THE EINSTEIN–YANG–MILLS EQUATIONS IN GAUGE SPACES OF ORDER k *

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In the $(k+1)n$ dimensional space P, where the transformation group is given by

$$y^{0a'} = y^{0a'}(y^{0a}),\ y^{1a'} = y^{1a'}(y^{0a}, y^{1a}), \dots, y^{ka'} = y^{ka'}(y^{0a}, y^{1a}, \dots, y^{ka})$$

the adapted basis in tangent and its dual space is constructed. The Einstein-Yang-Mills equations which give the extreme value of integral of action

$$I(\phi) = \int_\Omega \sqrt{g} L(\phi, \partial_{0a}\phi, \partial_{1a}\phi, \dots, \partial_{ka}\phi) d\omega$$

(L is Lagrangian, g is the determinant of the metric tensor) are determined. They are similar to those obtained from Gh. Munteanu in [22].

1. Natural and adapted bases

Let P be a $(k+1)n$ dimensional C^∞ manifold in which some point p in local chart (U, φ) has coordinates

$$\begin{aligned} &(y^{0a}, y^{1a}, y^{2a}, \dots, y^{ka}) = (y^{Aa}) \\ &A, B, C, D, E, \dots = 0, 1, 2, \dots, k, \quad a, b, c, d, e, \dots = 1, 2, \dots, n. \end{aligned} \tag{1.1}$$

If in another chart (U', φ') the same point p has coordinates $(y^{0i'}, y^{1i'}, y^{2i'}, \dots, y^{ki'})$, then the allowable coordinate transformations are

* *MSC 2000:* 53B40, 53C60, 53C80, 81T13, 53C07.
Keywords: gauge transformations of order k, Einstein–Yang–Mills equations.
† This research was partly supported by Science Fund of Serbia, grant number 1262.

given by

$$\begin{aligned} y^{0a'} &= y^{0a'}(y^{0a}), \\ y^{1a'} &= y^{1a'}(y^{0a}, y^{1a}), \\ y^{2a'} &= y^{2a'}(y^{0a}, y^{1a}, y^{2a}), \ldots, \\ y^{ka'} &= y^{ka'}(y^{0a}, y^{1a}, y^{2a}, \ldots, y^{ka}). \end{aligned} \tag{1.2}$$

Definition 1.1 The transformations of type (1.2) are called gauge transformations of order k.

The natural basis of $T^*(P)$ is

$$\bar{B}^* = \{dy^{0a}, dy^{1a}, \ldots, dy^{ka}\}. \tag{1.3}$$

Under the transformations of type (1.2) the elements of $\bar{B}^*$ are transforming in the following way:

$$\begin{aligned} dy^{0a'} &= (\partial_{0b} y^{0a'}) dy^{0b}, \\ dy^{1a'} &= (\partial_{0b} y^{1a'}) dy^{0b} + (\partial_{1b} y^{1a'}) dy^{1b}, \ldots, \\ dy^{ka'} &= (\partial_{0b} y^{ka'}) dy^{0b} + (\partial_{1b} y^{ka'}) dy^{1b} + \cdots + (\partial_{kb} y^{ka'}) dy^{kb}, \end{aligned} \tag{1.4}$$

where

$$\partial_{Ab} = \frac{\partial}{\partial y^{Ab}}, \qquad A = 0, 1, \ldots, k.$$

If we introduce the notations:

$$[dy^{Aa'}] = \begin{bmatrix} dy^{0a'} \\ dy^{1a'} \\ \vdots \\ dy^{ka'} \end{bmatrix}, \quad [dy^{Bb}] = \begin{bmatrix} dy^{0b} \\ dy^{1b} \\ \vdots \\ dy^{kb} \end{bmatrix}, \tag{1.5}$$

$$[J^{Aa'}_{Bb}] = \begin{bmatrix} \partial_{0b} y^{0a'} & 0 & \cdots & 0 \\ \partial_{0b} y^{1a'} & \partial_{1b} y^{1a'} & \cdots & 0 \\ \vdots & & & \\ \partial_{0b} y^{ka'} & \partial_{1b} y^{ka'} & \cdots & \partial_{kb} y^{ka'} \end{bmatrix}, \tag{1.6}$$

then (1.4) can be written in the form

$$[dy^{Aa'}] = [J^{Aa'}_{Bb}][dy^{Bb}]. \tag{1.7}$$

The elements of the Jacobian matrix are matrices of type $n \times n$. If

$$rank[J^{Aa'}_{Bb}] = n(k+1), \tag{1.8}$$

then exists inverse transformation of (1.2), which is the same type. The identity transformation is the special case of (1.2) and where the composition is defined, the product of two transformations is also of type (1.2).

Theorem 1.1 *The transformations of type (1.2) form a pseudo-group.*

The natural basis of $T(P)$ is

$$\bar{B} = \{\partial_{0a}, \partial_{1a}, \dots, \partial_{ka}\}. \tag{1.9}$$

The elements of $\bar{B}$ under transformations of type (1.2) are transforming as follows:

$$\begin{aligned} \partial_{0a} &= (\partial_{0a} y^{0b'})\partial_{0b'} + (\partial_{0a} y^{1b'})\partial_{1b'} + \dots + (\partial_{0a} y^{kb'})\partial_{kb'}, \\ \partial_{1a} &= (\partial_{1a} y^{1b'})\partial_{1b'} + \cdots + (\partial_{1a} y^{kb'})\partial_{kb'}, \dots, \\ \partial_{ka} &= (\partial_{ka} y^{kb'})\partial_{kb'}. \end{aligned} \tag{1.10}$$

