BIFURCATIONAL DIAGRAM AND DISCRIMINANT OF COMPLETELY INTEGRABLE SYSTEM

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In first part of this work we consider the set of polynomial first integrals $f_1, \dots, f_n$, which defines the bifurcational diagram Σ, and offer method how to construct new polynomial integral Φ, such as $\Sigma \in \{\Phi = 0\}$. We prove that, if we take this new integral instead of the old one, new bifurcational diagram will become the union of the plain and some fictitious singularities. Also in this work an explicit form of Φ for some classical cases is given. This polynomial is closely related to Lax form of integrable system and the conjecture is that Φ determines discriminant set for spectral curve of corresponding Lax pair.

1. Bifurcational diagram

1.1. *Liouville integrability and the momentum mapping*

Consider $2n$-dimensional symplectic manifold (M^{2n}, ω) and Hamiltonian system $v = \mathrm{sgrad}\ H$ with smooth Hamiltonian H.

Hamiltonian system v is called *completely integrable*, if there exist n functionally independent commutating (up to Poisson bracket) smooth integrals $f_1, \dots, f_n$, such as corresponding vector fields $\mathrm{sgrad}\, f_i$ are complete.

Since $f_1, \dots, f_n$ are preserved by flow of vector field v, their simultaneous levels are invariant. Decomposition of M into connected components of simultaneous levels is called *Liouville foliation*, corresponding the given system.

Using this set of commutating integrals $f_1, \dots, f_n$, we can define *momentum mapping* $\mathcal{F} : M^{2n} \longrightarrow \mathbb{R}^n$,

$$\mathcal{F}(x) = (f_1(x), \dots, f_n(x)).$$

Denote by $K \subset M^{2n}$ the set of critical points $\mathcal{F}$. Then, *bifurcational* *diagram* of momentum mapping is the set

$$\Sigma = \mathcal{F}(K).$$

According to Arnold-Liouville theorem[1], with supplementary condition of compactness, preimage of point $\xi \in \mathbb{R}^n \backslash \Sigma$ with momentum mapping $\mathcal{F}$ is the disjoint union of several n-dimensional tori.

By the Sard lemma Σ forms the null subset in $\mathbb{R}^n$, which is usually the manifold with singularities. If $n = 2$, Σ consists of smooth curve segments and, probably, isolated points. Bifurcational diagram divides $\mathbb{R}^n$ into connected regions. Preimages of points of the same region are diffeomorphic to disjoint unions equal number of tori. Preimage of point $\xi \in \Sigma$ is the singular fiber, which contains tori's bifurcation.

1.2. *Canonical first integral*

Because of Poisson bracket properties, each function, depending on commutating integrals $f_1, \ldots, f_n$, also will be the first integral, commutating with every f_i. Thus, the set of commutating integrals, momentum mapping and the bifurcational diagram of Hamiltonian system are not uniquely defined. However, since Liouville foliation is determined by integral trajectories of vector field v, the change of set of first integrals preserves foliation structure. Hence, the set of critical fibers of foliations where tori's bifurcation's occur are also invariant. Conventionally all first integrals are the algebraic functions of phase variables, so let us choose the special one.

Definition 1.1 Polynomial integral Φ is called *canonical integral* of given system if it's zero-level surface contains all tori's bifurcations of corresponding Liouville foliation.

This definition of canonical integral is nonconstructive. Though, if we already know some set of commutating integrals and it's bifurcational diagram then we can reduce the problem of describing critical tori to searching the algebraic function which determines surface Σ in $\mathbb{R}^n$. In classical examples of integrable systems such as Euler, Lagrange, Kovalewskaya and Sretenskii cases explicit form of this function is given in paragraph 3.

If we take canonical integral as one of the set of integrals using for Liouville integrability then corresponding bifurcational diagram will take on special form, as described in the theorem below.

