NON–FORMAL COMPACT MANIFOLDS WITH SMALL BETTI NUMBERS *

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We show that, for any $k \geq 1$, there exist non-formal compact orientable $(k-1)$-connected n-manifolds with k-th Betti number $b_k = b \geq 0$ if and only if $n \geq \max\{4k-1, 4k+3-2b\}$.

1. Introduction

Simply connected compact manifolds of dimension less than or equal to 6 are formal [10]. Moreover $(k-1)$-connected compact orientable manifolds of dimension less than or equal to $4k-2$ are formal [9, 5], for any $k \geq 1$. To check that this bound is optimal, a method to construct non-formal simply connected compact manifolds of any dimension $n \geq 7$ was given in [6] (see also [11] for a previous construction of an example of dimension $n = 7$). Later Dranishnikov and Rudyak [4] extended the result to give examples of $(k-1)$-connected non-formal manifolds of dimension $n \geq 4k-1$.

For any $k \geq 1$, there is an alternative approach used by Cavalcanti in [2] to construct $(k-1)$-connected compact orientable non-formal manifolds,

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which gives examples with small Betti number b_k. If one focusses in the case $k=1$, in [7] it is studied how the smallness of b_1 may force the formality of a manifold. More concretely, in [7] it is proved that any compact orientable n-dimensional manifold with first Betti number $b_1 = b$ is formal if $n \leq \max\{2, 6-2b\}$, and that this bound is sharp, i.e., there are non-formal examples of compact orientable manifolds if: (a) $b_k = 0$ and $n \geq 7$; (b) $b_k = 1$ and $n \geq 5$; or (c) $b_k \geq 2$ and $n \geq 3$. The examples constructed there follow the lines of those of [6]. For the general case $k \geq 1$, Cavalcanti [2] proves that $(k-1)$-connected compact orientable n-manifolds with $b_k = 1$ are formal whenever $n \leq 4k$.

The natural *geography* question that arises in this situation is the following:

> For which (n,k,b) with $n,k \geq 1$, $b \geq 0$, are there compact orientable $(k-1)$-connected manifolds of dimension n and with $b_k = b$ which are non-formal?

(Note that the orientability condition is only relevant if $k=1$.) In this paper we solve completely the above problem by proving the following main result.

Theorem 1.1 *For any $k \geq 1$, there exist non-formal compact orientable $(k-1)$-connected n-manifolds with k-th Betti number $b_k = b \geq 0$ if and only if $n \geq \max\{4k-1, 4k+3-2b\}$.*

The above result can be restated as follows: let M be a compact orientable $(k-1)$-connected manifold of dimension n. Then M is formal if:

(a) $b_k = 0$ and $n \leq 4k+2$;
(b) $b_k = 1$ and $n \leq 4k$; or
(c) $b_k \geq 2$ and $n \leq 4k-2$.

In all other situations, namely

(a') $b_k = 0$ and $n \geq 4k+3$;
(b') $b_k = 1$ and $n \geq 4k+1$; or
(c') $b_k \geq 2$ and $n \geq 4k-1$;

there are non-formal examples.

The examples that we construct in this paper follow the lines of [4] (see also Example 5 in [5] where the same construction is used). Some (alternative) examples of compact orientable $(k-1)$-connected n-manifold in the cases (b') and (c') are given in [2], but the list is not exhaustive (for instance it does not cover the case $b_k = 1$ and $n = 4k+1$, see remark 6.1).

2. Minimal models and formality

We recall some definitions and results about minimal models [8, 3]. Let (A, d) be a *differential algebra*, that is, A is a graded commutative algebra over the real numbers, with a differential d which is a derivation, i.e. $d(a \cdot b) = (da) \cdot b + (-1)^{\deg(a)} a \cdot (db)$, where $\deg(a)$ is the degree of a. A differential algebra (A, d) is said to be *minimal* if:

(1) A is free as an algebra, that is, A is the free algebra $\bigwedge V$ over a graded vector space $V = \oplus V^i$, and

(2) there exists a collection of generators $\{a_\tau, \tau \in I\}$, for some well ordered index set I, such that $\deg(a_\mu) \leq \deg(a_\tau)$ if $\mu < \tau$ and each da_τ is expressed in terms of preceding a_μ $(\mu < \tau)$. This implies that da_τ does not have a linear part, i.e., it lives in $\bigwedge V^{>0} \cdot \bigwedge V^{>0} \subset \bigwedge V$.