If we use the notations

$$[\partial_{Aa}] = [\partial_{0a}\partial_{1a} \dots \partial_{ka}], \qquad [\partial_{Bb'}] = [\partial_{0b'}\partial_{1b'} \dots \partial_{kb'}], \tag{1.11}$$

then (1.10) can be written in the form:

$$[\partial_{Aa}] = [\partial_{Bb'}][J^{Bb'}_{Aa}]. \tag{1.12}$$

Remark 1.1 The transformations of type (1.2) are more general as those in $Osc^k M$. The allowable transformations in $Osc^k M$ are the special cases of (1.2). The relations

$$\frac{\partial y^{0a'}}{\partial y^{0a}} = \frac{\partial y^{1a'}}{\partial y^{1a}} = \cdots = \frac{\partial y^{ka'}}{\partial y^{ka}}, \tag{1.13}$$

$$\frac{\partial y^{(A+B)a'}}{\partial y^{Ba}} = \binom{A+B}{B} \frac{\partial y^{Aa'}}{\partial y^{0a}},$$

which are valid in $Osc^k M$ ([6], [7]) in P are not valid in general case.
We shall construct the adapted basis

$$B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \dots, \delta y^{ka}\} \tag{1.14}$$

of $T^*(P)$ in the following way:

$$\begin{aligned} \delta y^{0a} &= dy^{0a}, \\ \delta y^{1a} &= dy^{1a} + M^{1a}_{0b} dy^{0b}, \\ \delta y^{2a} &= dy^{2a} + M^{2a}_{1b} dy^{1b} + M^{2a}_{0b} dy^{0b}, \dots, \\ \delta y^{ka} &= dy^{ka} + M^{ka}_{(k-1)b} dy^{(k-1)b} + \cdots + M^{ka}_{0b} dy^{0b}. \end{aligned} \tag{1.15}$$

If we introduce the notations

$$[\delta y^{Aa}] = \begin{bmatrix} \delta y^{0a} \\ \delta y^{1a} \\ \vdots \\ \delta y^{ka} \end{bmatrix}, \qquad [M^{Aa}_{Bb}] = \begin{bmatrix} \delta^a_b & 0 & 0 & \cdots & 0 \\ M^{1a}_{0b} & \delta^a_b & 0 & \cdots & 0 \\ M^{2a}_{0b} & M^{2a}_{1b} & \delta^a_b & \cdots & 0 \\ \vdots \\ M^{ka}_{0b} & M^{ka}_{1b} & M^{ka}_{2b} & \cdots & \delta^a_b \end{bmatrix}, \tag{1.16}$$

then (1.15) can be written in the form:

$$[\delta y^{Aa}] = [M^{Aa}_{Bb}][dy^{Bb}]. \tag{1.17}$$

We shall determine the functions M^{Aa}_{Bb} in such a way, that the elements of $B^* =$
$\{\delta y^{0a}, \delta y^{1a}, \ldots, \delta y^{ka}\}$ are transforming in the following way:

$$\delta y^{Aa'} = B^{Aa'}_{Aa} \delta y^{Aa}, \qquad B^{Aa'}_{Aa} = \frac{\partial y^{Aa'}}{\partial y^{Aa}},\ A = 0, 1, \ldots, k, \tag{1.18}$$

no summation over A.

Remark 1.2 The transformation (1.18) is more general, then the corresponding formula

$$\delta y^{Aa'} = B^{a'}_a \delta y^{Aa}, \quad B^{a'}_a = \frac{\partial x^{a'}}{\partial x^a} = \frac{\partial y^{0a'}}{\partial y^{0a}}$$

in $T^*(Osc^k M)$, because in $T^*(P)$ (1.13) is not necessarily satisfied.

We have:

Theorem 1.2 *The elements of the adapted basis B^* of $T^*(P)$ determined by (1.17) satisfy the transformation law (1.18) if the functions M^{Aa}_{Bb} are transforming in the following way:*

$$[B^{Aa'}_{Aa}][M^{Aa}_{Cc}] = [M^{Aa'}_{Bb'}][J^{Bb'}_{Cc}],\ \textit{(no summation over } A) \tag{1.19}$$

where

$$[B^{Aa'}_{Aa}] = \begin{bmatrix} \partial_{0a} y^{0a'} & 0 & \cdots & 0 \\ 0 & \partial_{1a} y^{1a'} & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & \partial_{ka} y^{ka'} \end{bmatrix}, \tag{1.20}$$

or in the explicite form

$$B^{(A+B)a'}_{(A+B)a} M^{(A+B)a}_{Ac} = M^{(A+B)a'}_{Ab'} \partial_{Ac} y^{Ab'} + M^{(A+B)a'}_{(A+1)b'} \partial_{Ac} y^{(A+1)b'} \tag{1.21}$$

$$+ \cdots + M^{(A+B)a'}_{(A+B-1)b'} \partial_{Ac} y^{(A+B-1)b'} + \partial_{Ac} y^{(A+B)a'}.$$

Proof. Using (1.7), (1.17) and (1.18) we can write:

$$[\delta y^{Aa'}] = [M^{Aa'}_{Bb'}][dy^{Bb'}] = [M^{Aa'}_{Bb'}][J^{Bb'}_{Cc}][dy^{Cc}],$$

$$[\delta y^{Aa'}] = [B^{Aa'}_{Aa}][\delta y^{Aa}] = [B^{Aa'}_{Aa}][M^{Aa}_{Cc}][dy^{Cc}],$$

no summation over A. From the above two equations it follows (1.19).
From (1.2) and (1.19) it follows, that

$$M^{(A+B)a}_{Ac} = M^{(A+B)a}_{Ac}(y^{0a}, y^{1a}, y^{2a}, \dots, y^{(A+B)a}),$$

for $0 \leq A \leq A + B \leq k$.
We shall construct the adapted basis

$$B = \{\delta_{0a}, \delta_{1a}, \dots, \delta_{ka}\} \tag{1.22}$$

of $T(P)$ in the following way

$$\begin{aligned} \delta_{0a} &= \partial_{0a} - N^{1b}_{0a}\partial_{1b} - N^{2b}_{0a}\partial_{2b} - \cdots - N^{kb}_{0a}\partial_{kb}, \\ \delta_{1a} &= \qquad \partial_{1a} \qquad - N^{2b}_{1a}\partial_{2b} - \cdots - N^{kb}_{1a}\partial_{kb}, \dots, \\ \delta_{ka} &= \qquad\qquad\qquad\qquad\qquad \partial_{ka}. \end{aligned} \tag{1.23}$$

