \tilde{d}_2, \tilde{c}_2, \tilde{b}_2, \tilde{a}_1, \tilde{d}_1, \tilde{c}_1, \tilde{b}_1\big),
\end{aligned}$$

and the result still preserves the symmetries.

6.2 Elements of $\mathcal{R}$ in coordinate representation

We consider the representation of canonical variables on a space of functions $G(\mathbf{x})$ of $\mathbf{x} = (x_1, x_2, x_3)$, where the "coordinate" $\hat{x}_k$ acts diagonally, i.e., $(\hat{x}_kG)(\mathbf{x}) = x_kG(\mathbf{x})$, $(\hat{p}_kG)(\mathbf{x}) =$

$-\frac{\mathrm{i}}{\pi}\frac{\partial G}{\partial x_k}(\mathbf{x})$. The functions $G(\mathbf{x})$ are actually supposed to belong to a subspace of $L^2(\mathbb{R}^3)$. See [7, Section 5.2] for details. We find it convenient to employ the bracket notation as $G(\mathbf{x}) = \langle \mathbf{x}|G\rangle$. Then the coordinate representation along with its dual can be summarized formally in a difference (exponential) form as follows:

$$\begin{aligned}
&\mathrm{e}^{\pi\mathfrak{b}\hat{x}_k}|\mathbf{x}\rangle = \mathrm{e}^{\pi\mathfrak{b}x_k}|\mathbf{x}\rangle, \qquad \mathrm{e}^{\pi\mathfrak{b}\hat{p}_k}|\mathbf{x}\rangle = |\mathbf{x}+\mathrm{i}\mathfrak{b}\mathbf{e}_k\rangle,\\
&\langle\mathbf{x}|\mathrm{e}^{\pi\mathfrak{b}\hat{x}_k} = \langle\mathbf{x}|\mathrm{e}^{\pi\mathfrak{b}x_k}, \qquad \langle\mathbf{x}|\mathrm{e}^{\pi\mathfrak{b}\hat{p}_k} = \langle\mathbf{x}-\mathrm{i}\mathfrak{b}\mathbf{e}_k| \qquad (6.11)
\end{aligned}$$

for $k=1,2,3$, where $|x_1,x_2,x_3\rangle$ (resp. $\langle x_1,x_2,x_3|$) is denoted by $|\mathbf{x}\rangle$ (resp. $\langle\mathbf{x}|$). The dual pairing is specified by $\langle\mathbf{x}|\mathbf{x}'\rangle = \delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3')$.

Theorem 6.1. *The matrix element $\mathcal{R}^{x_1,x_2,x_3}_{x_1',x_2',x_3'} = \langle x_1,x_2,x_3|\mathcal{R}|x_1',x_2',x_3'\rangle$ of (6.9) with f specified in (6.10) is given, up to an overall numerical factor, by*

$$\mathcal{R}^{x_1,x_2,x_3}_{x_1',x_2',x_3'} \equiv g(\tilde{a},\tilde{b},\tilde{c},\tilde{d})\delta(x_1+x_2-x_1'-x_2')\delta(x_2+x_3-x_2'-x_3')\mathrm{e}^{\mathrm{i}\pi\phi}I^{x_1,x_2,x_3}_{x_1',x_2',x_3'} \qquad (6.12)$$

$$\begin{aligned}
I^{x_1,x_2,x_3}_{x_1',x_2',x_3'} &= \int_{-\infty}^{\infty}\mathrm{d}z\,\mathrm{e}^{2\pi\mathrm{i}z(-x_2-\mathrm{i}\eta+\mathcal{C}_4)}\\
&\quad\times\frac{\Phi_\mathfrak{b}\big(z+\frac{1}{2}(x_1-x_3+\mathrm{i}\eta)+\mathcal{C}_5\big)\Phi_\mathfrak{b}\big(z+\frac{1}{2}(-x_1+x_3+\mathrm{i}\eta)+\mathcal{C}_6\big)}{\Phi_\mathfrak{b}\big(z+\frac{1}{2}(x_1+x_3-\mathrm{i}\eta)+\mathcal{C}_7\big)\Phi_\mathfrak{b}\big(z+\frac{1}{2}(-x_1'-x_3'-\mathrm{i}\eta)+\mathcal{C}_8\big)}, \qquad (6.13)
\end{aligned}$$

$$\phi = x_1'x_3' + \mathrm{i}\eta(x_1'+x_3'-x_2) + \mathcal{C}_1(x_1-x_3) + \mathcal{C}_2(x_1'+x_3') - 2\mathcal{C}_3x_2', \qquad (6.14)$$

$$g(\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = \exp\left(\mathrm{i}\pi(\mathcal{C}_4(\mathcal{C}_7+\mathcal{C}_8)+(\mathcal{C}_5-\mathcal{C}_7)(\mathcal{C}_6-\mathcal{C}_7)+\mathrm{i}\eta(\mathcal{C}_4-2\mathcal{C}_8))\right).$$

Proof. In order to calculate the matrix elements of $\mathcal{R}$, we insert appropriate complete bases between each factor in the expression (6.9) and use quantum dilogarithm identities.

Let us consider the matrix elements of the first quantum dilogarithm. Noting that $\hat{x}_1+\hat{p}_3$, $\hat{p}_2$ and $\hat{x}_3+\hat{p}_1$ in the argument commute with one another, we expand this quantum dilogarithm in the powers of these combinations of coordinates and momenta, sandwich the resulting series between $\langle x_1'',p_2,p_3|$ and $|p_1,p_2',x_3''\rangle$, and resum the series back to a quantum dilogarithm to get

$$\begin{aligned}
&\langle x_1'',p_2,p_3|\Phi_\mathfrak{b}\left(\frac{1}{2}(\hat{x}_1+\hat{x}_3+\hat{p}_1-\hat{p}_2+\hat{p}_3+2\mathcal{C}_7)\right)|p_1,p_2',x_3''\rangle\\
&\quad=\Phi_\mathfrak{b}\left(\frac{1}{2}(x_1''+x_3''+p_1-p_2+p_3+2\mathcal{C}_7)\right)\langle x_1'',p_2,p_3|p_1,p_2',x_3''\rangle.
\end{aligned}$$

Thus, the matrix elements are given, up to an overall numerical factor, by

$$\begin{aligned}
&\langle x_1,x_2,x_3|\Phi_\mathfrak{b}\left(\frac{1}{2}(\hat{x}_1+\hat{x}_3+\hat{p}_1-\hat{p}_2+\hat{p}_3+2\mathcal{C}_7)\right)|x_1',x_2',x_3'\rangle\\
&\quad\equiv\langle x_1,x_2,x_3|\int\mathrm{d}x_1''\mathrm{d}p_2\mathrm{d}p_3|x_1'',p_2,p_3\rangle\langle x_1'',p_2,p_3|\\
&\qquad\times\Phi_\mathfrak{b}\left(\frac{1}{2}(\hat{x}_1+\hat{x}_3+\hat{p}_1-\hat{p}_2+\hat{p}_3+2\mathcal{C}_7)\right)\\
&\qquad\times\int\mathrm{d}p_1\mathrm{d}p_2'\mathrm{d}x_3''|p_1,p_2',x_3''\rangle\langle p_1,p_2',x_3''|x_1',x_2',x_3'\rangle\\
&\quad\equiv\int\mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\Phi_\mathfrak{b}\left(\frac{1}{2}(x_1+x_3'+p_1-p_2+p_3+2\mathcal{C}_7)\right)\\
&\qquad\times\mathrm{e}^{\mathrm{i}\pi(x_2p_2+x_3p_3+x_1p_1-x_3'p_3-x_1'p_1-x_2'p_2)}.
\end{aligned}$$

Introducing $z_1=p_2+p_3-p_1$, $z_2=p_3+p_1-p_2$, $z_3=p_1+p_2-p_3$ and performing the integration over z_1 and z_3, we are left with

$$\langle x_1,x_2,x_3|\Phi_\mathfrak{b}\left(\frac{1}{2}(\hat{x}_1+\hat{x}_3+\hat{p}_1-\hat{p}_2+\hat{p}_3+2\mathcal{C}_7)\right)|x_1',x_2',x_3'\rangle$$

$$
\begin{aligned}
&\equiv \delta\big(x_1 - x_1' + x_2 - x_2'\big)\delta\big(x_2 - x_2' + x_3 - x_3'\big) \\
&\quad \times \int \mathrm{d}z_2 \Phi_{\mathfrak{b}}\left(\frac{1}{2}\big(x_1 + z_2 + x_3' + 2\mathcal{C}_7\big)\right) \mathrm{e}^{\mathrm{i}\frac{\pi}{2}z_2(x_1 - x_1' + x_3 - x_3')}.
\end{aligned}
$$

The matrix elements of the second quantum dilogarithm can be calculated in a similar manner. This time, $\hat{x}_1 - \hat{x}_3$, $\hat{p}_2$ and $\hat{p}_1 + \hat{p}_3$ in the argument mutually commute, so we can insert the completeness relation in the basis $\{|p_1, p_2, p_3\rangle\}$ and get

$$
\begin{aligned}
&\langle x_1, x_2, x_3|\Phi_{\mathfrak{b}}\left(\frac{1}{2}(\hat{x}_1 - \hat{x}_3 + \hat{p}_1 - \hat{p}_2 + \hat{p}_3 + 2\mathcal{C}_5)\right)\big|x_1', x_2', x_3'\big\rangle \\
&\quad \equiv \delta\big(x_1 - x_1' + x_2 - x_2'\big)\delta\big(x_2 - x_2' + x_3 - x_3'\big) \\
&\quad\quad \times \int \mathrm{d}z_2 \Phi_{\mathfrak{b}}\left(\frac{1}{2}(x_1 - x_3 + z_2 + 2\mathcal{C}_5)\right) \mathrm{e}^{\mathrm{i}\frac{\pi}{2}z_2(x_1 - x_1' + x_3 - x_3')}.
\end{aligned}
$$

To calculate the product of the above two matrices, we use the Fourier transform identity

$$
\int \mathrm{d}x \Phi_{\mathfrak{b}}(x)^{\pm 1} \mathrm{e}^{2\mathrm{i}\pi w x} = \mathrm{e}^{\mp \mathrm{i}\pi w^2 \pm \mathrm{i}\frac{\pi}{12}(1+4\eta^2)} \Phi_{\mathfrak{b}}(\pm w \pm \mathrm{i}\eta)^{\pm 1},
$$

which is a special case of (C.2) and (C.3). We find

$$
\begin{aligned}
&\langle x_1, x_2, x_3|\Phi_{\mathfrak{b}}\left(\frac{1}{2}(\hat{x}_1 + \hat{x}_3 + \hat{p}_1 - \hat{p}_2 + \hat{p}_3 + 2\mathcal{C}_7)\right) \\
&\quad \times \Phi_{\mathfrak{b}}\left(\frac{1}{2}(\hat{x}_1 - \hat{x}_3 + \hat{p}_1 - \hat{p}_2 + \hat{p}_3 + 2\mathcal{C}_5)\right)\big|x_1', x_2', x_3'\big\rangle \\
&\quad\quad \equiv \delta\big(x_1 + x_2 - x_1' - x_2'\big)\delta\big(x_2 + x_3 - x_2' - x_3'\big) \\
&\quad\quad\quad \times \int \mathrm{d}z_2 \Phi_{\mathfrak{b}}\left(-\frac{1}{2}(z_2 + x_1 + x_3 + 2\mathcal{C}_7) + \mathrm{i}\eta\right) \Phi_{\mathfrak{b}}\left(\frac{1}{2}(z_2 + x_1 - x_3 + 2\mathcal{C}_5)\right) \\
&\quad\quad\quad \times \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}(z_2 + x_1 + x_3 + 2\mathcal{C}_7)(z_2 + x_1 + x_3 + 2\mathcal{C}_7 - 4\mathrm{i}\eta) - \mathrm{i}\frac{\pi}{2}z_2(x_3' - x_3 + x_1' - x_1)}.
\end{aligned}
$$