We shall say that a minimal differential algebra $(\bigwedge V, d)$ is a *minimal model* for a connected differentiable manifold M if there exists a morphism of differential graded algebras $\rho \colon (\bigwedge V, d) \longrightarrow (\Omega M, d)$, where ΩM is the de Rham complex of differential forms on M, inducing an isomorphism $\rho^* \colon H^*(\bigwedge V) \longrightarrow H^*(\Omega M, d) = H^*(M)$ on cohomology.

If M is a simply connected manifold (or, more generally, if M is a nilpotent space, i.e., $\pi_1(M)$ is nilpotent and it acts nilpotently on $\pi_i(M)$ for $i \geq 2$), then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to V^i for any i. Halperin in [8] proved that any connected manifold M has a minimal model unique up to isomorphism, regardless of its fundamental group.

A minimal model $(\bigwedge V, d)$ of a manifold M is said to be *formal*, and M is said to be *formal*, if there is a morphism of differential algebras $\psi \colon (\bigwedge V, d) \longrightarrow (H^*(M), d = 0)$ that induces the identity on cohomology. An alternative way to look at this is the following: the above property means that $(\bigwedge V, d)$ is a minimal model of the differential algebra $(H^*(M), 0)$. Therefore $(\Omega M, d)$ and $(H^*(M), 0)$ share their minimal model, i.e., one can obtain the minimal model of M out of its real cohomology algebra. When M is nilpotent, the minimal model encodes its real homotopy type, so formality for M is equivalent to saying that its real homotopy type is determined by its real cohomology algebra.

In order to detect non-formality, we have Massey products. Let us recall its definition. Let M be a (not necessarily simply connected) manifold and let $a_i \in H^{p_i}(M)$, $1 \leq i \leq 3$, be three cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. Take forms α_i in M with $a_i = [\alpha_i]$ and write $\alpha_1 \wedge \alpha_2 = d\xi$,

$\alpha_2 \wedge \alpha_3 = d\eta$. The Massey product of the classes a_i is defined as

$$\begin{aligned}\langle a_1, a_2, a_3 \rangle &= [\alpha_1 \wedge \eta + (-1)^{p_1+1} \xi \wedge \alpha_3] \\ &\in \frac{H^{p_1+p_2+p_3-1}(M)}{a_1 \cup H^{p_2+p_3-1}(M) + H^{p_1+p_2-1}(M) \cup a_3} .\end{aligned}$$

We have the following result, for whose proof we refer to [3, 12].

Lemma 2.1 *If M has a non-trivial Massey product then M is non-formal.*

Therefore the existence of a non-zero Massey product is an obstruction to the formality.
In order to prove formality, we extract the following notion from [5].

Definition 2.1 Let $(\bigwedge V, d)$ be a minimal model of a differentiable manifold M. We say that $(\bigwedge V, d)$ is *s-formal*, or M is a *s-formal manifold* $(s \geq 0)$ if for each $i \leq s$ one can get a space of generators V^i of elements of degree i that decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:

(1) $d(C^i) = 0$,

(2) the differential map $d\colon N^i \longrightarrow \bigwedge V$ is injective,

(3) any closed element in the ideal $I_s = I(N^{\leq s})$, generated by $N^{\leq s}$ in $\bigwedge V^{\leq s}$, is exact in $\bigwedge V$.

The condition of s-formality is weaker than that of formality. However we have the following positive result proved in [5].

Theorem 2.1 *Let M be a connected and orientable compact differentiable manifold of dimension $2n$ or $(2n-1)$. Then M is formal if and only if is $(n-1)$-formal.*

This result is very useful because it allows us to check that a manifold M is formal by looking at its s-stage minimal model, that is, $\bigwedge V^{\leq s}$. In general, when computing the minimal model of M, after we pass the middle dimension, the number of generators starts to grow quite dramatically. This is due to the fact that Poincaré duality imposes that the Betti numbers do not grow and therefore there are a large number of cup products in cohomology vanishing, which must be killed in the minimal model by introducing elements in N^i, for i above the middle dimension. This makes Theorem 2.1 a very useful tool for checking formality in practice. For instance, we have the following results, whose proofs we include for completeness.

