Information

Multiplying after the Turn

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Abstract: In education multiplying is usually viewed as repeated joining together and dividing as repeated taking away or, which comes to the same thing, as an equal distribution. This presentation springs from Antiquity, when thought was mostly concrete. In modern mathematics we have relation-numbers instead of image-numbers and likewise multiplying is a facet of relational thinking. The view that children merely can learn through the concrete is often biassedly understood in the sense that the concrete has to be abstracted, which characterizes substantial thinking. However, in the case of relational thinking, learning through the concrete means that to achieve insight the mathematical activities have to be applied to reality, a crucial point, for most people have difficulties with applying multiplication, much more than with the inherent algorithms. It appears that they do not really know what multiplication is, particularly not its space structure. The more general the structure the more and wider the applications. This thesis infers that multiplying as multiple of classes is much less useful than multiplying as space form. Questing for the essence of multiplication is the major topic of this paper. Which changes has its structure undergone and how can education deal with them? At the end it is illustratively explained why probability, based on the established multiplication, is usually such a tough domain.

Kurzreferat: *Multiplizieren nach der Wende.* Im Unterricht bedeutet Multiplizieren gewöhnlich wiederholtes Zusammenfügen und Dividieren wiederholtes Wegnehmen, oder, was auf das Gleiche hinausläuft, gleichmäßiges Verteilen. Diese Darstellung stammt aus alten Zeiten gegenständlichen Denkens. In der modernen Mathematik hat man Zahlen-Relationen statt Zahlen-Bilder und entsprechend ist die Multiplikation ein Bereich relationalen Denkens. Die Ansicht, Kinder könnten nur durch Konkretes lernen wird oft in voreingenommener Weise so verstanden, daß das Konkrete abstrahiert werden solle, was gegenständliches Denken charakterisiert. Im Falle relationalen Denkens bedeutet Lernen am Konkreten jedoch, daß mathematische Aktivitäten auf die Realität angewandt werden müssen, um Einsichten zu erzeugen – ein kritischer Punkt, denn viele Leute haben mehr Schwierigkeiten mit dem Anwenden der Multiplikation als mit den zugehörigen Algorithmen. Anscheinend wissen sie nicht so genau, was Multiplikation ist, insbesondere kennen sie nicht deren räumliche Struktur. Je allgemeiner die Struktur ist, desto mehr und vielfältigere Anwendungen gibt es. Die These besagt, daß die Multiplikation als Vielfaches von Klassen weniger nützlich sei als die Multiplikation als räumliche Form. Die Suche nach dem Wesen der Multiplikation ist der Schwerpunkt dieses Beitrags. Welchen Veränderungen war ihre Struktur unterworfen und wie kann Unterricht damit umgehen? Abschließend wird veranschaulicht, weshalb die auf der etablierten Multiplikation basierende Wahrscheinlichkeit gewöhnlich solch ein schwieriger Bereich ist.

ZDM-Classification: E20, F30, H10

1. Multiplying as repeated adding of substances

Understanding traditional multiplication is apparently easy and more or less self-evident, which has its roots in the illustrative approach of the multiple. Most learning methods link up with the original visual meaning of multiples and start the learning of multiplication by skilfully adding symmetrical classes. Symmetrical materials abound: pennies, sequences of chairs, chocolate-bars, crates for bottles, an exercise-book with lines, and so on. Those plural objects must be concretely present or sensually represented. The task consists of handy fixing the quantity of elements, which goes on until memorisation, association and abstractions appear. Abstractions of the real or imaginary situations have to lead to multiples like three, six, nine ... A following abstracting phase is the representation of objects through points on a (number) line and the repeating through jumps on that line or simply through equal distances. Multiplication is thus presented as an *object*.

This illustrative approach to multiplication, i.e. the adding of the same shown cardinal numbers, dates back to Antiquity. At that time, numbers were multitudes of units, which in turn were abstractions of a concrete thing, mostly with a tally or a point as icon. Even now this happens sometimes in practice, also in education. Counting in this sense means scoring and numbers are multitudes of objects, like points on a dice.

Repeated adding, the multiple of classes, has a global holistic structure. The elements are not directly connected with each other, but put in equivalent classes and these classes are connected with each other. This process finds expression in the name-giving. The number of each class is called multiplicand and the number of classes the multiplier.

Thinking on the basis of connections observed in substances characterizes the illustrative approach. Humans are seen as nature dependent beings, programmed in their doings by their Creator.

2. The turn to the free-structuralist

However, in the practice of life humans appear to be really creators themselves and developers of their own culture.

