Mathematics Lessons with DERIVE
Developed by the CAVO Working Group

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Abstract: Since 1993 a group of Dutch mathematics teachers has been working on the development of teaching materials that use Derive. The group, called CAVO, published its first mathematics courseware collection in 1995. This paper reviews the content and the structure of this collection, gives the full text of one laboratory and delivers a report on the experience with these materials in the classroom. Besides providing this information, the paper also gives insight into the way CAVO works and into the process that led to this style of working.

Kurzreferat: *Mathematikstunden mit DERIVE, entwickelt von der Arbeitsgruppe CAVO.* Seit 1993 arbeitet eine Gruppe niederländischer Mathematiklehrer an der Entwicklung von Unterrichtsmaterialien für die Nutzung von DERIVE. Die CAVO genannte Gruppe veröffentlichte 1995 ihre erste Sammlung von Unterrichtssoftware. Dieser Bericht gibt einen Überblick über Inhalt und Struktur der Sammlung. Er enthält den vollen Text einer Unterrichtseinheit sowie einen Bericht über Erfahrungen mit diesem Material. Darüberhinaus wird Einblick in die Arbeitsweise der CAVO-Gruppe gegeben und in den Prozeß, der zu diesem Arbeitsstil führte.

ZDM-Classification: N80, R20, U50

1. Introduction
The working group "Computer Algebra in Secondary Education", in Dutch abbreviated to CAVO, has been in existence for about three years. CAVO, consisting of volunteering mathematics teachers, published its first product in 1995: a mathematics courseware collection (see Werkgroep CAVO 1995) for students at secondary level in The Netherlands (in Germany the upper SI and SII level).

In this article we (three CAVO members) describe how the courseware was developed, review its content and structure, give the full text of one laboratory and deliver a report on the experiences with this material in class. Finally, CAVO's plans for the future are discussed briefly. Besides providing this information, the aim of this paper is also to give insight into the way CAVO works, and into the process that led to this style of working.

2. The history of CAVO
In the summer of 1993 the Expertise Centre Computer Algebra Netherlands (CAN) gathered the names of approximately sixty mathematics teachers who had, on different occasions, shown interest in the use of computer Algebra in high school mathematics. At the initial meeting in December 1993, we were twenty to establish the working group Computer Algebra in Secondary Education (CAVO). The group works under the auspices of the Dutch Association of Mathematics Teachers and is supported by CAN and Delft University of Technology.

At the start, many of us already had some experience with computer algebra, ranging from performing demonstrations in our classes to organizing and supervising the use of existing computer algebra courseware by our students. Some of us had tested parts of the materials that were developed in 1991 by Paul Drijvers, and that resulted in the series "Learning Mathematics with DERIVE" (see Drijvers 1992).

However, there were two obstacles to using this booklet. First, while the series dealt with topics covered in the high school Mathematics B course (the formal mathematics course for the exact stream), the content did not always fit exactly. The Mathematics B curriculum being quite extensive, there is little time to deal with extra subject matter. Secondly, in some schools there was no computer algebra software (formally) available for the students.

The software problem could rapidly be solved. There was general agreement that the options offered by DERIVE, its ease of use, limited hardware requirements and low price make it the most appropriate computer algebra system for pretertiary mathematics courses. CAN managed to make DERIVE available to the schools of the participating teachers under favourable conditions.

And so the working group could really start to work.

3. The aims of CAVO
Our main goal is to develop ready-to-use instructional materials that can motivate less engaged colleagues to use DERIVE in their lessons. Other means of communicating our experiences with DERIVE, such as demonstrations or articles, are welcomed as well. Beside this, CAVO wants to be a stimulating platform for study, exchange and discussion, which will lead to the development of experience and expertise among its members.

Because CAVO consists of volunteers and hardly receives any financial support, the group is free to do what it considers to be good. The initiative is for the teachers, whose daily teaching experience guarantees materials that are not out of reality. The fact that there is no "top-down" steering makes CAVO an interesting phenomenon.

4. Initial agreements
The working group started developing practical lessons with DERIVE fitting into the current Mathematics A and Mathematics B curricula[1]. During the meetings (once every six weeks) we discussed the courseware designs, we reported on try-outs of assignment sheets with our students, and we exchanged new information gathered from the literature.