Theorem 2.2 *[9, 5] Let M be a $(k-1)$-connected compact orientable manifold of dimension less than or equal to $(4k-2)$, $k \geq 1$. Then M is formal.*

Proof. Since M is $(k-1)$-connected, a minimal model $(\bigwedge V, d)$ of M must satisfy $V^i = 0$ for $i \leq k-1$ and $V^k = C^k$ (i.e., $N^k = 0$). Therefore the first non-zero differential, being decomposable, must be $d : V^{2k-1} \longrightarrow V^k \cdot V^k$. This implies that $V^j = C^j$ (i.e., $N^j = 0$) for $k \leq j \leq (2k-2)$. Hence M is $2(k-1)$-formal. Now, using Theorem 2.1 we have that M is formal. □

Theorem 2.3 *[2] Let M be a $(k-1)$-connected compact manifold of dimension less than or equal to $4k$, $k > 1$, with $b_k = 1$. Then M is formal.*

Proof. The minimal model $(\bigwedge V, d)$ of M has $V^{<k} = 0$ and $V^k = C^k = \langle \xi \rangle$ one-dimensional, generated by a closed element ξ. The first non-zero differential is $d : V^{2k-1} \longrightarrow V^k \cdot V^k$. This implies that $N^j = 0$ for $k \leq j \leq (2k-2)$. Thus M is $(2k-2)$-formal. Now if k is odd, then $\xi \cdot \xi = 0$, so $N^{2k-1} = 0$. Hence M is $(2k-1)$-formal and, since $n \leq 4k$, Theorem 2.1 gives the formality of M.

If k is even, then either $N^{2k-1} = 0$ and M is formal as above, or $N^{2k-1} = \langle \eta \rangle$ with $d\eta = \xi^2$. In this case if $z \in I(N^{2k-1})$ is closed, write $z = \eta z_1$, $z_1 \in \bigwedge(C^{\leq (2k-1)})$, and $0 = dz = \xi^2 z_1$ which implies $z_1 = 0$ and hence $z = 0$. Therefore M is $(2k-1)$-formal and, applying Theorem 2.1 again, formal. □

Regarding the Massey products, we have the following refinement of Lemma 2.1, which follows from Lemma 2.9 in [5].

Lemma 2.2 *Let M be an s-formal manifold. Suppose that there are three cohomology classes $\alpha_i \in H^{p_i}(M)$, $1 \leq i \leq 3$, such that the Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is defined. If $p_1 + p_2 \leq s+1$ and $p_2 + p_3 \leq s+1$, then $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ vanishes.*

3. Non-formal examples with $b_k = 1$

Let $k \geq 1$. In this section, we are going to give a construction of a $(k-1)$-connected compact orientable non-formal manifold with $b_k = 1$ of any dimension $n \geq 4k+1$. The examples that we are going to construct follow the pattern of those in [4] (see also Example 5 in [5]).

Consider a wedge of two spheres $S^k \vee S^{k+1} \subset \mathbb{R}^{k+2}$. Let $a \in \pi_k(S^k \vee S^{k+1})$ be the image of the generator of $\pi_k(S^k)$ by the inclusion of $S^k \hookrightarrow S^k \vee S^{k+1}$

and let $b \in \pi_{k+1}(S^k \vee S^{k+1})$ be the image of the generator of $\pi_{k+1}(S^{k+1})$ by the inclusion of $S^{k+1} \hookrightarrow S^k \vee S^{k+1}$. The iterated Whitehead product $\gamma = [a,[a,b]] \in \pi_{3k-1}(S^k \vee S^{k+1})$ yields a map $\gamma : S^{3k-1} \longrightarrow S^k \vee S^{k+1}$. Let

$$C_k = (S^k \vee S^{k+1}) \cup_\gamma D^{3k}$$

be the mapping cone of γ, where D^{3k} is the $3k$-dimensional disk. Clearly C_k is $(k-1)$-connected and its homology groups are

$$\begin{cases} H_i(C_k) &= 0, \qquad 1 \le i \le k-1, \\ H_k(C_k) &= \langle a \rangle, \\ H_{k+1}(C_k) &= \langle b \rangle, \\ H_i(C_k) &= 0, \qquad k+2 \le i \le 3k-1, \\ H_{3k}(C_k) &= \langle c \rangle, \end{cases}$$

where a, b denote the images in $H_*(C_k)$ of the respective elements a, b under the map $\pi_*(S^k \vee S^{k+1}) \longrightarrow \pi_*(C_k) \longrightarrow H_*(C_k)$, where the last map is the Hurewicz homomorphism, and c is the element represented by the attached $3k$-cell to form C_k from the wedge of the two spheres.
Let $\tilde{a} \in H^k(C_k)$, $\tilde{b} \in H^{k+1}(C_k)$ and $\tilde{c} \in H^{3k}(C_k)$ be the cohomology classes dual to a, b, c, respectively. Let us describe the minimal model of C_k.

