JACOBI–TSANKOV MANIFOLDS WHICH ARE NOT 2–STEP NILPOTENT *

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There is a 14-dimensional algebraic curvature tensor which is Jacobi–Tsankov (i.e. $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ for all x, y) but which is not 2-step Jacobi nilpotent (i.e. $\mathcal{J}(x)\mathcal{J}(y) \neq 0$ for some x, y); the minimal dimension where this is possible is 14. We determine the group of symmetries of this tensor and show that it is geometrically realizable by a wide variety of pseudo-Riemannian manifolds which are geodesically complete and have vanishing scalar invariants. Some of the manifolds in the family are symmetric spaces. Some are 0-curvature homogeneous but not locally homogeneous.

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1. Introduction

Let ∇, $\mathcal{R}$, R, and $\mathcal{J}$ denote the Levi-Civita connection, the curvature operator, the curvature tensor, and the Jacobi operator, respectively, of a pseudo-Riemannian manifold $\mathcal{M} := (M, g)$:

$$\begin{aligned}&\mathcal{R}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]},\\&R(X,Y,Z,W) := g(\mathcal{R}(X,Y)Z,W), \quad \text{and}\\&\mathcal{J}(x)y := \mathcal{R}(y,x)x\,.\end{aligned}$$

The relationship between algebraic properties of the Jacobi operator and the underlying geometry of the manifold has been extensively studied in recent years. For example, $\mathcal{M}$ is said to be *Osserman* if the eigenvalues of $\mathcal{J}$ are constant on the pseudo-sphere bundles $S^{\pm}(M,g)$ of unit spacelike $(+)$ and timelike $(-)$ tangent vectors; we refer to [2, 6] for a further discussion in the pseudo-Riemannian context.

In this paper, we will focus our attention on a different algebraic property of the Jacobi operator. One says $\mathcal{M}$ is *Jacobi–Tsankov* if one has that $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ for all tangent vectors x, y. One says $\mathcal{J}$ is 2-*step Jacobi nilpotent* if $\mathcal{J}(x)\mathcal{J}(y) = 0$ for all tangent vectors x, y. The notation is motivated by the work of Tsankov [9].

It is convenient to work in the algebraic setting.

Definition 1.1 Let V be a finite dimensional vector space.

(1) $A \in \otimes^4 V^*$ is an *algebraic curvature tensor* if A has the symmetries of the curvature tensor:

  (a) $A(v_1,v_2,v_3,v_4) = -A(v_2,v_1,v_3,v_4) = A(v_3,v_4,v_1,v_2)$.

  (b) $A(v_1,v_2,v_3,v_4) + A(v_2,v_3,v_1,v_4) + A(v_3,v_1,v_2,v_4) = 0$.

  Let $\mathfrak{A}(V)$ be the set of algebraic curvature tensors on V.

(2) $\mathfrak{M} := (V, \langle\cdot,\cdot\rangle, A)$ is a 0-model if $\langle\cdot,\cdot\rangle$ is a non-degenerate inner product of signature (p,q) on V and if $A \in \mathfrak{A}(V)$. The associated *skew-symmetric curvature operator* $\mathcal{A}(x,y)$ is characterized by

$$\langle\mathcal{A}(x,y)z,w\rangle = A(x,y,z,w),$$

  and the associated *Jacobi operator* is given by $\mathcal{J}(x)y := \mathcal{A}(y,x)x$.

(3) $\mathfrak{M}$ is *Jacobi-Tsankov* if $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ for all $x, y \in V$.

(4) $\mathfrak{M}$ is 2-*step Jacobi nilpotent* if $\mathcal{J}(x)\mathcal{J}(y) = 0$ for all $x, y \in V$.

(5) $\mathfrak{M}$ is *skew-Tsankov* if $\mathcal{A}(x_1,x_2)\mathcal{A}(x_3,x_4) = \mathcal{A}(x_3,x_4)\mathcal{A}(x_1,x_2)$ for all $x_i \in V$. $\mathfrak{M}$ is 2-*step skew-curvature nilpotent* if $\mathcal{A}(x_1,x_2)\mathcal{A}(x_3,x_4) = 0$ for all $x_i \in V$. $\mathfrak{M}$ is *mixed-Tsankov* if

$\mathcal{A}(x_1,x_2)\mathcal{J}(x_3) = \mathcal{J}(x_3)\mathcal{A}(x_1,x_2)$ for all $x_i \in V$. $\mathfrak{M}$ is *mixed-nilpotent-Tsankov* if $\mathcal{A}(x_1,x_2)\mathcal{J}(x_3) = \mathcal{J}(x_3)\mathcal{A}(x_1,x_2) = 0$ for all $x_i \in V$.

(6) The 0-*model of* $\mathcal{M}$ *at* $P \in M$ is given by setting

$$\mathfrak{M}(\mathcal{M},P) := (T_PM, g_P, R_P)\,.$$

(7) We say that $\mathcal{M}$ is a *geometric realization* of $\mathfrak{M}$ and that $\mathcal{M}$ is 0-*curvature homogeneous with model* $\mathfrak{M}$ if for any point $P \in M$, $\mathfrak{M}(\mathcal{M},P)$ is isomorphic to $\mathfrak{M}$, i.e. if there exists an isomorphism $\Theta_P : T_PM \to V$ so that $\Theta_P^*\{\langle\cdot,\cdot\rangle\} = g_P$ and so that $\Theta_P^*A = R_P$.

The following results relate these concepts in the algebraic setting. They show in particular that any Jacobi–Tsankov Riemannian ($p = 0$) or Lorentzian ($p = 1$) manifold is flat:

Theorem 1.1 *Let* $\mathfrak{M} := (V, \langle\cdot,\cdot\rangle, A)$ *be a* 0*-model.*

(1) Let $\mathfrak{M}$ *be either Jacobi–Tsankov or mixed-Tsankov. Then one has that* $\mathcal{J}(x)^2 = 0$*. Furthermore, if* $p = 0$ *or if* $p = 1$*, then* $A = 0$*.*

(2) If $\mathfrak{M}$ *is Jacobi–Tsankov and if* $\dim(V) < 14$*,* $\mathfrak{M}$ *is* 2*-step Jacobi nilpotent.*

(3) The following conditions are equivalent if $\mathfrak{M}$ *is indecomposable:*

(a) $\mathfrak{M}$ *is* 2*-step Jacobi nilpotent.*

(b) $\mathfrak{M}$ *is* 2*-step skew-curvature nilpotent.*

(c) We can decompose $V = W \oplus \bar{W}$ *for* W *and* $\bar{W}$ *totally isotropic subspaces of* V *and for* $A = A_W \oplus 0$ *where the tensor* $A_W \in \mathfrak{A}(W)$ *is indecomposable.*

Theorem 1.1 is sharp. There is a 14-dimensional model $\mathfrak{M}_{14}$ which is Jacobi–Tsankov but which is not 2-step Jacobi nilpotent. This example will form the focus of our investigations in this paper. It may be defined as follows; it is essentially unique up to isomorphism.

Definition 1.2 Let $\{\alpha_i, \alpha_i^*, \beta_{i,1}, \beta_{i,2}, \beta_{4,1}, \beta_{4,2}\}$ be a basis for $\mathbb{R}^{14}$ for $1 \le i \le 3$. Let $\mathfrak{M}_{14} := (\mathbb{R}^{14}, \langle\cdot,\cdot\rangle, A)$ be the 0-model where the non-zero components of $\langle\cdot,\cdot\rangle$ and of A are given, up to the usual symmetries, by:

$$\langle\alpha_i, \alpha_i^*\rangle = \langle\beta_{i,1}, \beta_{i,2}\rangle = 1 \text{ for } 1 \le i \le 3,$$

$$\langle\beta_{4,1}, \beta_{4,1}\rangle = \langle\beta_{4,2}, \beta_{4,2}\rangle = -\tfrac{1}{2}, \quad \langle\beta_{4,1}, \beta_{4,2}\rangle = \tfrac{1}{4},$$

$$A(\alpha_2, \alpha_1, \alpha_1, \beta_{2,1}) = A(\alpha_3, \alpha_1, \alpha_1, \beta_{3,1}) = 1,$$

$$\begin{aligned}
&A(\alpha_3,\alpha_2,\alpha_2,\beta_{3,2}) = A(\alpha_1,\alpha_2,\alpha_2,\beta_{1,2}) = 1, \qquad (1)\\
&A(\alpha_1,\alpha_3,\alpha_3,\beta_{1,1}) = A(\alpha_2,\alpha_3,\alpha_3,\beta_{2,2}) = 1,\\
&A(\alpha_1,\alpha_2,\alpha_3,\beta_{4,1}) = A(\alpha_1,\alpha_3,\alpha_2,\beta_{4,1}) = -\tfrac{1}{2},\\
&A(\alpha_2,\alpha_3,\alpha_1,\beta_{4,2}) = A(\alpha_2,\alpha_1,\alpha_3,\beta_{4,2}) = -\tfrac{1}{2}\,.
\end{aligned}$$

Let $\mathcal{G}(\mathfrak{M}_{14})$ be the group of symmetries of the model:

$$\mathcal{G}(\mathfrak{M}_{14}) = \{T \in GL(\mathbb{R}^{14}) : T^*\{\langle\cdot,\cdot\rangle\} = \langle\cdot,\cdot\rangle,\ T^*A = A\}\,.$$

Let $SL_\pm(3) := \{A \in GL(\mathbb{R}^3) : \det(A) = \pm 1\}$. In Section 2, we will establish:

Theorem 1.2 *Let $\mathfrak{M}_{14}$ be as in Definition 1.2.*

(1) $\mathfrak{M}_{14}$ is Jacobi–Tsankov of signature $(8,6)$.

