8th International Conference on Geometry. Part 2*

Compiled and edited by H. Zeitler, Bayreuth

Lattice Geometry and Pythagorean Triangles[1]

Hans Walser, Basel (Switzerland)

Abstract: By drawing a Pythagorean triangle in a quadratic lattice and attaching a congruent lattice at the hypotenuse there will occur a Moiré effect with a new quadratic lattice of enlarged scale in the superposition. This new lattice is related to the parameterization of the Pythagorean triangle. A similar effect occurs with triangles with integer side lengths and an angle of 120° in a regular triangular lattice. We work with dot lattices on transparencies to visualize the optical effects.

Kurzreferat: *Gittergeometrie und pythagoreische Dreiecke.* Wird ein pythagoreisches Dreieck in ein quadratisches Raster gezeichnet und ein kongruenter Raster an die Hypotenuse angesetzt, erscheint in der Überlagerung ein Moiré-Effekt mit einem neuen, vergrößerten quadratischen Raster. Es wird ein Zusammenhang aufgezeigt zwischen diesem neuen Raster und der Parametrisierung der pythagoreischen Dreiecke. Ein ähnlicher Effekt erscheint bei Dreiecken mit ganzzahligen Seitenlängen und einem Winkel von 120° in einem regelmäßigen Dreiecksraster. Zur Visualisierung der optischen Effekte wird mit Punktrastern auf Klarsichtfolien gearbeitet.

ZDM-Classification: G40

1. The story

In 1991 there was an exposition to commemorate the 700th anniversary of the foundation of Switzerland in Zurich. In this exposition there was one section concerning the history of mathematics and science from the very beginnings up to day. It was a challenge to popularize some important topics of mathematics. To visualize the theorem of Pythagoras we built some puzzles where children – as well as adults – could rearrange the puzzle-parts from the two squares on the short sides of a right triangle in the square at the hypotenuse. Figure 1 depicts a possible dissection of the corresponding squares (see Baptist 1997, p. 59, or Fraedrich 1995, p. 25).

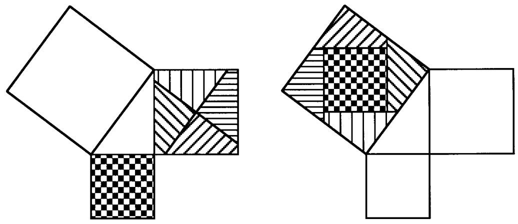
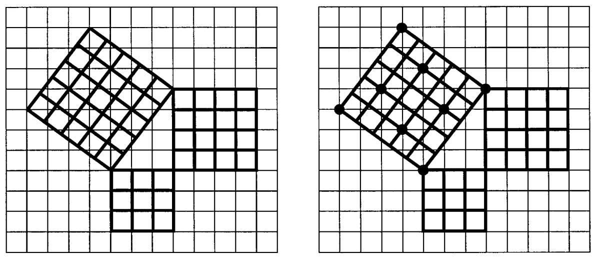


Fig. 1: Pythagoras' puzzle

We made also an example with the Pythagorean triangle with side lengths of 3, 4, and 5 units, where the puzzle simply consists of unit squares. I made a sketch of this example for the craftsmen. I used a sheet of paper with a square lattice as background and drew a corresponding square lattice in the square at the hypotenuse (Fig. 2a).



a) b)

Fig. 2: Sketch

Then I realized that the basic lattice and the lattice attached at the hypotenuse have some incident lattice points – they are marked by dots in the Figure 2b. Actually, this could have been seen already in a very old diagram dating from the Han dynasty in China (206 BC–220 AD) (Fig. 3).