Creating means that humans are able to position themselves above the real structures and themselves can relate or structure. The superposition is called the *turn* from object to subject, i.e., the conceptual activity[1]. I will consider this turn in more detail and also go into the consequences for our idea of multiplying. The major result of this turn is the concept of a conceptually created system, whose objects are known only through the relationships of the system. The objects are left unspecified. Any specification gives merely a sample or representation of the conceptual system. Modern arithmetic is such an example. In literature the opinion dominates that the turn from thinking in substances to thinking in relations took place around 1900, but this concerns the synthesis, the idea of an abstract hence conceptual system. But turning is undoubtedly a much older gradual process. After a certain development of any notion a turn is demonstrable in history. Turning is a more general event. It concerns the turn from object to activity, from abstracting to structuring. I will present here four cases: the unit, time, counting and, more general, structure itself.

First the unit. As already said, in Antiquity the number was a multitude of units. Socrates (469-399 B.C., Philebus 56) observed already that these units are dissimilar in the abstractionistic approach. Two oxen are never the same, their units differ consequently. People add up such unequal units, but a philosopher must know better and must apply the ideal equation $1 + 1 = 2$. Socrates thus distinguished mathematics for the people and mathematics for the scholars, in his time the philosophers. The underlying antithesis to Socrates' distinction was the mere thought of empirism-idealism.

Now consider time a little closer. Originally time is a notion abstracted from causal sequences, as Leibnitz argued. A global model is to be found in the sequence of day, night, day, night, day, night, day, night, ... The day and the night are here names, the terms do not matter, dag and nacht are naturally also right, eventually abbreviated into d, n, d, n, d, n, ... The substitution by 1, 2, 1, 2, 1, 2, ..., like Plato did, is nothing else than a name-numbers series. Here too the units are unequal, the lengths of the days differ from those of the nights and the days differ from each other. Gradually this discrete idea of time has been refined in our modern watches. There are still scientists who believe that time is essentially discrete, and have given the discrete unit the name chronon with a duration of 10^{-20} seconds. Others consider time to be continuous. Kant, for instance, assumed time to be a notion that precedes experience. He saw time (and space) as an "Anschauungsform" of man and in this manner made the turn from observer to thinker. Whether time is apriori, as Kant assumed, is not the main question. The answer is not settled yet. The point is the turn to the subject, the man who assumes time to be continuous.

The third example is counting. As we have stated, in Antiquity counting was scoring, notching or tallying. The change in the vein: name-giving, shortening, from notches like ////// to activity, i.e., the counting 1, 2, 3, 4, 5, 6, was not as simple as it appears just because of the twenty thousand years it took to put this step from icons to symbols (Klix, p. 191). The essence lies in the change from icon to symbol, what is called the turn from cardinal to ordinal nowadays, from representing to ordering, relating or structuring. The laborious development from icon to symbol corresponds with the development from object to conceptual activity. Is the turn from cardinal number to ordinal number essential in the preceding example? The answer depends on the objective, that is, the sense of knowledge. What is the sense, the object or the activity? Is the number or the counting, i.e., the application, the objective in our example? The turn from object to activity made clear to us that there is more than real structures and that there exists at the most a local covering between real object or phenomenon and subject.

The structure set another good example of the turn. In the original, particularly geometrically oriented, conception structure was an idea or form (morphe), conceived as something static and global. Later the Latin structure (struere = to build) became the vogue, i.e. a local dynamical, analytical notion. This change was a fundamental innovation, for structure is thus seen as an activity, properly to structure, i.e., a position of form with a change, motion, development or growth. A useful, simple example of the turning of an object in activity is the line. Its form is straight, the structure is the building method, the construction: a point with a direction. Still better the structure relates the points. Take a point and the direction goes to the following point, et cetera. The structure is activity, is the drawing or sketching of the line. The form is the drawing-model. The line as a structure knows a relationship of the points, knows a side and an opposite side. Suppose A and B are points of it then $A < B$ if B is on the side from A and reversed. It is only in the local structure that this relationship is apparently transitive. The circle does not know this ordination, because side and opposition meet each other.[2] In the application the line-structure is apparently much more general and suitable to order *arbitrarily* situated things or phenomenons. The line need not be straight anymore. Hence thinking does not cover reality, it creates its own reality. So the former straight form is now fictitious and extends the space conception. Observing occurs through the senses assimilating stimuli. The drawn line can be seen because it has thickness and length. Whatever we perceive or notice is conceived as things or phenomenons. Next to this there are the non-observable ideas, which spread our entire knowledge. It is not so easy to tell what they are precisely. The very large number of synonyms, such as form, structure, representation, concept, notion, mental activity, order, construction, composition and relationship point out fine shades of meaning. The noun structure concerns especially the building method and thus the interrelation of the parts or of part and whole, like the structure of an atom. Structure is not erection or result, but the underlying idea. Structure is strictly speaking never concrete, nevertheless we speak about real and mathematical structures. The activity structure may be concrete, like dividing a cake in equal pieces. The atom structure and in general structures of things or phenomenons are objective. Mathematical structures are subjective, they are pure activities of the thinking person, which we mostly call *structuring* or relating[3]. Studying and applying mathematics is in this manner in essence structuring, which leads to continually more complex systems. In the study of objective relationships or structures the finding of equal forms, or isomorphy, is central. The tracing of isomorphy belongs to morphology, the scientific study of form and structure, the study of omnipresent patterns in universe. Man has certainly knowledge of an invisible world of structures. For example, surely everybody knows a geometrical line, something abstract with merely length, consequently no thickness or width, without ever having seen such an object. The clearest elaboration of structuring is found not in philosophy but in mathematics. It is perhaps not really theoretical but practical. In particular in algebra structuring is in essence calculating. In the elementary arithmetic according to the four main operations, i.e. adding, subtracting, multiplying and dividing.