Calculation of the matrix elements of the last two quantum dilogarithms can be done analogously. A quick way to write down the result is to consider the case $\operatorname{Im}\mathfrak{b} = 0$ or $|\mathfrak{b}| = 1$, which allows us to make use of the unitarity $\overline{\Phi_{\mathfrak{b}}(z)} = \Phi_{\mathfrak{b}}(\bar{z})^{-1}$ and deduce

$$
\begin{aligned}
&\langle x_1, x_2, x_3|\Phi_{\mathfrak{b}}\left(\frac{1}{2}(\hat{x}_1 - \hat{x}_3 + \hat{p}_1 - \hat{p}_2 + \hat{p}_3 + \tilde{\alpha}_6)\right)^{-1} \\
&\quad \times \Phi_{\mathfrak{b}}\left(\frac{1}{2}(\hat{x}_1 + \hat{x}_3 + \hat{p}_1 - \hat{p}_2 + \hat{p}_3 + \tilde{\alpha}_8)\right)^{-1}\big|x_1', x_2', x_3'\big\rangle \\
&\quad\quad \equiv \delta\big(x_1 + x_2 - x_1' - x_2'\big)\delta\big(x_2 + x_3 - x_2' - x_3'\big) \\
&\quad\quad\quad \times \int \mathrm{d}z_2 \Phi_{\mathfrak{b}}\left(\frac{1}{2}(x_1 - x_3 + z_2 + \tilde{\alpha}_6)\right)^{-1} \Phi_{\mathfrak{b}}\left(-\frac{1}{2}\big(x_1' + x_3' + z_2 + \tilde{\alpha}_8\big) - \mathrm{i}\eta\right)^{-1} \\
&\quad\quad\quad \times \mathrm{e}^{\mathrm{i}\frac{\pi}{4}(z_2 + x_1' + x_3' + \tilde{\alpha}_8)(z_2 + x_1' + x_3' + \tilde{\alpha}_8 + 4\mathrm{i}\eta) + \mathrm{i}\frac{\pi}{2}z_2(x_1 - x_1' + x_3 - x_3')}.
\end{aligned}
$$

Finally, we can also easily calculate

$$
\begin{aligned}
&\langle x_1, x_2, x_3|\mathcal{P}\big|x_1', x_2', x_3'\big\rangle \\
&\quad \equiv \delta\big(x_1 + x_2 - x_3 - x_1' + \tilde{\lambda}_0\big)\delta\big(x_2 - x_3' + \tilde{\lambda}_0\big)\delta\big(x_3 - x_2' - \tilde{\lambda}_0\big)\mathrm{e}^{-\mathrm{i}\pi(\tilde{\lambda}_1 x_1' + \tilde{\lambda}_3 x_2' + \tilde{\lambda}_2 x_3')}.
\end{aligned}
$$

From the various matrix elements calculated above, we obtain

$$
\mathcal{R}^{x_1, x_2, x_3}_{x_1', x_2', x_3'} \equiv \mathrm{e}^{\mathrm{i}\pi(\mathcal{C}_4 - \mathcal{C}_5 + \mathcal{C}_7)(\mathcal{C}_5 + \mathcal{C}_6)} \delta\big(x_1 + x_2 - x_1' - x_2'\big)\delta\big(x_2 + x_3 - x_2' - x_3'\big)
$$

$$
\begin{aligned}
&\times \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}(x_1+x_3-x'_1-x'_3+2\mathcal{C}_7+2\mathcal{C}_8-4\mathrm{i}\eta)(x_1+x_3+x'_1+x'_3+2\mathcal{C}_7-2\mathcal{C}_8)-\mathrm{i}\pi(\tilde{\lambda}_1 x'_1+\tilde{\lambda}_3 x'_2+\tilde{\lambda}_2 x'_3)}\\
&\times \int \mathrm{d}z_2 \frac{\Phi_{\mathfrak{b}}\big(\frac{1}{2}(z_2+x_1-x_3+2\mathcal{C}_5)\big)\Phi_{\mathfrak{b}}\big(-\frac{1}{2}(z_2+x_1+x_3+2\mathcal{C}_7)+\mathrm{i}\eta\big)}{\Phi_{\mathfrak{b}}\big(-\frac{1}{2}(z_2-x'_1+x'_3+2\mathcal{C}_6)\big)}\\
&\times \frac{\mathrm{e}^{-\mathrm{i}\pi z_2(\tilde{\lambda}_0+x'_1+x'_2+\mathcal{C}_7-\mathcal{C}_8)}}{\Phi_{\mathfrak{b}}\big(\frac{1}{2}(z_2-x'_1-x'_3+2\mathcal{C}_8)-\mathrm{i}\eta\big)},
\end{aligned}
$$

where the first exponential factor comes from the function f in (6.10). Changing the integration variable to $z=(z_2-\mathrm{i}\eta)/2$ and using the identity (6.3), we arrive at the desired formula. ■

Under the transformation (4.26), $\mathcal{C}_k=(\pi\mathfrak{b})^{-1}C_k$ has the same symmetry as that for C_k mentioned in (5.7). Therefore, $\mathcal{R}$ is indeed invariant.

Remark 6.2. Comparison of (5.8) and (6.11) indicates the correspondence

$$
x_k=\mathrm{i}\mathfrak{b}n_k+\mathrm{i}\eta \tag{6.15}
$$

between the indices of R and $\mathcal{R}$. In fact, by using (5.3), (6.3) and (6.7), one can check that $R^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$ in Theorem 5.2 is transformed to $\mathcal{R}^{x_1,x_2,x_3}_{x'_1,x'_2,x'_3}$ in Theorem 6.1 up to normalization by replacing $\Psi_q\big(\mathrm{e}^{2\pi\mathfrak{b}z}\big)$ by $\Phi_{\mathfrak{b}}(z)^{-1}$ according to (6.8) and substituting (6.15). The strange normalization of $\mathrm{e}^{\mathsf{u}_k}$ in (5.8) is attributed to the second term of (6.15), which may be viewed as a modular double analogue of the "zero point energy".

6.3 Elements of $\mathcal{R}$ in momentum representation

Let us consider the modular $\mathcal{R}$ (6.9) in the "momentum representation" in which $\hat{p}_j$ becomes the diagonal operator of multiplying p_j as

$$
\begin{aligned}
&\mathrm{e}^{\pi\mathfrak{b}\hat{x}_k}|\mathbf{p}\rangle=|\mathbf{p}-\mathrm{i}\mathfrak{b}\mathbf{e}_k\rangle, && \mathrm{e}^{\pi\mathfrak{b}\hat{p}_k}|\mathbf{p}\rangle=\mathrm{e}^{\pi\mathfrak{b}p_k}|\mathbf{p}\rangle,\\
&\langle\mathbf{p}|\mathrm{e}^{\pi\mathfrak{b}\hat{x}_k}=\langle\mathbf{p}+\mathrm{i}\mathfrak{b}\mathbf{e}_k|, && \langle\mathbf{p}|\mathrm{e}^{\pi\mathfrak{b}\hat{p}_k}=\langle\mathbf{p}|\mathrm{e}^{\pi\mathfrak{b}p_k},
\end{aligned} \tag{6.16}
$$

where $k=1,2,3$ and $|p_1,p_2,p_3\rangle$ (resp. $\langle p_1,p_2,p_3|$) is denoted by $|\mathbf{p}\rangle$ (resp. $\langle\mathbf{p}|$). The dual pairing is specified by $\langle\mathbf{p}|\mathbf{p}'\rangle=\delta(p_1-p'_1)\delta(p_2-p'_2)\delta(p_3-p'_3)$.

From $\langle\mathbf{x}|\mathbf{p}\rangle\equiv\mathrm{e}^{\pi\mathrm{i}(p_1x_1+p_2x_2+p_3x_3)}$, its matrix element $\mathcal{S}^{p_1,p_2,p_3}_{p'_1,p'_2,p'_3}:=\langle p_1,p_2,p_3|\mathcal{R}|p'_1,p'_2,p'_3\rangle$ is obtained by taking the Fourier transformation

$$
\mathcal{S}^{p_1,p_2,p_3}_{p'_1,p'_2,p'_3}=\int_{\mathbb{R}^6}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x'_1\mathrm{d}x'_2\mathrm{d}x'_3\mathrm{e}^{\pi\mathrm{i}(p'_1x'_1+p'_2x'_2+p'_3x'_3-p_1x_1-p_2x_2-p_3x_3)}\mathcal{R}^{x_1,x_2,x_3}_{x'_1,x'_2,x'_3}, \tag{6.17}
$$

where $\mathcal{R}^{x_1,x_2,x_3}_{x'_1,x'_2,x'_3}$ is the coordinate representation given in Theorem 6.1.

Theorem 6.3. *Up to an overall factor depending on $\mathcal{C}_1,\dots,\mathcal{C}_8$, the following formula is valid:*

$$
\mathcal{S}^{p_1,p_2,p_3}_{p'_1,p'_2,p'_3}\equiv\mathrm{e}^{\mathrm{i}\pi(\alpha+\beta)}\frac{\Phi_{\mathfrak{b}}(z_1+\mathrm{i}\eta)\Phi_{\mathfrak{b}}(z_2+\mathrm{i}\eta)\Phi_{\mathfrak{b}}(z_3+\mathrm{i}\eta)\Phi_{\mathfrak{b}}(z_4+\mathrm{i}\eta)}{\Phi_{\mathfrak{b}}(z_3+z_4+\mathrm{i}\eta)}, \tag{6.18}
$$

$$
\begin{aligned}
&z_1=-\mathcal{C}_3-\frac{1}{2}(p_1+p_3-p'_2), \qquad z_3=-\mathcal{C}_1+\mathcal{C}_3-\mathcal{C}_5-\frac{1}{2}(-p_1-p_2+p_3+p'_1+p'_2-p'_3),\\
&z_2=\mathcal{C}_2+\mathcal{C}_8-\frac{1}{2}(p_2-p'_1-p'_3), \qquad z_4=\mathcal{C}_1+\mathcal{C}_3-\mathcal{C}_6-\frac{1}{2}(p_1-p_2-p_3-p'_1+p'_2+p'_3),\\
&\alpha=(z_1-z_2)(z_3+z_4)+z_3z_4-z_2^2-2\mathrm{i}\eta z_2,\\
&\beta=(\mathcal{C}_8-\mathcal{C}_7)(z_1+z_2)+(\mathcal{C}_8-\mathcal{C}_6-\mathcal{C}_4)z_3+(\mathcal{C}_8-\mathcal{C}_5-\mathcal{C}_4)z_4.
\end{aligned}
$$

Proof. Substitute (6.12) into (6.17) and eliminate x'_1 and x'_3 by the delta functions. With the shift $x_2 \to x_2 + x'_2$, the exponent of e in the result becomes *linear* in x'_2. In fact, up to an overall factor, (6.17) is equal to

$$\begin{aligned}
&\int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x'_2 \mathrm{d}z \mathrm{e}^{\mathrm{i}\pi x'_2(p'_2 - p_2 - 2z - \mathrm{i}\eta - 2\mathcal{C}_3) + \mathrm{i}\pi\alpha_1} \\
&\qquad \times \frac{\Phi_\mathfrak{b}\big(z + \frac{x_1 - x_3 + \mathrm{i}\eta}{2} + \mathcal{C}_5\big)\Phi_\mathfrak{b}\big(z + \frac{x_3 - x_1 + \mathrm{i}\eta}{2} + \mathcal{C}_6\big)}{\Phi_\mathfrak{b}\big(z + \frac{x_1 + x_3 - \mathrm{i}\eta}{2} + \mathcal{C}_7\big)\Phi_\mathfrak{b}\big(z + \frac{-x_1 - 2x_2 - x_3 - \mathrm{i}\eta}{2} + \mathcal{C}_8\big)}, \\
&\alpha_1 = -p_1 x_1 - p_2 x_2 - p_3 x_3 + p'_1(x_1 + x_2) + p'_3(x_2 + x_3) + (x_1 + x_2)(x_2 + x_3) \\
&\qquad + \mathrm{i}\eta(x_1 + x_2 + x_3) - 2(\mathrm{i}\eta + x_2)z + \mathcal{C}_1(x_1 - x_3) + \mathcal{C}_2(x_1 + 2x_2 + x_3) \\
&\qquad + \mathrm{i}\eta(\mathcal{C}_4 - 2\mathcal{C}_8) + 2\mathcal{C}_4 z.
\end{aligned}$$