Theorem 1.1 *Bifurcational diagram of momentum mapping*

$$\mathcal{F}' = f_1 \times \ldots f_{n-1} \times \Phi : M^{2n} \longrightarrow \mathbb{R}^n(\eta)$$

defining with canonical integral Φ *decomposes into two parts: the subset of coordinate plane* $\Sigma_0 \subset \{\eta_n (= \Phi(x)) = 0\}$ *and additional surface* Σ_Φ*, which represents the envelope of certain family of surfaces (in polynomial case — the discriminant surface of certain polynomial) and corresponds to set of only "fictitious" singularities — folds of some order of Liouville foliation.*

Here we use

Definition 1.2 Point P is said to be a *fold of order* $k \geq 2$ of map $\tau : M_1^n \longrightarrow M_2^n$, if there exist coordinate systems in neighborhood of P and $\tau(P)$ such as map τ in these coordinates takes form

$$\tau(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, x_n^k).$$

We need this simplest generalization of classical fold singularity definition[2] to cover cases with order more then 2.

Thus, using constructed canonical integral Φ we can define canonical form for bifurcational diagram of all completely integrable systems.

1.3. *Discriminant and bifurcational diagram*

Another important property of canonical integral is closely related to Lax representation (paragraph 4). Using matrix form of differential equations parameterized by complex spectral parameter we can determine new algebraic set in the image of momentum mapping — *discriminant of spectral curve.*

Theorem 1.2 *For integrable systems in Lagrange and Kovalewskaya cases zero-level surface of computed canonical integral, considered as a polynomial in complex variables* $f_1, \ldots, f_n$*, coincides with discriminant of spectral curve for corresponding Lax representation.*

And using results about discriminants (see prp. 2.5.1 in M.Audin's book[3]), we can conclude that in these cases real part (i.e. a set of points which have nonempty preimage by momentum mapping) of zero surface is precisely the bifurcational diagram.

Y. Brailov in his paper[6] also studies the relative position of discriminant and bifurcational diagram for integrable system given by argument shift method on $sl(n, \mathbb{C})$ and shows that they are coincide. Also he discusses argument shift method on the basic series of compact semisimple Lie algebras and their direct sums. For minimal representations of such algebras the following holds:

Theorem 1.3 *(Brailov) Bifurcational diagram lies in the intersection* $\mathcal{D} \cap \mathcal{F}(\mathrm{g})$ *of discriminant* $\mathcal{D}$ *and image of momentum mapping* $\mathcal{F}(\mathrm{g})$.

However, the issue about nonemptyness of the set $(\mathcal{D} \setminus \Sigma) \cap \mathcal{F}(\mathrm{g})$ remains open.

Thus, the problem of defining minimal algebraic integral correlates with the question of interconnections between algebraic (using Lax's pairs) and integrable (using Arnold-Liouville theorem) approaches to Hamiltonian systems.

2. Proof of theorem 1.1

Let us give more precise formulation of the theorem 1.1.

We are given Liouville foliation, corresponding to the system $v = \operatorname{sgrad} H$, and the number of first integrals $\mathcal{A} = \{f_1{=}H, \dots, f_n\}$. Let $\mathcal{F}$ be it's momentum mapping. Denote by K the set of singularities and by Σ — bifurcational diagram of $\mathcal{F}$.

Suppose there is a smooth function $\phi(\xi) : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\Sigma \subset \{\phi(\xi) = 0\}$. Let us determine function

$$\Phi(x) = \phi(f_1(x), \dots, f_n(x)) : M^{2n} \longrightarrow \mathbb{R}\,.$$

Then new set of functions $\mathcal{A}' = \{f_1, \dots, f_{n-1}, \Phi\}$ satisfies complete integrability conditions, so Arnold-Liouville theorem can be applied. And there is another momentum mapping $\mathcal{F}'$, based on this set.

Denote by K' the set of critical points of $\mathcal{F}'$ mapping:

$$K' = \{x \in M^{2n} : df_1, \dots, df_{n-1}, d\Phi \text{ are linearly dependent}\}.$$

But

$$d\Phi = \frac{\partial \Phi}{\partial f_1}\, df_1 + \dots + \frac{\partial \Phi}{\partial f_n}\, df_n,$$

so

$$K' = K \cup K_\Phi,$$

where K is the set of critical points of original mapping $\mathcal{F}$, and where $K_\Phi = \{x : \dfrac{\partial \Phi}{\partial f_n}(x) = 0\}$. Bifurcational diagram of the new momentum mapping is a union

$$\Sigma' = \mathcal{F}'(K') = \mathcal{F}'(K) \cup \mathcal{F}'(K_\Phi) = \Sigma_0 \cup \Sigma_\Phi,$$

where $\Sigma_0 = \mathcal{F}'(K) \subset \{\eta : \eta_n = 0\}$.