3. Definitions of some arithmetical operations as conceptual activities

Modern arithmetic begins with counting: $1,\ 2,\ 3,\ 4,$ ..., a completely linear order or structure.[4] Counting $1,\ 2,\ 3,\ 4,$ ... is an abbreviated manner of writing $1 < 2$, $2 < 3$, $3 < 4$, ..., which is per definition transitive and complete. An example of an incomplete linear order is $1,\ 3,\ 5,$ ..., an example of a partial order is $1,\ 2,\ 4;\ 1,\ 2$. Partial orders consist of two or more ordered pieces. In modern mathematics the number is thus an object of thought, represented by a symbol and specified in a fully linear structure. The linear order or line-structure is a relationship of symbols. Counting numbers are fixed in the counting sequence. For instance 6 is fixed in counting $1,\ 2,\ 3,\ 4,\ 5,\ 6$. The commas symbolize the relationship.

Six is also to be defined *holistically.* For example with $1,\ 2,\ 3,\ 4$ and $1,\ 2$. The relationship is then $1,\ 2,\ 3,\ 4,\ 1,\ 2$, a combination of two counting operations, represented by the comma between 4 and 1. The activity is now called an *addition.* We note down this addition as $4 + 2$.

There are also two- or more-dimensional orders. A couple, i.e. an element of a two-dimensional order, is symbolized by brackets and one comma (,), for example $(2, 3)$. For a four-dimensional element, one needs two brackets and three commas, for instance $(2,\ 2,\ 5,\ 4)$. We say a many-dimensional ordering is complete if no element can be put in between. A good example of a complete two-dimensional counting is: $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$. Actually two countings 1, 2 and 1, 2, 3 have been *elementaristically* related into a relationship of pairs. The relationship is complete, provided that all possible combinations are present. This complete two-dimensional counting is called a *multiplication,* noted down as 2×3. The ordering of $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$ occurs "alphabetically", only the alphabet has been replaced by the counting sequence. So first everything is ordered on the first term, then on the second one, etc.[5]

The counting itself is restricted, at least for mankind. Up to and including nine it will work, but the remembering of more names will produce difficulties in the end. With ten, in the decimal system, economical thinking will set up. After nine the numbers are additional. Thus linear ordering is supplied with the adding and with that intricacy increases, because the sequence and the distance among the symbols get also a meaning[6]. The sense of the activities multiplying, adding and counting is naturally in the applications. First I will consider some higher structurings in mathematics itself.

4. Further progressive structurings

To reach a higher thinking level one has to realize the one-to-one correspondence of forms. Multiplication and repeated addition correspond 1-1 to counting, although adding and multiplying differ as such essentially, they are not isomorphic. Adding is broken one-dimensional and multiplying is a many-dimensional ordering or structure. Two sets are only isomorph, similar or equal-structured when there is a bijection preserving the relationships. The counting up to and including 6 and the multiplication 2×3 are in a one-to-one correspondence, for there is a one-to-one-mapping and the "alphabetical" structuring is linear. That mapping is naturally $1 = (1, 1)$; $2 = (1, 2)$; $3 = (1, 3)$; $4 = (2, 1)$; $5 = (2, 2)$; $6 = (2, 3)$. It is two-sided: 1 becomes $(1, 1)$ and $(1, 1)$ becomes 1 and so on. The 1-1 correspondence of 6 and 2×3 is written as $6 = 2 \times 3$. Likewise the 1-1 correspondence of the addition $4 + 2$ with the counting 6 gives $6 = 4 + 2$. In this sense we have equations as $2 \times 3 = 3 + 3 = 6$ and $2\times 3+4\times 3 = (2+4)\times 3 = 6\times 3$. The distributive law is obtained by reversing the reasoning, which may simplify the calculations.

