CERTAIN CONTACT SUBMANIFOLDS OF COMPLEX SPACE FORMS *

MIRJANA DJORIĆ †
Faculty of Mathematics, University of Belgrade,
Studentski trg 16, pb. 550, 11000 Belgrade, Serbia
Email: mdjoric@matf.bg.ac.yu

MASAFUMI OKUMURA
5-25-25, Minami Ikuta, Tama-ku, Kawasaki, Japan
Email: mokumura@h8.dion.ne.jp

We study the differential geometric properties of n-dimensional real submanifolds M of complex space forms $\overline{M}$, whose maximal holomorphic tangent subspace is $(n-1)$-dimensional. On these manifolds there exists an almost contact structure F which is naturally induced from the ambient space. Using certain condition on the induced almost contact structure F and on the second fundamental form h of these submanifolds, which is sufficient for F to be the contact one, we give a classification of such submanifolds M and we obtain new characterizations of some model spaces in complex space forms.

1. Introduction

The study of real hypersurfaces of Kählerian manifolds has been an important subject in geometry of submanifolds, especially when the ambient space is a complex space form. One of the first results in this way (see [31]) was to state that any real hypersurface M of a complex space form $\overline{M}(c)$ with holomorphic sectional curvature $c \neq 0$ is not totally umbilical. This is a direct consequence of classical Codazzi's equation for such a hypersurface. On the other hand, Kon (resp. Montiel), in [17] (resp. [19]), stated that there are no Einstein real hypersurfaces in $\overline{M}(c)$ for $c > 0$ (resp.

* *MSC 2000:* 53C15, 53C40, 53B20.
Keywords: CR submanifold, complex space form, second fundamental form, almost contact metric manifold, contact metric manifold.
† Work supported by the Ministry of Science and Environmental Protection, project No *144032D*.

$c < 0$). Therefore, there exist problems in describing the differential geometric properties of hypersurfaces in complex space forms for $c \neq 0$. This situation has been improved since, for example, H. B. Lawson in [18] introduced the notion of "generalized equators" $M^C_{p,q}$ in a complex projective space, which naturally generalize the equatorial hypersurfaces of spheres. Following the idea of constructing a circle bundle over a real hypersurface, which is compatible with the Hopf fibration, he introduced the notion of $M^C_{p,q}$. Consequently, the study of hypersurfaces of complex space forms has involved finding sufficient conditions for a hypersurface to be one of the "standard examples". For example, Takagi in [27] and [28] classified connected complete real hypersurfaces in a complex projective space with two or three constant principal curvatures and Montiel in [19] gave a complete classification of the real hypersurfaces of complex hyperbolic space with at most two principal curvatures at each point. In this paper, in Section 4 we recall the construction of some of these hypersurfaces and in Sections 6 and 7 we derive new characterizations of these various examples.

Another important notion is that of (almost) contact manifold. A differential manifold M^n is said to be contact if it admits a linear functional η on the tangent bundle satisfying $\eta \wedge (d\eta)^{\frac{n-1}{2}} \neq 0$ (n is odd). The investigation of this as an intrinsic condition has received considerable study, see for example [2], [3]. In the case when M^n is a real hypersurface of an almost Hermitian manifold $\overline{M}$, the maximal holomorphic subspace is necessarily $(n-1)$–dimensional and M is equipped with an almost contact metric structure (φ, η, U) naturally induced by the almost Hermitian structure on $\overline{M}$. This fact was established by Tashiro in [29] and it was a fertile field for many authors ([6], [17], [26]). Moreover, it is natural to ask: when is a real hypersurface of a complex space form extrinsically contact? Such investigations have been carried out successfully for real hypersurfaces of complex Euclidean space ([22]), of complex projective space ([17]) and of complex hyperbolic space ([33]). Furthermore, in [22] the second author of this paper obtained one interesting algebraic condition for induced almost contact metric structure φ to be a contact metric structure, when the ambient space is a Kähler manifold: $A\varphi + \varphi A = 2\rho\varphi$, where A is the shape operator and ρ can be shown to be a constant.

However, for arbitrary codimension p, there are only a few recent results ([8], [11], [12], [24]). The purpose of the present paper is to generalize the problem by studying CR submanifolds of maximal CR dimension: if M^n is a real submanifold of the complex manifold $(\overline{M}^{n+p}, \overline{g})$ with complex structure J and the Hermitian metric $\overline{g}$, where $n > 1$, in [30] Tashiro showed that if the maximal holomorphic subspace of each tangent space of M^n is $(n-1)$-dimensional, the submanifold is necessarily odd-dimensional and it admits a naturally induced almost contact metric structure (F, u, U, g). Under this hypothesis, there exists a unit vector field ξ normal to M such that $JT_x(M) \subset T_x(M) \oplus span\{\xi_x\}$, for any $x \in M$ and M is called a CR submanifold of maximal CR dimension. Our purpose here is to study these submanifolds when $\overline{M}$ is a complex space form, which additionally satisfy the condition $h(FX, Y) - h(X, FY) = g(FX, Y)\eta$, $\eta \in T^{\perp}(M)$ on the naturally induced almost contact structure F and on the second fundamental form h. Hence we generalize the results which are valid for real hypersurfaces by giving a classification of such submanifolds. We also derive new characterizations of some spaces from the well-known Takagi's and Montiel's list. Moreover, recalling from [8] that the induced almost contact structure (F, u, U, g) is contact if and only if there exists function $\rho \neq 0$ which satisfies the relation $FA + AF = \rho F$, where A is the shape operator with respect to distinguished normal vector field ξ, it follows that the considered condition is also sufficient for M to be the contact manifold.

2. Almost contact metric manifold and contact metric manifold

A differentiable manifold M^{2m+1} is said to have an *almost contact structure* if it admits a (non-vanishing) vector field U (the so-called *characteristic vector field*), a one-form η and a $(1,1)$-tensor field φ (frequently considered as a field of endomorphisms on the tangent spaces at all points) satisfying

$$\eta(U) = 1, \quad \varphi^2 = -I + \eta \otimes U, \tag{1}$$

where I denotes the field of identity transformations of the tangent spaces at all points. These conditions imply that $\varphi U = 0$ and $\eta \circ \varphi = 0$, and that the endomorphism φ has rank $2m$ at every point in M. A manifold M, equipped with an almost contact structure (U, η, φ) is called an *almost contact manifold* and will be denoted by (M, U, η, φ).