Thus, theorem 1.1 is reduced to two statements.

Proposition 2.1 *Liouville foliation singularities, which corresponds to additional diagram Σ_Φ, are only folds of some order.*

Proof. Let $\eta \in \Sigma_\Phi$ $(\eta_n \neq 0)$, then

$$(\mathcal{F}')^{-1}(\eta) = \{x : f_1(x) = \eta_1, \dots, f_{n-1}(x) = \eta_{n-1}, \Phi(x) = \eta_n\} = \\ = \bigsqcup_{t \in \phi^{-1}(\eta_n)} \mathcal{F}^{-1}(\eta_1, \dots, \eta_{n-1}, t),$$

where ϕ is considered as function of only one variable:

$$\phi(\cdot) = \phi(\eta_1, \dots, \eta_{n-1}, \cdot).$$

For mentioned t point $(\eta_1, \dots, \eta_{n-1}, t)$ does not belong to Σ. So preimage of this point with mapping $\mathcal{F}$ is disjoint tori's union. According to Arnold-Liouville theorem[1] each torus has a neighborhood, where angle coordinates $\psi_1, \dots, \psi_n$ on torus are supplemented with coordinates $(f_1, \dots, f_n)$ up to nonsingular coordinate system. Using this coordinates mapping $\mathcal{F}'$ can be written as

$$\mathcal{F}' : M^{2n} \longrightarrow \mathbb{R}^n, \quad \mathcal{F}'(f_1, \dots, f_n, \psi_1, \dots \psi_n) = (f_1, \dots, f_{n-1}, \phi(f_1, \dots, f_n))$$

Denote by k the smallest natural number, such as

$$\left. \frac{\partial^k \phi}{(\partial f_n)^k} \right|_{(\eta_1, \dots, \eta_{n-1}, t)} \neq 0.$$

Then mapping $\mathcal{F}'$ is diffeomorphically equivalent, i.e. can be reduced using convenient change of basis, to the following:

$$\mathcal{G} : M^{2n} \longrightarrow \mathbb{R}^n, \quad \mathcal{G}(f_1, \dots, f_n, \psi_1, \dots \psi_n) = (f_1, \dots, f_{n-1}, f_n^k).$$

Hence critical point is a fold singularity of order k.
To formulate the second statement recall one algebraic definition.

Definition 2.1 *Discriminant* [4] of polynomial P in one variable x, $P(x) = a_m x^m + \dots + a_0$, is the function

$$\Delta_x(P) = \prod_{i<j} (x_i - x_j), \text{ where } x_i \text{ are the roots of polinomial } P.$$

Since it is a symmetric polynomial in roots P, discriminant can be written as a polynomial in terms of coefficients of P.

So the second part of theorem 1.1 is:

Proposition 2.2 *Additional bifurcation diagram Σ_Φ is the envelope of family of surfaces and, in algebraic case, represents discriminant of certain polynomial.*

Proof. Let us consider function

$$S_t(\eta) = \phi(\eta_1, \dots, \eta_{n-1}, t) - \eta_n.$$

And it follows from definition that $\eta \in \Sigma_\Phi$ if and only if

$$\begin{cases} S_t = 0, \\ \dfrac{\partial S_t}{\partial t} = 0. \end{cases}$$

These equations describe the envelope of parameterized family of surfaces $\Gamma_t = \{S_t(\eta) = 0\}$. Since S_t is polynomial in t, these conditions determine surface of multiple roots of function S_t as a polynomial in only one variable t.

Using definition 2.1 one can easily get that condition $\eta \in \Sigma_\Phi$ is equivalent to equality $\Delta_t(S_t(\eta)) = 0$, which is the equation on variables $\eta_1, \dots, \eta_n$.

Thus, in polynomial case additional diagram forms algebraic hypersurface in $\mathbb{R}^n(\eta)$.