Now something about the applying out of mathematics. Number six arbitrarily situated or appearing phenomenons $1,\ 2,\ 3,\ 4,\ 5,\ 6$. Then they are *fictitiously* linearly ordered. Instead of fictitious synonyms as assuming or hypothetical are often used. Six elements or phenomenons are to be ordered one-dimensionally with $1,\ 2,\ 3,\ 4,\ 5,\ 6$ or with $1,\ 2,\ 3,\ 4,\ 1,\ 2$ or two-dimensionally with $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$. In short, ordering through 6 or through $4 + 2$ or through the multiplication 2×3. Also other operations are possible. The position of the elements or phenomenons does not matter. A real structure, a concrete pattern or symmetry are not needed. *The ordering is free* although, of course, the number is defined: one cannot order for instance seven elements with 2×3. The mathematician structures then, i.e. he puts a spatial structure on the phenomenons, many- or one-dimensional. Please note that mathematicians structuralize freely, they may be quite unconcerned about the nature and position of phenomenons. In Verstappen 1995 I argued: "The product conception rests on strucure and not on experience". The freedom of ordering is at the root of my argument. In the liberation of mathematics from the deterministic reality, with its own internal structures, laws and rules, the turn from substantial thinking to relational thinking or from abstracting to structuring finds its expression. After this liberation the mathematical world does not cover locally, that is on each element, reality anymore. Understanding is henceforth structuring, i.e. conceiving through the general. So a block of cubes of sugar has or shows a product structure, but the mathematician can number also a sack of single cubes of sugar with his multiplying so that there arises a fictive, hypothetical block. This turn, from object to subject, or from substantial to relational, finds its expression in the terminology discovering and inventing. In former times mathematicians only discovered creation, nowadays they invent and create by themselves.

That inventing is maybe very well illustrated by generalizing, in this case through taking more dimensions. For example the activity $2 \times 3 \times 4 \times 5$ is a fully four-dimensional counting, imposing a four-dimensional structure. A difficult yet instructive task is to find out what a four-dimensional cube of sugar looks like and how it differs from a four-dimensional block of sugar.

The distinction between abstracting and structuring, between real and conceptual activity is essential. Initially mathematics was abstracting and resembled a translation of a strip-cartoon. As said before, the concrete underlies mathematics, however in a restricted form. Take combinatorial analysis which deals with the number of possible combinations of, e.g., three sweaters and four trousers. The elements are real but the combination as such is an idea, hence not real. The equations:

$$3 \times 4 = (1,1), ..., (3,4) = 1, 2, ..., 12 = 12$$

do not form the mathematical translation of the situation, on the contrary, they are applications. The presentation has to be well understood. It shows successively a space structure, a line structure and an object. Multiplying may be inspired by a real activity, but it is not identical to it. The confusion can be enhanced by the different meanings of the equal sign. The first equal sign expresses a definition, the second one a one-to-one correspondence and the third is essentially very strange, because it equates ordinals with a cardinal. There has been much philosophizing about the logical priority of the ordinal and the cardinal. Cassirer for instance puts the ordinal number first, Russell the cardinal number. Apart from the fact that there is no conclusive proof for either the one or the other, I notice that there is something complementary in it. Consider the sequence 1, 2, ..., 12. Every number here is an ordinal and is characterized through its position in the sequence. However, 12, the ultimate number, is special in that it is also characteristic of the whole and thus a cardinal number. The last object of a small piece counting is essentially what is understood by the *variable*, i.e. the synthesis of the counting activity. Generally in $1,\ 2,\ 3, \ldots, n$ the n is a variable. The activity has been objectified by the variable.

In fact mathematical structuring begins with relating two or more elements. For example the formation of the triple (a, b, c) wherein $(\ ,\ ,\ )$ represents the relationship. Although structuring is one of the many fundamental concepts that cannot be explained by other, simpler or clearer words, people do nevertheless understand such relationships or structures. If the elements have the same nature then (a, b, c) is an internal structure or else an external one. Think of a, b and c as numbers (internal) or of a as a number and b and c as points (external). Examples of internal structures are the operations of addition and multiplication in arithmetic or algebra. An example of an external relationship is the homothety in geometry. In a simple notation corresponding to $(a, b) = c$:

$$(2, \Delta) = \Delta$$

Through observing 1-1 correspondence between structures a higher level is reached. I restrict myself just to pairs as (a, b). The second level begins, as already remarked, with the equation $(a, b) = c$. Many equations are to be established or proved in this manner. The second level is thus the construction of c. The third level starts with the inverse operation, so c is known and also a or b, or c is known and a relationship between a and b, for example $a = 2b$. Here the hypothetical reasoning appears at full strength, because a or b need not exist in the original set. An existential thinking is asked then, that is the relationship $(a, b) = c$ acts as the definition of the unknown. The fourth level is an extension of the third in which two or more equations have been given with the unknowns a and b. For example $a + b = c$ and $ab = d$. A fifth level is reached through setting up the whole story for n-tuples, so $(a_1, a_2, ...a_n) = p$. The inverse of these linear equations yields a sixth level, which in principle forms the basis of linear algebra.