- Suppose $k \ge 3$ and odd. By degree reasons, all cup products in $H^*(C_k)$ are zero. Note that $\pi_{3k-1}(S^k \vee S^{k+1})$ is one-dimensional and generated by $\gamma = [a,[a,b]]$. Therefore $\pi_{3k-1}(C_k) = 0$, since γ contracts in C_k. The minimal model of C_k is thus

$$\begin{aligned} &(\textstyle\bigwedge(\langle \alpha, \beta, \eta, \xi \rangle \oplus V^{\ge 3k}), d), \\ &|\alpha| = k,\ |\beta| = k+1,\ |\eta| = 2k,\ |\xi| = 2k+1, \\ &d\alpha = d\beta = 0,\ d\eta = \alpha\,\beta,\ d\xi = \beta^2, \end{aligned}$$

  where $V^{\ge 3k}$ is the space generated by the generators of degree bigger than or equal to $3k$, $\tilde{a} = [\alpha]$, $\tilde{b} = [\beta]$. Note that $\alpha\,\eta$ is a closed non-exact element, i.e., we can write $\tilde{c} = [\alpha\,\eta]$. The triple Massey product

$$\langle \tilde{a}, \tilde{a}, \tilde{b} \rangle = \tilde{c} \ne 0 \in \frac{H^{3k}(C_k)}{[\alpha] \cup H^{2k}(C_k) + H^{2k-1}(C_k) \cup [\beta]} = H^{3k}(C_k).$$

  Therefore by Lemma 2.1, C_k is non-formal.
- Suppose k is even. In particular $k \ge 2$ and hence all cup products in $H^*(C_k)$ are zero. Again $\pi_{3k-1}(S^k \vee S^{k+1})$ is one-dimensional

and generated by $\gamma = [a,[a,b]]$, hence $\pi_{3k-1}(C_k) = 0$. The minimal model of C_k is thus

$$\begin{aligned}
&(\bigwedge(\langle \alpha, \beta, \eta, \xi \rangle \oplus V^{\geq 3k}), d),\\
&|\alpha| = k,\ |\beta| = k+1,\ |\eta| = 2k-1,\ |\xi| = 2k,\\
&d\alpha = d\beta = 0,\ d\eta = \alpha^2,\ d\xi = \alpha\,\beta,
\end{aligned}$$

where $\tilde{a} = [\alpha]$, $\tilde{b} = [\beta]$. Now $\alpha\,\xi + \beta\,\eta$ is a closed non-exact element of degree $3k$. So we can write $\tilde{c} = [\alpha\,\xi + \beta\,\eta]$. The triple Massey product

$$\langle \tilde{a}, \tilde{a}, \tilde{b} \rangle = \tilde{c} \neq 0 \in \frac{H^{3k}(C_k)}{[\alpha] \cup H^{2k}(C_k) + H^{2k-1}(C_k) \cup [\beta]} = H^{3k}(C_k),$$

and by Lemma 2.1, C_k is non-formal.

- The case $k = 1$ is slightly different. Here $\pi_2(S^1 \vee S^2)$ is infinitely generated by b, $[a,b]$, $[a,[a,b]]$, $[a,[a,[a,b]]], \ldots$ Therefore $\pi_2(C_1)$ is 2-dimensional generated by $b, [a,b]$. So the minimal model of C_1 is

$$\begin{aligned}
&(\bigwedge(\langle \alpha, \beta, \eta \rangle \oplus V^{\geq 3}), d),\\
&|\alpha| = 1,\ |\beta| = 2,\ |\eta| = 2,\\
&d\alpha = d\beta = 0,\ d\eta = \alpha\,\beta.
\end{aligned}$$

  Here $\tilde{c} = [\alpha\ \eta]$ and we have the non-vanishing triple Massey product

$$\langle \tilde{a}, \tilde{a}, \tilde{b} \rangle = \tilde{c} \neq 0 \in \frac{H^{3}(C_1)}{[\alpha] \cup H^{2}(C_1) + H^{1}(C_1) \cup [\beta]} = H^{3}(C_1),$$

  proving non-formality of C_1.

Now we aim to construct a differentiable compact manifold using C_k. For this, we use Corollary 2 in [4] to obtain a PL embedding $C_k \subset \mathbb{R}^{3k+(k+2)} = \mathbb{R}^{4k+2}$. Let W_k be a closed regular neighborhood of C_k in $\mathbb{R}^{4k+2}$ and let $Z_k = \partial W_k$ be its boundary. We can arrange easily that Z_k is a *smooth* manifold of dimension $4k+1$.