(2) $\mathfrak{M}_{14}$ is neither 2-step Jacobi nilpotent nor skew-Tsankov.

(3) There is a short exact sequence $1 \to \mathbb{R}^{21} \to \mathcal{G}(\mathfrak{M}_{14}) \to SL_\pm(3) \to 1$.

(4) $\mathfrak{M}_{14}$ is mixed–Tsankov.

In Section 3, we will show that the model $\mathfrak{M}_{14}$ is geometrically realizable. Thus there exist Jacobi–Tsankov manifolds which are not 2-step Jacobi nilpotent. We introduce the following notation.

Definition 1.3 Let $\{x_i, x_i^*, y_{i,1}, y_{i,2}, y_{4,1}, y_{4,2}\}$ for $1 \le i \le 3$ be coordinates on $\mathbb{R}^{14}$. Suppose given a collection of functions $\Phi := \{\phi_{i,j}\} \in C^\infty(\mathbb{R})$ such that $\phi'_{i,1}\phi'_{i,2} = 1$. Let $\mathcal{M}_\Phi := (\mathbb{R}^{14}, g_\Phi)$ where the non-zero components of g_Φ are, up to the usual $\mathbb{Z}_2$ symmetry, given by:

$$\begin{aligned}
&g_\Phi(\partial_{x_i},\partial_{x_i^*}) = g_\Phi(\partial_{y_{i,1}},\partial_{y_{i,2}}) = 1 \text{ for } 1 \le i \le 3,\\
&g_\Phi(\partial_{y_{4,1}},\partial_{y_{4,1}}) = g_\Phi(\partial_{y_{4,2}},\partial_{y_{4,2}}) = -\tfrac{1}{2}, \qquad g_\Phi(\partial_{y_{4,1}},\partial_{y_{4,2}}) = \tfrac{1}{4},\\
&g_\Phi(\partial_{x_1},\partial_{x_1}) = -2\phi_{2,1}(x_2)y_{2,1} - 2\phi_{3,1}(x_3)y_{3,1}, \quad g_\Phi(\partial_{x_2},\partial_{x_3}) = x_1 y_{4,1},\\
&g_\Phi(\partial_{x_2},\partial_{x_2}) = -2\phi_{3,2}(x_3)y_{3,2} - 2\phi_{1,2}(x_1)y_{1,2}, \quad g_\Phi(\partial_{x_1},\partial_{x_3}) = x_2 y_{4,2},\\
&g_\Phi(\partial_{x_3},\partial_{x_3}) = -2\phi_{1,1}(x_1)y_{1,1} - 2\phi_{2,2}(x_2)y_{2,2}\,.
\end{aligned}$$

Theorem 1.3 *Let $\mathcal{M}_\Phi := (\mathbb{R}^{14}, g_\Phi)$ be as in Definition 1.3.*

(1) $\mathcal{M}_\Phi$ is geodesically complete.

(2) For all $P \in \mathbb{R}^{14}$, $\exp_P$ is a diffeomorphism from $T_P(\mathbb{R}^{14})$ to $\mathbb{R}^{14}$.

(3) $\mathcal{M}_\Phi$ has 0-model $\mathfrak{M}_{14}$.

(4) $\mathcal{M}_\Phi$ is Jacobi–Tsankov but $\mathcal{M}_\Phi$ is not 2-step Jacobi nilpotent.

If we specialize the construction, we can say a bit more. We will establish the following result in Section 4:

Theorem 1.4 *Set* $\phi_{2,1}(x_2) = \phi_{2,2}(x_2) = x_2$ *and* $\phi_{3,1}(x_3) = \phi_{3,2}(x_3) = x_3$ *in Definition 1.3. Let* $\{\phi_{1,1}, \phi_{1,2}\}$ *be real analytic with* $\phi'_{1,1}\phi'_{1,2} = 1$ *and with* $\phi''_{1,j} \neq 0$. *Then*

(1) $\Xi := \{1 - \phi'_{1,1}\phi'''_{1,1}(\phi''_{1,1})^{-2}\}^2$ *is a local isometry invariant of* $\mathcal{M}_\Phi$.

(2) *If* $\phi'_{1,1}(x_1) \neq be^{cx_1}$, *then* Ξ *is not locally constant and hence* $\mathcal{M}_\Phi$ *is not locally homogeneous.*

There are symmetric spaces which have model $\mathfrak{M}_{14}$.

Definition 1.4 Let $\{x_i, x_i^*, y_{i,1}, y_{i,2}, y_{4,1}, y_{4,2}\}$ for $1 \leq i \leq 3$ be coordinates on $\mathbb{R}^{14}$. Let $A := \{a_{i,j}\}$ be a collection of real constants. Let $\mathcal{M}_A := (\mathbb{R}^{14}, g_A)$ where the non-zero components of g_A are given, up to the usual $\mathbb{Z}_2$ symmetry, by:

$$
\begin{aligned}
&g_A(\partial_{x_i}, \partial_{x_i^*}) = g_A(\partial_{y_{i,1}}, \partial_{y_{i,2}}) = 1 \text{ for } 1 \leq i \leq 3,\\
&g_A(\partial_{y_{4,1}}, \partial_{y_{4,1}}) = g_A(\partial_{y_{4,2}}, \partial_{y_{4,2}}) = -\tfrac{1}{2}, \qquad g_A(\partial_{y_{4,1}}, \partial_{y_{4,2}}) = \tfrac{1}{4},\\
&g_A(\partial_{x_1}, \partial_{x_1}) = -2a_{2,1}x_2y_{2,1} - 2a_{3,1}x_3y_{3,1},\\
&g_A(\partial_{x_2}, \partial_{x_2}) = -2a_{3,2}x_3y_{3,2} - 2a_{1,2}x_1y_{1,2},\\
&g_A(\partial_{x_3}, \partial_{x_3}) = -2a_{1,1}x_1y_{1,1} - 2a_{2,2}x_2y_{2,2},\\
&g_A(\partial_{x_1}, \partial_{x_2}) = 2(1 - a_{2,1})x_1y_{2,1} + 2(1 - a_{1,2})x_2y_{1,2}\\
&g_A(\partial_{x_2}, \partial_{x_3}) = x_1y_{4,1} + 2(1 - a_{3,2})x_2y_{3,2} + 2(1 - a_{2,2})x_3y_{2,2},\\
&g_A(\partial_{x_1}, \partial_{x_3}) = x_2y_{4,2} + 2(1 - a_{3,1})x_1y_{3,1} + 2(1 - a_{1,1})x_3y_{1,1}\,.
\end{aligned}
$$

We will establish the following result in Section 5:

Theorem 1.5 *Let* $\mathcal{M}_A$ *be described by Definition 1.4. Then* $\mathcal{M}_A$ *has* 0*-model* $\mathfrak{M}_{14}$. *Furthermore* $\mathcal{M}_A$ *is locally symmetric if and only if*

(1) $a_{1,1} + a_{2,2} + a_{3,1}a_{3,2} = 2$.

(2) $3a_{2,1} + 3a_{3,1} + 3a_{1,2}a_{1,1} = 4$.

(3) $3a_{1,2} + 3a_{3,2} + 3a_{2,1}a_{2,2} = 4$.

2. The model $\mathfrak{M}_{14}$

We study the algebraic properties of the model $\mathfrak{M}_{14}$. Introduce the polarization

$$\mathcal{J}(x_1, x_2) : y \to \tfrac{1}{2}(\mathcal{A}(y, x_1)x_2 + \mathcal{A}(y, x_2)x_1)\,.$$

Let $\{\beta_\nu\}$ be an enumeration of $\{\beta_{i,j}\}_{1\le i\le 4, 1\le j\le 2}$. The following spaces are invariantly defined:

$$V_{\beta,\alpha^*} := \mathrm{Span}_{\xi_i\in\mathbb{R}^{14}}\{\mathcal{J}(\xi_1)\xi_2\} = \mathrm{Span}\{\beta_\nu, \alpha_i^*\},$$
$$V_{\alpha^*} := \mathrm{Span}_{\xi_i\in\mathbb{R}^{14}}\{\mathcal{J}(\xi_1)\mathcal{J}(\xi_2)\xi_3\} = \mathrm{Span}\{\alpha_i^*\}\,.$$

Proof of Theorem 1.1. We have

$$\mathcal{J}(x) = \mathcal{J}(x,x) \quad \text{and} \quad \mathcal{J}(x,y)x = -\frac{1}{2}\,\mathcal{J}(x)y.$$

If $\mathfrak{M}$ is Jacobi–Tsankov, then $\mathcal{J}(x_1,x_2)\mathcal{J}(x_3,x_4) = \mathcal{J}(x_3,x_4)\mathcal{J}(x_1,x_2)$ for all x_i. We may show $\mathcal{J}(x)^2 = 0$ by computing:

$$0 = \mathcal{J}(x,y)\mathcal{J}(x)x = \mathcal{J}(x)\mathcal{J}(x,y)x = -\frac{1}{2}\,\mathcal{J}(x)\mathcal{J}(x)y\,.$$

Similarly, suppose that $\mathfrak{M}$ is mixed–Tsankov, i.e.

$$\mathcal{A}(x_1,x_2)\mathcal{J}(x_3) = \mathcal{J}(x_3)\mathcal{A}(x_1,x_2)$$

for all $x_i \in V$. We show $\mathcal{J}(x)^2 = 0$ in this setting as well by computing:

$$0 = \mathcal{A}(x,y)\mathcal{J}(x)x = \mathcal{J}(x)\mathcal{A}(x,y)x = -\mathcal{J}(x)\mathcal{J}(x)y\,.$$

We have shown that if $\mathfrak{M}$ is either Jacobi-Tsankov or mixed-Tsankov, then $J(x)^2 = 0$. Since the Jacobi operator is nilpotent, $\{0\}$ is the only eigenvalue of $\mathcal{J}$ so $\mathfrak{M}$ is Osserman. If $p = 0$, then $\mathcal{J}(x)$ is diagonalizable. Thus $\mathcal{J}(x)^2 = 0$ implies $\mathcal{J}(x) = 0$ for all x so $A = 0$. If $p = 1$, then $\mathfrak{M}$ is Osserman so $\mathfrak{M}$ has constant sectional curvature [1, 5]; $\mathcal{J}(x)^2 = 0$, $A = 0$. This establishes Assertion (1). Assertions (2) and (3) of Theorem 1.1 follow from results in [4].