The integral over x'_2 yields $2\delta(p'_2 - p_2 - 2z - \mathrm{i}\eta - 2\mathcal{C}_3)$. Further integral over z after shifting the contour leads, up to an overall factor, to

$$\begin{aligned}
&\int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{e}^{\mathrm{i}\pi\alpha_2} \frac{\Phi_\mathfrak{b}\big(\frac{x_1 - x_3 + p'_2 - p_2}{2} - \mathcal{C}_3 + \mathcal{C}_5\big)\Phi_\mathfrak{b}\big(\frac{x_3 - x_1 + p'_2 - p_2}{2} - \mathcal{C}_3 + \mathcal{C}_6\big)}{\Phi_\mathfrak{b}\big(\frac{x_1 + x_3 + p'_2 - p_2}{2} - \mathrm{i}\eta - \mathcal{C}_3 + \mathcal{C}_7\big)\Phi_\mathfrak{b}\big(\frac{-x_1 - 2x_2 - x_3 + p'_2 - p_2}{2} - \mathrm{i}\eta - \mathcal{C}_3 + \mathcal{C}_8\big)}, \\
&\alpha_2 = \alpha_1\big|_{z = \frac{p'_2 - p_2 - \mathrm{i}\eta}{2} - \mathcal{C}_3}.
\end{aligned}$$

Set $x_2 \to x_2 - (x_1 + x_3)/2$ and apply (6.3) to the second (right) $\Phi_\mathfrak{b}$ in the numerator and the denominator, which makes the power of e linear in all the integration variables. Up to an overall factor, the result reads

$$\begin{aligned}
&\int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{e}^{\mathrm{i}\pi\alpha_3} \frac{\Phi_\mathfrak{b}\big(\frac{x_1 - x_3 + p'_2 - p_2}{2} - \mathcal{C}_3 + \mathcal{C}_5\big)\Phi_\mathfrak{b}\big(x_2 + \frac{-p'_2 + p_2}{2} + \mathrm{i}\eta + \mathcal{C}_3 - \mathcal{C}_8\big)}{\Phi_\mathfrak{b}\big(\frac{x_1 - x_3 - p'_2 + p_2}{2} + \mathcal{C}_3 - \mathcal{C}_6\big)\Phi_\mathfrak{b}\big(\frac{x_1 + x_3 + p'_2 - p_2}{2} - \mathrm{i}\eta - \mathcal{C}_3 + \mathcal{C}_7\big)}, \\
&\alpha_3 = x_1\left(\mathcal{C}_1 - \mathcal{C}_6 - p_1 + \frac{p_2 + p'_1 - p'_3}{2}\right) + x_2\big(2\mathcal{C}_2 + 2\mathcal{C}_8 - p_2 + p'_1 + p'_3\big) \\
&\qquad + x_3\left(-\mathcal{C}_1 - 2\mathcal{C}_3 + \mathcal{C}_6 - p_3 + p'_2 + \frac{-p_2 - p'_1 + p'_3}{2}\right).
\end{aligned}$$

By setting $x_1 = s + t$, $x_3 = s - t$, this can be separated into three independent integrals as

$$\begin{aligned}
&\int \mathrm{d}s \frac{\mathrm{e}^{\mathrm{i}\pi s(-2\mathcal{C}_3 - p_1 - p_3 + p'_2)}}{\Phi_\mathfrak{b}\big(-\mathcal{C}_3 + \mathcal{C}_7 + s + \frac{p'_2 - p_2}{2} - \mathrm{i}\eta\big)} \\
&\qquad \times \int \mathrm{d}x_2 \mathrm{e}^{\mathrm{i}\pi x_2(2\mathcal{C}_2 + 2\mathcal{C}_8 - p_2 + p'_1 + p'_3)}\Phi_\mathfrak{b}\left(\mathcal{C}_3 - \mathcal{C}_8 + x_2 + \frac{p_2 - p'_2}{2} + \mathrm{i}\eta\right) \\
&\qquad \times \int \mathrm{d}t \mathrm{e}^{\pi\mathrm{i}t(2\mathcal{C}_1 + 2\mathcal{C}_3 - 2\mathcal{C}_6 - p_1 + p_2 + p_3 + p'_1 - p'_2 - p'_3)} \frac{\Phi_\mathfrak{b}\big(-\mathcal{C}_3 + \mathcal{C}_5 + t + \frac{p'_2 - p_2}{2}\big)}{\Phi_\mathfrak{b}\big(\mathcal{C}_3 - \mathcal{C}_6 + t + \frac{p_2 - p'_2}{2}\big)},
\end{aligned}$$

where the Jacobian value 2 has not been included. They can be evaluated by the formulas in Appendix C. After applying (6.3) again in the result, we obtain (6.18). ■

By construction, the R-matrix in the momentum representation $\big(\mathcal{S}^{p_1,p_2,p_3}_{p'_1,p'_2,p'_3}\big)$ in Theorem 6.3 also satisfies the tetrahedron equation.

Remark 6.4. From (5.15) and (6.16), one sees the correspondence $q^{n_k} \leftrightarrow \mathrm{e}^{\pi\mathfrak{b}p_k}$, i.e., $p_k \leftrightarrow \mathrm{i}\mathfrak{b}n_k$ in the w-diagonal/momentum representation. In fact, in the formula (6.18), replace $\Phi_\mathfrak{b}(z + \mathrm{i}\eta)$ according to

$$\Phi_\mathfrak{b}(z + \mathrm{i}\eta) \overset{(6.3)}{\equiv} \frac{\mathrm{e}^{\pi\mathrm{i}(z+\mathrm{i}\eta)^2}}{\Phi_\mathfrak{b}(-z - \mathrm{i}\eta)} \xrightarrow{(6.8)} \mathrm{e}^{\pi\mathrm{i}(z+\mathrm{i}\eta)^2}\Psi_q\big(\mathrm{e}^{-2\pi\mathfrak{b}(z+\mathrm{i}\eta)}\big) \overset{(2.5)}{=} \frac{\mathrm{e}^{\pi\mathrm{i}(z+\mathrm{i}\eta)^2}}{\big(\mathrm{e}^{-2\pi\mathfrak{b}z}\big)_\infty}.$$

Then under the identification $p_k = \mathrm{i}\mathfrak{b}n_k$, $p'_k = \mathrm{i}\mathfrak{b}n'_k$ $(k = 1,2,3)$ and $C_j = \pi\mathfrak{b}\mathcal{C}_j$ (6.7), the modular R-matrix $\mathcal{S}^{p_1,p_2,p_3}_{p'_1,p'_2,p'_3}$ in (6.18) is transformed to $S^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3}$ in (5.24) up to normalization. Taking Remark 5.6 into account, $\big(\mathcal{S}^{p_1,p_2,p_3}_{p'_1,p'_2,p'_3}\big)$ may be regarded as a modular version of R^{ZZZ} in [18].

7 Relation to quantized six-vertex model

In this section, we show that the R-matrix obtained in Section 4 satisfies the $RLLL = LLLR$ relation ($RLLL$ relation for short) for the quantized six-vertex model with full parameters [18]. This result is a quantum version of the observation made in [9] that the classical limit of the $RLLL$ relation arises from a mutation sequence of a symmetric butterfly quiver associated with a perfect network. We also provide a separate proof of the $RLLL$ relation for the modular $\mathcal{R}$ based on properties of the non-compact quantum dilogarithm.

7.1 3D L operator

Let $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$ be a two-dimensional vector space and $\mathcal{W}(p)$ be the p-Weyl algebra generated by $Z^{\pm1}$, $X^{\pm1}$ with the relation

$$ZX = pXZ. \tag{7.1}$$

We consider a $\mathcal{W}(p)$-valued operator

$$\mathcal{L}^p = \sum_{a,b,i,j=0,1} E_{ai} \otimes E_{bj} \otimes \mathcal{L}^{ab}_{ij} \in \mathrm{End}(V \otimes V) \otimes \mathcal{W}(p),$$
$$\mathcal{L}^{ab}_{ij} = 0 \qquad \text{unless} \qquad a + b = i + j, \tag{7.2}$$
$$\mathcal{L}^{00}_{00} = r, \qquad \mathcal{L}^{11}_{11} = s, \qquad \mathcal{L}^{10}_{10} = wX, \qquad \mathcal{L}^{01}_{01} = tX, \qquad \mathcal{L}^{10}_{01} = Z,$$
$$\mathcal{L}^{01}_{10} = rsZ^{-1} + twXZ^{-1}X. \tag{7.3}$$

Here r, s, t, w are parameters. Note that $\mathcal{L}^{ab}_{ij} = \mathcal{L}(r,s,t,w)^{ab}_{ij}$ depends also on p via (7.1). The symbol E_{ij} denotes the matrix unit on V acting on the basis as $E_{ij}v_k = \delta_{jk}v_i$. The operator $\mathcal{L}^p$ may be viewed as a quantized six-vertex model where the Boltzmann weights are $\mathcal{W}(p)$-valued. It is obtained from [18, Figure 1] by (i) gauge transformation $\mathcal{L}^{ab}_{ij} \to \alpha^{a-j}\mathcal{L}^{ab}_{ij}$ [17, Remark 3.23], preserving the $RLLL$ relation (7.4) described below, with $\alpha = \mathrm{i}q^{\frac{1}{2}}$, (ii) $t \to -\mathrm{i}q^{-\frac{1}{2}}t$, (iii) $w \to t^{-1}w$.[8] See Figure 7 for a graphical representation.

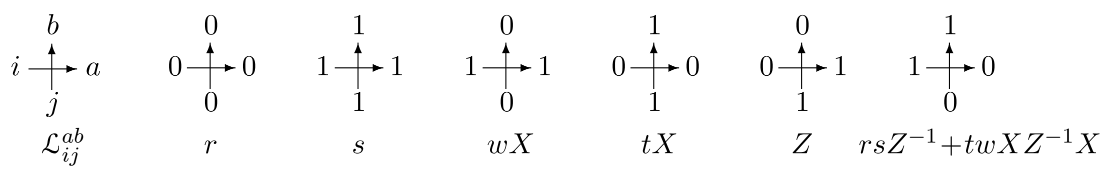


Figure 7. The operator $\mathcal{L}^p = \mathcal{L}^p(r,s,t,w)$ as a $\mathcal{W}(p)$-valued six-vertex model.

[8]The last term in (7.3) originates from $-t^2wZ^{-1}X^2$ in [18, equation (15)]. It is transformed to $q^{-1}twZ^{-1}X^2$ by (i)–(iii) and further to $twXZ^{-1}X$ by $XZ = qZX$ in [18, equation (6)] to eliminate the explicit q-dependence. The relation $XZ = qZX$ [18, equation (15)] corresponds to $p = q^{-1}$ and differs from the choice $p = q$ $\big($or $q^\vee\big)$ made in this paper.

7.2 *RLLL* relation

Consider the *RLLL* relation, which takes the form of the Yang–Baxter equation up to conjugation [2, 17]

$$\mathcal{R}_{456}\mathcal{L}^p_{236}\mathcal{L}^p_{135}\mathcal{L}^p_{124} = \mathcal{L}^p_{124}\mathcal{L}^p_{135}\mathcal{L}^p_{236}\mathcal{R}_{456}. \tag{7.4}$$

Here $\mathcal{R}_{456}$ is supposed to be an element of a group of operators whose adjoint action yield linear maps on $\mathcal{W}(p)^{\otimes 3}$. The indices denote the tensor components on which the operators act nontrivially. In terms of the components $\mathcal{L}^{ab}_{ij}$, the *RLLL* relation reads

$$\mathcal{R}\sum_{\alpha,\beta,\gamma=0,1}\left(\mathcal{L}^{\alpha\beta}_{ij}\otimes\mathcal{L}^{a\gamma}_{\alpha k}\otimes\mathcal{L}^{bc}_{\beta\gamma}\right) = \sum_{\alpha,\beta,\gamma=0,1}\left(\mathcal{L}^{ab}_{\alpha\beta}\otimes\mathcal{L}^{\alpha c}_{i\gamma}\otimes\mathcal{L}^{\beta\gamma}_{jk}\right)\mathcal{R} \tag{7.5}$$

for arbitrary $a, b, c, i, j, k \in \{0, 1\}$. See Figure 8.