3. Classical integrable cases

Examples of integrable systems came from mechanics. There is a standard method to reduce the law of motion of a rigid body to the Hamiltonian system on the manifold

$$TS^2 = \{ (S_1, S_2, S_3, R_1, R_2, R_3) \in \mathbb{R}^6 : \\ R_1^2 + R_2^2 + R_3^2 = 1, S_1R_1 + S_2R_2 + S_3R_3 = g\}.$$

For more details see, for example, Golubev's book[5]. In this 4-dimensional manifold we need only one additional integral K, commutating with given Hamiltonian H. Bifurcational diagrams of such cases are obtained (list of results is given in A. V. Bolsinov and A. T. Fomenko book[1]) and have sufficiently complicated structure — there are some curve segments on the plane $\mathbb{R}^2(h, k)$, given in parametrical form. How can we get ϕ?

3.1. *Method of calculations*

Suppose bifurcation diagram is given in parametric form

$$\Sigma = \{(h,k) \in \mathbb{R}^2 : h = P(t), k = Q(t)\},$$

where P and Q are the polynomials in t with degrees m and n respectively, and $m \geq n$. In fact we have to eliminate t from system of equations

$$\begin{cases} P_0(t,h,k) \stackrel{\text{def}}{=} P(t) - h = 0, \\ Q_0(t,h,k) \stackrel{\text{def}}{=} Q(t) - k = 0. \end{cases}$$

Let $P_0(t) = a_m^{(0)} t^m + \cdots + a_0^{(0)}$, $Q_0(t) = b_n^{(0)} t^n + \cdots + b_0^{(0)}$. Using transformations such as

$$P_1 = b_n^{(0)} P_0 - a_m^{(0)} t^{m-n} Q_0$$

we can decrease degrees of polynomials as long as we get polynomial P_r with degree 0 in t. With respect to variables h and k it is also the polynomial function. Since polynomial coefficients depend on h and k every transformation leads us to nonequivalent system. So set of solutions of equation $P_r(h,k) = 0$ contains not only bifurcational diagram. If we can decompose this set in union of algebraic varieties we have to reject unnecessary parts.

In some cases one can get required polynomial in more natural way, using some specific properties of given system.

3.2. *Examples*

3.2.1. *Euler case*

Hamiltonian function and additional integral are

$$H = \frac{S_1^2}{2A_1} + \frac{S_2^2}{2A_2} + \frac{S_3^2}{2A_3}, \quad K = S_1^2 + S_2^2 + S_3^2 \ , \quad \text{where } A_1 \geq A_2 \geq A_3.$$

Bifurcational diagram $\Sigma = \tau_0 \cup \tau_1 \cup \tau_2 \cup \tau_3$, where

$$\tau_0 = \{k = g^2, \frac{g^2}{2A_1} \leq h \leq \frac{g^2}{2A_3}\}, \qquad \tau_1 = \{k = 2A_1 h, k \geq g^2\},$$
$$\tau_2 = \{k = 2A_2 h, k \geq g^2\}, \qquad \tau_3 = \{k = 2A_3 h, k \geq g^2\}.$$

Since in this case bifurcational diagram contains parts of straight lines, minimal surface is given by the product of their equations.

$$\begin{aligned}\phi(h,k) &= (k-g^2)(k-2A_1h)(k-2A_2h)(k-2A_3h)\\ &= k^4 - 2(A_1+A_2+A_3)k^3h - g^2k^3 + 2g^2(A_1+A_2+A_3)k^2h\\ &\quad + 4(A_1A_2+A_2A_3+A_1A_3)k^2h^2 - 4g^2(A_1A_2+A_2A_3+A_1A_3)kh^2\\ &\quad - 8A_1A_2A_3kh^3 + 8g^2A_1A_2A_3h^3.\end{aligned}$$

3.2.2. *Lagrange top*

Hamiltonian and additional integral are

$$H = \frac{1}{2}(S_1^2 + S_2^2 + \frac{S_3^2}{\beta}) + R_3, \qquad K = S_3.$$

Bifurcational diagram Σ is the set

$$\{h = W_{g,k}(x) : W'_{g,k}(x) = 0, x \in (-1,1)\},$$
$$\text{where } W_{g,k}(x) = \frac{(g+k)^2}{4(1+x)} + \frac{(g-k)^2}{4(1-x)} + x + \frac{k^2(1-\beta)}{2\beta} \tag{1}$$

and two points $h_1 = \frac{g^2}{2\beta} + 1$, $k_1 = g$ and $h_2 = \frac{g^2}{2\beta} - 1$, $k_2 = -g$.