We soon agreed on the following guide-lines concerning the content and the structure of the student materials.

a. *New topics*
   In addition to lessons in which topics from the current curricula are introduced, extended or enriched using DERIVE, CAVO also will design lessons in which new topics or new applications of existing topics are made accessible to students. This extra material does not necessarily have to be tested with entire classes;

[1] Mathematics A prepares for studies in life sciences. Subjects are applications of calculus, statistics and matrices. Math A is linked to real life applications and does not focus on formal concepts. Mathematics B prepares for exact studies and has a more formal and traditional contence.

trials with two or three interested students also can provide valuable information.

b. *Algebra*
The lessons should take advantage not only of the graphics and numerical options available in DERIVE, but also of the symbolic and algebraic facilities.

c. *Separation of math and buttons*
Except for the introductory lesson that teaches students how to use DERIVE, little or no information about the technical aspects of working with DERIVE is presented on the students' assignment sheets. The necessary technical instructions are provided on a separate DERIVE-sheet. For these DERIVE-sheets we use the structure developed by Paul Drijvers in "Learning Mathematics with DERIVE": on the left a column entitled "What do you want to do?", and on the right a column entitled "How do you do that?" There are two reasons for this separation: first, we would like to make clear that the focus is on mathematics, and secondly, in case of new versions of DERIVE, only the DERIVE-sheet needs to be updated.

d. *Teacher information sheets*
A teacher's information sheet accompanies each lesson, setting out the goal of the session, the target group, the knowledge of mathematics and DERIVE required to carry out the session, and an estimation of the time required to do the problems. Practical and didactical hints are also provided.

5. CAVO's courseware collection

After two years, a collection of fifteen practical lessons for the final years of pretertiary high school mathematics (SII) was completed: "Mathematics Lessons with DERIVE", published by CAN in November 1995 (see Werkgroep CAVO 1995).

The collection integrates a range of quite diverse topics, some very closely related to the current curriculum and some quite different from it. Subject-matter which is part of the standard curriculum, subject-matter that extends and enriches it, and completely new subject-matter have all been included.

Standard topics with some extensions are found in "Plotting Graphs and Solving Equations", the only session besides "Introduction to DERIVE" intended for third year students, and in the three sessions for fourth year students among which are "Parabolas with a Parameter" (see appendix). Another topic example for advanced fourth year students is the session "Transformations" that deals with the standard transformations of graphs of functions. As an extension ellipses are translated and dilated along the x- and the y-axis, and, in each case, the equations for the pre-image and the image are compared.

Enrichment and application are used in three sessions for Mathematics A students in the final two years of pretertiary streams. For example, in the last one of these, "Redbreasts", the applied problem of the same name from the final examination for advanced Mathematics (May 1992) is treated in far more detail than is possible in a national written exam.

Most of the subject-matter in the collection, however, is not in the current curriculum, and it is noteworthy that most of the new subject-matter is dealt with in the seven lessons for Mathematics B students: some less well-known formulae for the volume of geometric figures, properties of sequences and series, a method for approximating roots, some number theory and the arc length of curves.

6. Testing the courseware

The testing of the courseware was not easy to organize and did not happen as often as was planned. In the end, half of the lessons were tested thoroughly. Owing to practical problems (the computer room was booked, there was no lesson time over after preparing students for tests or examinations, there was no time to prepare the lessons properly, the chapter treating the knowledge required for the lesson had not been completed) even our group of highly motivated teachers often had to give up their plans for a trial. The idea of working with a few students outside regular lesson time also did not come too much in practice. This is a pity because the few experiences of working this way were extremely positive. The lesson "Arc Length and Pi" was tried out completely independently by a few good students at one of the schools. The students enjoyed it, made constructive comments, and took an oral exam over the material with enthusiasm.

7. An example: Parabolas with a parameter

The appendix contains the full text of the chapter "Parabolas with a Parameter": the teacher information sheet, the assignment sheets for the students, and the DERIVE sheet.

The idea for this assignment stems from the following exercise in one of the common Dutch textbooks:

> Determine the value of p for which the equation
> $x^2 + px + 4 = 0$ has no solutions.

After the teacher had solved this equation using the discriminant, resulting in $p = 4$ or $p = -4$, one of the students said: "Now how is that? The equation should have no solutions, and now we have two!" Apparently, the student does not master the concept of a parameter. The assignment in the appendix is developed in order to improve the understanding of this phenomenon. Furthermore, we want to investigate whether the use of DERIVE can be fruitful for students of this age (15 years).

The student text has been revised twice, as a result of the discussion in the CAVO meeting and of the testing in the classroom.