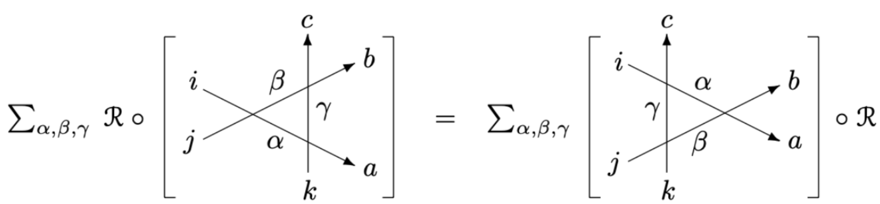


Figure 8. A pictorial representation of the quantized Yang–Baxter equation (7.5).

We take the parameters of $\mathcal{L}^p_{124}$, $\mathcal{L}^p_{135}$, $\mathcal{L}^p_{236}$ on both sides of (7.4) to be (r_1, s_1, t_1, w_1), (r_2, s_2, t_2, w_2), (r_3, s_3, t_3, w_3), respectively. From the conservation condition (7.2), the equation (7.5) becomes $0 = 0$ unless $a+b+c = i+j+k$. There are 20 choices of $(a, b, c, i, j, k) \in \{0, 1\}^6$ satisfying this condition. Among them, the cases $(0, 0, 0, 0, 0, 0)$ and $(1, 1, 1, 1, 1, 1)$ yield the trivial relation $\mathcal{R}(1\otimes 1\otimes 1) = (1\otimes 1\otimes 1)\mathcal{R}$. Thus, there are 18 nontrivial equations for (7.4). They are listed in Appendix D.

Theorem 7.1. *The RLLL relation* (7.4) *with* $p = q$ *holds for* $\mathcal{R} = R$ *in* (4.18) *under the identification*

$$X_i = \mathrm{e}^{\mathsf{u}_i}, \qquad Z_i = \mathrm{e}^{-\mathsf{w}_i}, \qquad r_i = \mathrm{e}^{c_i}, \qquad s_i = \mathrm{e}^{a_i}, \qquad t_i = \mathrm{e}^{-b_i}, \qquad w_i = \mathrm{e}^{-d_i}, \tag{7.6}$$

where $X_1 = X\otimes 1\otimes 1$, $X_2 = 1\otimes X\otimes 1$, $X_3 = 1\otimes 1\otimes X$, *and* Z_i *is defined similarly.*[9]

Proof. Write the *RLLL* relation as $\widehat{R}^{\mathsf{uw}}_{456}\left(\mathcal{L}^q_{236}\mathcal{L}^q_{135}\mathcal{L}^q_{124}\right) = \mathcal{L}^q_{124}\mathcal{L}^q_{135}\mathcal{L}^q_{236}$. The symmetries (4.27)–(4.28) of $\widehat{R}^{\mathsf{uw}}$ relate the component equations (D.1)–(D.18) by the following three transformations:

- $r_1 \leftrightarrow r_3$, $s_1 \leftrightarrow s_3$, $t_1 \leftrightarrow w_3$, $w_1 \leftrightarrow t_3$, $t_2 \leftrightarrow w_2$, $X_1 \leftrightarrow X_3$, $Z_1 \leftrightarrow Z_3$,
- $r_i \leftrightarrow s_i$, $t_i \leftrightarrow w_i$, $\mathcal{R} \leftrightarrow \mathcal{R}^{-1}$,
- $q \mapsto q^{-1}$, $r_i \leftrightarrow s_i$, $t_i \leftrightarrow w_i$, $Z_i \leftrightarrow Y_i$,

where Y_i is defined in Appendix D. Accordingly, it suffices to check one equation in each of the following four groups:

1) (D.1), (D.7), (D.12), (D.18),
2) (D.3), (D.9), (D.10), (D.16),

[9] The parameter w_i should not be confused with the canonical variable w_i.

3) (D.2), (D.4), (D.6), (D.8), (D.11), (D.13), (D.15), (D.17),
4) (D.5), (D.14).

The equations in (1) follow from (4.5) and (4.6). Equation (D.10) is equivalent to $\widehat{R}^{\mathsf{uw}}_{123} \circ \phi'_{\mathrm{SB}}\big(Y_1'^{-1}\big) = \phi_{\mathrm{SB}} \circ \widehat{R}_{123}\big(Y_1'^{-1}\big)$, as can be seen from (4.3), (4.4) and Proposition 3.4. One can reduce (D.4) to $\widehat{R}^{\mathsf{uw}}_{123} \circ \phi'_{\mathrm{SB}}\big(Y_6'^{-1}\big) = \phi_{\mathrm{SB}} \circ \widehat{R}_{123}\big(Y_6'^{-1}\big)$ by multiplying it by (D.10). Finally, to verify (D.5), one can check that the relation (D.10)(D.5) $= r_1r_2r_3$(D.16) $+ q$(D.4)(D.11) holds whether the left-hand sides or the right-hand sides of the equations are used. ■

The relation (7.6) between parameters agrees with (5.28).

In [18], it has been shown that the solutions R to the $RLLL$ relation for the present L are unique up to normalization within appropriate parity sectors [18]. Thus, Theorem 7.1 effectively identifies the concrete R-matrices obtained in [18] with the images of (4.15) in the corresponding representations of the canonical variables. Moreover, Theorem 4.3 verifies the validity of the various tetrahedron equations of the form $RRRR = RRRR$ for these R-matrices as conjectured in [18, Section 6.2].

7.3 $RLLL$ relation for the modular $\mathcal{R}$

Let $\mathcal{V}$ be the space of ket vectors $|x\rangle (x \in \mathbb{C})$ (cf. Section 6.2) and consider the joint representations of $\mathcal{W}(q)$ and $\mathcal{W}\big(q^\vee\big)$ on $\mathcal{V}$ given by

$$\begin{aligned}
&\pi_q\colon\ \mathcal{W}(q) \to \mathrm{End}(\mathcal{V})\colon\ X|x\rangle = \mathrm{e}^{\pi\mathfrak{b}x}|x\rangle, \qquad Z|x\rangle = |x - \mathrm{i}\mathfrak{b}\rangle,\\
&\pi_{q^\vee}\colon\ \mathcal{W}\big(q^\vee\big) \to \mathrm{End}(\mathcal{V})\colon\ X|x\rangle = \mathrm{e}^{\pi\mathfrak{b}^{-1}x}|x\rangle, \quad Z|x\rangle = \big|x - \mathrm{i}\mathfrak{b}^{-1}\big\rangle.
\end{aligned}$$

See (6.1) for the relations between the parameters q, $q^\vee$ and $\mathfrak{b}$. We introduce two L-operators that are modular dual to each other as follows:

$$\begin{aligned}
&\mathcal{L}^q = (1 \otimes 1 \otimes \pi_q)(\mathcal{L}^q) \in \mathrm{End}(V \otimes V \otimes \mathcal{V}),\\
&\mathcal{L}^{q^\vee} = (1 \otimes 1 \otimes \pi_{q^\vee})\big(\mathcal{L}^{q^\vee}\big) \in \mathrm{End}(V \otimes V \otimes \mathcal{V}).
\end{aligned}$$

Since there is no explicit dependence on p in (7.3) or in Figure 7, Theorem 7.1 implies the following.

Corollary 7.2. *The R matrix $\mathcal{R}$ in Theorem 6.1 satisfies the RLLL relation*

$$\mathcal{R}_{456}\mathcal{L}^p_{236}\mathcal{L}^p_{135}\mathcal{L}^p_{124} = \mathcal{L}^p_{124}\mathcal{L}^p_{135}\mathcal{L}^p_{236}\mathcal{R}_{456} \tag{7.7}$$

for $p = \exp\big(\mathrm{i}\pi\mathfrak{b}^{\pm 2}\big) = \big(\begin{smallmatrix} q \\ q^\vee \end{smallmatrix}\big)$ and the parameters $\mathcal{C}_1, \dots, \mathcal{C}_8$ given by

$$\begin{aligned}
&\mathrm{e}^{\pi\mathfrak{b}^{\pm1}\mathcal{C}_1} = \sqrt{\frac{r_1t_2w_3}{r_3t_1w_2}}, \qquad \mathrm{e}^{\pi\mathfrak{b}^{\pm1}\mathcal{C}_2} = \sqrt{\frac{r_2t_1w_3}{r_1r_3s_2}}, \qquad \mathrm{e}^{\pi\mathfrak{b}^{\pm1}\mathcal{C}_3} = \sqrt{\frac{r_1r_3}{r_2}},\\
&\mathrm{e}^{\pi\mathfrak{b}^{\pm1}\mathcal{C}_4} = \sqrt{\frac{r_2s_2}{t_2w_2}}, \qquad \mathrm{e}^{\pi\mathfrak{b}^{\pm1}\mathcal{C}_5} = \sqrt{\frac{r_3s_3w_1}{r_2w_3}}, \qquad \mathrm{e}^{\pi\mathfrak{b}^{\pm1}\mathcal{C}_6} = \sqrt{\frac{r_1s_1t_3}{r_2t_1}},\\
&\mathrm{e}^{\pi\mathfrak{b}^{\pm1}\mathcal{C}_7} = \sqrt{\frac{t_3w_1}{r_2}}, \qquad \mathrm{e}^{\pi\mathfrak{b}^{\pm1}\mathcal{C}_8} = \sqrt{\frac{r_1r_3s_1s_3}{r_2t_1w_3}}.
\end{aligned} \tag{7.8}$$

The upper choice of parameters in (7.8) is consistent with (7.6) and (5.28). (Recall the rescaling (6.7).) The parameters (7.8) satisfy the constraint (5.2).

In the rest of this subsection, we illustrate an independent check of (7.7) at the level of matrix elements in the strong coupling regime assuming that $\mathcal{C}_1, \dots, \mathcal{C}_8$ are all real. Thanks to the modular duality, it suffices to consider the $p = q$ case. Note also $\eta = \mathrm{Im}(i\mathfrak{b})$. There are three cases (i), (ii), (iii).

(i) Trivial: (D.1), (D.7), (D.12) and (D.18). They are satisfied due to the presence of the two delta functions in (6.12).

(ii) Easy: (D.4), (D.10), (D.11), (D.13), (D.16), (D.17). They are shown by direct substitution and the recursion relation (6.4). Let us illustrate the calculation along the example of (D.17). The matrix elements of LHS − RHS for the transition $\left|x_1', x_2', x_3' + \mathrm{i}\mathfrak{b}\right\rangle \mapsto |x_1, x_2, x_3\rangle$ is

$$w_3 \mathrm{e}^{\pi\mathfrak{b}(x_3'+\mathrm{i}\mathfrak{b})}\big(r_1 s_1 + t_1 w_1 \mathrm{e}^{\pi\mathfrak{b}(2x_1'+\mathrm{i}\mathfrak{b})}\big) R^{x_1,x_2,x_3}_{x_1'+\mathrm{i}\mathfrak{b},x_2'-\mathrm{i}\mathfrak{b},x_3'+\mathrm{i}\mathfrak{b}} + s_2 w_1 \mathrm{e}^{\pi\mathfrak{b}x_1'} R^{x_1,x_2,x_3}_{x_1',x_2',x_3'} \\ - s_1 w_2 \mathrm{e}^{\pi\mathfrak{b}x_2} R^{x_1,x_2,x_3+\mathrm{i}\mathfrak{b}}_{x_1',x_2',x_3'+\mathrm{i}\mathfrak{b}}.$$