Condition (1) means the existence on interval $(-1,1)$ multiple root of function $h - W_{g,k}$ considered as function of one variable x. Let us consider instead of rational function the polynomial $V(h,k,x) = (1-x)(1+x)(h - W_{g,k})$. Existence of multiple root of this polynomial can be written in terms of discriminant.

Hence

$$\Sigma \subset \{(h,k) : \Delta_x(V(h,k,x)) = 0\}.$$

Δ is a polynomial, so we can take it for ϕ.

$$\begin{aligned}\phi_\beta(h,k) = \Delta_x(V) &= -k^8 - g^2k^6\beta + 8hk^6\beta + 3k^8\beta + 8k^4\beta^2 + 6g^2hk^4\beta^2\\ &\quad - 24h^2k^4\beta^2 + 20gk^5\beta^2 + 2g^2k^6\beta^2 - 18hk^6\beta^2 - 3k^8\beta^2 + 36g^2k^2\beta^3\\ &\quad - 32hk^2\beta^3 - 12g^2h^2k^2\beta^3 + 32h^3k^2\beta^3 + 18g^3k^3\beta^3 - 80ghk^3\beta^3\\ &\quad + 20k^4\beta^3 - 8g^2hk^4\beta^3 + 36h^2k^4\beta^3 - 22gk^5\beta^3 - g^2k^6\beta^3 + 12hk^6\beta^3\\ &\quad + k^8\beta^3 - 16\beta^4 + 27g^4\beta^4 - 72g^2h\beta^4 + 32h^2\beta^4 + 8g^2h^3\beta^4 - 16h^4\beta^4\\ &\quad + 48gk\beta^4 - 36g^3hk\beta^4 + 80gh^2k\beta^4 - 30g^2k^2\beta^4 - 40hk^2\beta^4\\ &\quad + 8g^2h^2k^2\beta^4 - 24h^3k^2\beta^4 - 2g^3k^3\beta^4 + 44ghk^3\beta^4 - k^4\beta^4\\ &\quad + 2g^2hk^4\beta^4 - 12h^2k^4\beta^4 + 2gk^5\beta^4 - 2hk^6\beta^4.\end{aligned}$$

The direct method described in 3.1 give us the same function.

3.2.3. *Kovalewskaya case*

Hamiltonian and additional integral are

$$H = \frac{1}{2}(S_1^2 + S_2^2 + 2S_3^2) + R_1, \qquad K = (\frac{S_1^2 - S_2^2}{2} - R_1)^2 + (S_1 S_2 - R_2)^2.$$

Bifurcational diagram $\Sigma = \tau_1 \cup \tau_2 \cup \tau_3$, where

$$\begin{aligned}
\tau_1 &= \{k = 0, h > g^2\}, \\
\tau_2 &= \{k = (h - g^2)^2, \frac{g^2}{2} - 1 \le h \le g^2 + \frac{1}{2g^2}\}, \\
\tau_3 &= \{k = 1 + tg + \frac{t^4}{4}, \, h = \frac{t^2}{2} - \frac{g}{t}, \, t \in (-\infty, 0) \cup (g, \infty)\}.
\end{aligned}$$

$$\begin{aligned}
\phi_g(h,k) = k \cdot (k - (h - g^2)^2) \cdot (&4 + 27g^4 - 36g^2 h + 8h^2 - 4g^2 h^3 + 4h^4 \\
&- 12k + 36g^2 hk - 16h^2 k - 4h^4 k + 12k^2 + 8h^2 k^2 - 4k^3).
\end{aligned}$$

3.2.4. *Sretenskii case*

Hamiltonian and additional integral are

$$H = \frac{1}{2}(S_1^2 + S_2^2 + 4(S_3 + \lambda)^2) + R_1, \qquad K = (S_3 + 2\lambda)(S_1^2 + S_2^2) - S_1 R_3.$$

Bifurcational diagram $\Sigma = \tau_1 \cup \tau_2 \cup \tau_3$, where

$$\begin{aligned}
\tau_1 &= \{k = 0, h \ge -1\}, \\
\tau_2 &= \{h = \frac{3t^2}{2} + 4\lambda t + 2\lambda^2 + 1, \, k = t^3 + 2\lambda t^2\}, \\
\tau_3 &= \{h = \frac{3t^2}{2} + 4\lambda t + 2\lambda^2 - 1, \, k = t^3 + 2\lambda t^2\}.
\end{aligned}$$