Proof of Theorem 1.2 (1,2). It is immediate from the definition that

$$\mathcal{J}(\alpha_3)\mathcal{J}(\alpha_2)\alpha_1 = \mathcal{J}(\alpha_3)\beta_{1,1} = \alpha_1^*$$

so $\mathfrak{M}_{14}$ is not 2-step Jacobi nilpotent.

We define $\beta_{4,1}^*$ and $\beta_{4,2}^*$ by the relations: $\langle\beta_{4,i}^*, \beta_{4,j}\rangle = \delta_{ij}$. We then have:

$$\beta_{4,1}^* = -\tfrac{8}{3}\beta_{4,1} - \tfrac{4}{3}\beta_{4,2}, \quad \beta_{4,2}^* = -\tfrac{4}{3}\beta_{4,1} - \tfrac{8}{3}\beta_{4,2}\,.$$

Let $\mathcal{A}_{ij} := \mathcal{A}(\alpha_i, \alpha_j)$. We show that $\mathfrak{M}_{14}$ is not skew-Tsankov by computing:

$$\mathcal{A}_{12}\mathcal{A}_{13}\alpha_3 = \mathcal{A}_{12}\beta_{1,2} = -\alpha_2^*,$$
$$\mathcal{A}_{13}\mathcal{A}_{12}\alpha_3 = -\tfrac{1}{2}\mathcal{A}_{13}\{\beta_{4,1}^* - \beta_{4,2}^*\} = \mathcal{A}_{13}\{\tfrac{2}{3}\beta_{4,1} - \tfrac{2}{3}\beta_{4,2}\} = \tfrac{1}{3}\alpha_2^*\,.$$

If $\xi \in \mathbb{R}^{14}$, then $\mathcal{J}(\xi)\alpha_i \subset V_{\beta,\alpha^*}$, $\mathcal{J}(\xi)\beta_\nu \subset V_{\alpha^*}$, and $\mathcal{J}(\xi)\alpha_i^* = 0$. Thus to show $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ for all x, y, it suffices to show

$$\mathcal{J}(x)\mathcal{J}(y)\alpha_i = \mathcal{J}(y)\mathcal{J}(x)\alpha_i$$

for all x, y, i. Since $\mathcal{J}(x)\mathcal{J}(y)\alpha_i \in V_{\alpha^*}$, this can be done by establishing:

$$\langle \mathcal{J}(x)\alpha_i, \mathcal{J}(y)\alpha_j \rangle = \langle \mathcal{J}(y)\alpha_i, \mathcal{J}(x)\alpha_j \rangle$$

for all x, y, i, j. Since $\mathcal{J}(x_1, x_2)\alpha_i \in V_{\alpha^*}$ if either x_1 or $x_2 \in V_{\beta,\alpha^*}$, we may take $x_1 = \alpha_i$ and $x_2 = \alpha_j$. Let $\mathcal{J}_{ijk} := \mathcal{J}(\alpha_i, \alpha_j)\alpha_k$. We must show:

$$\langle \mathcal{J}_{i_1 i_2 i_3}, \mathcal{J}_{j_1 j_2 j_3} \rangle = \langle \mathcal{J}_{i_1 i_2 j_3}, \mathcal{J}_{j_1 j_2 i_3} \rangle \quad \forall i_1 i_2 i_3 j_1 j_2 j_3 .$$

The non-zero components of $\mathcal{J}_{ijk} = \mathcal{J}_{jik}$ are:

$$\begin{array}{llll}
\mathcal{J}_{112} = \beta_{2,2}, & \mathcal{J}_{113} = \beta_{3,2}, & \mathcal{J}_{221} = \beta_{1,1}, & \mathcal{J}_{223} = \beta_{3,1}, \\
\mathcal{J}_{331} = \beta_{1,2}, & \mathcal{J}_{332} = \beta_{2,1}, & \mathcal{J}_{121} = -\frac{1}{2}\beta_{2,2}, & \mathcal{J}_{122} = -\frac{1}{2}\beta_{1,1}, \\
\mathcal{J}_{131} = -\frac{1}{2}\beta_{3,2}, & \mathcal{J}_{133} = -\frac{1}{2}\beta_{1,2}, & \mathcal{J}_{232} = -\frac{1}{2}\beta_{3,1}, & \mathcal{J}_{233} = -\frac{1}{2}\beta_{2,1},
\end{array}$$

$$\mathcal{J}_{132} = \tfrac{1}{4}\beta^*_{4,1} - \tfrac{1}{2}\beta^*_{4,2} = \beta_{4,2}, \qquad \mathcal{J}_{231} = -\tfrac{1}{2}\beta^*_{4,1} + \tfrac{1}{4}\beta^*_{4,2} = \beta_{4,1},$$

$$\mathcal{J}_{123} = \tfrac{1}{4}\beta^*_{4,1} + \tfrac{1}{4}\beta^*_{4,2} = -\beta_{4,1} - \beta_{4,2} .$$

The non-zero inner products are:

$$\begin{array}{llll}
\langle \mathcal{J}_{112}, \mathcal{J}_{332} \rangle = 1, & \langle \mathcal{J}_{112}, \mathcal{J}_{233} \rangle = -\frac{1}{2}, & \langle \mathcal{J}_{121}, \mathcal{J}_{332} \rangle = -\frac{1}{2}, & \langle \mathcal{J}_{121}, \mathcal{J}_{233} \rangle = \frac{1}{4}, \\
\langle \mathcal{J}_{113}, \mathcal{J}_{223} \rangle = 1, & \langle \mathcal{J}_{113}, \mathcal{J}_{232} \rangle = -\frac{1}{2}, & \langle \mathcal{J}_{131}, \mathcal{J}_{223} \rangle = -\frac{1}{2}, & \langle \mathcal{J}_{232}, \mathcal{J}_{131} \rangle = \frac{1}{4}, \\
\langle \mathcal{J}_{221}, \mathcal{J}_{331} \rangle = 1, & \langle \mathcal{J}_{221}, \mathcal{J}_{133} \rangle = -\frac{1}{2}, & \langle \mathcal{J}_{122}, \mathcal{J}_{331} \rangle = -\frac{1}{2}, & \langle \mathcal{J}_{122}, \mathcal{J}_{133} \rangle = \frac{1}{4}, \\
\langle \mathcal{J}_{123}, \mathcal{J}_{123} \rangle = \star, & \langle \mathcal{J}_{123}, \mathcal{J}_{132} \rangle = \frac{1}{4}, & \langle \mathcal{J}_{123}, \mathcal{J}_{231} \rangle = \frac{1}{4}, & \langle \mathcal{J}_{132}, \mathcal{J}_{132} \rangle = \star, \\
\langle \mathcal{J}_{132}, \mathcal{J}_{231} \rangle = \frac{1}{4}, & \langle \mathcal{J}_{231}, \mathcal{J}_{231} \rangle = \star . & &
\end{array}$$

The desired symmetries are now immediate:

$$\begin{array}{ll}
\langle \mathcal{J}_{112}, \mathcal{J}_{233} \rangle = -\frac{1}{2} = \langle \mathcal{J}_{113}, \mathcal{J}_{232} \rangle, & \langle \mathcal{J}_{123}, \mathcal{J}_{132} \rangle = \frac{1}{4} = \langle \mathcal{J}_{122}, \mathcal{J}_{133} \rangle, \\
\langle \mathcal{J}_{121}, \mathcal{J}_{332} \rangle = -\frac{1}{2} = \langle \mathcal{J}_{122}, \mathcal{J}_{331} \rangle, & \langle \mathcal{J}_{123}, \mathcal{J}_{231} \rangle = \frac{1}{4} = \langle \mathcal{J}_{121}, \mathcal{J}_{233} \rangle, \\
\langle \mathcal{J}_{131}, \mathcal{J}_{223} \rangle = -\frac{1}{2} = \langle \mathcal{J}_{133}, \mathcal{J}_{221} \rangle, & \langle \mathcal{J}_{132}, \mathcal{J}_{231} \rangle = \frac{1}{4} = \langle \mathcal{J}_{131}, \mathcal{J}_{232} \rangle .
\end{array}$$

Proof of Theorem 1.2 (3,4). Let $\mathcal{G} = \mathcal{G}(\mathfrak{M}_{14})$ be the group of symmetries of the model $\mathfrak{M}_{14}$. Note that the spaces V_{β,α^*} and V_{α^*} are preserved by $\mathcal{G}$, i.e.