Upon substitution of (6.12)–(6.14) and (6.4), this is equal, up to an overall factor, to

$$\int_{-\infty}^{\infty} \mathrm{d}z \mathrm{e}^{2\pi\mathrm{i}z(-x_2-\mathrm{i}\eta+\mathcal{C}_4)} \frac{\Phi_\mathfrak{b}\big(z+\frac{1}{2}(x_1-x_3+\mathrm{i}\eta)+\mathcal{C}_5\big)\Phi_\mathfrak{b}\big(z+\frac{1}{2}(-x_1+x_3+\mathrm{i}\eta)+\mathcal{C}_6\big)}{\Phi_\mathfrak{b}\big(z+\frac{1}{2}(x_1+x_3-\mathrm{i}\eta)+\mathcal{C}_7\big)\Phi_\mathfrak{b}\big(z+\frac{1}{2}\big(-x_1'-x_3'-\mathrm{i}\eta\big)-\mathrm{i}\mathfrak{b}+\mathcal{C}_8\big)}\mathcal{D},$$

$$\mathcal{D} = \mathrm{e}^{\pi\mathfrak{b}(-2\mathcal{C}_2-2\mathcal{C}_3-x_1'-2\mathrm{i}\eta)}\big(r_1 s_1 + t_1 w_1 \mathrm{e}^{\pi\mathfrak{b}(2x_1'+\mathrm{i}\mathfrak{b})}\big) w_3 \\ + \mathrm{e}^{\pi\mathfrak{b}x_1'}\big(1 + \mathrm{e}^{\pi\mathfrak{b}(2\mathcal{C}_8-\mathrm{i}\mathfrak{b}-x_1'-x_3'-\mathrm{i}\eta+2z)}\big) s_2 w_1 \\ - \mathrm{e}^{\pi\mathfrak{b}(\mathcal{C}_1-\mathcal{C}_2+\mathcal{C}_4-x_1'-2\mathrm{i}\eta)}\big(1 + \mathrm{e}^{\pi\mathfrak{b}(2\mathcal{C}_5-\mathrm{i}\mathfrak{b}+x_1-x_3+\mathrm{i}\eta+2z)}\big) s_1 w_2.$$

Under the constraint $x_1 - x_3 = x_1' - x_3'$ deduced from the two delta functions, $\mathcal{D} = 0$ amounts to three equalities

$$\mathrm{e}^{\pi\mathfrak{b}(\mathcal{C}_2+2\mathcal{C}_8)} s_2 w_1 = \mathrm{e}^{\pi\mathfrak{b}(\mathcal{C}_1+\mathcal{C}_4+2\mathcal{C}_5)} s_1 w_2, \qquad \mathrm{e}^{\pi\mathfrak{b}(\mathcal{C}_1+\mathcal{C}_2+2\mathcal{C}_3+\mathcal{C}_4)} w_2 = r_1 w_3, \\ \mathrm{e}^{\pi\mathfrak{b}(2\mathcal{C}_2)}\mathrm{e}^{\pi\mathfrak{b}(2\mathcal{C}_3+2\mathrm{i}\eta)} s_2 + \mathrm{e}^{\pi\mathrm{i}\mathfrak{b}^2} t_1 w_3 = 0.$$

They can be confirmed by using (7.8).

(iii) The remaining cases (D.2), (D.3), (D.5), (D.6), (D.8), (D.9), (D.14), (D.15). Direct substitution of the formula (6.12) with appropriate shift of the integration variable z and the application of (6.4) lead to LHS − RHS $= \int_{\mathbb{R}+\mathrm{i}f} \mathrm{d}z \Xi(z)$, where $f \in \mathbb{R}$ represents a freedom to shift the integration contour. Although $\Xi(z)$ is not identically vanishing, one can always find $\tilde{\Xi}(z)$ such that $\Xi(z) = \tilde{\Xi}(z+\mathrm{i}\mathfrak{b}) - \tilde{\Xi}(z)$.[10] Thus, the claim reduces to the analyticity of $\tilde{\Xi}(z)$ in $f < \mathrm{Im}\, z < f + \eta$ and the damping in $\mathrm{Re}(z) \to \pm\infty$ in this strip. As an example, consider (D.6), whose elements of LHS − RHS for the transition $\left|x_1' - \mathrm{i}\mathfrak{b}, x_2', x_3'\right\rangle \mapsto |x_1, x_2, x_3\rangle$ read

$$w_1 \mathrm{e}^{\pi\mathfrak{b}(x_1'-\mathrm{i}\mathfrak{b})}\big(r_2 s_2 + t_2 w_2 \mathrm{e}^{\pi\mathfrak{b}(2x_2'+\mathrm{i}\mathfrak{b})}\big) R^{x_1,x_2,x_3}_{x_1'-\mathrm{i}\mathfrak{b},x_2'+\mathrm{i}\mathfrak{b},x_3'-\mathrm{i}\mathfrak{b}} \\ + r_2 w_3 \mathrm{e}^{\pi\mathfrak{b}x_3'}\big(r_1 s_1 + t_1 w_1 \mathrm{e}^{\pi\mathfrak{b}(2x_1'-\mathrm{i}\mathfrak{b})}\big) R^{x_1,x_2,x_3}_{x_1',x_2',x_3'} \\ - r_3 w_2 \mathrm{e}^{\pi\mathfrak{b}x_2}\big(r_1 s_1 + t_1 w_1 \mathrm{e}^{\pi\mathfrak{b}(2x_1-\mathrm{i}\mathfrak{b})}\big) R^{x_1-\mathrm{i}\mathfrak{b},x_2,x_3}_{x_1'-\mathrm{i}\mathfrak{b},x_2',x_3'}, \tag{7.9}$$

where $x_1 + x_2 = x_1' + x_2'$ and $x_2 + x_3 = x_2' + x_3'$ are assumed in view of the two delta functions in (6.12). After shifting the integration variable z for the last term to $z + \mathrm{i}\mathfrak{b}/2$, the integrand for this expression can be shown to be proportional to $\tilde{\Xi}(z+\mathrm{i}\mathfrak{b}) - \tilde{\Xi}(z)$ with

$$\tilde{\Xi}(z) = \mathrm{e}^{2\pi\mathrm{i}z(-x_2-\mathrm{i}\eta+\mathcal{C}_4)} \frac{\Phi_\mathfrak{b}\big(z+\frac{1}{2}(x_1-x_3+\mathrm{i}\eta)+\mathcal{C}_5-\mathrm{i}\mathfrak{b}\big)\Phi_\mathfrak{b}\big(z+\frac{1}{2}(-x_1+x_3+\mathrm{i}\eta)+\mathcal{C}_6\big)}{\Phi_\mathfrak{b}\big(z+\frac{1}{2}(x_1+x_3-\mathrm{i}\eta)+\mathcal{C}_7\big)\Phi_\mathfrak{b}\big(z+\frac{1}{2}\big(-x_1'-x_3'-\mathrm{i}\eta\big)+\mathcal{C}_8\big)}.$$

Consider the region $-\eta < \mathrm{Im}\, x_2 = \mathrm{Im}\, x_2' < 0$ with $x_1, x_3, x_1', x_3' \in \mathbb{R}$, which is compatible with the above mentioned condition. From (6.6), $\tilde{\Xi}(z)$ is analytic in the strip $-\eta/2 < \mathrm{Im}\, z < \eta/2$.

[10] An analogous treatment can also be found in the proof of [17, Theorem 3.18].

Moreover, from (6.5) and $\mathcal{C}_5 + \mathcal{C}_6 = \mathcal{C}_7 + \mathcal{C}_8$, $\tilde{\tilde{\Xi}}(z)$ asymptotically tends to $\mathrm{e}^{2\pi\mathrm{i}z(-x_2-\mathrm{i}\eta+\mathcal{C}_4)}$ for $\operatorname{Re} z \to -\infty$ and to $\mathrm{e}^{2\pi\mathrm{i}z(-x_2'+\mathrm{i}\eta-\mathrm{i}\mathfrak{b}+\mathcal{C}_4)}$ for $\operatorname{Re} z \to +\infty$. Therefore, from $0 < \eta < 1$, they are both decaying if $-\eta < \operatorname{Im} x_2 = \operatorname{Im} x_2' < 0$ as long as the limit $|\operatorname{Re} z| \to \infty$ is taken in the strip $-\eta/2 < \operatorname{Im} z < \eta/2$. (This corresponds to the choice $f = -\eta/2$.) This verifies (D.2).

As seen in this example, the precise region of x_i, x_i' which validates such a check is sensitive to how they appear shifted as the indices of R's as in (7.9), and they indeed vary case by case. We have checked, for all the equations of (iii), that there is $\tilde{\tilde{\Xi}}(z)$ having a similar 'factorized' form, and there is a subregion of $-\eta < \operatorname{Im} x_2 = \operatorname{Im} x_2' < \eta$ with $x_1, x_3, x_1', x_3' \in \mathbb{R}$ which assures that $\tilde{\tilde{\Xi}}(z)$ possesses a strip $f < \operatorname{Im} z < f + \eta$ where it behaves in the same manner as the above example.

Remark 7.3. As a corollary of Theorem 7.1, it is evident that postulating the $RLLL$ relation (7.7) for $p = q$ and $q^\vee$ *simultaneously* compels $\mathcal{C}_1 = \cdots = \mathcal{C}_8 = 0$. The resulting parameter-free (except $\mathfrak{b}$) R-matrix (6.12)–(6.14) exactly reproduces [2, equation (51)]. We refer to this particular case as the *modular double* $\mathcal{R}$.

In Section 6, the term "modular $\mathcal{R}$" (without "double") is deliberately used to distinctively describe the results with full parameters. Note on the other hand that the condition $\mathcal{C}_1 = \cdots = \mathcal{C}_8 = 0$ still leaves five free parameters among $(r_j, s_j, t_j, w_j)_{j=1,2,3}$.

8 Reduction to the Fock–Goncharov quiver

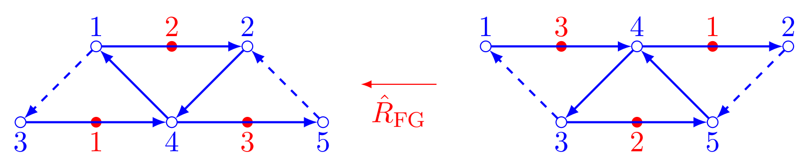
In this section, we explain that our R-matrix (4.14) for the symmetric butterfly (SB) quiver reduces to that for the Fock–Goncharov (FG) quiver [11] in a certain limit of parameters.

8.1 *R*-matrix for the FG quiver

Let p_i, u_i $(i = 1, 2, 3)$ be canonical variables obeying $[\mathrm{p}_i, \mathrm{u}_j] = \delta_{ij}\hbar$, $[\mathrm{p}_i, \mathrm{p}_j] = [\mathrm{u}_i, \mathrm{u}_j] = 0$. Recall the R-matrix in [11, equation (4.14)][11] given as

$$R_{\mathrm{FG}} = \Psi_q\big(\mathrm{e}^{\theta_1-\theta_3+\mathrm{p}_1+\mathrm{u}_1+\mathrm{p}_3-\mathrm{u}_3-\mathrm{p}_2}\big)P_{\mathrm{FG}}, \qquad P_{\mathrm{FG}} = \rho_{23}\mathrm{e}^{\frac{1}{\hbar}\mathrm{p}_1(\mathrm{u}_3-\mathrm{u}_2)}\mathrm{e}^{\frac{\theta_2-\theta_3}{\hbar}(\mathrm{u}_3-\mathrm{u}_1)}, \tag{8.1}$$

where θ_i' s are parameters. It has been deduced from $\hat{R}_{\mathrm{FG}} = \operatorname{Ad}(R_{\mathrm{FG}})$, where $\hat{R}_{\mathrm{FG}}$ is the cluster transformation corresponding to μ_4^* in the FG quivers depicted as follows:



(8.2)

As in Figure 4, the dots marked 1, 2, 3 in red signify the crossings of the associated wiring diagram, to which the canonical variables are attached.

Let B_{FG} and B_{FG}' be (the exchange matrices of) the left and the right quivers in (8.2), respectively. Let $\mathcal{Y}(B_{\mathrm{FG}})$ be the skew field generated by $\mathcal{Y}_1, \dots, \mathcal{Y}_5$ which are attached to the vertices of B_{FG} and obey the commutation relation (2.1) where b_{ij} is taken to be the elements of B_{FG}. Define $\mathcal{Y}(B_{\mathrm{FG}}')$ generated by $\mathcal{Y}_1', \dots, \mathcal{Y}_5'$ from B_{FG}' similarly.