Required polynomial is a product $\phi = \phi_1 \cdot \phi_2 \cdot \phi_3$, where

$$\begin{aligned}
\phi_1 &= k, \\
\phi_2 &= -8 + 24h - 24h^2 + 8h^3 - 27k^2 + 72k\lambda - 72hk\lambda \\
&\quad - 32\lambda^2 + 64h\lambda^2 - 32h^2\lambda^2 + 16k\lambda^3 - 32\lambda^4 + 32h\lambda^4, \\
\phi_3 &= 8 + 24h + 24h^2 + 8h^3 - 27k^2 - 72k\lambda - 72hk\lambda \\
&\quad - 32\lambda^2 - 64h\lambda^2 - 32h^2\lambda^2 + 16k\lambda^3 + 32\lambda^4 + 32h\lambda^4.
\end{aligned}$$

Thus, we reduce problem of canonical representation of bifurcational diagram to choice of function Φ, such as it's zero level surface include's all singular fibers of given foliation. The supplementary condition for Φ is the polynomial form. These conditions do not determine unique function, so we have to find the "minimal" one (in some way).

4. Lax representation

There is another approach to study singularities of integrable systems. It also give us construction of algebraic set in the image of momentum mapping. It is based on the special form of equations.

4.1. *Lax equations*

Let's leave the integrability question for a while.

Definition 4.1 *Lax equations* is the system

$$\frac{d}{dt} A_\lambda = [A_\lambda, B_\lambda],$$

where A_λ and B_λ are matrices smoothly depend on *spectral parameter* λ, and $[\,\cdot\,,\cdot\,]$ is usual matrix bracket.

This equation is equivalent to existence of invertible matrix $U(t)$ such as A_λ can be written in form

$$A_\lambda(t) = U(t)A_\lambda(t_0)(U(t))^{-1}.$$

System like that has evident polynomial first integrals $f_1, \ldots, f_m$ — coefficients of characteristic polynomial A_λ or, for example, functions $\det A_\lambda$, $\det A_\lambda^2, \ldots$.

Consider characteristic polynomial $P(\lambda,\mu) = \det(A_\lambda - \mu E)$. For every point of origin configuration space equation $P(\lambda,\mu) = 0$ determine certain algebraic curve in $\mathbb{C}^2(\lambda,\mu)$, which can be extended for curve C on $\mathbf{CP}^1 \times \mathbf{CP}^1$. It's called *spectral curve* of given Lax equation. Coefficients of spectral curve are the first integrals of system, so each simultaneous level surface goes with only one curve. Let C_ξ be the spectral curve corresponding to surface $T_\xi = \{x : f_1(x) = \xi_1, \ldots, f_m(x) = \xi_m\}$.

Definition 4.2 *Discriminant* of spectral curve is the set

$$\mathcal{D} = \{\xi \in \mathbb{C}^m : C_\xi \text{ has singularities}\}.$$

Case of completely integrable systems

Now suppose system has both Lax pair and sufficient number of first integrals. In over words, the set of non-trivial integrals $f_1, \ldots, f_n$, obtained as coefficients of characteristic polynomial, satisfies conditions (1) – (4) of complete integrability and n equals to the half of phase space dimension. Then one can consider momentum mapping

$$\mathcal{F} = f_1 \times \cdots \times f_n : M^{2n} \longrightarrow \mathbb{R}^n.$$

There are two special sets in the image of momentum mapping:

$$\mathcal{D}_{\mathbb{R}} = \{\xi \in \mathbb{R}^n : \exists (\lambda, \mu) \in \mathbf{CP}^1 \times \mathbf{CP}^1,$$
$$P_\xi(\lambda,\mu) = \frac{\partial P_\xi}{\partial \lambda}(\lambda,\mu) = \frac{\partial P_\xi}{\partial \mu}(\lambda,\mu) = 0\}$$
$$\Sigma = \{\xi \in \mathbb{R}^n : \exists\, x \in \mathcal{F}^{-1}(\xi),\quad d_x f_1, \ldots, d_x f_n \text{ linearly dependent}\}.$$

We are interested in their positional relationship.