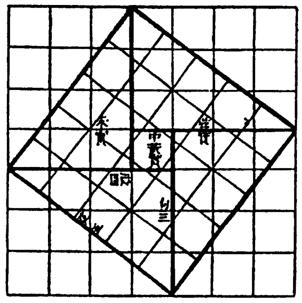


Fig. 3: Han dynasty, China

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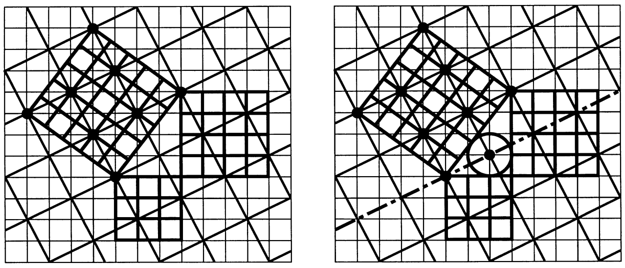
But here is more: these common points are grid points of a new and larger square lattice (Fig. 4a). One observes that the center of the incircle is one of the grid points of this new lattice and that the bisector of one of the angles of the triangle is a line of this lattice (Fig. 4b).

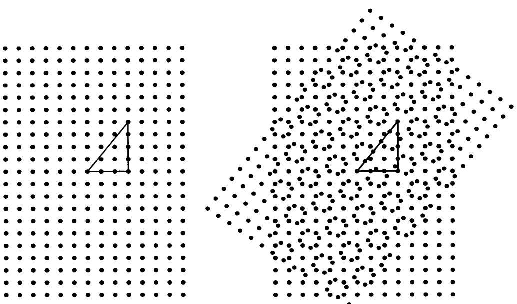


a) b)

Fig. 4: A new square lattice

2. Didactic approach

Of course one tries now to investigate whether these phenomena hold in every Pythagorean triangle. To do this, we need a new tool, the dot lattice. From this dot lattice we make two transparent copies in different colors, e.g. in blue and red. In one lattice – e.g. the basic blue lattice – we may sketch the Pythagorean triangle (Fig. 5a), and then we can attach the red lattice at the hypotenuse (Fig. 5b). On the overhead projector we see now a picture with three different colors: the "horizontal" lattice in blue, the slanting lattice in red, and – new – a larger lattice in purple, consisting of the common points of the blue and the red lattice. Figure 6 indicates the larger square lattice.



a) b)

Fig. 5: Dot lattices

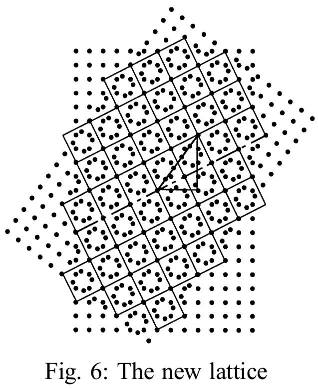


Fig. 6: The new lattice

For the students I made sets of two black and white lattices: One copy on paper, the other on a OHP-transparency. These dot lattices have to be of great precision. Therefore use a graphic software with a constrained grid. The printer is usually precise enough, but with the copy machines I made bad experiences. Very often they give insights into the topic of affine transformations with different eigenvalues.

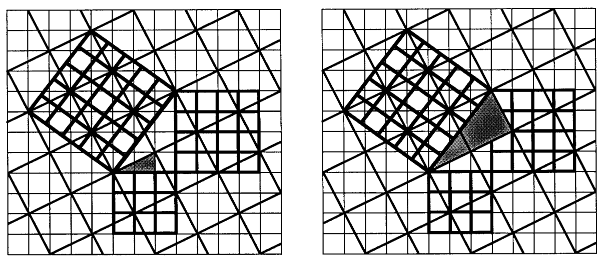
3. Phenomena and results

Using these dot lattices it is easy to show, that there will occur the above phenomena also in other Pythagorean triangles, e.g. in the triangle with side lengths of 5, 12, and 13 units. The center of the incircle is always a point of the new lattice. Therefore the radius of the incircle of a Pythagorean triangle must be integer (see Baptist 1982, Peters 1956/57). One of the angle bisectors is a lattice line.