The students have been working with DERIVE for one hour before they started with this assignment. They have some familiarity with the commands **A**uthor, **S**implify, appro****X****, **M**anage **S**ubstitute, so**L**ve and **P**lot.

Let us quote some observations and comments from the teachers' reports.

Exercise 3

Comment: Some students tried to plot $x^2 + 5x + p$. DERIVE then comes up with a 3D plot window. It takes some efforts to explain to the students what has happened and how they can get back to the "normal" situation.

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Fig. 1 (included in print version only)

Exercise 5b

Some students use soLve and find two complex roots.

Comment: I said to the students: “DERIVE uses $\hat{\imath}$ when it encounters $\sqrt{(-1)}$ during its calculations. The solution thus does not exist. One has to keep on thinking while using DERIVE.” Does anyone have a better suggestion?

Exercise 11b

Student1:

0 intersection points: $p > 2 - 2\sqrt{2}$ or $p > 2 + 2\sqrt{2}$
1 intersection point: $p = 2 - 2\sqrt{2}$ or $p = 2 + 2\sqrt{2}$
2 intersection points: $p < 2 - 2\sqrt{2}$ or $p < 2 + 2\sqrt{2}$

Comment: The inequality-signs seem to be placed at random!

Student2:

0 intersection points: $p > -0.8$
1 intersection point: $p = -0.8$
2 intersection points: $p < -0.8$

Comment: Found by trying. The second critical value is not detected.

Student2 solved exercise 12a and b correctly, but did not notice the errors in 11.

Comment: The computer enables “running through” the exercises and does not provoque reflection.

Exercise 14b

Comment: Students found the right numbers. But is there any real understanding?

The conclusion of the report of one of the teachers:

“The students were enthusiastic and enjoyed not having to draw the graphs by hand. Studying a ‘family’ of parabolas was not easy for most students. When they were asked to give an explanation, they had problems in phrasing the answer. Students are better in calculations than in formulations.

Now that much work can be left to the computer, time for reflection is lacking. Students quickly pass on to the next question. For some students only the final classroom discussion revealed what they had been calculating. This discussion is very important in effectuating the reflection that we want.

Some weeks after this experience, the students have been working on exercises concerning the number of points of intersection of a family of straight lines with a parabola. In former years, this used to be a hard subject. This year’s group, who did the DERIVE assignment, recognized the problem and were able to solve it easily. My conclusion is that the use of (a small part of) DERIVE was fruitful to these students.”

Besides the interesting conclusions, this teacher’s report also gives insight into some of the problems one can encounter when using DERIVE with young students. Furthermore, it gives some insight into the way one can exchange experiences by means of open, informal reports.

8. Reflections on the role of DERIVE

The description of the content of the collection (see 5.) shows that while the lessons for the older students deal, as intended, with topics from both Mathematics A and Mathematics B, Mathematics A topics are underrepresented. This is a consequence of the decision to make use of the algebraic capacities of the program. Instruction in many of the topics in the Mathematics A curriculum does not benefit from the opportunities DERIVE provides to manipulate formulae and to perform symbolic calculations.

A glance through the assignment sheets shows how frequently the graphics and numerical options available are used. Some lessons use the manipulation of formulae and symbolic calculation options only once. On the other hand there are lessons that would not be possible without these options. Understandably, these are mainly the lessons in which new subject-matter is dealt with. Two examples are “The Volume of a Truncated Cone” and “Prime Numbers”, both intended for Mathematics B students in the final two years of the pretertiary streams. These lessons show clearly that some topics considered too difficult for pretertiary courses until now can be presented in a way that makes them accessible for the students.

CAVO made no agreements as to the types of mathematical activities for which DERIVE would be used. In retrospect obviously the following five activities are supported by DERIVE:

– checking answers first worked out using a different method
– experimenting
– doing research

– deriving new formulae
– proving (only infrequently).

Let us briefly comment on these points.

DERIVE is used not only to *check answers* worked out without extra help but also after the student has reached a conclusion or derived a formula based on graphs produced, or calculations carried out, by DERIVE.

Experimenting is used mainly by Mathematics A students. These students must manipulate boundary conditions and then check the effect of the resulting change in value of some elements in the matrix.

A good example of *research* is provided in the previously mentioned lesson "Prime Numbers". Here students are asked to determine whether prime numbers are found less numbers.