4.2. *Proof of Theorem 1.2*

The proof of the theorem in fact consists in direct computation respective discriminant curves and comparing them with results of paragraphs 3.2.2 and 3.2.3. Discriminants for both cases are computed by M. Audin, [3]. Let us consider, for example, Lagrange case.

Motion of symmetric top is described by the following system of equations on $\mathbb{R}^6((S_1, S_2, S_3, R_1, R_2, R_3))$

$$\begin{cases} \dot{\Gamma} = [\Gamma, \Omega\,], \\ \dot{M} = [M, \Omega\,] + [\Gamma, L\,], \end{cases} \qquad (*)$$

where

$\Gamma = (R_1, R_2, R_3)^T$ is a unit vertical vector, taken in moving frame,
$\mathcal{J}$ is the tensor of inertia, which is considered to have form
$\mathcal{J} = diag\{1, 1, \beta\}$,
$M = (-S_1, -S_2, -S_3)^T$ is angular momentum, which is equal to
$M = \mathcal{J}(\Omega)$,
L is constant vector, determines axis of the top, $L = (0, 0, 1)^T$.

It is the Hamiltonian system with Hamiltonian

$$H = \frac{1}{2}\, M \cdot \Omega + \Gamma \cdot L$$

which has on $M^4_{1,g} = \{R_1^2 + R_2^2 + R_3^2 = 1, S_1R_1 + S_2R_2 + S_3R_3 = g\}$ an additional integral

$$K = M \cdot L.$$

So it represents classical Lagrange case of integrability. But there exists also Lax pair.

Proposition 4.1 *(T. Ratiu, P. van Moerbeke, [7]) Equations* $(*)$ *are equivalent to the following:*

$$\dot{\overbrace{\Gamma \cdot \lambda^{-1} + M + L \cdot \lambda}} = [\Gamma \cdot \lambda^{-1} + M + L \cdot \lambda, \, \Omega + L \cdot \lambda].$$

Proof. It is sufficient to open the brackets and to compute coefficients of respective powers of λ.

From this form of equations we can get that spectral curve equation can be written as

$$P(\lambda, \mu) = \mu(\mu^2 + Q(\lambda)) = 0\,, \text{ where } Q(\lambda) = \|\Gamma \cdot \lambda^{-1} + M + L \cdot \lambda\|^2.$$

In terms of integrals $Q(\lambda)$ takes on form

$$Q(\lambda) = \lambda^{-2} + 2g\lambda^{-1} + (2h + (1 - \frac{1}{\beta})k^2) + 2k\lambda + \lambda^2.$$

Spectral curve equation determines curve on $\mathbb{C}P^1(\mu) \times \mathbb{C}P^1(\lambda)$ so one can shift exponents of λ^2. Discriminant of spectral curve $\mathcal{D}$ is the set of points $(h, k) \in \mathbb{C}^2$, such as for these values spectral curve has singularities. In our case spectral curve has singularities if and only if $\lambda^2 Q(\lambda)$ has multiple root. Using definition 2.1 for discriminant of one-variable polynomial one can find precisely the same function as we compute in 3.2.2, in other words $\mathcal{D} = \{\, \phi(h, k) = 0\}$, where ϕ is considering as a polynomial in complex variables.

5. Conclusion

Finally, polynomial, constructed using the set of arbitrary first integrals, serves as internal characteristic of Liouville foliation corresponding given system. It defines integral surface containing all critical tori's bifurcations. Bifurcational diagram form given by theorem 1.1 can be considered as it's canonical representation.

In most cases of completely integrable systems bifurcational diagram is defined as discriminant of some polynomial. For systems having Lax representation A. T. Fomenko and Y. Brailov set up a conjecture:

Conjecture 5.1 *(for 2-dimensional diagrams) The set of points of spectral curve discriminant such as they have nonempty pre-image is equal to bifurcational diagram.*

In Lagrange, Kovalewskaya and $sl(n, \mathbb{C})$ cases this statement is proved. There exists contrary instance for systems with three degrees of freedom. Though discriminant and diagram can be regarded as cell complexes and in terms of cells the statement takes on following form:

Conjecture 5.2 *(for an arbitrary dimension) Spectral curve discriminant coincide with bifurcational diagram in the cells of maximum dimension.*

Also the open question is what does canonical integral represent in case when integrable system does not allow Lax form.

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