It is wrong to think that getting in on a high abstract level is favourable for comprehension and application. The reason for this is that the applications differ highly qua structure. Otte (1994, p. 76) refers to the group for these differences. Take the group of two elements a and b, with $aa = a$, $ab = b$, $ba = b$ and $bb = a$ as the activity. The three applications 1, -1 or I, S or I, R where S is reflection and R is a half rotation, are essentially different. I, R and I, R differ for instance qua orientation. The group structure tells nothing about the activity, except for some properties, roughly it is a reversible, associative and commutative map $V \times V \to V$. The algorithm itself is thus left out of consideration. The crux is the activity, especially the reversibility. Suppose $(a, b) \to c$. Are a, b and c known then the activity is fixed. If the activity is known then two elements have to be filled in and the third can be calculated. Let me underline the unknown. There are three possibilities: $(a, b) \to \underline{c}$; $(a, \underline{b}) \to c$; $(\underline{a}, b) \to c$. The point is that the activity is only defined for $(a, b) \to \underline{c}$, and for the other two cases it is supposed to be the same. That implies the supposition of the existence of the $\underline{a}$ and the $\underline{b}$. In the combinatorial definition of multiplication the numbers are natural. The reversal is then not always possible, unless the existence of the rational numbers is assumed. Thus the genuine nature of multiplication has been lost, just as its application. In other words, there is no real example of the division. Likewise there is no concrete example of the multiplication of negative numbers.

In the three examples given the activity is the common multiplication and the composition of transformations. The group structure tells us little about the possible applications. Qua group structure addition and multiplication are isomorph, however as relationship they diverge, especially qua dimension.

The four-dimensional activity/structure/combination $2 \times 3 \times 4 \times 5$ differs fundamentally from $2 \times (3 \times (4 \times 5))$, the multiple of a multiple of a multiple, which has a one-dimensional structure. Both equate 120, but $2 \times 3 \times 4 \times 5$ and $2 \times (3 \times (4 \times 5))$ have totally distinct applications. The same goes for the extension to the rational numbers. $1/2 \times 1/4$ as product differs essentially from the half of a fourth, so from the part of a part.

In short, we have the crux: *multiplying is spatial thinking*. In this sense the turn from substantial to relational thinking is generalizing from linear to spatial, from holistic to elementaristic and from discrete to continuous. So far I have discussed the combinatorial conception of multiplication, which is discrete. The extension of the number system, through further relational thinking, leads to the continuous multiplication $a \times b = ((0,0), ..., (a,b)]$. $(0,0)$ does not belong to it, but (a,b) does. This definition is to be represented visually as a rectangle and in general as an n-dimensional block in the n-dimensional space.

In short, originally $a \times b = b + b + \ldots$ (a times), that is a discrete and linear conception. After the turn we have $a \times b = ((0,0), ..., (a,b)]$, that is a continuous and spatial conception, a result of relational thinking. Four dices do not tell us much. A throw of them yields an element of a relationship, which has a mathematical meaning. All possible throws is a following mental synthesis, forms a discrete four-dimensional block. Its extension constitutes a continuous block, that is, $a \times b \times c \times d = ((0,0,0,0), ..., (a,b,c,d)]$. A solid four-dimensional mental block would be a model of this multiplication. For instance an element can consist of the times four people A, B, C and D got up. Please note that the size of this four-dimensional multiplication, if it exists, has little to do with the space. The multiplication itself defines this space. In fact multiplying is much more general than the real space, whatever the latter may be.

5. Well begun is half done

Obviously multiplication and counting are related activities. Hence their didactical set up rests on the same foundations. Let me first make some remarks on counting, and above all consider educational consequences of the turn.

The present-day mathematicians do not worry anymore about the essence of numbers and they abandon certainly the reality of numbers. Through each extension of the system the character of the numbers have changed (Cassirer, p. 72). It is therefore senseless to stick to the original nature. In 1899 Peano axiomatized the arithmetic of the cardinal numbers of that time. He took five axioms with three indefinite terms: number, zero and immediate successor. From these axioms he deduced propositions[7]. In fact Peano gave the number an ordinal character. It is an example of the development of the historical mathematical subject.[8] Perhaps more convincing is the development of a proof. The original proof of a proposition is usually long and complicated[9]. Mathematicians constantly shorten it and improve and sometimes find different methods.