$$TV_{\alpha^*} \subset V_{\alpha^*} \quad \text{and} \quad TV_{\beta,\alpha^*} \subset V_{\beta,\alpha^*} \quad \text{if} \quad T \in \mathcal{G} . \tag{2}$$

Let $\tau : \mathcal{G} \to GL(3)$ be the restriction of T to $V_{\alpha^*} = \mathbb{R}^3$. We will prove Theorem 1.2 (3) by showing:

$$SL_\pm(3) = \tau(\mathcal{G}) \quad \text{and} \quad \ker(\tau) = \mathbb{R}^{21}\,.$$

We argue as follows to show $SL_\pm(3) \subset \tau(\mathcal{G})$. Let $\beta_{4,3} := -\beta_{4,1} - \beta_{4,2}$. One may interchange the first two coordinates by setting:

$$\begin{aligned}&T : \alpha_1 \leftrightarrow \alpha_2, \quad T : \alpha_3 \leftrightarrow \alpha_3, \quad T : \alpha_1^* \leftrightarrow \alpha_2^*, \quad T : \alpha_3^* \leftrightarrow \alpha_3^*,\\ &T : \beta_{1,1} \leftrightarrow \beta_{2,2}, T : \beta_{1,2} \leftrightarrow \beta_{2,1}, T : \beta_{3,1} \leftrightarrow \beta_{3,2}, T : \beta_{4,1} \leftrightarrow \beta_{4,2}\,.\end{aligned}$$

One may interchange the first and third coordinates by setting:

$$\begin{aligned}&T : \alpha_1 \leftrightarrow \alpha_3, \quad T : \alpha_2 \leftrightarrow \alpha_2, \quad T : \alpha_1^* \leftrightarrow \alpha_3^*, \quad T : \alpha_2^* \leftrightarrow \alpha_2^*,\\ &T : \beta_{1,1} \leftrightarrow \beta_{3,1}, T : \beta_{1,2} \leftrightarrow \beta_{3,2}, T : \beta_{2,1} \leftrightarrow \beta_{2,2}, T : \beta_{4,1} \leftrightarrow \beta_{4,3},\\ &T : \beta_{4,2} \leftrightarrow \beta_{4,2}\,.\end{aligned}$$

To form a rotation in the first two coordinates, we set

$$\begin{aligned}&T_\theta : \alpha_1 \to \cos\theta\alpha_1 + \sin\theta\alpha_2, \qquad T_\theta : \alpha_2 \to -\sin\theta\alpha_1 + \cos\theta\alpha_2,\\ &T_\theta : \alpha_1^* \to \cos\theta\alpha_1^* + \sin\theta\alpha_2^*, \qquad T_\theta : \alpha_2^* \to -\sin\theta\alpha_1^* + \cos\theta\alpha_2^*,\\ &T_\theta : \alpha_3 \to \alpha_3, \qquad T_\theta : \alpha_3^* \to \alpha_3^*,\\ &T_\theta : \beta_{1,1} \to \cos\theta\beta_{1,1} + \sin\theta\beta_{2,2}, \quad T_\theta : \beta_{1,2} \to \cos\theta\beta_{1,2} + \sin\theta\beta_{2,1},\\ &T_\theta : \beta_{2,1} \to -\sin\theta\beta_{1,2} + \cos\theta\beta_{2,1}, T_\theta : \beta_{2,2} \to -\sin\theta\beta_{1,1} + \cos\theta\beta_{2,2},\\ &T_\theta : \beta_{3,1} \to \sin^2\theta\beta_{3,2} - 2\sin\theta\cos\theta\beta_{4,3} + \cos^2\theta\beta_{3,1},\\ &T_\theta : \beta_{3,2} \to \cos^2\theta\beta_{3,2} + 2\cos\theta\sin\theta\beta_{4,3} + \sin^2\theta\beta_{3,1},\\ &T_\theta : \beta_{4,1} \to \tfrac{1}{2}\sin\theta\cos\theta\beta_{3,2} - \tfrac{1}{2}\sin\theta\cos\theta\beta_{3,1} - \sin^2\theta\beta_{4,2} + \cos^2\theta\beta_{4,1},\\ &T_\theta : \beta_{4,2} \to \tfrac{1}{2}\sin\theta\cos\theta\beta_{3,2} - \tfrac{1}{2}\sin\theta\cos\theta\beta_{3,1} + \cos^2\theta\beta_{4,2} - \sin^2\theta\beta_{4,1}\,.\end{aligned}$$

Finally, we show that the dilatations of determinant 1 belong to Range$\{\tau\}$. Suppose $a_1a_2a_3 = 1$. We set

$$\begin{aligned}&T\alpha_1 = a_1\alpha_1, \quad T\alpha_2 = a_2\alpha_2, \quad T\alpha_3 = a_3\alpha_3, \quad T\alpha_1^* = \tfrac{1}{a_1}\alpha_1^*,\\ &T\alpha_2^* = \tfrac{1}{a_2}\alpha_2^*, \quad T\alpha_3^* = \tfrac{1}{a_3}\alpha_3^*, \quad T\beta_{1,1} = \tfrac{a_2}{a_3}\beta_{1,1}, T\beta_{1,2} = \tfrac{a_3}{a_2}\beta_{1,2},\\ &T\beta_{2,1} = \tfrac{a_3}{a_1}\beta_{2,1}, T\beta_{2,2} = \tfrac{a_1}{a_3}\beta_{2,2}, T\beta_{3,1} = \tfrac{a_2}{a_1}\beta_{3,1}, T\beta_{3,2} = \tfrac{a_1}{a_2}\beta_{3,2},\\ &T\beta_{4,1} = \beta_{4,1}, \quad T\beta_{4,2} = \beta_{4,2}\,.\end{aligned}$$

Since these elements acting on V_{α^*} generate $SL_\pm(3)$, $SL_\pm(3) \subset \tau(\mathcal{G})$. Conversely, let $T \in \mathcal{G}$. We must show $\tau(T) \in SL_\pm(3)$. As $SL_\pm(3) \subset \text{Range}(\tau)$, there exists $S \in \mathcal{G}$ so that $\tau(TS)$ is diagonal. Thus without loss of generality, we may assume $\tau(T)$ is diagonal and hence:

$$T\alpha_i = a_i\alpha_i + \textstyle\sum_\nu b_i^\nu\beta_\nu + \sum_j c_i^j\alpha_j^*, \quad T\beta_\nu = b_\nu\beta_\nu + \sum_i d_\nu^i\alpha_i^*, \quad T\alpha_i^* = a_i^{-1}\alpha_i^*\,.$$

The relations

$$-\tfrac{1}{2} = A(T\alpha_1, T\alpha_2, T\alpha_3, T\beta_{4,1}) = -\tfrac{1}{2} a_1 a_2 a_3 b_{4,1},$$
$$-\tfrac{1}{2} = \langle T\beta_{4,1}, T\beta_{4,1}\rangle = -\tfrac{1}{2} b_{4,1} b_{4,1}$$

show that $b_{4,1}^2 = 1$ and thus $a_1 a_2 a_3 = \pm 1$. Thus $\mathrm{Range}(\tau) = SL_\pm(3)$.

We complete the proof of Assertion (3) by studying $\ker(\tau)$. If one has $T \in \ker(\tau)$, then

$$T\alpha_i = \alpha_i + \textstyle\sum_\nu b_i^\nu \beta_\nu + \sum_j c_i^j \alpha_j^*, \quad T\beta_\nu = \beta_\nu + \sum_i d_\nu^i \alpha_i^*, \quad T\alpha_i^* = \alpha_i^* .$$

Using the relations $A(\alpha_i, \alpha_j, \alpha_k, \alpha_l) = 0$ then leads to the following 6 linear equations the coefficients b_i^ν must satisfy:

$$\begin{aligned}
0 &= A(T\alpha_2, T\alpha_1, T\alpha_1, T\alpha_2)\\
&= 2A(b_2^{2,1}\beta_{2,1}, \alpha_1, \alpha_1, \alpha_2) + 2A(b_1^{1,2}\beta_{1,2}, \alpha_2, \alpha_2, \alpha_1) = 2b_2^{2,1} + 2b_1^{1,2},\\
0 &= A(T\alpha_3, T\alpha_1, T\alpha_1, T\alpha_3)\\
&= 2A(b_3^{3,1}\beta_{3,1}, \alpha_1, \alpha_1, \alpha_3) + 2A(b_1^{1,1}\beta_{1,1}, \alpha_3, \alpha_3, \alpha_1) = 2b_3^{3,1} + 2b_1^{1,1},\\
0 &= A(T\alpha_3, T\alpha_2, T\alpha_2, T\alpha_3)\\
&= 2A(b_3^{3,2}\beta_{3,2}, \alpha_2, \alpha_2, \alpha_3) + 2A(b_2^{2,2}\beta_{2,2}, \alpha_3, \alpha_3, \alpha_2) = 2b_3^{3,2} + 2b_2^{2,2},\\
0 &= A(T\alpha_2, T\alpha_1, T\alpha_1, T\alpha_3)\\
&= A(b_2^{3,1}\beta_{3,1}, \alpha_1, \alpha_1, \alpha_3) + A(\alpha_2, \alpha_1, \alpha_1, b_3^{2,1}\beta_{2,1})\\
&\quad + A(\alpha_2, b_1^{4,1}\beta_{4,1} + b_1^{4,2}\beta_{4,2}, \alpha_1, \alpha_3) + A(\alpha_2, \alpha_1, b_1^{4,1}\beta_{4,1} + b_1^{4,2}\beta_{4,2}, \alpha_3)\\
&= b_2^{3,1} + b_3^{2,1} - \tfrac{1}{2}b_1^{4,1} - \tfrac{1}{2}b_1^{4,1} + \tfrac{1}{2}b_1^{4,2},\\
0 &= A(T\alpha_1, T\alpha_2, T\alpha_2, T\alpha_3)\\
&= A(b_1^{3,2}\beta_{3,2}, \alpha_2, \alpha_2, \alpha_3) + A(\alpha_1, \alpha_2, \alpha_2, b_3^{1,2}\beta_{1,2})\\
&\quad + A(\alpha_1, b_2^{4,1}\beta_{4,1} + b_2^{4,2}\beta_{4,2}, \alpha_2, \alpha_3) + A(\alpha_1, \alpha_2, b_2^{4,1}\beta_{4,1} + b_2^{4,2}\beta_{4,2}, \alpha_3)\\
&= b_1^{3,2} + b_3^{1,2} - \tfrac{1}{2}b_2^{4,2} + \tfrac{1}{2}b_2^{4,1} - \tfrac{1}{2}b_2^{4,2},\\
0 &= A(T\alpha_1, T\alpha_3, T\alpha_3, T\alpha_2)\\
&= A(b_1^{2,2}\beta_{2,2}, \alpha_3, \alpha_3, \alpha_2) + A(\alpha_1, \alpha_3, \alpha_3, b_2^{1,1}\beta_{1,1})\\
&\quad + A(\alpha_1, b_3^{4,1}\beta_{4,1} + b_3^{4,2}\beta_{4,2}, \alpha_3, \alpha_2) + A(\alpha_1, \alpha_3, b_3^{4,1}\beta_{4,1} + b_3^{4,2}\beta_{4,2}, \alpha_2)\\
&= b_1^{2,2} + b_2^{1,1} + \tfrac{1}{2}b_3^{4,2} + \tfrac{1}{2}b_3^{4,1}.
\end{aligned}$$