If we introduce the notations

$$[\delta_{Aa}] = [\delta_{0a}\delta_{1a}\dots\delta_{ka}], \quad [\delta_{Aa'}] = [\delta_{0a'}\delta_{1a'}\dots\delta_{ka'}], \tag{1.24}$$

$$[N^{Bb}_{Aa}] = \begin{bmatrix} \delta^b_a & 0 & \cdots & 0 \\ -N^{1b}_{0a} & \delta^b_a & \cdots & 0 \\ -N^{2b}_{0a} & -N^{2b}_{1a} & \cdots & 0 \\ \vdots & & & \\ -N^{kb}_{0a} & -N^{kb}_{1a} & \cdots & \delta^b_a \end{bmatrix} \tag{1.25}$$

then (1.23) can be written in the form:

$$[\delta_{Aa}] = [\partial_{Bb}][N^{Bb}_{Aa}]. \tag{1.26}$$

We shall determine the transformation law of functions N^{Bb}_{Aa} under conditions that the elements of B are transforming in the following way:

$$\delta_{Aa} = B^{Aa'}_{Aa}\delta_{Aa'} \quad A = 0, 1, 2, \dots, k, \tag{1.27}$$

no summation over A. (1.27) in the matrix form can be written as follows

$$[\delta_{Aa}] = [\delta_{Aa'}][B^{Aa'}_{Aa}]. \tag{1.28}$$

We have

Theorem 1.3 *The elements of the adapted basis* B *of* $T(P)$ *satisfy the transformation law (1.27) if the functions* N^{Bb}_{Aa} *are transforming in the following way:*

$$[N^{Cc'}_{Aa'}][B^{Aa'}_{Aa}] = [J^{Cc'}_{Bb}][N^{Bb}_{Aa}], \tag{1.29}$$

or in the explicite form

$$\begin{aligned} B^{(A+B)a'}_{(A+B)a} N^{(A+B)c'}_{Aa'} = & N^{(A+B)b}_{Aa} \partial_{(A+B)b} y^{(A+B)c'} + \\ & N^{(A+B-1)b}_{Aa} \partial_{(A+B-1)b} y^{(A+B)c'} + \cdots + \\ & N^{(B+1)b}_{Aa} \partial_{(B+1)b} y^{(A+B)c'} - \partial_{Aa} y^{(A+B)c'}. \end{aligned} \tag{1.30}$$

Proof. From (1.12) and (1.23) and (1.28) it follows

$$\begin{aligned} [\delta_{Aa}] &= [\partial_{Bb}][N^{Bb}_{Aa}] = [\partial_{Cc'}][J^{Cc'}_{Bb}][N^{Bb}_{Aa}], \\ [\delta_{Aa}] &= [\delta_{Aa'}][B^{Aa'}_{Aa}] = [\partial_{Cc'}][N^{Cc'}_{Aa'}][B^{Aa'}_{Aa}]. \end{aligned}$$

From the above two equations it follows (1.29).

From (1.29) and (1.30) we have

$$N^{(A+B)b}_{Aa} = N^{(A+B)b}_{Aa}(y^{0a}, y^{1a}, y^{2a}, \ldots, y^{(A+B)a}) \tag{1.31}$$

for $0 \leq A \leq A + B \leq k$.

Theorem 1.4 *The necessary and sufficient condition that the adapted basis* B^* *is dual to the adapted basis* B*, when* $\bar{B}^*$ *is dual to* $\bar{B}$*, i.e.*

$$[\delta y^{Bb}][\delta_{Aa}] = \delta^B_A \delta^b_a I \tag{1.32}$$

is the following equation:

$$[M^{Bb}_{Aa}][N^{Aa}_{Cc}] = \delta^B_C \delta^b_c I. \tag{1.33}$$

The explicit form of (1.33) is:

$$\begin{aligned} N^{(A+B)b}_{Aa} = & M^{(A+B)b}_{Aa} - M^{(A+B)b}_{(A+1)c} N^{(A+1)c}_{Aa} - \\ & M^{(A+B)b}_{(A+2)c} N^{(A+2)c}_{Aa} - \cdots - M^{(A+B)b}_{(A+B-1)c} N^{(A+B-1)c}_{Aa} \end{aligned}$$

Proof. The substitution of (1.17) and (1.26) into (1.32) gives

$$[\delta y^{Bb}][\delta_{Cc}] = [M^{Bb}_{Aa}][dy^{Aa}][\partial_{Dd}][N^{Dd}_{Cc}] =$$

$$[M^{Bb}_{Aa}]\delta^A_D\delta^a_d I[N^{Dd}_{Cc}] = [M^{Bb}_{Aa}][N^{Aa}_{Cc}] = \delta^B_C\delta^b_c I.$$

(1.33) means, that the matrices $[M^{Bb}_{Aa}]$ and $[N^{Aa}_{Cc}]$ are inverse matrices.

Theorem 1.5 *The elements of natural basis $\bar{B}$ are connected with the elements of the adapted basis B in the following way:*

$$[\partial_{Aa}] = [\delta_{Bb}][M^{Bb}_{Aa}]. \tag{1.34}$$

The explicit form of (1.34) is

$$\begin{aligned}
\partial_{0a} &= \delta_{0a} + M^{1b}_{0a}\delta_{1b} + M^{2b}_{0a}\delta_{2b} + \ldots + M^{kb}_{0a}\delta_{kb},\\
\partial_{1a} &= \qquad \delta_{1a} \qquad + M^{2b}_{1a}\delta_{2b} + \cdots + M^{kb}_{1a}\delta_{kb}, \ldots,\\
\partial_{ka} &= \qquad\qquad\qquad\qquad\qquad\qquad \delta_{ka}.
\end{aligned}$$

Proof. The multiplication of (1.26) with the matrix $[M^{Aa}_{Cc}]$ and (1.33) results

$$[\delta_{Aa}][M^{Aa}_{Cc}] = [\partial_{Bb}][N^{Bb}_{Aa}][M^{Aa}_{Cc}] = [\partial_{Bb}]\delta^B_C\delta^b_c I = [\partial_{Cc}].$$

The above equation is equivalent to (1.34).