Deriving new formulae is the main topic in the lesson "The Volume of a Truncated Cone", but it is also used in one of the problems in "About Polynomial Functions" for the fourth year of the advanced mathematics course. In this problem, using the formula for the x-coordinate for the top of a parabola, known by the students, the formula for the "y-top" is derived. This last lesson also provides an excellent, and for the students difficult, example of *proving* with DERIVE. The student has to show that for every function of degree four whose graph has two points of inflexion the following assertion is true: The x-coordinate of the intersection of the tangents at the points of inflexion is a zero of the third derivative of the function.

9. Looking into future

Since "Mathematics with DERIVE" was published, CAVO has continued its activities. One item on the current agenda, not surprising in the light of the comments above, is to try out a number of lessons from the present collection with students in class or outside regular lesson time.

Another item is to develop instructional materials that could be used with the new mathematics programs being developed as part of the revision of all curricula for the final years of the pretertiary streams of secondary education in the Netherlands. CAVO is also working on subject-matter of instruction with DERIVE that may be included in the new curricula as optional material.

Finally, we hope that CAVO can play a role in the discussion on the use of technology in assessment, and will continue to support the acquisition of expertise on the use of computer algebra in mathematics education.

10. References

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Appendix: *Teaching and learning materials for the lesson "Parabolas with a parameter"*

Parabolas with a parameter
teacher information sheet

Outline

This instruction is developed for students at upper secondary level, 15 or 16 years old.
The aim of the instruction is threefold:
– to visualize "families" of parabolas,
– to investigate some of these families, and
– to analyse the relation between the value of the parameter and the number of intersection points of the parabola with the x-axis.

Practical information

Before starting this instruction, the students should be able to
– graph parabolas;
– solve quadratic equations using factorisation, isolating the square and the discriminant method;
– use the formula for the top of a parabola: $x_{\text{top}} = -b/(2a)$.

Considering a family of parabolas is completely new to the students.
As far as the use of Derive is concerned, the students should be familiar with
– the commands **A**uthor, **S**implify, appro**X**, **M**anage **S**ubstitute, so**L**ve, **P**lot;
– with the handling of the Plot-window;
– with the handling of the cursor keys.

The instruction takes two lessons of fifty minutes. In a third lesson, the results can be discussed.

Didactical information

Not all students realize what the different questions have in common. They are often not able to write down their explanations in a clear way. Classroom discussion and evaluation can be very fruitful.
It can be confusing to students to find complex roots when Derive solves an equation. The program gives two solutions, while looking at the graph there doesn't seem to be one.
During the classroom experiment with this instruction, students often tried to plot expressions such as $x^2 + 5x + p$. It took some time to get back to the "normal" screen (using **W**indow **C**lose) and to explain to the students what happened. They found the 3D graphs interesting, though.

Parabolas with a parameter
assignment sheet 1

In this instruction we will investigate parabolas with equations that depend on a parameter. We will see that different values of this parameter result in different parabolas. In each case we will consider the position of the parabola relative to the x-axis.

A. $\mathbf{y = x^2 + 5x + p}$

We will investigate for which values of p does the parabola $y = x^2 + 5x + p$ have respectively 0, 1 or 2 intersection points with the x-axis. In order to find the number of intersection points with the x-axis, the parabolas will be drawn.

1 a Enter: $x\hat{}2 + 5x + p$.
b Substitute for p the value 4 and let Derive draw the graph of $y = x^2 + 5x + 4$.
c How many intersection points with the x-axis does the parabola $y = x^2 + 5x + 4$ have?
d Calculate manually the x-coordinates of these points.

2 a Substitute for p the value 5 and graph the parabola with equation $y = x^2 + 5x + 5$.
b How many intersection points does the parabola $y = x^2 + 5x + 5$ have with the x-axis?
c Calculate the x-coordinates of these intersection points using the Derive command So**L**ve.

3 a Repeat exercise 2 for different values of p and investigate for what values of p the parabola "touches" the x-axis.
b What is the influence of the change of p's value on the graph?

B. $\mathbf{y = x^2 + px + 4}$

Clear both windows.

4 a Enter: $x\hat{}2 + px + 4$.
b As a start, substitute the value 6 for p.
c Let Derive graph $y = x^2 + 6x + 4$.
d How many intersection points does the parabola $y = x^2 + 6x + 4$ have with the x-axis?

5 a Substitute 3 for p and graph the parabola.
b How many intersection points with the x-axis does the parabola have now ?

Parabolas with a parameter
assignment sheet 2

6 a Substitute -3 for p and graph the parabola.
b How many intersection points with the x-axis does the parabola have now ?