There is a strong relationship between these visual phenomena and the parameterization of the Pythagorean triangles.

You can find primitive Pythagorean triangles – the sides of a primitive Pythagorean triangle have no common divider – by using two parameters u and v with $u, v \in N$, $u > v$, $(u, v) = 1$ (i.e. u and v have no common divider), $u \neq v \bmod 2$ (u and v have different parity, not both even or both odd). Then set $a := u^2 - v^2$, $b := 2uv$, $c := u^2 + v^2$. By computing you can check that a, b, and c are sides of a Pythagorean triangle. One can even prove, that there are for every primitive Pythagorean triangles matching parameters u and v (see Rademacher; Toeplitz 1968).

For $u = 2$ and $v = 1$ we get $a = 3$, $b = 4$, and $c = 5$, the example we have already studied. In Figure 7 a we see that the new large square lattice has slope $\frac{v}{u} = \frac{1}{2}$, relatively to the basic horizontal lattice and the hypotenuse of the triangle has again slope $\frac{v}{u} = \frac{1}{2}$ relatively to the new lattice.



a) b)

Fig. 7: Slopes

In general one can prove (see Walser 1995), that in a primitive Pythagorean triangle with parameters u and v we get by attaching a lattice at the hypotenuse a larger lattice with slope $\frac{v}{u}$, and relatively to this new lattice the hypotenuse has the same slope $\frac{v}{u}$. One lattice line with slope $\frac{v}{u}$ is an angle bisector of the Pythagorean triangle. Therefore we can obtain the red lattice attached at the hypotenuse simply by reflecting the basic blue lattice at a line with slope $\frac{v}{u}$.

Now let's consider a bad example, $u = 3$ and $v = 1$. Both parameters are odd. We get $a = 8$, $b = 6$, and $c = 10$. This is a Pythagorean triangle, but not primitive. In Figure 8 we see that we get again a new large square lattice in this case, but the quotient $\frac{v}{u}$ is not the slope of the lattice itself, but the slope of the diagonals of this lattice.

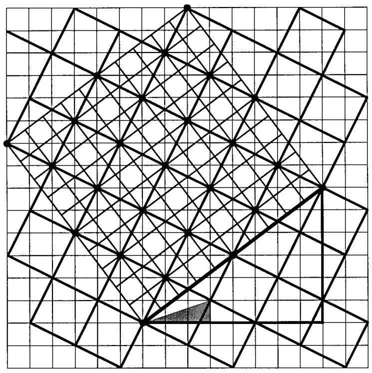


Fig. 8: The bad example

4. Triangular lattices

Instead of studying right triangles in a square lattice we can also deal with triangles with one angle of 60° or 120° in a *regular triangular lattice* (Fig. 9).

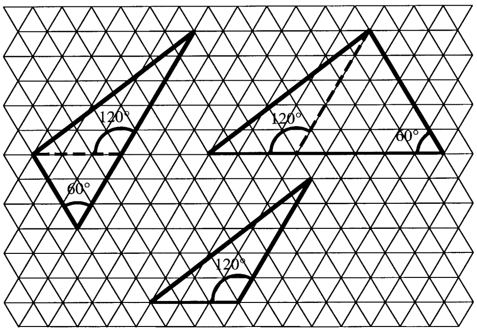


Fig. 9: Triangular lattice

Since we can dissect an equilateral triangle with integer side length from every triangle with one angle of 60° such that the remaining triangle contains an angle of 120°, it is sufficient to consider only triangles with an angle of 120°. Are there such triangles with two sides (sides a and b) matching in the triangular lattice such that the third side c has an integer length as well? From the law of cosine we get in our case of an angle of 120° opposite to c:

$$c^2 = a^2 + b^2 + ab.$$

An example of an integer solution of this equation is $a = 3$, $b = 5$, and $c = 7$, which is the 120°-triangle depicted in Figure 9. Now we can attach a triangular lattice at side c (Fig. 10) and we see that we have again common points of the basic triangular lattice and the attached triangular lattice. These common points are obviously points of a new larger triangular lattice.