Theorem 1.6 *The elements of the natural basis $\bar{B}^*$ are connected with the elements of the adapted basis B^* in the following way*

$$[dy^{Aa}] = [N^{Aa}_{Bb}][\delta y^{Bb}]. \tag{1.35}$$

The explicit form of (1.35) is

$$\begin{aligned}
dy^{0a} &= \delta y^{0a}\\
dy^{1a} &= \delta y^{1a} - N^{1a}_{0b}\delta y^{0b}\\
dy^{2a} &= \delta y^{2a} - N^{2a}_{1b}\delta y^{1b} - N^{2a}_{0b}\delta y^{0b}, \ldots,\\
dy^{ka} &= \delta y^{ka} - N^{ka}_{(k-1)b}\delta y^{(k-1)b} - \cdots - N^{ka}_{0b}\delta y^{0b}.
\end{aligned}$$

Proof. The multiplication of (1.17) with $[N^{Cc}_{Aa}]$ and (1.33) results

$$[N^{Cc}_{Aa}][\delta y^{Aa}] = [N^{Cc}_{Aa}][M^{Aa}_{Bb}][dy^{Bb}] = \delta^C_B\delta^c_b I[dy^{Bb}] = [dy^{Cc}].$$

The above equation is equivalent to (1.35).

Definition 1.2 Some tensor T in the space $T(P)\otimes T^*(P)$ is a d-tensor if it can be expressed in the form

$$T = T^{Aa}{}_{Bb}\delta_{Aa}\otimes\delta y^{Bb} \tag{1.36}$$

and the components of T under (1.2) are transforming as follows

$$T^{Aa'}{}_{Bb'} = T^{Aa}{}_{Bb}B^{Aa'}_{Aa}B^{Bb}_{Bb'}, \tag{1.37}$$

no summation over A and B.

This transformation is good defined, because

$$\begin{aligned}T &= T^{Aa'}{}_{Bb'}\delta_{Aa'}\otimes\delta y^{Bb'} = T^{Aa}{}_{Bb}B^{Aa'}_{Aa}B^{Bb}_{Bb'}\delta_{Aa'}\otimes\delta y^{Bb'}\\ &= T^{Aa}{}_{Bb}\delta_{Aa}\otimes\delta y^{Bb}.\end{aligned}$$

In the above equation (1.18) and (1.27) was used.

2. The Einstein-Yang-Mills equations in gauge spaces of order *k*

The metric tensor in P is a symmetric, positive definite tensor of type $(0,2)$. In the natural basis $\bar{B}^*$ of $T^*(P)$, G has the form

$$G = \bar{g}_{AaBb}dy^{Aa}\otimes dy^{Bb} = [dy^{Aa}]^T[\bar{g}_{AaBb}]\otimes[dy^{Bb}],\ (T-\text{ transposed }). \tag{2.1}$$

From (1.35) we have

$$[dy^{Bb}] = [N^{Bb}_{Cc}][\delta y^{Cc}],\ [dy^{Aa}]^T = [\delta y^{Dd}]^T[N^{Aa}_{Dd}]^T,$$

so (2.1) can be written in the form

$$G = [\delta y^{Dd}]^T[g_{DdCc}]\otimes[\delta y^{Cc}] = g_{DdCc}\delta y^{Dd}\otimes\delta y^{Cc} \tag{2.2}$$

where

$$[g_{DdCc}] = [N^{Aa}_{Dd}]^T[\bar{g}_{AaBb}][N^{Bb}_{Cc}]. \tag{2.3}$$

Let us denote by $T^*_0, T^*_1, T^*_2, \ldots, T^*_k$ the subspaces of $T^*(P)$ generated by $\{\delta y^{0a}\}, \{\delta y^{1a}\}, \{\delta y^{2a}\},\ \ldots, \{\delta y^{ka}\}$ respectively. If they are mutually orthogonal to each other with respect to the metric tensor G, then all blocks in matrix $[g_{DdCc}]$ for which $D\neq C$ are equal to zero.

Theorem 2.1 *The subspaces $T^*_0, T^*_1, T^*_2, \ldots, T^*_k$ of $T^*(P)$ are mutually orthogonal to each other with respect to G, if and only if*

$$G = g_{0a0b}\delta y^{0a}\otimes\delta y^{0b} + g_{1a1b}\delta y^{1a}\otimes\delta y^{1b} + \cdots + g_{kakb}\delta y^{ka}\otimes\delta y^{kb}. \tag{2.4}$$

Let $L(y^{0a}, y^{1a}, \dots, y^{ka})$ be a Lagrangian defined on the compact set $\Omega \subset R^{n(k+1)}$. As it is a scalar field we have

$$L(y^{0a}, y^{1a}, \dots, y^{ka}) = L(y^{0a'}, y^{1a'}, \dots, y^{ka'}). \tag{2.5}$$

We shall suppose that the adapted basis $B = \{\delta_{0a}, \delta_{1a}, \dots, \delta y^{ka}\}$ are chosen in such a way, that $T_0^*, T_1^*, \dots, T_k^*$ are mutually orthogonal to G, i.e. when (2.4) is satisfied. We have

$$g = det|g_{AaBb}| = det|g_{0a0b}|det|g_{1a1b}| \dots det|g_{kakb}|. \tag{2.6}$$

and

$$g_{AaAb} = g_{Aa'Ab'} B^{Aa'}_{Aa} B^{Ab'}_{Ab}, \ A = 0, 1, \dots, k. \tag{2.7}$$