Suppose that M^{2m+1} is a manifold carrying an almost contact structure. A Riemannian metric g on M satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all vector fields X and Y is called *compatible* with (or *associated* to) the almost contact structure, and (U, η, φ, g) is said to be an *almost contact metric structure* on M. It is known that an almost contact manifold always admits at least one compatible metric. Note that putting $Y = U$ in (2) yields

$$\eta(X) = g(X, U) \tag{3}$$

for all vector fields X tangent to M, which means that η is the metric dual of the characteristic vector field U. Further, substituting $X = U$ in (3), we see from (1) that U is a vector field of unit length.

It can be shown (see for example [2], [3], [25]) that the conditions (1) are equivalent to the fact that the structural group of the tangent bundle TM of M is reducible to $\mathcal{U}(n) \times 1$, i.e., that one can construct an open covering $\{U_\alpha\}_{\alpha \in I}$ of M^{2m+1}, together with orthonormal frame fields which transform in the intersections $U_\alpha \cap U_\beta$ by the action of $\mathcal{U}(n) \times 1$. (This alternative definition of almost contact manifolds was first given by J. Gray in [15].)

If (M, U, η, φ, g) is an almost contact metric manifold, we can define a two-form ϕ on M by

$$\phi(X, Y) = g(X, \varphi Y) \tag{4}$$

for all vector fields X and Y on M. This two-form ϕ is called the *fundamental two-form* or *Sasaki form* of (M, U, η, φ, g). The Sasaki form of any almost contact metric manifold satisfies

$$\eta \wedge \phi^m \neq 0. \tag{5}$$

It turns out that condition (5) is characteristic for almost contact manifolds. This yields a third definition of an almost contact manifold: a manifold M of dimension $2m+1$ carries an almost contact structure if and only if it admits a global one-form η and a global two-form ϕ satisfying (5) everywhere on M.

A manifold M^{2m+1} is said to be a *contact manifold* if it carries a global one-form η such that

$$\eta \wedge (d\eta)^m \neq 0 \tag{6}$$

everywhere on M. The one-form η is called the *contact form.* It is obvious from the above discussion that a contact manifold can always be equipped with an almost contact structure (U, η, φ). For a proof, we refer to [2] or [3].

Let g be a Riemannian metric on M which is compatible with this almost contact structure and define the Sasaki form ϕ as in (4). If ϕ satisfies the equation

$$\phi = d\eta, \tag{7}$$

then $(\xi, \eta, \varphi, g, \phi)$ is called a *contact metric structure* and $(M, \xi, \eta, \varphi, g, \phi)$ is a *contact metric manifold*. We note that there always exists such a compatible metric.

3. Real hypersurfaces in complex space forms and induced almost contact structure

Since in the present paper we use some results from the real hypersurface theory of complex space forms and moreover, as a real hypersurface of almost Hermitian manifold is a typical example of CR submanifolds of maximal CR dimension, in this section we review some fundamental definitions and necessary results on real hypersurfaces of complex space forms. For more details and proofs we refer to [7] and [20].

Let $\overline{M}(c)$ be a space of constant holomorphic sectional curvature $4c$ with complex dimension m (real dimension $2m$), with almost complex structure J and Levi-Civita connection $\overline{\nabla}$. For an immersed manifold $\imath : M^{2m-1} \to \overline{M}$, the Levi-Civita connection ∇ of the induced metric g and the shape operator A of the immersion are characterized respectively by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + g(AX, Y)\xi\,, \\ \overline{\nabla}_X \xi &= -AX, \end{aligned}$$

for a local choice of a unit normal ξ, where we omit to mention $\imath$, for brevity of notation. Let us define a skew symmetric $(1,1)$-tensor field φ from the tangential projection of J by

$$JX = \varphi X + g(X, U)\xi\,,$$

for any vector field X tangent to M, where we put $J\xi = -U$. Using this relation and the Hermitian property, it follows

$$\varphi^2 X = -X + g(X, U)U.$$

Moreover, it is easy to check that

$$g(\varphi X, \varphi Y) = g(X, Y) - g(X, U)g(Y, U), \quad \varphi U = 0.$$

Noting that $\varphi^2 = -I$ on $U^\perp = \{X \in TM : g(X, U) = 0\}$ we see that φ has rank $2m-2$ and that $ker\varphi = span\{U\}$. Such a φ determines an almost contact metric structure described in Section 2 (see [2], [3], [29] for more details) and $U^\perp$ is called the holomorphic distribution.

Of course, in general, one cannot expect that this induced almost contact metric structure (φ, ξ, η, g) is a contact metric structure. One of the conditions, when the ambient space is Kähler, was obtained by the second author of this paper in [22]:

Theorem 3.1 *Let M^{2m-1} be a hypersurface of a Kähler manifold $\overline{M}^{2m}$, (φ, ξ, η) its induced almost contact structure and A its Weingarten map. Then (φ, ξ, η) is a contact structure if and only if there exists a non-zero valued function ρ such that $A\varphi + \varphi A = 2\rho\varphi$.*

It can be shown that ρ is constant. Contact metric hypersurfaces are contact hypersurfaces on which $\rho = 1$.

In the next section we give a detailed construction of some important examples of contact hypersurfaces in complex space forms.

4. Certain examples of contact hypersurfaces in complex space forms

Since one of the purposes of this paper is to give new characterizations of some "model spaces" ("standard examples") of hypersurfaces in complex space forms, whose naturally induced almost contact structure is contact, in this section we explain their construction and we recall some properties of these spaces. These examples are so important that they have a standard nomenclature. In complex projective space they divide into five types, A-E, while complex hyperbolic space has just two types; and types are further subdivided. All of these examples are tubes of some sort and we will use these descriptive names for the purposes of identification, without a justification of these names (see [6], [20], [34] for more details). In complex projective space $\mathbb{C}P^m$, the list is as follows

($A1$) Geodesic spheres.

($A2$) Tubes over totally geodesic complex projective spaces $\mathbb{C}\,P^k$, where $1 \leq k \leq m-2$.

(B) Tubes over the complex quadrics.

(C) Tubes over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^l$ where $2l+1 = m$ and $m \geq 5$.

(D) Tubes over the Plücker embedding of the complex Grassmann manifold $G_{2,5}$, which occur only for $m = 9$.

(E) Tubes over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$, which occur only for $m = 15$.