Let $\mathcal{W}_3^{\mathrm{pu}}$ be the direct product of the q-Weyl algebras generated by $\mathrm{e}^{\pm\mathrm{p}_i}$, $\mathrm{e}^{\pm\mathrm{u}_i}$ for $i = 1, 2, 3$. The fractional field of $\mathcal{W}_3^{\mathrm{pu}}$ is denoted by $\mathcal{A}_3^{\mathrm{pu}}$. We denote the isomorphism π_{123} of $\mathcal{W}_3^{\mathrm{pu}}$ [11,

[11]The parameter λ_i in [11, equation (3.5)] is denoted by θ_i here.

equation (3.7)] (in the sense of exponentials) by π_{FG} and recall the embeddings $\phi_{\mathrm{FG}}\colon \mathcal{Y}(B_{\mathrm{FG}}) \hookrightarrow \mathcal{A}_3^{\mathrm{pu}}$ and $\phi'_{\mathrm{FG}}\colon \mathcal{Y}(B'_{\mathrm{FG}}) \hookrightarrow \mathcal{A}_3^{\mathrm{pu}}$ [11, equation (3.6)] given as

$$\pi_{\mathrm{FG}}\colon \begin{cases} \mathrm{p}_1 \mapsto \mathrm{p}_1 + \theta_2 - \theta_3, & \mathrm{p}_2 \mapsto \mathrm{p}_1 + \mathrm{p}_3, \quad \mathrm{p}_3 \mapsto \mathrm{p}_2 - \mathrm{p}_1 - \theta_2 + \theta_3, \\ \mathrm{u}_1 \mapsto \mathrm{u}_1 + \mathrm{u}_2 - \mathrm{u}_3, & \mathrm{u}_2 \mapsto \mathrm{u}_3, \quad \mathrm{u}_3 \mapsto \mathrm{u}_2, \end{cases}$$

$$\phi_{\mathrm{FG}}\colon \begin{cases} \mathcal{Y}_1 \mapsto \mathrm{e}^{-\theta_2+\mathrm{p}_2-\mathrm{u}_2-\mathrm{p}_1}, \\ \mathcal{Y}_2 \mapsto \mathrm{e}^{\theta_2+\mathrm{p}_2+\mathrm{u}_2-\mathrm{p}_3}, \\ \mathcal{Y}_3 \mapsto \mathrm{e}^{-\theta_1+\mathrm{p}_1-\mathrm{u}_1}, \\ \mathcal{Y}_4 \mapsto \mathrm{e}^{\theta_1-\theta_3+\mathrm{p}_1+\mathrm{u}_1+\mathrm{p}_3-\mathrm{u}_3-\mathrm{p}_2}, \\ \mathcal{Y}_5 \mapsto \mathrm{e}^{\theta_3+\mathrm{p}_3+\mathrm{u}_3}, \end{cases} \qquad \phi'_{\mathrm{FG}}\colon \begin{cases} \mathcal{Y}'_1 \mapsto \mathrm{e}^{-\theta_3+\mathrm{p}_3-\mathrm{u}_3}, \\ \mathcal{Y}'_2 \mapsto \mathrm{e}^{\theta_1+\mathrm{p}_1+\mathrm{u}_1}, \\ \mathcal{Y}'_3 \mapsto \mathrm{e}^{-\theta_2+\mathrm{p}_2-\mathrm{u}_2-\mathrm{p}_3}, \\ \mathcal{Y}'_4 \mapsto \mathrm{e}^{-\theta_1+\theta_3+\mathrm{p}_3+\mathrm{u}_3+\mathrm{p}_1-\mathrm{u}_1-\mathrm{p}_2}, \\ \mathcal{Y}'_5 \mapsto \mathrm{e}^{\theta_2+\mathrm{p}_2+\mathrm{u}_2-\mathrm{p}_1}. \end{cases}$$

One has $\pi_{\mathrm{FG}} = \mathrm{Ad}(P_{\mathrm{FG}})$. With these notation, the R-matrix (8.1) is rephrased as

$$R_{\mathrm{FG}} = \Psi_q(\phi_{\mathrm{FG}}(\mathcal{Y}_4))P_{\mathrm{FG}}.$$

8.2 Embedding FG into SB

We employ a parallel notation $\mathcal{Y}(B_{\mathrm{SB}})$ and $\mathcal{Y}(B'_{\mathrm{SB}})$ to signify the skew fields corresponding to the left and the right quivers in Figure 4. It is easy to see that the following maps yield morphisms of the skew fields $\alpha\colon \mathcal{Y}(B_{\mathrm{FG}}) \to \mathcal{Y}(B_{\mathrm{SB}})$ and $\alpha'\colon \mathcal{Y}(B'_{\mathrm{FG}}) \to \mathcal{Y}(B'_{\mathrm{SB}})$

$$\alpha\colon \begin{cases} \mathcal{Y}_1 \mapsto Y_9, \\ \mathcal{Y}_2 \mapsto qY_8Y_7, \\ \mathcal{Y}_3 \mapsto Y_6, \\ \mathcal{Y}_4 \mapsto qY_5Y_4, \\ \mathcal{Y}_5 \mapsto qY_3Y_2, \end{cases} \qquad \alpha'\colon \begin{cases} \mathcal{Y}'_1 \mapsto Y'_9, \\ \mathcal{Y}'_2 \mapsto qY'_5Y'_7, \\ \mathcal{Y}'_3 \mapsto Y'_6, \\ \mathcal{Y}'_4 \mapsto qY'_3Y'_4, \\ \mathcal{Y}'_5 \mapsto qY'_8Y'_2. \end{cases} \tag{8.3}$$

Recall that $\mathcal{A}_3$ defined after Figure 6 for the SB quiver is a fractional field of $\mathcal{W}_3$ in which $\mathrm{e}^{\mathsf{u}_i}\mathrm{e}^{\mathsf{w}_j} = q^{\delta_{ij}\hbar}\mathrm{e}^{\mathsf{w}_j}\mathrm{e}^{\mathsf{u}_i}$. On the other hand, $\mathcal{A}_3^{\mathrm{pu}}$ for the FG quiver in the previous subsection is a fractional field of $\mathcal{W}_3^{\mathrm{pu}}$ in which $\mathrm{e}^{\mathrm{p}_i}\mathrm{e}^{\mathrm{u}_i} = q^{\delta_{ij}\hbar}\mathrm{e}^{\mathrm{u}_i}\mathrm{e}^{\mathrm{p}_i}$. Thus there is an isomorphism $\beta\colon \mathcal{W}_3^{\mathrm{pu}} \to \mathcal{W}_3$ given by

$$\beta\colon \ \mathrm{p}_i \mapsto -\mathsf{w}_i, \qquad \mathrm{u}_i \mapsto \mathsf{u}_i, \qquad i = 1, 2, 3, \tag{8.4}$$

in the sense of exponentials. We consider the diagram

$$\begin{tikzcd} \mathcal{Y}(B_{\mathrm{FG}}) \arrow[r, "\alpha"] \arrow[d, "\phi_{\mathrm{FG}}"'] & \mathcal{Y}(B_{\mathrm{SB}}) \arrow[d, "\phi_{\mathrm{SB}}"] \\ \mathcal{W}_3^{\mathrm{pu}} \arrow[r, "\beta"] \arrow[d, "\pi_{\mathrm{FG}}"'] & \mathcal{W}_3 \arrow[d, "\tau^{\mathsf{uw}}_{--++}"] \\ \mathcal{W}_3^{\mathrm{pu}} \arrow[r, "\beta"] & \mathcal{W}_3 \\ \mathcal{Y}(B'_{\mathrm{FG}}) \arrow[r, "\alpha'"] \arrow[u, "\phi'_{\mathrm{FG}}"] & \mathcal{Y}(B'_{\mathrm{SB}}), \arrow[u, "\phi'_{\mathrm{SB}}"'] \end{tikzcd} \tag{8.5}$$

where ϕ_{SB}, ϕ'_{SB} and $\tau^{\mathsf{uw}}_{--++}$ are defined in (4.3), (4.4) and (4.7), respectively.

Proposition 8.1. *The diagram* (8.5) *is commutative if and only if the parameters θ_i $(i=1,2,3)$ and (a_i,b_i,c_i,d_i,e_i) subject to* (4.2) *satisfy the relations*

$$e_2=e_3, \qquad a_1=-a_3=c_3=-c_1, \qquad a_2=c_2=0, \tag{8.6}$$

$$\theta_1=-b_1, \qquad \theta_2=-a_1-b_2, \qquad \theta_3=d_3+e_3. \tag{8.7}$$

Proof. For the top square, it suffices to consider the image of $\mathcal{Y}_i \in \mathcal{Y}(B_{\mathrm{FG}})$ $(i=1,\dots,5)$. For instance, one has $\phi_{\mathrm{SB}}\circ\alpha(\mathcal{Y}_1)=\phi_{\mathrm{SB}}(Y_9)=\mathrm{e}^{a_1+b_2+\mathsf{w}_1-\mathsf{u}_2-\mathsf{w}_2}$ and $\beta\circ\phi_{\mathrm{FG}}(\mathcal{Y}_1)=\beta\big(\mathrm{e}^{-\theta_2+\mathrm{p}_2-\mathrm{u}_2-\mathrm{p}_1}\big)=\mathrm{e}^{-\theta_2-\mathsf{w}_2-\mathsf{u}_2+\mathsf{w}_1}$, hence the commutativity requires $\theta_2=-a_1-b_2$. A similar calculation leads to

$$\theta_2=-a_1-b_2=e_2+d_2+a_3, \qquad \theta_1=-b_1, \qquad \theta_1-\theta_3=e_1+d_1+c_2+b_3,$$
$$\theta_3=e_3+d_3.$$

For the middle square, it suffices to consider the image of p_i, u_i, $(i=1,2,3)$. The commutativity leads to

$$\theta_2-\theta_3=-\lambda_1=\lambda_2, \qquad \lambda_0=\lambda_3=0,$$

where λ_i's are specified in (4.8). These nine relations are equivalent to (8.6) and (8.7). The commutativity of the bottom square follows from them. ■

8.3 R_{FG} as a limit of R_{SB}

Let R_{SB} be the R-matrix (4.14) for the SB quiver under the specialization of the parameters (8.6) and (8.7). Explicitly, we have

$$\begin{aligned}R_{\mathrm{SB}}={}&\Psi_q\big(\mathrm{e}^{-\Lambda+e_1+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)^{-1}\Psi_q\big(\mathrm{e}^{-\Lambda-e_3+e_1+\mathsf{u}_1-\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)^{-1}\\&\times\Psi_q\big(\mathrm{e}^{\theta_1-\theta_3+\mathsf{u}_1-\mathsf{u}_3-\mathsf{w}_1+\mathsf{w}_2-\mathsf{w}_3}\big)\Psi_q\big(\mathrm{e}^{\Lambda+e_2+\mathsf{u}_1+2\mathsf{u}_2-\mathsf{u}_3-\mathsf{w}_1+\mathsf{w}_2-\mathsf{w}_3}\big)P_{\mathrm{SB}},\\P_{\mathrm{SB}}={}&\mathrm{e}^{\frac{1}{\hbar}(\mathsf{u}_3-\mathsf{u}_2)\mathsf{w}_1}\mathrm{e}^{\frac{\theta_3-\theta_2}{\hbar}(\mathsf{u}_1-\mathsf{u}_2)}\rho_{23}=\rho_{23}\mathrm{e}^{\frac{1}{\hbar}(\mathsf{u}_2-\mathsf{u}_3)\mathsf{w}_1}\mathrm{e}^{\frac{\theta_3-\theta_2}{\hbar}(\mathsf{u}_1-\mathsf{u}_3)},\end{aligned}$$

where $\Lambda=\theta_1-\theta_3=d_1+e_1+c_2+b_3$. The above formula for P_{SB} follows from (4.10) under the specialization.