From (1.6) it follows that the Jacobian matrix

$$[J^{Aa'}_{Bb}] = \frac{D(y^{0a'}, y^{1a'}, \dots y^{ka'})}{D(y^{0b}, y^{1b}, \dots, y^{kb})} \tag{2.8}$$

is a triangle matrix which has blocks $B^{0a'}_{0b}, B^{1a'}_{1b}, \dots, B^{ka'}_{kb}$ on the main diagonal, so

$$|J| = |B^{0a'}_{0a}||B^{1a'}_{1a}| \dots |B^{ka'}_{ka}|. \tag{2.9}$$

From (2.6) and (2.7) it follows

$$\begin{aligned} &|g_{AaAb}| = |g_{Aa'Ab'}||B^{Aa'}_{Aa}||B^{Ab'}_{Ab}| = |g_{Aa'Ab'}||B^{Aa'}_{Aa}|^2, \\ &A = 0, 1, 2, \dots, k. \end{aligned} \tag{2.10}$$

The substitution of (2.7), (2.9) and (2.10) into (2.6) gives

$$g = det|g_{AaBb}| = det|g_{Aa'Bb'}||J|^2 = g'|J|^2. \tag{2.11}$$

Let us define the Lagrangian density by

$$\mathcal{L}(y^{0a}, y^{1a}, \dots, y^{ka}) = L(y^{0a}, y^{1a}, \dots, y^{ka})\sqrt{g}. \tag{2.12}$$

The substitution of (2.5) and (2.11) into (2.12) results

$$\begin{aligned} \mathcal{L}(y^{0a}, y^{1a}, \dots, y^{ka}) &= L(y^{0a'}, y^{1a'}, \dots, y^{ka'})\sqrt{g'}|J| \\ &= \mathcal{L}(y^{0a'}, y^{1a'}, \dots, y^{ka'})|J|. \end{aligned} \tag{2.13}$$

The elementary volume element $d\omega$ in Ω under coordinate transformation (1.2) is transforming in known way:

$$d\omega(y^{0a'}, y^{1a'}, \dots, y^{ka'}) = |J|d\omega(y^{0a}, y^{1a}, \dots, y^{ka}). \tag{2.14}$$

Proposition 2.1 *The integral of action*

$$I = \int\limits_{\Omega} \mathcal{L}(y^{0a}, y^{1a}, \ldots, y^{ka}) d\omega \tag{2.15}$$

does not depend on coordinate system if and only if $\mathcal{L}(y^{0a}, y^{1a}, \ldots, y^{ka})$ *satisfies the relation*

$$\mathcal{L}(y^{0a}, y^{1a}, \ldots, y^{ka}) = |J| \mathcal{L}(y^{0a'}, y^{1a'}, \ldots, y^{ka'}). \tag{2.16}$$

Proof. I is invariant if

$$\begin{aligned} \mathcal{L}(y^{0a}, y^{1a}, \ldots, y^{ka}) d\omega(y^{0a}, y^{1a}, \ldots, y^{ka}) = \\ \mathcal{L}(y^{0a'}, y^{1a'}, \ldots, y^{ka'}) d\omega(y^{0a'}, y^{1a'}, \ldots, y^{ka'}). \end{aligned} \tag{2.17}$$

The substitution of (2.14) into (2.17) results (2.16). The proof in the opposite direction is obvious.

From (2.12) and (2.13) we obtain that one example for $\mathcal{L}$, which gives coordinate invariant action is

$$\mathcal{L}(y^{0a}, y^{1a}, \ldots, y^{ka}) = \sqrt{g} L(y^{0a}, y^{1a}, \ldots, y^{ka}).$$

Proposition 2.2 *For arbitrary* C^2 *function* $\mathcal{L}(y^{0a}, y^{1a}, \ldots, y^{ka})$ *the following relation is valid:*

$$d\mathcal{L} = (\delta_{0a}\mathcal{L})\delta y^{0a} + (\delta_{1a}\mathcal{L})\delta y^{1a} + \cdots + (\delta_{ka}\mathcal{L})\delta y^{ka}. \tag{2.18}$$

Proof. As $\mathcal{L} = \mathcal{L}(y^{0a}, y^{1a}, \ldots, y^{ka})$ we have

$$d\mathcal{L} = (\partial_{0a}\mathcal{L}) dy^{0a} + (\partial_{1a}\mathcal{L}) dy^{1a} + \cdots + (\partial_{ka}\mathcal{L}) dy^{ka}. \tag{2.19}$$

The substitution of (1.34) into (2.19) gives:

$$\begin{aligned} d\mathcal{L} = & (\delta_{0a} + M_{0a}^{1b}\delta_{1b} + M_{0a}^{2b}\delta_{2b} + \cdots + M_{0a}^{kb}\delta_{kb})\mathcal{L} dy^{0a} \\ & + (\delta_{1a} + M_{1a}^{2b}\delta_{2b} + \cdots + M_{1a}^{kb}\delta_{kb})\mathcal{L} dy^{1a} + \cdots + \delta_{ka}\mathcal{L} dy^{ka} \\ = & \ (\delta_{0a}\mathcal{L}) dy^{0a} + (\delta_{1a}\mathcal{L})(dy^{1a} + M_{0b}^{1a} dy^{0b}) + \cdots \\ & + (\delta_{ka}\mathcal{L})(dy^{ka} + \cdots + M_{1b}^{ka} dy^{1b} + M_{0b}^{ka} dy^{0b}). \end{aligned} \tag{2.20}$$

From (1.17) and (2.20) it follows (2.18).