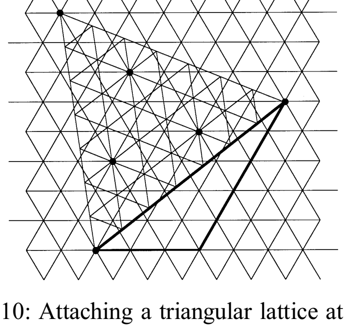


Fig. 10: Attaching a triangular lattice at side c

Again it is helpful to use dot lattices in different colors. This yields beautiful flowerlike visual effects. Figure 11 shows a black and white version.

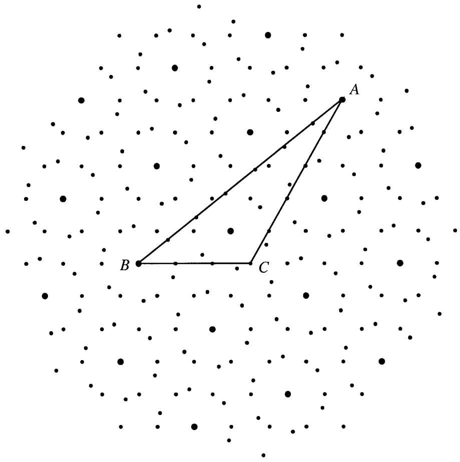


Fig. 11: Dot lattices

Like the Pythagorean right triangles we can parameterize the primitive 120° triangles with integer side lengths (see Dickson 1920, Hasse 1977): We use two natural numbers u and v with $u > v$, $(u, v) = 1$, $u \neq v \bmod 3$ and set $a := u^2 - v^2$, $b := 2uv + v^2$, and $c := u^2 + v^2 + uv$. For $u = 2$ and $v = 1$ we get $a : 3$, $b = 5$, and $c = 7$, the example we have already found. The quotient $\frac{v}{u}$ gives again the slope of the new large triangular lattice in terms of the basic lattice, where u measures the horizontal distance, and v the slanting distance according to Figure 12.

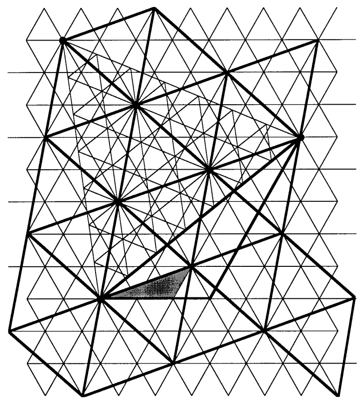


Fig. 12: Parameters and slope

5. References

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Author
Walser, Hans, Dr., Mathematisches Institut, Universität Basel, Rheinsprung 21, CH-4051 Basel.
E-mail: hwalser@bluewin.ch

Vorschau auf Analysethemen der nächsten Hefte

Für die Analysen der Jahrgänge 32 (2000) bis 33 (2001) sind folgende Themen geplant:

- Computergestütztes Lösen offener Probleme im Mathematikunterricht
- Mathematikdidaktische Forschung im Primarbereich
- Mathematik an Hochschulen lehren und lernen
- Analysis an Hochschulen
- Mathematik in der Ingenieurausbildung
- Theoretische Betrachtungen zu Schulbuchanalysen.

Vorschläge für Beiträge zu o.g. Themen erbitten wir an die Schriftleitung.

Outlook on Future Topics

The following subjects are intended for the analysis sections of Vol. 32 (2000) to Vol. 33 (2001):

- Computer-aided solution of open problems in mathematics teaching
- Research in primary mathematics education
- Teaching and learning mathematics at university level
- Calculus at universities
- Mathematics and engineering education
- Concepts and issues in textbook analyses.

Suggestions for contributions to these subjects are welcome and should be addressed to the editor.