Now for learning the question arises: do we have to follow the historical subject or do we learn the present-day mathematical subject? Do we follow historio-didactics or not? For the learning of a language the answer is absolutely clear. Unquestionably the evolution passed from representation, icons or hieroglyphs to letters or symbol-writing. Some languages, like the Chinese and Japanese, still use hieroglyphs. But no Western school begins writing hieroglyphically, to proceed gradually to letter writing. That would be a very difficult and fully superfluous way. In practice, children do not have problems using symbols from the very start. Nevertheless historio-didactics dominates in teaching activities, which leads to obstacles inherent to the transition from object to conceptual activity. As yet education considers the turn as unimportant, which finds expression in keeping off the modern conception of counting and multiplying and the inherent applications.

Learning mathematics must lead to insight, that is being able to structuralize. How are the structures acquired? The theoretical understanding of learning shows two extremities: learning is abstracting the real and learning is connecting or relating, structuring, eventually the real. The distinction emerges also from the learning object, that is a task in abstractionism and a given in structuralism. An activity like counting or multiplying is solely to be carried out when the objects are given. Precisely that is the case: numbers are there nowadays, just like letters in language. *My thesis: now that the objects exist, learning ought to proceed from them.* Consequently the pupil must not develop counting and multiplying, but apply it. Both are purely conceptual activities and the main problem is their application. Where are they to be used?

Of course, as already said, the evolution of human knowledge and skills happened in the beginning certainly with the aid of illustrations. Archaic thinking just as constructing is illustrative, in fact a strip cartoon. Even many proofs of Pythagoras' theorem are such strip cartoons. Only later does substantial thinking turn into relational thinking. Relationships cannot be seen, therefore it cannot be sensible or desirable in education to learn according to the historical line.

Furthermore the historical evolution passed off very laboriously, because the real character of activity has long been maintained. The evolution from icon to symbol, generally from the substantial to the relational required an enormous mental exertion of our ancestors. We are not able to experience this, because our thinking cannot detach from its knowledge. Even though we use their quantity signs, we transpose them unconsciously into our symbols[10]. It is very fundamental for the above thesis that we conceive something much faster than its inventors[11]. The point is that we have the developed symbols at our disposal and that we may use the inventors' last activity, which is essentially different from their first impulse. Besides there is much more going on than making use of symbols; namely, the *activity itself has changed.* The ancient people added and represented. Their basic idea was the number as collection of units: / // /// //// ..., strictly speaking it is mapping, while we count, which is a different activity in principle. Counting is only possible when names or symbols are available. The application of counting means giving objects a name and thus ordering them. Today the mental process is precisely reversed, it is structuralistic and not empiristic anymore. Children need not make the turn from adding to counting and from repeated adding to multiplying, which boils down to the development from line(ar) order to arbitrariness, respectively from pattern or symmetry to ad random. It is preferable to start by directly learning the right activity.

Mostly children do not see the connection between scoring and counting. With scoring they automatically make a one-one correspondence, but they do not do that with counting. Through assuming fundamental ideas and laws, previous to experience, limits are set to the sense of perception. The pupil who can count need not be able to apply it without mistakes. He can adjudge a number twice to an object, which happens often when the objects are in a circle. He can also pass over an object. He need not be aware that the size exists and that the arrangement of counting does not influence the ultimate number. In short, applying has to be learned. Applying without such supplementary knowledge does not go right. By winking at Kant's aphorism: application without thinking is blind and activity without applying is empty.[12]

The same holds, perhaps in a large measure, for multiplying. The activity can be carried out since the objects, pairs, triples and so on of numbers, exist. The application is again the principal problem. First the general structure must be known, second it becomes very varied in its applications. We have to avoid the traditional course, which is for the greater part oriented on size. For instance we saw the plane form $3 \times 4 = 1, 2, 3 \times 1, 2, 3, 4 = (1, 1), ..., (3, 4) = 1, 2, 3, ..., 11, 12 = 12$. In the present practice one pays only attention to $3 \times 4 = 12$. The activity in between is left out, a result of the mechanistical training. In this manner pupils are no judge of the application. In the sequence $(1, 1), (1, 2), \dots, (3, 4)$ ordinality and cardinality are unified.

The 12 is not as important as 3×4, for the object does not define the activity. One has also for instance:

1, 2, 3, . . . , 11, 12 = 12	line form
8 + 4 = 1, 2, 3, 4, 5, 6, 7, 8, 1, 2, 3, 4 =	
= 1, 2, 3, . . . , 11, 12 = 12	line form
6 + 6 = 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6 =	
= 1, 2, 3, . . . , 11, 12 = 12	line form
2 × 2 × 3 = (1,1,1), (1,1,2), . . . , (2,2,3) =	
= 1, 2, 3, . . . , 11, 12 = 12	space form.