We shall suppose that the Lagrangian $L(y^{0a}, y^{1a}, \ldots, y^{ka})$, is the function of $\phi^{\alpha}(y^{0a}, y^{1a}, \ldots, y^{ka})$, $\partial_{0a}\phi^{\alpha}(y^{0a}, y^{1a}, \ldots, y^{ka})$, $\partial_{1a}\phi^{\alpha}(y^{0a}, y^{1a}, \ldots, y^{ka})$, $\ldots, \partial_{ka}\phi^{\alpha}(y^{0a}, y^{1a}, \ldots, y^{ka})$, where $\alpha = 1, 2, \ldots, p$.

For the simplification we shall consider only one function $\phi = \phi(y^{0a}, y^{1a}, \ldots, y^{ka})$ and use the abbreviations

$$\partial_{Aa}\phi = \frac{\partial\phi(y^{0a}, y^{1a}, \ldots, \partial y^{ka})}{\partial y^{Aa}},\ A = 0, 1, \ldots, k.$$

Now we have

$$\begin{aligned}\mathcal{L}(y^{0a}, y^{1a}, \ldots, y^{ka}) &= \mathcal{L}(\phi, \partial_{0a}\phi, \partial_{1a}\phi, \ldots, \partial_{ka}\phi) \\ &= \sqrt{g}L(\phi, \partial_{0a}\phi, \partial_{1a}\phi, \ldots, \partial_{ka}\phi)\end{aligned} \tag{2.21}$$

and the integral of action has the form:

$$I(\phi) = \int_{\Omega} \mathcal{L}(\phi, \partial_{0a}\phi, \partial_{1a}\phi, \ldots, \partial_{ka}\phi) d\omega. \tag{2.22}$$

We are looking for such function ϕ, for which $I(\phi)$ has maximal or minimal value, i.e. for which $\delta I(\phi) = 0$.

For the simplicity we shall suppose that Ω is the $(k+1)n$ dimensional rectangle, such that

$$\int_{\Omega} d\omega = \int_{a^{0a}}^{b^{0a}} dy^{0a} \int_{a^{1a}}^{b^{1a}} dy^{1a} \cdots \int_{a^{ka}}^{b^{ka}} dy^{ka}, \tag{2.23}$$

where

$$\int_{a^{Aa}}^{b^{Aa}} dy^{Aa} = \int_{a^{A1}}^{b^{A1}} dy^{A1} \int_{a^{A2}}^{b^{A2}} dy^{A2} \cdots \int_{a^{An}}^{b^{An}} dy^{An},\ A = 0, 1, 2, \ldots, k. \tag{2.24}$$

The variation of ϕ on the boundary of Ω is equal to zero, i.e.

$$\begin{aligned}&\delta\phi(b^{0a}, y^{1a}, \ldots, y^{ka}) = \delta\phi(a^{0a}, y^{1a}, \ldots, y^{ka}) = 0, \ldots, \\ &\delta\phi(y^{0a}, \ldots, y^{(k-1)a}, b^{ka}) = \delta\phi(y^{0a}, \ldots, y^{(k-1)a}, a^{ka}) = 0, \\ &\text{for } a = 1, 2, \ldots, n,\end{aligned} \tag{2.25}$$

where for instance

$$\begin{aligned}&\delta\phi(y^{0a}, \ldots, b^{A2}, \ldots, y^{ka}) = \\ &\delta\phi(y^{01}, \ldots, y^{0n}, \ldots, y^{A1}b^{A2}, y^{A3}, \ldots, y^{An}, \ldots, y^{k1}, \ldots, y^{kn}), \\ &A = 0, 1, 2, \ldots, k.\end{aligned} \tag{2.26}$$

From the variation principle we have

$$\delta I(\phi) = \int\limits_{\Omega} \delta\mathcal{L}(y^{0a}, y^{1a}, \dots, y^{ka}) d\omega \tag{2.27}$$
$$= \int\limits_{\Omega} \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_{0a}\phi)}\delta(\partial_{0a}\phi) + \cdots + \frac{\partial \mathcal{L}}{\partial(\partial_{ka}\phi)}\delta(\partial_{ka}\phi) \right] d\omega.$$

From (2.21) it can be seen, that $\mathcal{L}$ is a function of independent variables $\phi, \partial_{0a}\phi, \dots, \partial_{ka}\phi$. To express this fact in (2.28) we shall write

$$\frac{\partial \mathcal{L}}{\partial(\partial_{0a}\phi)} = \frac{d\mathcal{L}}{d(\partial_{0a}\phi)}, \cdots, \frac{\partial \mathcal{L}}{\partial(\partial_{ka}\phi)} = \frac{d\mathcal{L}}{d(\partial_{ka}\phi)}. \tag{2.28}$$

From (2.18) it follows

$$\mathcal{L}(\phi, \partial_{0a}\phi, \partial_{1a}\phi, \dots, \partial_{ka}\phi) = \mathcal{L}(\phi, \delta_{0a}\phi, \delta_{1a}\phi, \dots, \delta_{ka}\phi), \tag{2.29}$$

where (see (1.23)

$$\begin{aligned} \delta_{0b}\phi &= \partial_{0b}\phi - N^{1a}_{0b}\partial_{1a}\phi - N^{2a}_{0b}\partial_{2a}\phi - \cdots - N^{ka}_{0b}\partial_{ka}\phi, \\ \delta_{1b}\phi &= \partial_{1b}\phi - N^{2a}_{1b}\partial_{2a}\phi - \cdots - N^{ka}_{1b}\partial_{ka}\phi, \dots, \\ \delta_{ka}\phi &= \partial_{ka}\phi. \end{aligned} \tag{2.30}$$