These equations are linearly independent so there are 18 degrees of freedom in choosing the b's. Once the b's are known, the coefficients d_ν^i are determined

$$0 = \langle T\alpha_i, T\beta_\nu\rangle = d_\nu^i + \textstyle\sum_\mu \langle \beta_\nu, \beta_\mu\rangle b_i^\mu .$$

The relation $\langle T\alpha_i, T\alpha_j\rangle = \delta_{ij}$ implies $c_i^j + c_j^i = 0$; this creates an additional 3 degrees of freedom. Thus $\ker(\tau)$ is isomorphic to the additive group $\mathbb{R}^{21}$.

Let $\xi_i \in V$. Since $\mathcal{R}(\xi_1,\xi_2)\mathcal{J}(\xi_3) = \mathcal{J}(\xi_3)\mathcal{R}(\xi_1,\xi_2) = 0$ if any of the $\xi_i \in V_{\beta,\alpha^*}$, we may work modulo V_{β,α^*} and suppose that $\xi_i \in \text{Span}\{\alpha_i\}$. Since $\mathcal{R}(\xi_1,\xi_2) = 0$ if the ξ_i are linearly dependent, we suppose ξ_1 and ξ_2 are linearly independent.

There are 2 cases to be considered. We first suppose $\xi_3 \in \text{Span}\{\xi_1,\xi_2\}$. The argument given above shows that a subgroup of $\mathcal{G}$ isomorphic to $SL_\pm(3)$ acts $\text{Span}\{\alpha_i\}$. Thus we may suppose $\text{Span}\{\xi_1,\xi_2\} = \text{Span}\{\alpha_1,\alpha_2\}$ and that $\xi_3 = \alpha_1$. Since $\mathcal{A}(\xi_1,\xi_2) = c\mathcal{A}(\alpha_1,\alpha_2)$, we may also assume $\xi_1 = \alpha_1$ and $\xi_2 = \alpha_2$. Let $\mathcal{A}_{ij} := \mathcal{A}(\alpha_i,\alpha_i)$ and $\mathcal{J}_k := \mathcal{J}(\alpha_k)$. We establish the desired result by computing:

$$\begin{aligned}
&\mathcal{A}_{12}\mathcal{J}_1\alpha_1 = 0, && \mathcal{J}_1\mathcal{A}_{12}\alpha_1 = -\mathcal{J}_1\beta_{2,2} = 0,\\
&\mathcal{A}_{12}\mathcal{J}_1\alpha_2 = \mathcal{A}_{12}\beta_{2,2} = 0, && \mathcal{J}_1\mathcal{A}_{12}\alpha_2 = \mathcal{J}_1\beta_{1,1} = 0,\\
&\mathcal{A}_{12}\mathcal{J}_1\alpha_3 = \mathcal{A}_{12}\beta_{3,2} = 0, && \mathcal{J}_1\mathcal{A}_{12}\alpha_3 = \tfrac{1}{2}\mathcal{J}_1(-\beta_{4,1}^* + \beta_{4,2}^*) = 0\,.
\end{aligned}$$

On the other hand, if $\{\xi_1,\xi_2,\xi_3\}$ are linearly independent, we can apply a symmetry in $\mathcal{G}$ and rescale to assume $\xi_i = \alpha_i$. We complete the proof of Theorem 1.2 by computing:

$$\begin{aligned}
&\mathcal{A}_{12}\mathcal{J}_3\alpha_1 = \mathcal{A}_{12}\beta_{1,2} = -\alpha_2^*, && \mathcal{J}_3\mathcal{A}_{12}\alpha_1 = -\mathcal{J}_3\beta_{2,2} = -\alpha_2^*,\\
&\mathcal{A}_{12}\mathcal{J}_3\alpha_2 = \mathcal{A}_{12}\beta_{2,1} = \alpha_1^*, && \mathcal{J}_3\mathcal{A}_{12}\alpha_2 = \mathcal{J}_3\beta_{1,1} = \alpha_1^*,\\
&\mathcal{A}_{1,2}\mathcal{J}_3\alpha_3 = 0, && \mathcal{J}_3\mathcal{A}_{1,2}\alpha_3 = \tfrac{1}{2}\mathcal{J}_3(-\beta_{4,1}^* + \beta_{4,2}^*) = 0\,.
\end{aligned}$$

Remark 2.1 If $\{e_1,e_2\}$ is an oriented orthonormal basis for a non-degenerate 2-plane π, one may define $\mathcal{R}(\pi) := \mathcal{R}(e_1,e_2)$ and one may define $\mathcal{J}(\pi) := \langle e_1,e_1\rangle\mathcal{J}(e_1) + \langle e_2,e_2\rangle\mathcal{J}(e_2)$. These operators are independent of the particular orthonormal basis chosen. Stanilov and Videv [8] have shown that if $\mathcal{M}$ is a 4-dimensional Riemannian manifold, then $\mathcal{R}(\pi)\mathcal{J}(\pi) = \mathcal{J}(\pi)\mathcal{R}(\pi)$ for all oriented 2-planes π if and only if $\mathcal{M}$ is Einstein. Assertion (4) of Theorem 1.2 shows $\mathfrak{M}_{14}$ has this property.

3. A geometric realization of $\mathfrak{M}$

We begin the proof of Theorem 1.3 with a general construction:

Definition 3.1 Let $\{x_i, x_i^*, y_\mu\}$ be coordinates on $\mathbb{R}^{2a+b}$ where $1 \le i \le a$ and $1 \le \mu \le b$. We suppose given a non-degenerate symmetric matrix $C_{\mu\nu}$ and smooth functions $\psi_{ij\mu} = \psi_{ij\mu}(\vec{x})$ with $\psi_{ij\mu} = \psi_{ji\mu}$. Consider the pseudo-Riemannian manifold $\mathcal{M}_{C,\psi} := (\mathbb{R}^{2a+b}, g_{C,\psi})$, where:

$$g_{C,\psi}(\partial_{x_i}, \partial_{x_j}) = 2\sum_k y_\mu \psi_{ij\mu}, \quad g_{C,\psi}(\partial_{x_i}, \partial_{x_i^*}) = 1, \quad g_{C,\psi}(\partial_{y_\mu}, \partial_{y_\nu}) = C_{\mu\nu}\,.$$

Lemma 3.1 *Let $\mathcal{M}_{C,\psi} = (\mathbb{R}^{2a+b}, g_{C,\psi})$ be as in Definition 3.1. Then*

(1) $\mathcal{M}_{C,\psi}$ is geodesically complete.

(2) For all $P \in \mathbb{R}^{2a+b}$, $\exp_P$ is a diffeomorphism from $T_P(\mathbb{R}^{2a+b})$ to $\mathbb{R}^{2a+b}$.