This list consists precisely of the homogeneous real hypersurfaces in $\mathbb{C}P^m$ as determined by Takagi [26], and is often referred to as "Takagi's list".

Moreover, in [17] Kon proved

Theorem 4.1 *Let M^{2m-1} be a connected complete real hypersurface in complex projective space $\mathbb{C}\,P^m$, $m \geq 3$. If $\varphi A + A\varphi = k\varphi$ for some constant $k \neq 0$, then M is congruent to a geodesic sphere or to a tube over the complex quadric.*

Using Theorem 3.1, it follows that these hypersurfaces (Type $A1$ and Type B) are contact.

In complex hyperbolic space the list is as follows:

($A0$) Horospheres.

($A1$) Geodesic spheres and tubes over totally geodesic complex hyperbolic hyperplanes.

($A2$) Tubes over totally geodesic $\mathbb{C}H^k$, where $1 \leq k \leq m-2$.

(B) Tubes over totally real hyperbolic space $\mathbb{R}H^m$.

These hypersurfaces are homogeneous, but there is yet no classification theorem for homogeneous hypersurfaces in $\mathbb{C}H^m$. This classification was begun by S. Montiel [19] and we refer to the list as "Montiel's list".

Let us recall one of the classification theorems, proved by Vernon in [33], which gives a characterization of contact hypersurfaces in $\mathbb{C}H^m$:

Theorem 4.2 *Let M be a complete connected contact hypersurface of $\mathbb{C}H^m(-4)$, $m \geq 3$. Then M is congruent to one of the following:*

(i) A tube of radius $r > 0$ around a totally geodesic, totally real hyperbolic space form $H^m(-1)$;

(ii) A tube of radius $r > 0$ around a totally geodesic complex hyperbolic space form $\mathbb{C}H^{m-1}(-4)$;

(iii) A geodesic hypersphere of radius $r > 0$, or

(iv) A horosphere.

Since in Section 6 and Section 7 we will characterize certain subsets of these lists, we continue this section with the construction and some features of these spaces.

Complex projective space $\mathbb{C}P^m$ can be regarded as a projection from the sphere S^{2m+1} with the fibre S^1. H.B. Lawson ([18]) was first to exploit this idea to study a hypersurface in $\mathbb{C}P^m$ by lifting it to an S^1-invariant hypersurface of the sphere. Therefore, let us consider a Hermitian bilinear form on the complex vector space $\mathbb{C}^{m+1}$ given by

$$G(z, w) = \sum_{k=0}^{m} z_k \bar{w}_k$$

for all $z = (z_0, z_1, \dots, z_m)$ and $w = (w_0, w_1, \dots, w_m)$ in $\mathbb{C}^{m+1}$ and let $< z, w >$ be the real part of $G(z, w)$. Let then π be the canonical projection of the $(2m+1)$-sphere $S^{2m+1}(r)$ of radius $r > 0$, defined by

$$S^{2m+1}(r) = \{z \in \mathbb{C}^{m+1} | < z, z >= r^2\},$$

to complex projective space $\mathbb{C}P^m$, $\pi : S^{2m+1} \to \mathbb{C}P^m$. We now consider a hypersurface M in $\mathbb{C}P^m$. Then $M' = \pi^{-1}M$ is an S^1-invariant hypersurface in S^{2m+1}. More information about the geometry of hypersurfaces and their lifts and the relationship between them can be found in [20] and its references.

In particular, we now discuss the Type $A1$ and Type B hypersurfaces in complex projective space. Let r be a positive constant and $c = \frac{1}{r^2}$. Let us choose b so that $0 < b < r$ and

$$M' = \{z = (z_1, z_2) \in \mathbb{C}^{m+1} : G_1(z_1, z_1) = r^2 - b^2,\ G_2(z_2, z_2) = b^2\},$$

where G_1 and G_2 are the restrictions of G to $\mathbb{C}^{p+1}$ and $\mathbb{C}^{q+1}$, respectively, where $\mathbb{C}^{m+1} = \mathbb{C}^{p+1} \times \mathbb{C}^{q+1}$, $p, q \geq 0$ and $p + q = m - 1 > 0$. Then M' is the Cartesian product of spheres whose radii have been chosen so that M' lies in S^{2m+1}, i.e. $M'^{2m} = S^{2p+1}((r^2 - b^2)^{\frac{1}{2}}) \times S^{2q+1}(b)$. One can prove that $\pi M'$ is a hypersurface in $\mathbb{C}P^m$, denoted by $M_{2p+1,2q+1}$. If we write $b = r \sin u$, we can choose u so that $0 < u < \frac{\pi}{2}$. There is only one kind of Type $A1$ hypersurface since tubes over complex projective hyperplanes are also geodesic spheres. Namely, the geodesic spheres (Type $A1$) in complex projective space have two distinct principal curvatures: $\frac{1}{r}\cot u$ of multiplicity $2m-2$ and $\frac{2}{r}\cot 2u$ of multiplicity 1. The Type $A2$ hypersurface in complex projective space have three distinct principal curvatures: $-\frac{1}{r}\tan u$ of multiplicity $2p$, $\frac{1}{r}\cot u$ of multiplicity $2q$ and $\frac{2}{r}\cot 2u$ of multiplicity 1, where $p, q \geq 0$ and $p + q = m - 1 > 0$. Type B hypersurfaces, i.e. tubes around the complex quadric, form a one-parameter family and are defined as $M = \pi M'$ where

$$M' = \{z \in \mathbb{C}^{m+1} : \langle z, z \rangle = r^2,\ |G(z, \bar{z})|^2 = t\}.$$

Type B hypersurfaces are also tubes over totally geodesic real projective spaces $\mathbb{R}P^m$. The parameter u is chosen so that the tubes have radius ru. Then the tubes over the complex quadric have radius $r(\frac{\pi}{4} - u)$. Type B hypersurfaces in complex projective space have three distinct principal curvatures: $-\frac{1}{r}\cot u$ of multiplicity $m-1$, $\frac{1}{r}\tan u$ of multiplicity $m-1$ and $\frac{2}{r}\tan 2u$ of multiplicity 1.