We have

$$A_0 = \frac{d\mathcal{L}}{d(\partial_{0a}\phi)} = \frac{\partial \mathcal{L}}{\partial(\delta_{0b}\phi)}\frac{\partial(\delta_{0b}\phi)}{\partial(\partial_{0a}\phi)} = \frac{\partial \mathcal{L}}{\partial(\delta_{0a}\phi)} \tag{2.31}$$

$$A_1 = \frac{d\mathcal{L}}{d(\partial_{1a}\phi)} = \frac{\partial \mathcal{L}}{\partial(\delta_{0b}\phi)}\frac{\partial(\delta_{0b}\phi)}{\partial(\partial_{1a}\phi)} + \frac{\partial \mathcal{L}}{\partial(\delta_{1b}\phi)}\frac{\partial(\delta_{1b}\phi)}{\partial(\partial_{1a}\phi)}$$
$$= \frac{\partial \mathcal{L}}{\partial(\delta_{1a}\phi)} - N^{1a}_{0b}\frac{\partial \mathcal{L}}{\partial(\delta_{0b}\phi)},$$

$$A_2 = \frac{d\mathcal{L}}{d(\partial_{2a}\phi)} = \frac{\partial \mathcal{L}}{\partial(\delta_{2a}\phi)} - N^{2a}_{1b}\frac{\partial \mathcal{L}}{\partial(\delta_{1b}\phi)} - N^{2a}_{0b}\frac{\partial \mathcal{L}}{\partial(\delta_{0b}\phi)},$$

$$A_3 = \frac{d\mathcal{L}}{d(\partial_{3a}\phi)} = \frac{\partial \mathcal{L}}{\partial(\delta_{3a}\phi)} - N^{3a}_{2b}\frac{\partial \mathcal{L}}{\partial(\delta_{2b}\phi)} - N^{3a}_{1b}\frac{\partial \mathcal{L}}{\partial(\delta_{1b}\phi)} - N^{3a}_{0b}\frac{\partial \mathcal{L}}{\partial(\delta_{0b}\phi)},$$

....

$$A_k = \frac{d\mathcal{L}}{d(\partial_{ka}\phi)} = \frac{\partial \mathcal{L}}{\partial(\delta_{ka}\phi)} - N^{ka}_{(k-1)b}\frac{\partial \mathcal{L}}{\partial(\delta_{(k-1)b})} - \cdots - N^{ka}_{1b}\frac{\partial \mathcal{L}}{\partial(\delta_{1b}\phi)}$$
$$- N^{ka}_{0b}\frac{\partial \mathcal{L}}{\partial(\delta_{0b}\phi)}.$$

We shall suppose, that

$$\delta(\partial_{0a}\phi) = (\partial_{0a}\delta\phi), \delta(\partial_{1a}\phi) = \partial_{1a}(\delta\phi), \dots, \delta(\partial_{ka}\phi) = \partial_{ka}(\delta\phi). \tag{2.32}$$

From (2.28) and (2.31) we get

$$\begin{aligned}\delta\mathcal{L} = {} & \frac{d\mathcal{L}}{d\phi}\delta\phi + \frac{d\mathcal{L}}{d(\partial_{0a}\phi)}\delta(\partial_{0a}\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_{1a}\phi)}\delta(\partial_{1a}\phi) + \dots + \\ & + \frac{\partial\mathcal{L}}{\partial(\partial_{ka}\phi)}\delta(\partial_{ka}\phi) = \\ & = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + A_0\delta(\partial_{0a}\phi) + A_1\delta(\partial_{1a}\phi) + \dots + A_k\delta(\partial_{ka}\phi) \\ & = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + A_0\partial_{0a}(\delta\phi) + A_1\partial_{1a}(\delta\phi) + \dots + A_k\partial_{ka}(\delta\phi) \\ & = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \partial_{0a}(A_0\delta\phi) + \partial_{1a}(A_1\delta\phi) + \dots + \partial_{ka}(A_k\delta\phi) \\ & - (\partial_{0a}A_0)\delta\phi - (\partial_{1a}A_1)\delta\phi - \dots - (\partial_{ka}A_k)\delta\phi.\end{aligned} \tag{2.33}$$

Using (2.24) and (2.25) we have

$$\begin{aligned}\int\limits_{\Omega}\partial_{0a}(A_0\delta\phi)d\omega &= \int\limits_{a^{1a}}^{b^{1a}} dy^{1a}\int\limits_{a^{2a}}^{b^{2a}} dy^{2a}\cdots\int\limits_{a^{ka}}^{b^{ka}} dy^{ka}\int\limits_{a^{0a}}^{b^{0a}}\partial_{0a}(A_0\delta\phi)dy^{0a} \\ &= \int\limits_{a^{1a}}^{b^{1a}} dy^{1a}\int\limits_{a^{2a}}^{b^{2a}} dy^{2a}\cdots\int\limits_{a^{ka}}^{b^{ka}} dy^{ka}(A_0\delta\phi)\Big|_{a^{0a}}^{b^{0a}} = 0.\end{aligned} \tag{2.34}$$

In the similar way we obtain

$$\int\limits_{\Omega}\partial_{1a}(A_1\delta\phi)d\omega = 0, \dots, \int\limits_{\Omega}\partial_{ka}(A_k\delta\phi)d\omega = 0. \tag{2.35}$$

From (2.33)–(2.35) we obtain

$$\delta I(\phi) = \int\limits_{\Omega}(\delta\mathcal{L})d\omega = \int\limits_{\Omega}\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_{0a}A_0 - \partial_{1a}A_1 - \dots - \partial_{ka}A_k\right)\delta\phi d\omega. \tag{2.36}$$

The extreme value of integral of action is obtained when $\delta\mathcal{L} = 0$. If we in (2.36) substitute $A_0, A_1, \ldots, A_k$ from (2.31) we get