(3) The possibly non-zero components of the curvature tensor are, up to the usual $\mathbb{Z}_2$ symmetries given by:

$$\begin{aligned}
&R(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}, \partial_{y_\nu}) = -\partial_{x_i}\psi_{jk\nu} + \partial_{x_j}\psi_{ik\nu},\\
&R(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}, \partial_{x_l}) = \textstyle\sum_{\nu\mu} C^{\nu\mu}\{\psi_{ik\mu}\psi_{jl\nu} - \psi_{il\mu}\psi_{jk\nu}\}\\
&\quad + \textstyle\sum_\nu y_\nu\left\{\partial_{x_i}\partial_{x_k}\psi_{jl\nu} + \partial_{x_j}\partial_{x_l}\psi_{ik\nu} - \partial_{x_i}\partial_{x_l}\psi_{jk\nu} - \partial_{x_j}\partial_{x_k}\psi_{il\nu}\right\}\,.
\end{aligned}$$

Proof. The non-zero Christoffel symbols of the first kind are given by:

$$\begin{aligned}
&g(\nabla_{\partial_{x_i}}\partial_{x_j}, \partial_{x_k}) = \textstyle\sum_\mu\{\partial_{x_i}\psi_{jk\mu} + \partial_{x_j}\psi_{ik\mu} - \partial_{x_k}\psi_{ij\mu}\}y_\mu,\\
&g(\nabla_{\partial_{x_i}}\partial_{x_j}, \partial_{y_\nu}) = -\psi_{ij\nu},\\
&g(\nabla_{\partial_{x_i}}\partial_{y_\nu}, \partial_{x_k}) = g(\nabla_{\partial_{y_\nu}}\partial_{x_i}, \partial_{x_k}) = \psi_{ik\nu},
\end{aligned}$$

and the non-zero Christoffel symbols of the second kind are given by:

$$\begin{aligned}
&\nabla_{\partial_{x_i}}\partial_{x_j} = \textstyle\sum_\mu y_\mu\{\partial_{x_i}\psi_{jk\mu} + \partial_{x_j}\psi_{ik\mu} - \partial_{x_k}\psi_{ij\mu}\}\partial_{x_k^*} - \sum_{\mu\nu} C^{\nu\mu}\psi_{ij\nu}\partial_{y_\mu},\\
&\nabla_{\partial_{x_i}}\partial_{y_\nu} = \nabla_{\partial_{y_\nu}}\partial_{x_i} = \textstyle\sum_k \psi_{ik\nu}\partial_{x_k^*}\,.
\end{aligned}$$

This shows that $\mathcal{M}$ is a generalized plane wave manifold; Assertions (1) and (2) then follow from results in [7]. Assertion (3) now follows by a direct calculation. □

Proof of Theorem 1.3 (1)-(3) Assertions (1) and (2) of Theorem 1.3 follow by specializing the corresponding results of Lemma 3.1. We use Assertion (3) of Lemma 3.1 to see that the possibly non-zero components of the curvature tensor defined by the metric of Definition 1.3 are:

$$\begin{aligned}
&R(\partial_{x_{i_1}}, \partial_{x_{i_2}}, \partial_{x_{i_3}}, \partial_{x_{i_4}}) = \star,\\
&R(\partial_{x_1}, \partial_{x_2}, \partial_{y_{2,1}}, \partial_{x_1}) = \partial_{x_2}\phi_{2,1}, \quad R(\partial_{x_1}, \partial_{x_3}, \partial_{y_{3,1}}, \partial_{x_1}) = \partial_{x_3}\phi_{3,1},\\
&R(\partial_{x_2}, \partial_{x_3}, \partial_{y_{3,2}}, \partial_{x_2}) = \partial_{x_3}\phi_{3,2}, \quad R(\partial_{x_2}, \partial_{x_1}, \partial_{y_{1,2}}, \partial_{x_2}) = \partial_{x_1}\phi_{1,2},\\
&R(\partial_{x_3}, \partial_{x_1}, \partial_{y_{1,1}}, \partial_{x_3}) = \partial_{x_1}\phi_{1,1}, \quad R(\partial_{x_3}, \partial_{x_2}, \partial_{y_{2,2}}, \partial_{x_3}) = \partial_{x_2}\phi_{2,2},\\
&R(\partial_{x_2}, \partial_{x_1}, \partial_{y_{4,1}}, \partial_{x_3}) = R(\partial_{x_3}, \partial_{x_1}, \partial_{y_{4,1}}, \partial_{x_2}) = -\tfrac{1}{2},\\
&R(\partial_{x_1}, \partial_{x_2}, \partial_{y_{4,2}}, \partial_{x_3}) = R(\partial_{x_3}, \partial_{x_2}, \partial_{y_{4,2}}, \partial_{x_1}) = -\tfrac{1}{2}.
\end{aligned}$$

We introduce the following basis as a first step in the proof of Assertion (3). Let the index i range from 1 to 3 and the index j run from 1 to 2. Set:

$$\bar\alpha_i := \partial_{x_i}, \quad \alpha_i^* := \partial_{x_i^*}, \quad \bar\beta_{4,j} := \partial_{y_{4,j}}, \quad \bar\beta_{i,j} := \{\phi'_{i,j}\}^{-1}\partial_{y_{i,j}}. \tag{3}$$

Since $\phi'_{i,1}\cdot\phi'_{i,2} = 1$, the relations of Equation (1) are satisfied. However, we still have the following potentially non-zero terms to deal with:

$$g(\bar\alpha_i,\bar\alpha_j) = \star \quad \text{and} \quad R(\bar\alpha_i,\bar\alpha_j,\bar\alpha_k,\bar\alpha_l) = \star\,.$$

To deal with the extra curvature terms, we introduce a modified basis setting:

$$\begin{aligned}
&\tilde\alpha_1 := \bar\alpha_1 + R(\bar\alpha_1,\bar\alpha_2,\bar\alpha_3,\bar\alpha_1)\bar\beta_{4,1} - \tfrac12 R(\bar\alpha_1,\bar\alpha_2,\bar\alpha_2,\bar\alpha_1)\bar\beta_{1,2},\\
&\tilde\alpha_2 := \bar\alpha_2 + R(\bar\alpha_2,\bar\alpha_1,\bar\alpha_3,\bar\alpha_2)\bar\beta_{4,2} - \tfrac12 R(\bar\alpha_2,\bar\alpha_3,\bar\alpha_3,\bar\alpha_2)\bar\beta_{2,2},\\
&\tilde\alpha_3 := \bar\alpha_3 - 2R(\bar\alpha_3,\bar\alpha_1,\bar\alpha_2,\bar\alpha_3)\bar\beta_{4,1} - \tfrac12 R(\bar\alpha_1,\bar\alpha_3,\bar\alpha_3,\bar\alpha_1)\bar\beta_{3,1},\\
&\beta_{1,1} := \bar\beta_{1,1} + \tfrac12 R(\bar\alpha_1,\bar\alpha_2,\bar\alpha_2,\bar\alpha_1)\alpha_1^*, \qquad \beta_{1,2} := \bar\beta_{1,2}\\
&\beta_{2,1} := \bar\beta_{2,1} + \tfrac12 R(\bar\alpha_2,\bar\alpha_3,\bar\alpha_3,\bar\alpha_2)\alpha_2^*, \qquad \beta_{2,2} := \bar\beta_{2,2},\\
&\beta_{3,2} := \bar\beta_{3,2} + \tfrac12 R(\bar\alpha_1,\bar\alpha_3,\bar\alpha_3,\bar\alpha_1)\alpha_3^*, \qquad \beta_{3,1} := \bar\beta_{3,1},\\
&\beta_{4,1} := \bar\beta_{4,1} + \tfrac12 R(\bar\alpha_1,\bar\alpha_2,\bar\alpha_3,\bar\alpha_1)\alpha_1^* - \tfrac14 R(\bar\alpha_2,\bar\alpha_1,\bar\alpha_3,\bar\alpha_2)\alpha_2^*\\
&\qquad\qquad - R(\bar\alpha_3,\bar\alpha_1,\bar\alpha_2,\bar\alpha_3)\alpha_3^*,\\
&\beta_{4,2} := \bar\beta_{4,2} - \tfrac14 R(\bar\alpha_1,\bar\alpha_2,\bar\alpha_3,\bar\alpha_1)\alpha_1^* + \tfrac12 R(\bar\alpha_2,\bar\alpha_1,\bar\alpha_3,\bar\alpha_2)\alpha_2^*\\
&\qquad\qquad + \tfrac12 R(\bar\alpha_3,\bar\alpha_1,\bar\alpha_2,\bar\alpha_3)\alpha_3^*\,.
\end{aligned} \tag{4}$$

All the normalizations of Equation (1) are satisfied except for the unwanted metric terms $g(\tilde\alpha_i,\tilde\alpha_j)$. To eliminate these terms and to exhibit a basis with the required normalizations, we set:

$$\alpha_i := \tilde\alpha_i - \tfrac12 \textstyle\sum_j g(\tilde\alpha_i,\tilde\alpha_j)\alpha_j^*\,. \tag{5}$$

4. Isometry Invariants

We now turn to the task of constructing invariants.

Lemma 4.1 *Adopt the assumptions of Theorem 1.4. Let $\{\alpha_i,\beta_\nu,\alpha_i^*\}$ be defined by Equations (3)-(5). Set $\phi_1 := \phi'_{1,1}$ and $\phi_2 := \phi'_{1,2}$.*

(1) $\nabla R(v_1,v_2,v_3,v_4;v_5) = 0$ if at least one of the $v_i \in V_{\alpha^*}$.

(2) $\nabla R(v_1,v_2,v_3,v_4;v_5) = 0$ if at least two of the $v_i \in V_{\beta,\alpha^*}$.

(3) $\nabla^k R(\alpha_1,\alpha_2,\alpha_2,\beta_{1,2};\alpha_1,...,\alpha_1)=\phi_2^{-1}\phi_2^{(k)}$.

(4) $\nabla^k R(\alpha_1,\alpha_3,\alpha_3,\beta_{1,1};\alpha_1,...,\alpha_1)=\phi_1^{-1}\phi_1^{(k)}$.