7 a Choose some other values for p and find in the graph the number of intersection points of parabola and x-axis.
b For which values of p does the parabola $y = x^2 + px + 4$ have
0 intersection points with the x-axis?
1 intersection point with the x-axis?
2 intersection points with the x-axis?
c The parabolas that are drawn all seem to encounter one point. Which point is it, and how can you prove that this is always the case, no matter what value for p one chooses?

C. $\mathbf{y = -x^2 + px - 1}$

8 a Enter: $-x\hat{}2 + px - p$.
b Substitute different values p and find in the graph the number of intersection points of parabola and x-axis.
c For which values of p does the parabola $y = -x^2 + px - p$ have
0 intersection points with the x-axis?
1 intersection point with the x-axis?
2 intersection points with the x-axis?
d Again, the parabolas that are drawn all seem to encounter one point. Which point is it, and how can you prove that this is always the case, no matter what value for p one chooses?

D. $\mathbf{y = x^2 + px + p}$

9 a Enter the expression $x^2 + px + p$.
b Substitute different values p and find in the graph the number of intersection points of parabola and x-axis.
c For which values of p does the parabola $y = x^2 + px + p$ have
0 intersection points with the x-axis?
1 intersection point with the x-axis?
2 intersection points with the x-axis?
d Again, the parabolas that are drawn all seem to encounter one point. Which point is it, and how can you prove that this is always the case, no matter what value for p one chooses?

Parabolas with a parameter
assignment sheet 3

10 The discriminant of $x^2 + px + p$ equals $p^2 - 4p$.
a Verify this by writing down the discriminant of this quadratic expression.
b Find the values of p for which this discriminant equals zero.
c What does the answer to b have to do with the results of exercise 9?

E. $\mathbf{y = x^2 + px + p + 1}$

11 a Again, find the number of intersection points of this parabola with the x-axis for different values of p. Clean up the screen before you start.
b For which values of p does the parabola $y = x^2 + px + p + 1$ have
0 intersection points with the x-axis?
1 intersection point with the x-axis?
2 intersection points with the x-axis?

12 The discriminant of $x^2 + px + p + 1$ is equal to $p^2 - 4(p + 1)$.
a Verify this.
b Calculate for which values of p the discriminant is zero.
You can do this manually, or leave it to Derive by entering $p^2 - 4(p + 1)$ and choosing So**L**ve.
c The answers of b can be used to verify your results of 11. Are they correct?

Using Derive, there is a different way to calculate the values of p for which the parabola has only one intersection point with the x-axis.

13 a Enter the expression $x\hat{}2 + px + p + 1$.
b Solve x in $x^2 + px + p + 1 = 0$ using Derive's

SoLve-command; enter x as variable and press ENTER again.
Derive now gives two solutions, both containing the parameter p. The parabola has only one intersection point with the x-axis if these solutions are equal. Build the equation that puts both solutions equal and solve it.

c Verify the result using the solution of the previous exercise.

d An easier method consists of putting the determinant equal to zero. When the discriminant is zero, the parabola has only one intersection point with the x-axis. Use this method to find the values of p for which the graph of $y = x^2 + px + p + 1$ has exactly one intersection point with the x-axis.

14 a Investigate for which values of p the parabola $y = x^2 + (p-1)x + p$ has only one intersection point with the x-axis.

b Do all these parabolas also meet in one point?

Parabolas with a parameter
DERIVE sheet

The information given below can be useful in this instruction. The bold capitals indicate the commands that should be given one after another, departing from the main algebra or plot menu.

What do you want?	How do you do that?
Split up the screen in an algebra window and a plot window	**P**lot **B**eside 40
Substitute for the parameter p in the expression $x^2 + 5x + p$ the value 7	highlight $x^2 + 5x + p$ **M**anage **S**ubstitute Enter Enter (the x remains unchanged) type a 7 instead of the p
Draw the parabola with equation $y = x^2 + 5x + 2$	**A**uthor enter x^$2 + 5x + 2$ Enter **P**lot **P**lot
Shift between algebra window and plot window	press function key F1
Zoom in at a smaller part of the graph	in the plot window: press F9
Zoom out at a larger part of the graph	in the plot window: press F10
Clear the plot window	**D**elete **A**ll
Clear the algebra window	**T**ransfer **C**lear **Y**es
Select a subexpression	use ↑ and ↓ to highlight the expression. use → and ← to select the subexpression. (eventually press F6 before)
Copy a (sub-) expression into a new expression	**A**uthor select the (sub-)expression press function key F3 or F4