We continue this section with examples of real hypersurfaces in complex hyperbolic space. For $z = (z_0, z_1, \ldots, z_m)$, $w = (w_0, w_1, \ldots, w_m)$, in $\mathbb{C}^{m+1}$, let us consider the Hermitian form F given by

$$F(z, w) = -z_0\bar{w}_0 + \sum_{k=1}^{m} z_k\bar{w}_k.$$

Then the inner product $< z, w >= \operatorname{Re} F(z, w)$ is an indefinite metric of index 2 on $\mathbb{C}^{m+1}$. The hypersurface H_1^{2m+1} defined by

$$H_1^{2m+1}(r) = \{z \in \mathbb{C}^{m+1} | < z, z >= -r^2\}$$

is the well-known anti-De Sitter space of radius r in $\mathbb{C}^{m+1}$ and we will denote it by $\mathbb{H}$. Further, we denote by $\mathbb{C}H^m$ the image of H_1^{2m+1} by the canonical projection π to complex projective space, $\pi : H_1^{2m+1}(r) \to \mathbb{C}H^m \subset \mathbb{C}P^m$. Thus, topologically, $\mathbb{C}H^m$ is an open subset of $\mathbb{C}P^m$. However, as Riemannian manifolds, they have quite different structures. It is well-known that H_1^{2m+1} is a principal S^1-bundle over $\mathbb{C}H^m$ with projection π.

Let us recall that if M is a hypersurface in $\mathbb{C}H^m$, then $M' = \pi^{-1}M$ is a S^1-invariant hypersurface in $\mathbb{H}$. Further, let r be a positive number and let the holomorphic curvature of $\mathbb{C}H^m$ be $4c = -\frac{4}{r^2}$. First, let us introduce the "horospheres", which form a one-parameter family, parametrized by $t > 0$. Since it can be verified that

$$M' = \{z \in \mathbb{C}^{m+1} : \langle z, z\rangle = -r^2,\ |z_0 - z_1| = t\}$$

is a Lorentz hypersurface of $\mathbb{H}$, then the *horosphere* (Type $A0$) is defined by $M = \pi M'$ and it is a hypersurface in $\mathbb{C}H^m$ and a routine calculation yields that it has two distinct principal curvatures: $\frac{1}{r}$ of multiplicity $2m-2$ and $\frac{2}{r}$ of multiplicity 1. Further, *tubes around complex hyperbolic spaces*, (Types $A1$, $A2$), also form a one-parameter family, parametrized initially by $b > 0$ and later by u. Let

$$M' = \{z = (z_1, z_2) \in \mathbb{C}^{m+1} : F_1(z_1, z_1) = -(b^2 + r^2),\ F_2(z_2, z_2) = b^2\},$$

where F_1 and F_2 are the restrictions of F to $\mathbb{C}^{p+1}$ and $\mathbb{C}^{q+1}$, respectively, where $\mathbb{C}^{m+1} = \mathbb{C}^{p+1}\times\mathbb{C}^{q+1}$, $p, q \geq 0$ and $p+q = m-1 > 0$. Then M' is the Cartesian product of an anti-de Sitter space and a sphere whose radii have been chosen so that M' lies in $\mathbb{H}$, i.e. $M'^{2m} = H_1^{2p+1}((b^2+r^2)^{\frac{1}{2}})\times S^{2q+1}(b)$. The hypersurface $\pi M'$ is denoted by $M_{2p+1,2q+1}$. Let $b = r\sinh u$. When $p = 0$, M is a geodesic hypersphere with principal curvatures $\frac{1}{r}\coth u$, of multiplicity $2m-2$, and $\alpha = \frac{2}{r}\coth 2u$ of multiplicity 1, where the radius of the sphere is ru. When $q = 0$, M is a tube of radius ru over a complex hyperbolic hyperplane with two principal curvatures $\frac{1}{r}\tanh u$, of multiplicity $2m-2$, and $\frac{2}{r}\coth 2u$ of multiplicity 1. These are the Type $A1$ hypersurfaces.

The Type $A2$ hypersurfaces in complex hyperbolic space have three distinct principal curvatures: $\frac{1}{r}\tanh u$ of multiplicity $2p$, $\frac{1}{r}\coth u$ of multiplicity $2q$ and $\frac{2}{r}\coth 2u$ of multiplicity 1. They are tubes of radius ru about complex hyperbolic spaces of codimension greater than 1.

Further, tubes around real hyperbolic space $\mathbb{R}H^m$, known as Type B hypersurfaces, form a one-parameter family, parametrized by $t > r^4$. Namely,

$$M' = \{z \in \mathbb{C}^{m+1} : \langle z, z\rangle = -r^2,\ |F(z, \bar{z})|^2 = t\}$$

is a hypersurface in $\mathbb{H}$ and it can be verified that $\pi M'$ is a hypersurface in $\mathbb{C}H^m$. A routine calculation yields that it has three principal curvatures: $\frac{1}{r}\coth u$ of multiplicity $m-1$, $\frac{1}{r}\tanh u$ of multiplicity $m-1$ and $\frac{2}{r}\tanh 2u$ of multiplicity 1.

Finally, let $\overline{M}$ be a Euclidean space $\mathbb{E}^{2m}$, i.e. a Kähler manifold with vanishing holomorphic sectional curvature and let us consider contact hypersurfaces of a Euclidean space. In [22] the second author of this paper showed that a hypersphere S^{2m-1} of $\mathbb{C}^m$ and the product manifold of a $(m-1)$ dimensional sphere S^{m-1} with a m-dimensional Euclidean space $\mathbb{E}^m$ are contact hypersurfaces in $\mathbb{E}^{2m}$. A natural question is to determine in which other cases the induced almost contact metric structure of hypersurfaces is contact metric and a complete classification is given in [22]. First, it was shown that a contact hypersurface of a Euclidean space admits at most two distinct principal curvatures. If the hypersurface admits only one principal curvature, it is a totally umbilical hypersurface and therefore it is a portion of a $(2m-1)$-dimensional sphere. The case with two distinct principal curvatures implies that the contact hypersurface is locally isometric with an open submanifold of $S^{m-1}\times\mathbb{E}^m$. Consequently, for a suitable orthonormal frame, the shape operators of these hypersurfaces have the following matrix representation: $diag(\lambda I_{2m-1})$, $diag(\lambda I_{m-1}, 0)$, respectively.

5. A class of submanifolds of an almost Hermitian manifold

In this section we recall some general preliminary facts concerning CR submanifolds.