In all cases the activities are different and still 12 is the synthesis of the activities. The second point of difficulties is that application goes with many varieties. The formation of fictitious structures or activities is not in accordance with intuition or real activities. Think for instance of the sack of sugar fictitiously ordered as a block. The various models of multiplication have apparently little in common, hence the diversity is likewise very large. The meaning of the formal is no open book.

Multiplication is elementaristic. Here a relationship exists of all elements. The numbers have the same meaning, so there is no distinction between multiplicand and multiplier. The structure of multiplication is the space form. For the calculation of the space form, hence the definition of the size, one can relapse into the line form. On the other hand for the application the line form has to be extended into the space form. The space form is between the line form and the group. Space is an application of thinking.

There are plenty suitable models, however, especially at level one, e.g. throwing dice and crossing of ways. The latter has the advantage that it is easier to extend to the continuous space forms.

6. An application

At the end I present an example of a task that tends to be tough. The cause is in the traditional understanding of multiplication. Chance or probability is the number that multiplied with the total of possibilities gives the number of favourable possibilities. The total of possibilities is a product space, the favourable possibilities constitute a subspace. The probability is perhaps the best example in which hypothetical, so existentional thinking takes place and sometimes comes over as common-sense-thinking, which is certainly not the case, with all didactical problems this entails. Take the following example:

> Two persons always go shopping in their lunch interval, that is from 12 till 13 hour, in a neighbouring supermarket. Usually each of them spends ten minutes on this. What is the probability that they are together in the supermarket? How long is their mean meeting time?

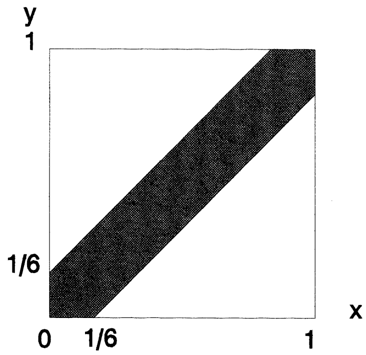

Fig. 1

Ten minutes of every hour infers for both a probability of $1/6$. The relationship or model that the probability of being together at the same time indicates is certainly not a matter of mere common sense. The product of the probabilities $1/36$ is wrong. For the right answer it is necessary to investigate the mathematical space in which everything takes place. First suppose x and y are the arrival times of both persons, reduced to the interval $[0, 1]$. Secondly, x and y are independent variables, consequently we have to do with all possible combinations, that is, the *multiplication* 1×1. Third the meeting implies a relationship namely $|x - y| < 1/6$. The grid is a suitable means to conceive that, which is historically a relatively young instrument. In the figure, depiction of the multiplication 1×1, a point (x, y) with $0 < x < 1$ and $0 < y < 1$ is within the unit square. Inside the shaded part, the expression of $|x - y| < 1/6$, both persons are shopping. Its probability is the area of this hexagon. It is not difficult to calculate that this area equals $11/36$.

The meeting is noticed by the relationship $z = 1/6 - |x - y|$. The point (x, y, z) lies inside the tent-shaped domain, limited by the roof $z = 1/6 - |x - y|$ and the base, the shaded hexagon. The mean time of being present at the same time, the expectation-value, is the volume $= 17/648$ divided by the basal area $11/36$, which is in minutes $170/33$. Properly it is the integral of the probability density $1/6 - |x - y|$ over the hexagon, divided by the base. (The problem is easy to extend, e.g. to three or more persons, execution times and so on.)

7. Notes

[1] In the philosophical jargon we speak about the relationship between O and S, say the reality and the thinking subject, for us in particular the relationship between the mathematical reality and the (apprentice) mathematician. Everywhere the question discharges into the two mainstreams empirism/positivism and idealism/rationalism and into the uniting stream: complementarism. At first in gnoseology, a discipline of philosophy, the thought dominated that knowledge is representing the objects. Later the idea became more critical and one realized that knowledge is the result of mental activities. So we have the turn from O to S. Rationalists put in all cases the initiative into the thinking human. They consider the distinction between what O gave us and what S made of it essential. Complementarists take the sequence $O \rightarrow S \rightarrow O$. The O has thus a double function namely object and means. Man wants to know what O is and uses for that O itself, as a kind of intuitive given. The quintessence therefore is the query whether the turn from $O \rightarrow S$ to $S \rightarrow O$ is essential or not?

[2] The equal division refines that ordering through dividing the line in identical line segments. The endpoints correspond then with a counting structure, of which the differences are one. The mathematicians next gave a direct algebraical definition to the geometrical.

[3] Nevertheless the idea can spring from reality.