Theorem 8.2. *The R-matrix R_{FG} is reproduced from the specialized R-matrix R_{SB} as*

$$\lim R_{\mathrm{SB}}=\beta(R_{\mathrm{FG}}),$$

where the limit is taken as

$$\begin{aligned}&e_1\to-\infty, \qquad e_2=e_3\to-\infty, \qquad e_1-e_3\to-\infty,\\&e_i+d_i=\text{finite}, \qquad i=1,2,3.\end{aligned} \tag{8.8}$$

Proof. Since Λ remains finite in the limit, one has $\lim R_{\mathrm{SB}}=\Psi_q\big(\mathrm{e}^{\theta_1-\theta_3+\mathsf{u}_1-\mathsf{u}_3-\mathsf{w}_1+\mathsf{w}_2-\mathsf{w}_3}\big)P_{\mathrm{SB}}$. By comparing this with (8.1), the claim is checked easily. ■

Remark 8.3. Parallel results which fit the formula (B.2) can also be formulated. One replaces (8.3) with

$$\alpha\colon\begin{cases}\mathcal{Y}_1\mapsto Y_7,\\\mathcal{Y}_2\mapsto qY_8Y_9,\\\mathcal{Y}_3\mapsto Y_2,\\\mathcal{Y}_4\mapsto qY_3Y_4,\\\mathcal{Y}_5\mapsto qY_5Y_6,\end{cases}\qquad \alpha'\colon\begin{cases}\mathcal{Y}'_1\mapsto Y'_7,\\\mathcal{Y}'_2\mapsto qY'_3Y'_9,\\\mathcal{Y}'_3\mapsto Y'_2,\\\mathcal{Y}'_4\mapsto qY'_5Y'_4,\\\mathcal{Y}'_5\mapsto qY'_8Y'_6,\end{cases}$$

and (8.4) with $\beta\colon(\mathrm{p}_1,\mathrm{p}_2,\mathrm{p}_3,\mathrm{u}_1,\mathrm{u}_2,\mathrm{u}_3)\mapsto(-\mathsf{w}_3,-\mathsf{w}_2,-\mathsf{w}_1,\mathsf{u}_3,\mathsf{u}_2,\mathsf{u}_1)$. Then the diagram (8.5) in which $\tau^{\mathsf{uw}}_{--++}$ is replaced with $\tau^{\mathsf{uw}}_{-+-+}$ (B.1) becomes commutative if and only if (8.6) and $(\theta_1,\theta_2,\theta_3)=(-d_3,-d_2-a_3,b_1+e_1)$ hold.

8.4 A limiting procedure for modular R

Let us demonstrate an explicit limiting procedure that essentially corresponds to Theorem 8.2 in the context of the modular R in the coordinate representation. In the strong coupling regime $0<\eta<1$, one has $|\mathfrak{b}|=1$ and $\operatorname{Re}\mathfrak{b}>0$. Thus the product representation (6.2) indicates

$$\lim_{z\to-\infty}\Phi_{\mathfrak{b}}(z)=1. \tag{8.9}$$

Proposition 8.4. *Specialize the parameters in $\mathcal{C}_1,\dots,\mathcal{C}_8$ in Theorem 6.1 as*

$$\mathcal{C}_1=\frac{1}{2}(\zeta-\xi)\ell_{21}-\frac{1}{2}\zeta\ell_{31},\qquad \mathcal{C}_2=-\mathcal{C}_8=-T-\frac{1}{2}\zeta\ell_{31},\qquad \mathcal{C}_3=0,$$
$$\mathcal{C}_4=T+\frac{1}{2}(\xi+\zeta)\ell_{21},\qquad \mathcal{C}_5=\frac{1}{2}\zeta\ell_{31},\qquad \mathcal{C}_7=\mathcal{C}_5+\mathcal{C}_6-\mathcal{C}_8,$$

where $\ell_{ij}=\ell_i-\ell_j$. Then the elements of the R-matrix associated with the FG quiver in [11, Proposition 7.4], with the exchange of components $1\leftrightarrow 3$, are reproduced as a limit of (6.12) as follows:

$$\begin{aligned}&\lim_{\substack{T\to\infty\\ \mathcal{C}_6,\mathcal{C}_7\to-\infty}} \mathrm{e}^{-\mathrm{i}\pi\gamma}g\big(\tilde{a},\tilde{b},\tilde{c},\tilde{d}\big)^{-1}\mathcal{R}^{x_1,x_2,x_3}_{x_1',x_2',x_3'}\\ &\qquad=\delta\big(x_1+x_2-x_1'-x_2'\big)\delta\big(x_2+x_3-x_2'-x_3'\big)\\ &\qquad\quad\times\Phi_{\mathfrak{b}}\left(x_2-x_1'+\mathrm{i}\eta-\frac{1}{2}(\xi+\zeta)\ell_{21}\right)\mathrm{e}^{\mathrm{i}\pi(x_2-x_1'-\xi\ell_{21})(x_1-x_3+\zeta\ell_{31}-2\mathrm{i}\eta)},\end{aligned}$$

where $\gamma=-T^2+(\mathrm{i}\eta-\zeta\ell_{31})T-\frac{1}{2}\mathrm{i}\eta\ell_{21}(3\xi-\zeta)+\frac{1}{2}\ell_{31}\zeta(\ell_{21}(\xi-\zeta)+2\mathrm{i}\eta)+\frac{1+4\eta^2}{12}$.

Proof. Due to (8.9), in the limit $\mathcal{C}_6,\mathcal{C}_7\to-\infty$, the integral $I^{x_1,x_2,x_3}_{x_1',x_2',x_3'}$ in (6.13) simplifies to

$$\begin{aligned}I^{x_1,x_2,x_3}_{x_1',x_2',x_3'}&\to\int_{-\infty}^{\infty}\mathrm{d}z\mathrm{e}^{2\pi\mathrm{i}z(-x_2-\mathrm{i}\eta+\mathcal{C}_4)}\frac{\Phi_{\mathfrak{b}}\big(z+\frac{1}{2}(x_1-x_3+\mathrm{i}\eta)+\mathcal{C}_5\big)}{\Phi_{\mathfrak{b}}\big(z+\frac{1}{2}(-x_1'-x_3'-\mathrm{i}\eta)+\mathcal{C}_8\big)}\\ &=\mathrm{e}^{\mathrm{i}\pi\nu}\frac{\Phi_{\mathfrak{b}}\big(x_2-x_1'+\mathrm{i}\eta-\mathcal{C}_4-\mathcal{C}_5+\mathcal{C}_8\big)\Phi_{\mathfrak{b}}\big(x_1'+\mathcal{C}_5-\mathcal{C}_8\big)}{\Phi_{\mathfrak{b}}(x_2-\mathcal{C}_4)},\end{aligned}$$

where we have evaluated the integral by (C.1), set $x_1=x_1'-x_3'+x_3$ and then applied (6.3) in the result. We omit the messy explicit form of the power ν. Noting that $-\mathcal{C}_4-\mathcal{C}_5+\mathcal{C}_8=-\frac{1}{2}(\xi+\zeta)\ell_{21}$, the rest is straightforward. ■

In view of the symmetry (4.30) and the comment after Theorem 6.1, one can also reproduce the original form of [11, equation (7.12)] without the exchange of components $1\leftrightarrow 3$ by specializing the parameters in (6.13) as

$$\begin{aligned}&\mathcal{C}_1=\frac{1}{2}(\xi-\zeta)\ell_{23}+\frac{1}{2}\zeta\ell_{13},\qquad \mathcal{C}_2=-\mathcal{C}_8=-T-\frac{1}{2}\zeta\ell_{13},\qquad \mathcal{C}_3=0,\\ &\mathcal{C}_4=T+\frac{1}{2}(\xi+\zeta)\ell_{23},\qquad \mathcal{C}_6=\frac{1}{2}\zeta\ell_{13},\qquad \mathcal{C}_7=\mathcal{C}_5+\mathcal{C}_6-\mathcal{C}_8,\end{aligned} \tag{8.10}$$

and taking the limit $-T$, $\mathcal{C}_5$, $\mathcal{C}_7\to-\infty$.

These results may be regarded as modular R versions of Theorem 8.2 at the level of matrix elements. In fact, under the specialization (8.6), one has

$$\mathcal{C}_1=\frac{1}{2}\big(\tilde{b}_1-\tilde{b}_2+2\tilde{c}_1+\tilde{d}_2-\tilde{d}_3\big),\qquad \mathcal{C}_2=-\frac{1}{2}\big(\tilde{b}_1+\tilde{d}_3\big),\qquad \mathcal{C}_3=0,\qquad \mathcal{C}_4=\frac{1}{2}\big(\tilde{b}_2+\tilde{d}_2\big),$$
$$\mathcal{C}_5=\frac{1}{2}\big(-\tilde{d}_1+\tilde{d}_3\big),\qquad \mathcal{C}_6=\frac{1}{2}\big(\tilde{b}_1-\tilde{b}_3\big),\qquad \mathcal{C}_7=\frac{1}{2}\big(-\tilde{d}_1-\tilde{b}_3\big),\qquad \mathcal{C}_8=\frac{1}{2}\big(\tilde{b}_1+\tilde{d}_3\big).$$

See (5.1) and (6.7). Then the limit (8.8) with e_i, d_i replaced with $\tilde{e}_i$, $\tilde{d}_i$ can be identified with $-T,\mathcal{C}_5,\mathcal{C}_7\to-\infty$ for (8.10).

A Supplement to Section 3.4

The two sides of the inhomogeneous twisted tetrahedron equation (3.13) yield the following monomial transformation:

$$\begin{aligned}
&Y_1^{(22)} \mapsto Y_1, \qquad Y_2^{(22)} \mapsto Y_2, \qquad Y_3^{(22)} \mapsto qY_7^{-1}Y_{12}^{-1}Y_{13}^{-1}Y_{14}^{-1}, \qquad Y_4^{(22)} \mapsto Y_4,\\
&Y_5^{(22)} \mapsto q^{-1}Y_5Y_6, \qquad Y_6^{(22)} \mapsto qY_{13}Y_{14}, \qquad Y_7^{(22)} \mapsto Y_{13}^{-1}, \qquad Y_8^{(22)} \mapsto q^{-1}Y_{12}Y_{13},\\
&Y_9^{(22)} \mapsto qY_8Y_9, \qquad Y_{10}^{(22)} \mapsto q^{-1}Y_{10}Y_{11}, \qquad Y_{11}^{(22)} \mapsto Y_3Y_6Y_7Y_{11}Y_{12},\\
&Y_{12}^{(22)} \mapsto q^{-3}Y_3^{-1}Y_6^{-1}Y_7^{-1}Y_8^{-1}Y_{14}^{-1}Y_{15}^{-1}, \qquad Y_{13}^{(22)} \mapsto q^{-1}Y_3Y_6Y_7Y_8,\\
&Y_{14}^{(22)} \mapsto q^{-1}Y_3^{-1}Y_6^{-1}Y_7^{-1}Y_8^{-1}Y_{11}^{-1}Y_{12}^{-1}, \qquad Y_{15}^{(22)} \mapsto q^{-4}Y_3Y_7Y_8Y_{14}Y_{15},\\
&Y_{16}^{(22)} \mapsto qY_{15}Y_{16}, \qquad Y_{17}^{(22)} \mapsto q^2Y_7Y_{12}Y_{13}Y_{14}Y_{17}.
\end{aligned} \tag{A.1}$$

Let us describe the monomial parts τ_{-+++} and τ_{---+} in (3.3) mentioned in Proposition 3.8

$$\tau_{-+++}\colon \begin{cases} Y_1' \mapsto Y_1, \qquad Y_2' \mapsto Y_2Y_3Y_4, \qquad Y_3' \mapsto q^2Y_3Y_4Y_8,\\ \qquad Y_4' \mapsto q^2Y_3^{-1}Y_4^{-2}Y_5^{-1}Y_8^{-1},\\ Y_5' \mapsto Y_4Y_5Y_8, \qquad Y_6' \mapsto Y_4Y_5Y_6, \qquad Y_7' \mapsto Y_7, \qquad Y_8' \mapsto Y_4^{-1},\\ Y_9' \mapsto Y_9, \qquad Y_{10}' \mapsto Y_{10}, \end{cases}$$

$$\tau_{---+}\colon \begin{cases} Y_1' \mapsto Y_1, \qquad Y_2' \mapsto Y_2, \qquad Y_3' \mapsto q^{-2}Y_4^{-1}Y_5^{-1}Y_8, \qquad Y_4' \mapsto Y_8^{-1},\\ Y_5' \mapsto Y_3^{-1}Y_4^{-1}Y_8, \qquad Y_6' \mapsto Y_6, \qquad Y_7' \mapsto Y_3Y_4Y_7, \qquad Y_8' \mapsto Y_3Y_4Y_5,\\ Y_9' \mapsto q^{-2}Y_4Y_5Y_9, \qquad Y_{10}' \mapsto Y_{10}. \end{cases}$$

Note that the image is not necessarily sign coherent.