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi} - (\partial_{0a} - N_{0a}^{1b}\partial_{1b} - \cdots - N_{0a}^{kb}\partial_{kb})\frac{\partial\mathcal{L}}{\partial(\delta_{0a}\phi)} \tag{2.37}$$

$$-(\partial_{1a} - N_{1a}^{2b}\partial_{2b} - \cdots - N_{1a}^{kb}\partial_{kb})\frac{\partial\mathcal{L}}{\partial(\delta_{1a}\phi)} - \cdots -$$

$$-(\partial_{ka})\frac{\partial\mathcal{L}}{\partial(\delta_{ka})} + B_0 + B_1 + B_2 + \cdots + B_{(k-1)},$$

where

$$B_0 = \frac{\partial\mathcal{L}}{\partial(\delta_{0b}\phi)}[\partial_{1a}N_{0b}^{1a} + \partial_{2a}N_{0b}^{2a} + \cdots + \partial_{ka}N_{0b}^{ka}]$$

$$B_1 = \frac{\partial\mathcal{L}}{\partial(\delta_{1b}\phi)}[\partial_{2a}N_{1b}^{2a} + \partial_{3a}N_{1b}^{3a} + \cdots + \partial_{ka}N_{1b}^{ka}]$$

$$B_2 = \frac{\partial\mathcal{L}}{\partial(\delta_{2b}\phi)}[\partial_{3a}N_{2b}^{3a} + \cdots + \partial_{ka}N_{2b}^{ka}]$$

$$B_{(k-1)} = \frac{\partial\mathcal{L}}{\partial(\delta_{(k-1)b}\phi)}\partial_{ka}N_{(k-1)b}^{ka}.$$

Using (1.23) we have

Theorem 2.2 *The extreme value of integral of action is obtained, when* $\delta\mathcal{L} = 0$, *i.e.*

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi} - \delta_{0a}\frac{\partial\mathcal{L}}{\partial(\delta_{0a}\phi)} - \delta_{1a}\frac{\partial\mathcal{L}}{\partial(\delta_{1a}\phi)} - \cdots - \delta_{ka}\frac{\partial\mathcal{L}}{\partial(\delta_{ka}\phi)} \tag{2.38}$$

$$+ B_0 + B_1 + B_2 + \cdots + B_{(k-1)} = 0,$$

In $Osc^k M$ we have

$$N_{Ab}^{(A+B)a} = N_{Ab}^{(A+B)a}(y^{0a}, y^{1a}, \ldots, y^{Ba}) \Rightarrow \tag{2.39}$$

$$B_1 = 0, B_2 = 0, \ldots, B_{k-1} = 0. \tag{2.40}$$

Proposition 2.3 *In* $Osc^k M$ *the integral of action has extreme value if*

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi} - \delta_{0a}\frac{\partial\mathcal{L}}{\partial(\delta_{0a}\phi)} - \delta_{1a}\frac{\partial\mathcal{L}}{\partial(\delta_{1a}\phi)} - \cdots - \delta_{ka}\frac{\partial\mathcal{L}}{\partial(\delta_{ka}\phi)} + B_0 = 0. \tag{2.41}$$

As from (2.19) we have $\mathcal{L} = \sqrt{g}L$ and $\sqrt{g}$ is not the function of $\phi, \partial_{0a}\phi, \partial_{1a}\phi, \ldots, \partial_{ka}\phi$, so from (2.38) we obtain

Theorem 2.3 *The Einstein-Yang-Mills equation for the gauge transformation of order k given by (1.2) expressed as function of the Lagrangian L and the metric function g is given by*

$$\begin{aligned}\delta\mathcal{L} = \sqrt{g}\Bigg[&\left(\frac{\partial}{\partial\phi} - \delta_{0a}\frac{\partial}{\partial(\delta_{0a}\phi)} - \delta_{1a}\frac{\partial}{\partial(\delta_{1a}\phi)} - \cdots - \delta_{ka}\frac{\partial}{\partial(\delta_{ka}\phi)}\right)L \\ &+ \frac{\partial L}{\partial(\delta_{0b}\phi)}[\partial_{1a}N^{1a}_{0b} + \partial_{2a}N^{2a}_{0b} + \cdots + \partial_{ka}N^{ka}_{0b}] \\ &+ \frac{\partial L}{\partial(\delta_{1b}\phi)}[\partial_{2a}N^{2a}_{1b} + \partial_{3a}N^{3a}_{1b} + \cdots + \partial_{ka}N^{ka}_{1b}] \\ &+ \frac{\partial L}{\partial(\delta_{2b}\phi)}[\partial_{3a}N^{3a}_{2b} + \cdots + \partial_{ka}N^{ka}_{2b}] + \cdots + \\ &+ \frac{\partial L}{\partial(\delta_{(k-1)b}\phi)}\partial_{ka}N^{ka}_{(k-1)b}\Bigg] \\ &- \left[\frac{\partial L}{\partial(\delta_{0a}\phi)}\delta_{0a} + \frac{\partial L}{\partial(\delta_{1a}\phi)}\delta_{1a} + \cdots + \frac{\partial L}{\partial(\delta_{ka}\phi)}\delta_{ka}\right]\sqrt{g} = 0.\end{aligned} \tag{2.42}$$

Proposition 2.4 *In Osc^kM the integral of action has extreme value if*

$$\begin{aligned}\delta\mathcal{L} = \sqrt{g}\Bigg[&\left(\frac{\partial L}{\partial\phi} - \delta_{0a}\frac{\partial L}{\partial(\delta_{0a}\phi)} - \delta_{1a}\frac{\partial L}{\partial(\delta_{1a}\phi)} - \cdots - \delta_{ka}\frac{\partial L}{\partial(\delta_{ka}\phi)}\right) \\ &+ \frac{\partial L}{\partial(\delta_{0b}\phi)}[\partial_{1a}N^{1a}_{0b} + \partial_{2a}N^{2a}_{0b} + \cdots + \partial_{ka}N^{ka}_{0b}]\Bigg] \\ &- \left[\frac{\partial L}{\partial(\delta_{0a}\phi)}\delta_{0a} + \frac{\partial L}{\partial(\delta_{1a}\phi)}\delta_{1a} + \cdots + \frac{\partial L}{\partial(\delta_{ka}\phi)}\delta_{ka}\right]\sqrt{g} = 0.\end{aligned} \tag{2.43}$$

Proof. The proof follows from Theorem 2.3 and equation 2.40.

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