Let $(\overline{M}, \bar{g}, J)$ be an $(n+p)$–dimensional almost Hermitian manifold and let M be a connected n–dimensional submanifold of $\overline{M}$ with induced metric g and $n > 1$. For $x \in M$ we denote by T_xM and $T_x^{\perp}M$ the tangent space and the normal space of M at x, respectively. Further, let the maximal J-invariant subspace of the tangent space $T_x(M)$ at $x \in M$, called the holomorphic tangent space at x, has constant dimension for any $x \in M$. Then the submanifold M is called the Cauchy-Riemann submanifold or briefly CR submanifold and the constant complex dimension of the holomorphic tangent space is called the CR dimension of M (see [21], [32]). Another definition of a CR submanifold was given by Bejancu in [1]: A submanifold M of $(\overline{M}, J)$ is called a CR submanifold if it is endowed with a pair of mutually orthogonal and complementary distributions $(\Delta, \Delta^\perp)$ such that for any $x \in M$ we have $J\Delta_x = \Delta_x$ and $J\Delta_x^\perp \subset T_x^\perp(M)$. It is easily seen that if M is a CR submanifold in the sense of Bejancu, M is also a CR submanifold in the sense of original definition. In the case when M is a CR submanifold of CR dimension $\frac{n-1}{2}$, these definitions coincide, $\dim \Delta^\perp = 1$ and M is called a CR submanifold of maximal CR dimension. On the other hand, see [9], when the CR dimension is less than $\frac{n-1}{2}$, the converse is wrong.

Let us mention here that the notion of a CR-manifold is also very important. The geometry of CR-manifolds goes back to Poincaré and received a great attention in works of É. Cartan, Tanaka, Moser, Chern and others. Let M be an n-dimensional C^∞ manifold and $T^{\mathbb{C}}M$ its complexified tangent bundle, i.e., $T_p^{\mathbb{C}}M = T_pM \times_{\mathbb{R}} \mathbb{C} \simeq T_pM \oplus iT_pM$. Let $\mathcal{H}$ be a C^∞ complex subbundle of complex dimension l. A CR-manifold, as introduced by Greenfield in [14], of real dimension n and CR-dimension l is a pair $(M, \mathcal{H})$ such that $\mathcal{H}_p \cap \overline{\mathcal{H}}_p = 0$ and $\mathcal{H}$ is involutive, i.e., for vector fields $X, Y \in \mathcal{H}$, $[X, Y] \in \mathcal{H}$. It is interesting to renew here the relation between CR-manifold and CR submanifold. Namely, let M be a submanifold of a Hermitian manifold $(\overline{M}, J, g)$, i.e. the almost complex structure J is integrable. Then the theorem of Blair and Chen in [4] states that if $\overline{M}$ is a Hermitian manifold and M a CR-submanifold, in the sense of Bejancu, then M is a CR-manifold.

Further, let M be a CR submanifold of CR dimension $\frac{n-1}{2}$, that is, at each point x of M the tangent space $T_x(M)$ satisfies $\dim(JT_x(M) \cap T_x(M)) = n-1$. It is easy to verify that the following spaces are examples of CR submanifolds of maximal CR dimension of complex manifolds $\overline{M}$:

-real hypersurfaces of almost Hermitian manifolds $\overline{M}$;

-real hypersurfaces M of complex submanifolds M' of almost Hermitian manifolds $\overline{M}$;

-F'-invariant submanifolds of real hypersurfaces M' of almost Hermitian manifolds $\overline{M}$, where F' is an almost contact metric structure naturally induced by the almost Hermitian structure on $\overline{M}$.

In the rest of the paper we discuss CR submanifolds of maximal CR dimension. This implies that M is odd–dimensional and that there exists a unit vector field ξ normal to M such that $JT_x(M) \subset T_x(M) \oplus \mathrm{span}\{\xi_x\}$, for any $x \in M$. Hence, for any $X \in T(M)$, choosing a local orthonormal basis $\xi, \xi_1, \ldots, \xi_{p-1}$ of vectors normal to M, we have the following decomposition into tangential and normal components:

$$J\imath X = \imath FX + u(X)\xi\,, \tag{8}$$

$$J\xi = -\imath U + P\xi, \tag{9}$$

$$J\xi_a = -\imath U_a + P\xi_a \qquad (a = 1, \ldots, p-1), \tag{10}$$

where F and P are skew–symmetric endomorphisms acting on the tangent bundle $T(M)$ and on the normal bundle $T^{\perp}(M)$, respectively, U, U_a, $a = 1, \ldots, p-1$ are tangent vector fields and u is one-form on M. Furthermore, using (8)- (10), the Hermitian property of J implies

$$g(U, X) = u(X)\,, \quad U_a = 0 \quad (a = 1, \ldots, p-1), \tag{11}$$

$$F^2 X = -X + u(X)U, \tag{12}$$

$$u(FX) = 0\,, \quad FU = 0\,, \quad P\xi = 0\,. \tag{13}$$

Hence, relations (9) and (10) may be written in the form

$$J\xi = -\imath U\,, \quad J\xi_a = P\xi_a \quad (a = 1, \ldots, p-1)\,. \tag{14}$$

Moreover, these relations imply that (F, u, U, g) defines an almost contact metric structure on M (see Section 2 and [2], [3], [30]).

Now, we suppose that the ambient manifold $\overline{M}$ is a Kähler manifold. As stated before, the submanifold M is odd–dimensional and say dim $M = n = 2l + 1$. If on M there exists a function ρ which takes a value zero nowhere, satisfying

$$du(X, Y) = \rho g(FX, Y), \tag{15}$$

for any tangent vector fields X, Y, that is, for the Kähler form ω of $\overline{M}$

$$du(X, Y) = \rho\omega(\imath X, \imath Y) = \rho(\omega \circ \imath)(X, Y)\,,$$

then, since F has rank $2l$, we easily obtain

$$u \wedge \underbrace{du \wedge \cdots \wedge du}_{l} \neq 0\,,$$

which shows that u is a contact form of M and M is a contact manifold. In this sense we call the submanifold M, whose induced almost contact structure (F, u, U, g) satisfies (15), a contact submanifold. From now on we suppose that the dimension of the contact submanifold M is greater than 3. Then, using (15), we conclude (see [8] and [22] for the hypersurface case) that the almost contact structure (F, u, U, g) is contact if and only if there exists a function ρ which takes a value zero nowhere and satisfies relation:

$$FA + AF = \rho F\,. \tag{16}$$

6. Certain condition and classification theorems

In this section we treat CR submanifolds M^n of maximal CR dimension of complex space forms $\overline{M}^{n+p}$. Using the certain condition on the naturally induced almost contact structure F and on the second fundamental form h of these submanifolds, we obtain new characterizations of model spaces discussed in section 4. Moreover, we prove that this condition is sufficient for F to be the contact structure and we determine all CR submanifolds which satisfy this condition. Namely, we study CR submanifolds M^n of CR dimension $\frac{n-1}{2}$ of complex space forms $\overline{M}^{n+p}$ under the assumption that

$$h(FX, Y) - h(X, FY) = g(FX, Y)\eta, \quad \eta \in T^{\perp}(M) \tag{17}$$

for all $X, Y \in T(M)$. Since F is a skew-symmetric endomorphism acting on $T(M)$, using relation $h(X, Y) = g(AX, Y)\xi + \sum_{a=1}^{p-1} g(A_a X, Y)\xi_a$ and setting

$$\eta = \rho\xi + \sum_{a=1}^{p-1} \rho^a \xi_a,$$

it follows that relation (17) is equivalent to

$$AFX + FAX = \rho FX, \tag{18}$$

$$A_a FX + FA_a X = \rho^a FX, \tag{19}$$

for all $a = 1, \dots, p-1$, where A, A_a are the shape operators for the normals ξ, ξ_a, respectively.

Using relation (16) it follows that CR submanifolds of maximal CR dimension of Kähler manifolds which satisfy the condition (17) are contact submanifolds. Moreover, in [8], Lemma 3.1., we proved that $\rho \neq 0$ is constant. In the rest of the paper we will assume that $\rho \neq 0$, since the case $\rho = 0$ reduces the condition (17) to $h(FX, Y) - h(X, FY) = 0$, which we considered in [10]. Further, it follows that if the condition (17) is satisfied, then $\rho^a = 0$, $a = 1, \dots, p-1$.

Using relations (12) and (13), it is easy to check that if the condition (18) is satisfied, then U is the eigenvector of the shape operator A with respect to distinguished normal vector field ξ, at any point of M. Moreover, if the ambient space is a complex Euclidean space and the condition (17) is satisfied, then this eigenvalue is constant.

Further, using Gauss and Weingarten formulae, Gauss equation, a routine, but long, calculation yields the following important information on the vector field ξ:

Lemma 6.1 *Let M be a complete CR submanifold of maximal CR dimension of a complex space form. If the condition (17) is satisfied, then the distinguished normal vector field ξ is parallel with respect to the normal connection.*

For the complete proof of this lemma we refer to [11] and [12]. Moreover, if $\rho \neq 2\lambda$, where λ is the eigenvalue corresponding to the eigenvector of the shape operator A, orthogonal to U, the condition (17) together with the fact that the distinguished normal vector field ξ is parallel with respect to the normal connection, implies

Lemma 6.2 *Let M be a complete n–dimensional CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex projective space (resp. complex Euclidean space). If the condition (17) is satisfied and $\rho \neq 2\lambda$, where λ is the eigenvalue corresponding to the eigenvector of the shape operator A, orthogonal to U, then $A_a = 0$, $a = 1, \ldots, p-1$, where A, A_a are the shape operators for the normals ξ, ξ_a, respectively.*

Therefore, it follows $FA_a = 0$, $a = 1, \ldots, p-1$ and the orthogonal complement of $JN_0(x) \cap N_0(x)$ in $T^\perp(M)$, where $N_0(x) = \{\xi \in T_x^\perp(M) | A_\xi = 0\}$, is spanned by ξ. Since ξ is parallel with respect to the normal connection, by Lemma 6.1, we can apply the codimension reduction theorem for real submanifolds of complex projective space, [23], (resp. complex Euclidean space, [13]) and conclude that there exists real $(n+1)$-dimensional totally geodesic complex projective subspace (resp. totally geodesic complex Euclidean space) of $\mathbb{C}P^{\frac{n+p}{2}}$ (resp. $\mathbb{C}^{\frac{n+p}{2}}$), such that M is its real hypersurface:

Theorem 6.1 *([11], [12]) Let M be a complete n–dimensional CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex projective space $\mathbb{C}P^{\frac{n+p}{2}}$ (resp. complex Euclidean space $\mathbb{C}^{\frac{n+p}{2}}$). If the condition (17) is satisfied and $\rho \neq 2\lambda$, where λ is the eigenvalue corresponding to the eigenvector of the shape* *operator A, orthogonal to U, then there exists a totally geodesic complex projective subspace $\mathbb{C}P^{\frac{n+1}{2}}$ of $\mathbb{C}P^{\frac{n+p}{2}}$ (complex Euclidean subspace $\mathbb{C}^{\frac{n+1}{2}}$ of $\mathbb{C}^{\frac{n+p}{2}}$) such that M is its real hypersurface.*

This codimension reduction theorem and the "classical" facts from the hypersurface theory are the main tools in the proof of our results, since, using Theorem 6.1, CR submanifolds M^n of maximal CR dimension of complex projective space $\mathbb{C}P^{\frac{n+p}{2}}$ (resp. complex Euclidean space $\mathbb{C}^{\frac{n+p}{2}}$) which satisfy the condition (17) for $\rho \neq 2\lambda$, can be regarded as real hypersurfaces of $\mathbb{C}P^{\frac{n+1}{2}}$ (resp. $\mathbb{C}^{\frac{n+1}{2}}$), which are totally geodesic. Therefore, by utilizing the well-known formulas for the immersion and using the corresponding results for real hypersurfaces of complex projective space (Theorem 4.1) and of complex Euclidean space (M. Okumura [22] and É. Cartan [5]), we conclude:

Theorem 6.2 *[11] Let M be a complete n–dimensional CR submanifold of maximal CR dimension of a complex projective space $\mathbb{C}P^{\frac{n+p}{2}}$. If the condition*

$$h(FX,Y) - h(X,FY) = g(FX,Y)\eta, \ \eta \in T^{\perp}(M)$$

is satisfied, where F and h are the induced almost contact structure and the second fundamental form of M, respectively, then F is a contact structure and M is congruent to a geodesic sphere or to a tube over the complex quadric, or there exists a geodesic hypersphere of $\mathbb{C}P^{\frac{n+p}{2}}$ such that M is its invariant submanifold.