(5) $\nabla R(\alpha_i,\alpha_j,\alpha_k,\beta_\nu;\alpha_{l_1},...,\alpha_{l_k})=0$ *in cases other than those given in (3) and (4) up to the usual* $\mathbb{Z}_2$ *symmetry in the first 2 entries.*

Proof. Let v_i be coordinate vector fields. To prove Assertion (1), we suppose some $v_i\in V_{\alpha^*}$. We may use the second Bianchi identity and the other curvature symmetries to assume without loss of generality $v_1\in V_{\alpha^*}$. Since $\nabla_{v_5}v_1=0$ and since $R(v_1,\cdot,\cdot,\cdot)=0$, Assertion (1) follows. The proof of the second assertion is similar and uses the fact that $R(\cdot,\cdot,\cdot,\cdot)=0$ if 2-entries belong to V_{β,α^*}. The proof of the remaining assertions is similar and uses the particular form of the warping functions $\phi_{i,j}$; the factor of $\phi_{1,j}^{-1}$ arising from the normalization in Equation (3). □

Definition 4.1 We say that a basis $\tilde{\mathcal{B}}:=\{\tilde\alpha_i,\tilde\beta_\nu,\tilde\alpha_i^*\}$ is 0-*normalized* if the normalizations of Equation (1) are satisfied and 1-*normalized* if it is 0-normalized and if additionally

$$\begin{aligned}&\nabla R(\tilde\alpha_1,\tilde\alpha_3,\tilde\alpha_3,\tilde\beta_{1,1};\tilde\alpha_1)=-\nabla R(\tilde\alpha_3,\tilde\alpha_1,\tilde\alpha_3,\tilde\beta_{1,1};\tilde\alpha_1)\neq 0,\\&\nabla R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1)=-\nabla R(\tilde\alpha_2,\tilde\alpha_1,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1)\neq 0,\\&\nabla R(\tilde\alpha_i,\tilde\alpha_j,\tilde\alpha_k,\tilde\beta_\nu;\tilde\alpha_l)=0\quad\text{otherwise}\,.\end{aligned}$$

Lemma 4.2 *Adopt the assumptions of Theorem 1.4. Then:*

(1) There exists a 1*-normalized basis.*

(2) If $\tilde{\mathcal{B}}$ *is a* 1*-normalized basis, then there exist constants* a_i *so* $a_1a_2a_3=\varepsilon$ *for* $\varepsilon=\pm1$ *and so that exactly one of the following conditions holds:*

(a) $\tilde\alpha_1=a_1\alpha_1$, $\tilde\alpha_2=a_2\alpha_2$, $\tilde\alpha_3=a_3\alpha_3$, $\tilde\beta_{1,1}=\varepsilon\frac{a_2}{a_3}\beta_{1,1}$, $\tilde\beta_{1,2}=\varepsilon\frac{a_3}{a_2}\beta_{1,2}$.

(b) $\tilde\alpha_1=a_1\alpha_1$, $\tilde\alpha_2=a_3\alpha_3$, $\tilde\alpha_3=a_2\alpha_2$, $\tilde\beta_{1,1}=\varepsilon\frac{a_3}{a_2}\beta_{1,2}$, $\tilde\beta_{1,2}=\varepsilon\frac{a_2}{a_3}\beta_{1,1}$.

Proof. We use Equations (3), (4), and (5) to construct a 0-normalized basis and then apply Lemma 4.1 to see this basis is 1-normalized. On the other hand, if $\tilde{\mathcal{B}}$ is a 1-normalized basis, we may expand:

$$\begin{aligned}&\tilde\alpha_1=a_{11}\alpha_1+a_{12}\alpha_2+a_{13}\alpha_3+...,\\&\tilde\alpha_2=a_{21}\alpha_1+a_{22}\alpha_2+a_{23}\alpha_3+...,\quad\tilde\beta_{1,2}=b_{21}\beta_{1,1}+b_{22}\beta_{1,2}+...\\&\tilde\alpha_3=a_{31}\alpha_1+a_{32}\alpha_2+a_{33}\alpha_3+...,\quad\tilde\beta_{1,1}=b_{11}\beta_{1,1}+b_{12}\beta_{1,2}+....\end{aligned}$$

Because

$$\begin{aligned} 0 &\neq \nabla R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1)\\ &= a_{11}\left\{(a_{11}a_{22}-a_{12}a_{21})a_{22}b_{22})\phi_2^{-1}\phi_2'\right.\\ &\qquad \left.+(a_{11}a_{33}-a_{13}a_{31})a_{33}b_{21})\phi_1^{-1}\phi_1'\right\}, \end{aligned}$$

we have $a_{11}\neq 0$. Because

$$0=\nabla R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_2)=\tfrac{a_{21}}{a_{11}}\nabla R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1),$$

we have $a_{21}=0$; similarly $a_{31}=0$. Since $\operatorname{Span}\{\alpha_i\}=\operatorname{Span}\{\tilde\alpha_i\}$ mod $V_{\beta,\alpha*}$,

$$a_{22}a_{33}-a_{23}a_{32}\neq 0\,.$$

By hypothesis $R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_3,\beta;\tilde\alpha_1)=0$ if $\beta\in\operatorname{Span}\{\tilde\beta_\nu,\tilde\alpha_i^*\}=V_{\beta,\alpha^*}$ so

$$\begin{aligned} 0&=R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_3,\beta_{1,2};\tilde\alpha_1)=a_{11}^2a_{22}a_{32}\phi_2^{-1}\phi_2',\\ 0&=R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_3,\beta_{1,1};\tilde\alpha_1)=a_{11}^2a_{23}a_{33}\phi_1^{-1}\phi_1'\,. \end{aligned}$$

Suppose that $a_{22}\neq 0$. Since $a_{11}^2a_{22}a_{32}=0$ and $a_{11}\neq 0$, $a_{32}=0$. Since $a_{22}a_{33}-a_{23}a_{32}\neq 0$, $a_{33}\neq 0$. Since $a_{11}^2a_{23}a_{33}=0$, we also have $a_{23}=0$. Since the basis is also 0-normalized, $diag(a_{11}^{-1},a_{22}^{-1},a_{33}^{-1})\in SL_\pm(3)$ from the discussion in Section 2. Thus $\varepsilon:=a_{11}a_{22}a_{33}=\pm1$, $b_{11}=\varepsilon\frac{a_{33}}{a_{22}}$, and $b_{22}=\varepsilon\frac{a_{22}}{a_{33}}$. These are the relations of Assertion (2a). The argument is similar if $a_{32}\neq 0$; we simply reverse the roles of $\tilde\alpha_2$ and $\tilde\alpha_3$ to establish the relations of Assertion (2b). □

Proof of Theorem 1.4. Let

$$\Xi(\mathcal{B}):=\frac14\left\{\frac{\nabla^2R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1,\tilde\alpha_1)}{\{\nabla R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1)\}^2}-\frac{\nabla^2R(\tilde\alpha_1,\tilde\alpha_3,\tilde\alpha_3,\tilde\beta_{1,1};\tilde\alpha_1,\tilde\alpha_1)}{\{\nabla R(\tilde\alpha_1,\tilde\alpha_3,\tilde\alpha_3,\tilde\beta_{1,1};\tilde\alpha_1)\}^2}\right\}^2$$

We apply Lemma 4.2. Suppose the conditions of Assertion (2a) hold. Then:

$$\begin{aligned} &\nabla R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1)=a_1\phi_2^{-1}\phi_2',\\ &\nabla^2R(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1,\tilde\alpha_1)=a_1^2\phi_2^{-1}\phi_2'',\\ &\nabla R(\tilde\alpha_1,\tilde\alpha_3,\tilde\alpha_3,\tilde\beta_{1,1};\tilde\alpha_1)=a_1\phi_1^{-1}\phi_1',\\ &\nabla^2R(\tilde\alpha_1,\tilde\alpha_3,\tilde\alpha_3,\tilde\beta_{1,1};\tilde\alpha_1,\tilde\alpha_1)=a_1^2\phi_1^{-1}\phi_1'', \end{aligned}$$

$$\Xi(\mathcal{B})=\frac14\left\{\frac{\phi_2\phi_2''}{\phi_2'\phi_2'}-\frac{\phi_1\phi_1''}{\phi_1'\phi_1'}\right\}^2\,.$$

The roles of ϕ_1 and ϕ_2 are reversed if Assertion (2b) holds. It now follows that Ξ is a local isometry invariant. Since $\phi_2=\phi_1^{-1}$, $\phi_2'=-\phi_1^{-2}\phi_1'$,

$\phi_2'' = 2\phi_1^{-3}\phi_1'\phi_1' - \phi_1^{-2}\phi_1''$, we may establish Assertion (1) of Theorem 1.4 by computing

$$\frac{\phi_2\phi_2''}{\phi_2'\phi_2'} = \frac{\phi_1^{-1}(2\phi_1^{-3}\phi_1'\phi_1' - \phi_1^{-2}\phi_1'')}{\phi_1^{-4}\phi_1'\phi_1'} = 2 - \frac{\phi_1\phi_1''}{\phi_1'\phi_1'}\,.$$

Consequently

$$\Xi = \frac{1}{4}\left\{2 - 2\frac{\phi_1\phi_1''}{\phi_1'\phi_1'}\right\}^2\,.$$

If $\mathcal{M}_\Phi$ is locally homogeneous, then Ξ must be constant. Conversely, if Ξ is constant, then $\phi_1\phi_1'' = k\phi_1'\phi_1'$ for some $k \in \mathbb{R}$. The solutions to this ordinary differential equation take the form $\phi_1(t) = a(t+b)^c$ if $k \neq 1$ and $\phi_1(t) = ae^{bt}$ if $k = 1$ for suitably chosen constants a and b and for $c = c(k)$. The first family is ruled out as ϕ_1 and ϕ_1' must be invertible for all t. Thus $\phi_1(t)$ is a pure exponential; Assertion (2) of Theorem 1.4 follows.