[4] If it holds for every pair that either $m < n$ or $m = n$ or $m > n$ and if the ordering is transitive (if $m < n$ and $n < p$ then $m < p$) then we speak of a linear ordering. If there are pairs for which one of the relations $\langle ; =; \rangle$ does not hold then the ordering is called partial.

[5] More precise for three-dimensional order: $(a, b, c) < (a', b', c')$ if $a < a'$ or if $a = a'$ and $b < b'$ or if $a = a'$, $b = b'$ and $c < c'$.

[6] The ciphering with numbers without a position system functions certainly, but is laborious. What is the genuine difficulty that Archimedes, who knew the power formula $n^s \cdot n^t = n^{s+t}$, nevertheless did not find the position system? (Klix, p. 228) Strangely enough the Babylonians knew already the sexagenary system.

[7] This everywhere well known given does not hold for applied mathematics. There the axioms have to be true and thus verifiable. The truth of the axioms and the exactness of the deductions thus give a true result. Please note that only the axioms are verified, the rest is relating. Mathematics itself is thus never empirical.

[8] The point is that the O itself changes from nature to cultivated world. Cultivation is a result of human's thinking and handling. The same holds for S. S is changing oneself through his thinking and handling.

[9] To give an example, Gauss needed about thirty pages for his proof of the fundamental theorem of algebra. Mathematicians reduced it to a few lines. Their proofs rest essentially on two approaches: the regularity of the inverse polynomial or the adjunction of the roots. (See v. d. Waerden, Algebra §71)

[10] I have tried for example to follow the Babylonian solution of the quadratic equation, and observed how laborious that was. It was only after transposing their method into our symbols that I understood directly how they did it.

[11] But unfortunately by further developing it, we again progress laboriously.

[12] We impose, according to Kant, upon experience our concepts causality, space and time. A position with great implications for learning. Later philosophers and scientists came to the same tenet, only without the assumption of apriori concepts like space and time. For example, Poincaré argued that we do not know the space, only the bodies. Einstein (Relativity, Note to the fifteenth edition): "I wished to show that space-time is not necessary something to which one can ascribe a separate existence, independently of the actual objects of physical reality." His space depends on the gravitation field hence on the collection of matter. With that Einstein goes against Kant, whose space-time exists as category, so without reality. Likewise we do not know time, only the change of bodies, as the burning candle, the rotating earth and the vibrating atom. We do not know the physical length, merely the relation of two or more bodies. Transposing a multiple in a unit and reversed are only possible when the dimension is known, but we do not know the dimension as such and with that we end up in a paradoxical situation, that perfectly led itself to philosophical disputes, which are often yet explanatory.
In this way Reichenbach opposed in his *The theory of motion* from 1924 (In Reichenbach, H.; Selected Writings, 1909–1953, Dordrecht: Reidel, 1987) Leibniz's opinion against Kant: "Leibniz's characterization of time as the general structure of causal sequences is superior to Kant's unfortunate characterization of time as the form of intuition presupposed by causality." For Leibniz space is "the order of co-existing things." "The Place of A is a concept which characterizes a certain state of things A, C, E, F, G, ... Leibniz continues "And that which comprehends *all those Places,* is called Space (ce qui comprend toutes ces places, est appelé Espace), ... it is sufficient to consider these *Relations,* and the Rules of their Changes, without needing to fancy any absolute Reality *out* of the Things whose situation we consider". Quoted from Samuel Clarke, A collection of papers, which passed between the late Learned Mr Leibnitz and Dr Clarke in the years 1715 and 1716, London 1717 p. 197–199. Leibniz uses the concept situation, we would prefer structure. Within a situation of things then the concept place of a thing exists and the space enclosed all those places. One need not work with this absolute space. The relations or tranformation rules are important and these are described. Leibniz's space and time are a framework in which things have been ordered. They are no things. I think Leibniz's example of the family genealogical tree is rather explanatory. The genealogical tree constitutes a conceptual ordering and describes at the same time a real situation of persons. Not the persons themselves are described, only the affinities. Reichenbach remarks then rightly: "One must, however, avoid the familiar epistemological mistake of taking 'description' to mean 'copy of reality'." The genealogical tree is not merely a duplicate of reality, the structure has been brought in. The point at issue is that Kant's conception of space and time implies that they have not been rooted in the nature of the things, just like other conceptual schemes. Kant's describing happens with non-real relations and Leibniz's with connections directly rooted in reality. What is the difference qua structure between a, b, d, e, ... and c, t, u, ...? Answer, the first series contains closed elements, the other the open. Has this been rooted somewhere? We are acquainted with real open and closed things, however, that does not hold for the products of existential thinking. The problem is what to do with the means which are not to be rooted in reality like infinite, infinitesimal and formulas.

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