B Formulas for $\tau^{\mathsf{uw}}_{-+-+}$, P_{-+-+} and R_{123} for $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-,+,-,+)$

Let τ_{-+-+} be the one in Example 3.3. Under the parametrization by q-Weyl algebra generators (4.3)–(4.4), it is translated into the transformation of the canonical variables as

$$\tau^{\mathsf{uw}}_{-+-+}\colon \begin{cases} \mathsf{u}_1 \mapsto \mathsf{u}_2 + \kappa_0, & \mathsf{w}_1 \mapsto \mathsf{w}_2 - \mathsf{w}_3 + \kappa_2,\\ \mathsf{u}_2 \mapsto \mathsf{u}_1 - \kappa_0, & \mathsf{w}_2 \mapsto \mathsf{w}_1 + \mathsf{w}_3 + \kappa_1,\\ \mathsf{u}_3 \mapsto -\mathsf{u}_1 + \mathsf{u}_2 + \mathsf{u}_3 + \kappa_0, & \mathsf{w}_3 \mapsto \mathsf{w}_3 + \kappa_3, \end{cases} \tag{B.1}$$

where $\kappa_r = \kappa_r(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ for $r = 0,1,2,3$ is defined, under the condition (4.2), by

$$\begin{aligned}
&\kappa_0 = \frac{e_2 - e_1}{2}, \qquad \kappa_1 = c_1 - c_2 + c_3, \qquad \kappa_2 = d_1 - d_2 - a_3 - \kappa_0,\\
&\kappa_3 = b_1 + c_1 - b_2 - c_2 - \kappa_0.
\end{aligned}$$

This is realized as an adjoint action as

$$\begin{aligned}
&\tau^{\mathsf{uw}}_{-+-+} = \mathrm{Ad}(P_{-+-+}),\\
&P_{-+-+} = \mathrm{e}^{\frac{1}{\hbar}(\mathsf{u}_1-\mathsf{u}_2)\mathsf{w}_3}\mathrm{e}^{\frac{\kappa_0}{\hbar}(\mathsf{w}_1-\mathsf{w}_2-\mathsf{w}_3)}\mathrm{e}^{\frac{1}{\hbar}(\kappa_1\mathsf{u}_1+\kappa_2\mathsf{u}_2+\kappa_3\mathsf{u}_3)}\rho_{12} \in N_3 \rtimes \mathfrak{S}_3.
\end{aligned}$$

From (3.6) and (3.7), the formulas analogous to (4.14) and (4.15) become as follows:

$$\Psi_q\big(\mathrm{e}^{-d_1-c_2-b_3+\mathsf{u}_1+\mathsf{u}_3+\mathsf{w}_1-\mathsf{w}_2+\mathsf{w}_3}\big)^{-1}\Psi_q\big(\mathrm{e}^{d_1+c_2+b_3+e_3-\mathsf{u}_1+\mathsf{u}_3-\mathsf{w}_1+\mathsf{w}_2-\mathsf{w}_3}\big)$$

$$
\begin{aligned}
&\times\Psi_q\big(\mathrm{e}^{-d_1-e_1-c_2-b_3-\mathrm{u}_1+\mathrm{u}_3+\mathrm{w}_1-\mathrm{w}_2+\mathrm{w}_3}\big)^{-1}\\
&\times\Psi_q\big(\mathrm{e}^{d_1+c_2+e_2+b_3+e_3-\mathrm{u}_1+2\mathrm{u}_2+\mathrm{u}_3-\mathrm{w}_1+\mathrm{w}_2-\mathrm{w}_3}\big)P_{-+-+}\\
&\quad=\Psi_q\big(\mathrm{e}^{-d_1-c_2-b_3+\mathrm{u}_1+\mathrm{u}_3+\mathrm{w}_1-\mathrm{w}_2+\mathrm{w}_3}\big)^{-1}\Psi_q\big(\mathrm{e}^{d_1+c_2+b_3+e_3-\mathrm{u}_1+\mathrm{u}_3-\mathrm{w}_1+\mathrm{w}_2-\mathrm{w}_3}\big)P_{-+-+}\\
&\qquad\times\Psi_q\big(\mathrm{e}^{b_1+a_2+d_3+e_3-\mathrm{u}_1+\mathrm{u}_3-\mathrm{w}_1+\mathrm{w}_2-\mathrm{w}_3}\big)^{-1}\Psi_q\big(\mathrm{e}^{-b_1-a_2-d_3+\mathrm{u}_1+\mathrm{u}_3+\mathrm{w}_1-\mathrm{w}_2+\mathrm{w}_3}\big).
\end{aligned}\tag{B.2}
$$

C Integral formula involving non-compact quantum dilogarithm

The following is known as a modular double analogue of the Ramanujan ${}_1\Psi_1$-sum

$$
\begin{aligned}
\int \mathrm{d}t\frac{\Phi_{\mathfrak{b}}(t+u)}{\Phi_{\mathfrak{b}}(t+v)}\mathrm{e}^{2\pi\mathrm{i}wt}&=\frac{\Phi_{\mathfrak{b}}(u-v-\mathrm{i}\eta)\Phi_{\mathfrak{b}}(w+\mathrm{i}\eta)}{K\Phi_{\mathfrak{b}}(u-v+w-\mathrm{i}\eta)}\mathrm{e}^{-2\pi\mathrm{i}w(v+\mathrm{i}\eta)}\\
&=\frac{K\Phi_{\mathfrak{b}}(v-u-w+\mathrm{i}\eta)}{\Phi_{\mathfrak{b}}(v-u+\mathrm{i}\eta)\Phi_{\mathfrak{b}}(-w-\mathrm{i}\eta)}\mathrm{e}^{-2\pi\mathrm{i}w(u-\mathrm{i}\eta)},
\end{aligned}\tag{C.1}
$$

where $K=\mathrm{e}^{-\mathrm{i}\pi(4\eta^2+1)/12}$. See [4, Section 6.3] for the condition concerning the validity of the integrals. From $\Phi_{\mathfrak{b}}(u)|_{u\to-\infty}\to 1$, their limit $u,v\to-\infty$ reduces to

$$
\int \mathrm{d}t\frac{\mathrm{e}^{2\pi\mathrm{i}wt}}{\Phi_{\mathfrak{b}}(t+v)}=\frac{\Phi_{\mathfrak{b}}(w+\mathrm{i}\eta)}{K}\mathrm{e}^{-2\pi\mathrm{i}w(v+\mathrm{i}\eta)},\tag{C.2}
$$

$$
\int \mathrm{d}t\Phi_{\mathfrak{b}}(t+u)\mathrm{e}^{2\pi\mathrm{i}wt}=\frac{K}{\Phi_{\mathfrak{b}}(-w-\mathrm{i}\eta)}\mathrm{e}^{-2\pi\mathrm{i}w(u-\mathrm{i}\eta)}.\tag{C.3}
$$

D Explicit form of *RLLL* relation (7.4)

We write down the explicit form of (7.4) together with the corresponding choice of $(abcijk)$ in (7.5) or in Figure 8. As mentioned after Figure 8, there are 18 non-trivial cases. To save the space, we write $Y_\alpha=r_\alpha s_\alpha Z^{-1}+t_\alpha w_\alpha XZ^{-1}X$,

$$(001001):\ \mathcal{R}(1\otimes X\otimes X)=(1\otimes X\otimes X)\mathcal{R},\tag{D.1}$$

$$(001010):\ \mathcal{R}\big(r_2t_1X\otimes 1\otimes Y_3+t_3Z\otimes Y_2\otimes X\big)=r_1t_2\big(1\otimes X\otimes Y_3\big)\mathcal{R},\tag{D.2}$$

$$(001100):\ \mathcal{R}\big(t_3w_1X\otimes Y_2\otimes X+r_2Y_1\otimes 1\otimes Y_3\big)=r_1r_3\big(1\otimes Y_2\otimes 1\big)\mathcal{R},\tag{D.3}$$

$$(010001):\ r_1t_2\mathcal{R}(1\otimes X\otimes Z)=\big(r_2t_1X\otimes 1\otimes Z+t_3Y_1\otimes Z\otimes X\big)\mathcal{R},\tag{D.4}$$

$$
\begin{aligned}
(010010):\ &\mathcal{R}\big(r_2t_1w_3X\otimes 1\otimes X+Z\otimes Y_2\otimes Z\big)\\
&\qquad=\big(r_2t_1w_3X\otimes 1\otimes X+Y_1\otimes Z\otimes Y_3\big)\mathcal{R},
\end{aligned}\tag{D.5}
$$

$$(010100):\ \mathcal{R}\big(w_1X\otimes Y_2\otimes Z+r_2w_3Y_1\otimes 1\otimes X\big)=r_3w_2\big(Y_1\otimes X\otimes 1\big)\mathcal{R},\tag{D.6}$$

$$(011011):\ \mathcal{R}(X\otimes X\otimes 1)=(X\otimes X\otimes 1)\mathcal{R},\tag{D.7}$$

$$(011101):\ s_3t_2\mathcal{R}\big(Y_1\otimes X\otimes 1\big)=\big(t_1X\otimes Y_2\otimes Z+s_2t_3Y_1\otimes 1\otimes X\big)\mathcal{R},\tag{D.8}$$

$$(011110):\ s_1s_3\mathcal{R}\big(1\otimes Y_2\otimes 1\big)=\big(t_1w_3X\otimes Y_2\otimes X+s_2Y_1\otimes 1\otimes Y_3\big)\mathcal{R},\tag{D.9}$$

$$(100001):\ r_1r_3\mathcal{R}(1\otimes Z\otimes 1)=(t_3w_1X\otimes Z\otimes X+r_2Z\otimes 1\otimes Z)\mathcal{R},\tag{D.10}$$

$$(100010):\ r_3w_2\mathcal{R}(Z\otimes X\otimes 1)=\big(w_1X\otimes Z\otimes Y_3+r_2w_3Z\otimes 1\otimes X\big)\mathcal{R},\tag{D.11}$$

$$(100100):\ \mathcal{R}(X\otimes X\otimes 1)=(X\otimes X\otimes 1)\mathcal{R},\tag{D.12}$$

$$(101011):\ \mathcal{R}(t_1X\otimes Z\otimes Y_3+s_2t_3Z\otimes 1\otimes X)=s_3t_2(Z\otimes X\otimes 1)\mathcal{R},\tag{D.13}$$

$$
\begin{aligned}
(101101):\ &\mathcal{R}\big(s_2t_3w_1X\otimes 1\otimes X+Y_1\otimes Z\otimes Y_3\big)\\
&\qquad=\big(s_2t_3w_1X\otimes 1\otimes X+Z\otimes Y_2\otimes Z\big)\mathcal{R},
\end{aligned}\tag{D.14}
$$

$$(101110):\ s_1w_2\mathcal{R}\big(1\otimes X\otimes Y_3\big)=\big(s_2w_1X\otimes 1\otimes Y_3+w_3Z\otimes Y_2\otimes X\big)\mathcal{R},\tag{D.15}$$

$$(110011):\ \mathcal{R}(t_1w_3X\otimes Z\otimes X+s_2Z\otimes 1\otimes Z)=s_1s_3(1\otimes Z\otimes 1)\mathcal{R},\tag{D.16}$$

$$(110101):\ \mathcal{R}\big(w_3Y_1\otimes Z\otimes X+s_2w_1X\otimes 1\otimes Z\big)=s_1w_2(1\otimes X\otimes Z)\mathcal{R}, \tag{D.17}$$

$$(110110):\ \mathcal{R}(1\otimes X\otimes X)=(1\otimes X\otimes X)\mathcal{R}. \tag{D.18}$$

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