5. A symmetric space with model $\mathfrak{M}_{14}$

We give the proof of Theorem 1.5 as follows. Let $\mathcal{M}_A$ be as described in Definition 1.4. By Lemma 3.1 one has that:

$$\begin{aligned}
&R(\partial_{x_2},\partial_{x_1},\partial_{x_1},\partial_{y_{2,1}}) = R(\partial_{x_3},\partial_{x_1},\partial_{x_1},\partial_{y_{3,1}}) = 1,\\
&R(\partial_{x_3},\partial_{x_2},\partial_{x_2},\partial_{y_{3,2}}) = R(\partial_{x_1},\partial_{x_2},\partial_{x_2},\partial_{y_{1,2}}) = 1,\\
&R(\partial_{x_1},\partial_{x_3},\partial_{x_3},\partial_{y_{1,1}}) = R(\partial_{x_2},\partial_{x_3},\partial_{x_3},\partial_{y_{2,2}}) = 1,\\
&R(\partial_{x_1},\partial_{x_2},\partial_{x_3},\partial_{y_{4,1}}) = R(\partial_{x_1},\partial_{x_3},\partial_{x_2},\partial_{y_{4,1}}) = -\tfrac{1}{2},\\
&R(\partial_{x_2},\partial_{x_3},\partial_{x_1},\partial_{y_{4,2}}) = R(\partial_{x_2},\partial_{x_1},\partial_{x_3},\partial_{y_{4,2}}) = -\tfrac{1}{2}.
\end{aligned}$$

The same argument constructing a 0-normalized basis which was given in the proof of Theorem 1.2 can then be used to construct a 0-normalized basis in this setting and establish that $\mathcal{M}_{\mathcal{A}}$ has 0-model $\mathfrak{M}_{14}$.

We can also apply Lemma 3.1 to see:

$$\begin{aligned}
&R(\partial_{x_1},\partial_{x_2},\partial_{x_2},\partial_{x_1}) = -a_{3,1}a_{3,2}x_3^2,\\
&R(\partial_{x_1},\partial_{x_3},\partial_{x_3},\partial_{x_1}) = -\tfrac{1}{3}(2+3a_{2,1}a_{2,2})x_2^2,\\
&R(\partial_{x_3},\partial_{x_2},\partial_{x_2},\partial_{x_3}) = -\tfrac{1}{3}(2+3a_{1,1}a_{1,2})x_1^2,\\
&R(\partial_{x_2},\partial_{x_1},\partial_{x_1},\partial_{x_3}) = (1 - a_{1,1} - a_{1,2} + a_{1,1}a_{1,2} + a_{2,1}\\
&\quad -a_{2,1}a_{2,2} + a_{3,1} - a_{3,1}a_{3,2})x_2x_3,\\
&R(\partial_{x_1},\partial_{x_2},\partial_{x_2},\partial_{x_3}) = (1 + a_{1,2} - a_{2,1} - a_{1,1}a_{1,2} - a_{2,2}\\
&\quad +a_{2,1}a_{2,2} + a_{3,2} - a_{3,1}a_{3,2})x_1x_3,\\
&R(\partial_{x_1},\partial_{x_3},\partial_{x_3},\partial_{x_2}) = (\tfrac{2}{3} + a_{1,1} - a_{1,1}a_{1,2} + a_{2,2}\\
&\quad -a_{2,1}a_{2,2} - a_{3,1} - a_{3,2} + a_{3,1}a_{3,2})x_1x_2.
\end{aligned}$$

The Christoffel symbols describing $\nabla_{\partial_{x_i}}\partial_{x_j}$ are given by:

$$\nabla_{\partial_{x_1}}\partial_{x_1} = (2-a_{2,1})y_{2,1}\partial_{x_2^*} + (2-a_{3,1})y_{3,1}\partial_{x_3^*} + a_{2,1}x_2\partial_{y_{2,2}} + a_{3,1}x_3\partial_{y_{3,2}},$$

$$\nabla_{\partial_{x_2}}\partial_{x_2} = (2-a_{1,2})y_{1,2}\partial_{x_1^*} + (2-a_{3,2})y_{3,2}\partial_{x_3^*} + a_{1,2}x_1\partial_{y_{1,1}} + a_{3,2}x_3\partial_{y_{3,1}},$$

$$\nabla_{\partial_{x_3}}\partial_{x_3} = (2-a_{1,1})y_{1,1}\partial_{x_1^*} + (2-a_{2,2})y_{2,2}\partial_{x_2^*} + a_{2,2}x_2\partial_{y_{2,1}} + a_{1,1}x_1\partial_{y_{1,2}},$$

$$\nabla_{\partial_{x_1}}\partial_{x_2} = -a_{2,1}y_{2,1}\partial_{x_1^*} - a_{1,2}y_{1,2}\partial_{x_2^*} + \tfrac{y_{4,1}+y_{4,2}}{2}\partial_{x_3^*} + (a_{1,2}-1)x_2\partial_{y_{1,1}} + (a_{2,1}-1)x_1\partial_{y_{2,2}},$$

$$\nabla_{\partial_{x_1}}\partial_{x_3} = -a_{3,1}y_{3,1}\partial_{x_1^*} + \tfrac{y_{4,1}-y_{4,2}}{2}\partial_{x_2^*} - a_{1,1}y_{1,1}\partial_{x_3^*} + (a_{1,1}-1)x_3\partial_{y_{1,2}} + (a_{3,1}-1)x_1\partial_{y_{3,2}} + \tfrac{2x_2}{3}\partial_{y_{4,1}} + \tfrac{4x_2}{3}\partial_{y_{4,2}},$$

$$\nabla_{\partial_{x_2}}\partial_{x_3} = \tfrac{-y_{4,1}+y_{4,2}}{2}\partial_{x_1^*} - a_{3,2}y_{3,2}\partial_{x_2^*} - a_{2,2}y_{2,2}\partial_{x_3^*} + (a_{2,2}-1)x_3\partial_{y_{2,1}} + (a_{3,2}-1)x_2\partial_{y_{3,1}} + \tfrac{4x_1}{3}\partial_{y_{4,1}} + \tfrac{2x_1}{3}\partial_{y_{4,2}}.$$

It is now easy to show that the non-zero components of ∇R are:

$$\nabla R(\partial_{x_1},\partial_{x_2},\partial_{x_2},\partial_{x_1};\partial_{x_3}) = -2(-2+a_{1,1}+a_{2,2}+a_{3,1}a_{3,2})x_3,$$

$$\nabla R(\partial_{x_1},\partial_{x_3},\partial_{x_3},\partial_{x_1};\partial_{x_2}) = -\tfrac{2}{3}(-4+3a_{1,2}+3a_{3,2}+3a_{2,1}a_{2,2})x_2,$$

$$\nabla R(\partial_{x_2},\partial_{x_3},\partial_{x_3},\partial_{x_2};\partial_{x_1}) = -\tfrac{2}{3}(-4+3a_{2,1}+3a_{3,1}+3a_{1,1}a_{1,2})x_1,$$

$$\nabla R(\partial_{x_2},\partial_{x_1},\partial_{x_1},\partial_{x_3};\partial_{x_2}) = (2-a_{1,1}-a_{1,2}+a_{2,1}-a_{2,2} + a_{3,1}-a_{3,2}+a_{1,1}a_{1,2}-a_{2,1}a_{2,2}-a_{3,1}a_{3,2})x_3,$$

$$\nabla R(\partial_{x_2},\partial_{x_1},\partial_{x_1},\partial_{x_3};\partial_{x_3}) = (2-a_{1,1}-a_{1,2}+a_{2,1}-a_{2,2} + a_{3,1}-a_{3,2}+a_{1,1}a_{1,2}-a_{2,1}a_{2,2}-a_{3,1}a_{3,2})x_2,$$

$$\nabla R(\partial_{x_1},\partial_{x_2},\partial_{x_2},\partial_{x_3};\partial_{x_1}) = (2-a_{1,1}+a_{1,2}-a_{2,1}-a_{2,2} - a_{3,1}+a_{3,2}-a_{1,1}a_{1,2}+a_{2,1}a_{2,2}-a_{3,1}a_{3,2})x_3,$$

$$\nabla R(\partial_{x_1},\partial_{x_2},\partial_{x_2},\partial_{x_3};\partial_{x_3}) = (2-a_{1,1}+a_{1,2}-a_{2,1}-a_{2,2} - a_{3,1}+a_{3,2}-a_{1,1}a_{1,2}+a_{2,1}a_{2,2}-a_{3,1}a_{3,2})x_1,$$

$$\nabla R(\partial_{x_1},\partial_{x_3},\partial_{x_3},\partial_{x_2};\partial_{x_1}) = (\tfrac{2}{3}+a_{1,1}-a_{1,2}-a_{2,1}+a_{2,2} - a_{3,1}-a_{3,2}-a_{1,1}a_{1,2}-a_{2,1}a_{2,2}+a_{3,1}a_{3,2})x_2,$$

$$\nabla R(\partial_{x_1},\partial_{x_3},\partial_{x_3},\partial_{x_2};\partial_{x_2}) = (\tfrac{2}{3}+a_{1,1}-a_{1,2}-a_{2,1}+a_{2,2} - a_{3,1}-a_{3,2}-a_{1,1}a_{1,2}-a_{2,1}a_{2,2}+a_{3,1}a_{3,2})x_1.$$

We set $\nabla R = 0$ to obtain the desired equations of Theorem 1.5; the first 3 equations generate the last 6.

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