Theorem 6.3 *[12] Let M be a complete n–dimensional CR submanifold of maximal CR dimension of a complex Euclidean space $\mathbb{C}^{\frac{n+p}{2}}$. If the condition*

$$h(FX,Y) - h(X,FY) = g(FX,Y)\eta, \ \eta \in T^{\perp}(M)$$

is satisfied, where F is the induced almost contact structure and h is the second fundamental form of M, then F is a contact structure and there exists a geodesic hypersphere $S^{n+p-1}(\frac{1}{|\alpha|})$ of $\mathbb{C}^{\frac{n+p}{2}}$ such that M is an invariant submanifold of S, or M is congruent to one of the following:

$$S^n, \ S^{\frac{n-1}{2}} \times \mathbb{E}^{\frac{n+1}{2}},$$

where S^n denotes an n-dimensional sphere and $\mathbb{E}^n$ is an n-dimensional Euclidean space.

For the proofs in the case when $\rho = 2\lambda$, we refer to [11] and [12].

7. CR submanifolds of complex hyperbolic space satisfying certain condition

In this section we continue our study by considering the case when M is a complete n–dimensional CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex hyperbolic space $\mathbb{C}H^{\frac{n+p}{2}}$ which satisfies the condition (17). We use the results obtained in [11] and [12], which are recalled in section 6: if the condition (17) is satisfied, then the naturally induced almost contact structure F is a contact structure, the distinguished normal vector field ξ is parallel with respect to the normal connection and U is an eigenvector of the shape operator A with respect to ξ, at any point of M. In this section we classify all such submanifolds of complex hyperbolic space which satisfy the condition (17), for $\rho \neq 2\lambda$, where λ is the eigenvalue corresponding to the eigenvector of the shape operator A, orthogonal to U. However, according to our knowledge, the case $\rho = 2\lambda$ is still unsolved.

Let M^n be a CR submanifold of CR dimension $\frac{n-1}{2}$ of complex hyperbolic space $\mathbb{C}H^{\frac{n+p}{2}}$ which satisfies the condition (17) and let $\rho \neq 2\lambda$. Using the same computations as in the case of complex projective space and complex Euclidean space, we conclude that $A_a = 0$, $a = 1, \ldots, p-1$, and hence, the following lemma holds:

Lemma 7.1 *Let M be a complete n–dimensional CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex hyperbolic space $\mathbb{C}H^{\frac{n+p}{2}}$. If the the condition (17) is satisfied and $\rho \neq 2\lambda$, where λ is the eigenvalue corresponding to the eigenvector of the shape operator A, orthogonal to U, then $A_a = 0$, $a = 1, \ldots, p-1$, where A_a, are the shape operators for the normals ξ_a.*

Making use of this result, we prove

Theorem 7.1 *Let M be a complete n–dimensional CR submanifold of maximal CR dimension of a complex hyperbolic space $\mathbb{C}H^{\frac{n+p}{2}}$. If the condition (17) is satisfied and $\rho \neq 2\lambda$, where λ is the eigenvalue corresponding to the eigenvector of the shape operator A, orthogonal to U, then there exists a totally geodesic complex hyperbolic subspace $\mathbb{C}H^{\frac{n+1}{2}}$ of $\mathbb{C}H^{\frac{n+p}{2}}$ such that M is a real hypersurface of $\mathbb{C}H^{\frac{n+1}{2}}$.*

Proof. For $N_0(x) = \{\xi \in T_x^{\perp}(M) | A_\xi = 0\}$, Lemma 7.1 implies that $N_0(x) = \operatorname{span}\{\xi_1(x), \ldots, \xi_{p-1}(x)\}$. If $H_0(x)$ denotes the maximal J-invariant subspace of $N_0(x)$, that is, $H_0(x) = JN_0(x) \cap N_0(x)$, by the second equation of (14), we obtain $JN_0(x) = N_0(x)$ and consequently, $H_0(x) = \text{span}\{\xi_1(x), \ldots, \xi_{p-1}(x)\}$. Hence the orthogonal complement $H_1(x)$ of $H_0(x)$ in $T^\perp(M)$ is spanned by ξ, which is parallel with respect to the normal connection. Therefore, we can apply the codimension reduction theorem for real submanifolds of complex hyperbolic space ([16]) and conclude that there exists real $(n+1)$-dimensional totally geodesic complex hyperbolic space $\mathbb{C}H^{\frac{n+1}{2}}$, such that M is its real hypersurface.

Further, using Theorem 7.1, the submanifold M can be regarded as a real hypersurface of $\mathbb{C}H^{\frac{n+1}{2}}$, which is totally geodesic submanifold in $\mathbb{C}H^{\frac{n+p}{2}}$. Denoting by $\imath_1$ the immersion of M into $\mathbb{C}H^{\frac{n+1}{2}}$, and by $\imath_2$ the totally geodesic immersion of $\mathbb{C}H^{\frac{n+1}{2}}$ into $\mathbb{C}H^{\frac{n+p}{2}}$, from the Gauss equation, it follows that

$$\nabla'_{\imath_1 X}\imath_1 Y = \imath_1\nabla_X Y + g(A'X, Y)\xi',$$

where A' is the corresponding shape operator and ξ' is a unit normal vector field to M in $\mathbb{C}H^{\frac{n+1}{2}}$. By composing the totally geodesic immersion $\imath_2$ with the immersion $\imath_1$ and using the Gauss equation we compute

$$\overline{\nabla}_{\imath_2\cdot\imath_1 X}\imath_2\cdot\imath_1 Y = \imath_2(\imath_1\nabla_X Y + g(A'X, Y)\xi'), \tag{20}$$

since $\mathbb{C}H^{\frac{n+1}{2}}$ is totally geodesic in $\mathbb{C}H^{\frac{n+p}{2}}$. Further, comparing relation (20) with the well-known Gauss formula, we conclude that $\xi = \imath_2\xi'$ and $A = A'$. Since $\mathbb{C}H^{\frac{n+1}{2}}$ is a complex submanifold of $\mathbb{C}H^{\frac{n+p}{2}}$, with the induced complex structure J', we have $J\imath_2 X' = \imath_2 J'X'$, $X' \in T(\mathbb{C}H^{\frac{n+1}{2}})$. Thus, from (8) it follows that

$$J\imath X = \imath_2 J'\imath_1 X = \imath F'X + \nu'(X)\imath_2\xi' = \imath F'X + \nu'(X)\xi$$

and therefore, we conclude that $F = F'$ and $\nu' = u$. As the condition (18) implies that F is a contact structure , M^n can be regarded as a contact hypersurface of $\mathbb{C}H^{\frac{n+1}{2}}$ and Theorem 4.2 (see also [33], pg. 221) implies the following classification:

Theorem 7.2 *Let M be a complete n–dimensional CR submanifold of maximal CR dimension of a complex hyperbolic space $\mathbb{C}H^{\frac{n+p}{2}}$. If the condition (17) is satisfied and $\rho \neq 2\lambda$, where λ is the eigenvalue corresponding to the eigenvector of the shape operator A, orthogonal to U, and F and h are the induced almost contact structure and the second fundamental form of M, respectively, then F is a contact structure and M is congruent to one of the following:*

(i) A tube of radius $r > 0$ around a totally geodesic, totally real hyperbolic space form $H^{\frac{n+1}{2}}(-1)$;

(ii) A tube of radius $r > 0$ *around a totally geodesic complex hyperbolic space form* $\mathbb{C}H^{\frac{n-1}{2}}(-4)$*;*

(iii) A geodesic hypersphere of radius $r > 0$*, or*

(iv) A horosphere.

References

1. A. Bejancu, *CR-submanifolds of a Kähler manifold I*, In: Proc. Amer. Math. Soc., **69** (1978), 135–142.
2. D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., **509** Springer, Berlin, (1976).
3. D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, **203**, Birkhauser Boston-Basel-Berlin, (2001).
4. D.E. Blair and B.Y. Chen, *On CR-submanifolds of Hermitian manifolds*, Israel J. Math., **34** (1979) 353–363.
5. É. Cartan, *Sur quelques familles remarquables d'hypersurfaces*, C. R. Congrés Math. Liége 30–41, (1939); Oeuvres completes Tom III. vol. 2, 1481-1492.
6. T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc., **269** (1982), 481–499.
7. B.Y. Chen, *Geometry of submanifolds*, Pure Appl. Math. **22**, Marcel Dekker, New York, 1973.
8. M. Djorić and M. Okumura, *On contact submanifolds in complex projective space*, Math. Nachr., **202** (1999), 17–28.
9. M. Djorić and M. Okumura, *CR submanifolds of maximal CR dimension in complex manifolds*, In: PDE's, Submanifolds and Affine Differential Geometry, Banach center publications, Institute of Mathematics, Polish Academy of Sciences, Warsawa 2002., **57** (2002), 89–99.
10. M. Djorić and M. Okumura, *CR submanifolds of maximal CR dimension in complex space forms and second fundamental form*, In: Proceedings of the Workshop Contemporary Geometry and Related Topics, Belgrade, May 15-21, 2002, (2004), 105–116.
11. M. Djorić and M. Okumura, *Certain condition on the second fundamental form of CR submanifolds of maximal CR dimension of complex projective space*, submitted.
12. M. Djorić, M. Okumura, *Certain condition on the second fundamental form of CR submanifolds of maximal CR dimension of complex Euclidean space*, Ann. Glob. Anal Geom, **30** (2006), 383–396.
13. J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differential Geometry, **5** (1971), 333–340.
14. S. Greenfield, *Cauchy-Riemann equations in several variables*, Ann. Sc. Norm. Sup. Pisa, **22**, (1968), 275–314.
15. J. Gray, *Some global properties of contact structures*, Ann. Math., **69** (1959), 421–450.
16. S. Kawamoto, *Codimension reduction for real submanifolds of complex hy-*

perbolic space, In: Revista Matematica de la Universidad Complutense de Madrid, **7** (1994), 119–128.
17. M. Kon, *Pseudo-Einstein real hypersurfaces in complex space forms*, J. Diff. Geom. **14** (1979), 339–354.
18. H. B. Lawson, Jr., *Rigidity theorems in rank-1 symmetric spaces*, J. Diff. Geom., **4**, (1970), 349–357.
19. S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan, **37** (1985), 515–535.
20. R. Niebergall and P.J. Ryan, *Real hypersurfaces in complex space forms*, In: Tight and taut submanifolds, *(eds. T.E. Cecil and S.-s. Chern), Math. Sciences Res. Inst. Publ. 32, Cambridge Univ. Press, Cambridge*, (1997), 233–305.
21. R. Nirenberg and R.O. Wells, Jr., *Approximation theorems on differentiable submanifolds of a complex manifold*, Trans. Amer. Math. Soc., **142** (1969), 15–35.
22. M. Okumura, *Contact hypersurfaces in certain Kaehlerian manifolds*, Tôhoku Math. J., **18** (1966), 74–102.
23. M. Okumura, *Codimension reduction problem for real submanifolds of complex projective space*, Colloq. Math. Soc. János Bolyai, **56** (1989), 574–585.
24. M. Okumura and L. Vanhecke, *A class of normal almost contact CR–submanifolds in* $\mathbb{C}^q$, Rend. Sem. Mat. Univ. Polotec. Torino, **52** (1994), 359–369.
25. S. Sasaki, *Almost contact manifolds* I, II, III, Lecture Notes, Mathematical Institute, Tôhoku University, 1965, 1967 and 1968.
26. R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math., **10** (1973) 495–506.
27. R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures*, J. Math. Soc. Japan, **27** (1975), 43–53.
28. R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures II*, J. Math. Soc. Japan, **27** (1975), 507–516.
29. Y.Tashiro, *On contact structure of hypersurfaces in complex manifold I*, Tôhoku Math. J., **15** (1963), 62–78.
30. Y. Tashiro, *Relations between almost complex spaces and almost contact spaces*, Sûgaku, **16** (1964), 34–61.
31. Y. Tashiro and S. Tachibana, *On Fubinian and C-Fubinian manifolds*, Kōdai Math.Sem.Rep., **15** (1963), 176–183.
32. A.E. Tumanov, *The geometry of CR manifolds*, In: Encyclopedia of Math. Sci. 9 VI, Several complex variables III, Springer-Verlag, (1986), 201–221.
33. M.H. Vernon, *Contact hypersurfaces of a complex hyperbolic space*, Tôhoku Math. J., **39** (1987), 215–222.
34. M.H. Vernon, *Some families of isoparametric hypersurfaces and rigidity in a complex hyperbolic space*, Trans. Amer. Math. Soc., **312** No. 1 (1989) 237–256.