Theorem 3.1 *Suppose $k \geq 2$. Then Z_k is a $(k-1)$-connected compact (orientable) non-formal $(4k+1)$-dimensional manifold with $b_k(Z_k) = 1$.*

Proof. Suppose $k \geq 1$ by now. The first observation is that $\pi_i(Z_k) \cong \pi_i(W_k - C_k) \cong \pi_i(W_k) \cong \pi_i(C_k) = 0$, for $1 \leq i \leq k-1$, where the first and last isomorphism are by retraction deformation, and the middle one because C_k has codimension $k+2$ in W_k. Therefore Z_k is $(k-1)$-connected. Moreover $\pi_k(Z_k) \cong \pi_k(C_k) \cong \mathbb{Z}$, hence $b_k(Z_k) = 1$.

Now let us see that Z_k is non-formal. For this, we need to compute its cohomology. There is a long exact sequence

$$\cdots \longrightarrow H^i(W_k, Z_k) \xrightarrow{j^*} H^i(W_k) \xrightarrow{i^*} H^i(Z_k) \xrightarrow{\partial^*} H^{i+1}(W_k, Z_k) \longrightarrow \cdots,$$

where $i : Z_k \to W_k$ and $j : W_k \to (W_k, Z_k)$ are the inclusions. Using that $H^*(C_k) \cong H^*(W_k)$ and $H^*(W_k, Z_k) \cong H_{4k+2-*}(W_k) \cong H_{4k+2-*}(C_k)$, the first isomorphism by Poincaré duality, we rewrite the above sequence as

$$\cdots \longrightarrow H_{4k+2-i}(C_k) \xrightarrow{j^*} H^i(C_k) \xrightarrow{i^*} H^i(Z_k) \xrightarrow{\partial^*} H_{4k+1-i}(C_k) \longrightarrow \cdots$$

From this it follows easily that i^* is always injective. The only non-trivial case is $k = 1$ where there is a map $H_3(C_1) \longrightarrow H^3(C_1)$, but this is an antisymmetric map between rank one spaces, hence the zero map.

We deduce the following cohomology groups for any $k \geq 1$:

$$\begin{cases} H^i(Z_k) & = 0, \qquad 1 \leq i \leq k-1, \\ H^k(Z_k) & = \langle \breve{a} \rangle, \\ H^{k+1}(Z_k) & = \langle \breve{b}, \hat{c} \rangle, \\ H^i(Z_k) & = 0, \qquad k+2 \leq i \leq 3k-1, \\ H^{3k}(Z_k) & = \langle \breve{c}, \hat{b} \rangle, \\ H^{3k+1}(Z_k) & = \langle \hat{a} \rangle, \\ H^i(Z_k) & = 0, \qquad 3k+2 \leq i \leq 4k, \\ H^{4k+1}(Z_k) & = \langle [Z_k] \rangle, \end{cases}$$

where $\breve{a}, \breve{b}, \breve{c}$ denote the images of $\tilde{a}, \tilde{b}, \tilde{c} \in H^*(C_k)$ under i^*, and $\hat{a}, \hat{b}, \hat{c}$ denote the preimages of $\tilde{a}, \tilde{b}, \tilde{c} \in H^*(C_k)$ under ∂^*, and $[Z_k]$ is the fundamental class of Z_k.

Now suppose that $k \geq 2$. Let us see that Z_k has a non-vanishing Massey product. As $i^* : H^*(C_k) \longrightarrow H^*(Z_k)$ is injective, there is a Massey product $\langle i^*\tilde{a}, i^*\tilde{a}, i^*\tilde{b} \rangle = i^*\tilde{c}$ and this is non-zero in

$$\frac{H^{3k}(Z_k)}{\breve{a} \cup H^{2k}(Z_k) + H^{2k-1}(Z_k) \cup \breve{b}} = H^{3k}(Z_k),$$

using that $H^{2k}(Z_k) = 0$ and $H^{2k-1}(Z_k) = 0$ for $k \geq 2$ (this is the only place where the assumption $k \geq 2$ is used). This completes the proof. □

For constructing higher dimensional examples, we may use Lemma 6.2 below, but a more direct way is available, as follows.

Theorem 3.2 *Let $k \geq 1$. There are $(k-1)$-connected compact non-formal n-dimensional manifolds $Z_{k,n}$ with $b_k(Z_{k,n}) = 1$, for any $n \geq 4k+1$, with $n \neq 5k$.*

Proof. For any $n \geq 4k+1$, embed $C_k \subset \mathbb{R}^{4k+2} \subset \mathbb{R}^{n+1}$. Take a tubular neighborhood $W_{k,n}$ of C_k in $\mathbb{R}^{n+1}$ and let $Z_{k,n} = \partial W_{k,n}$ be its boundary. The same argument as in the proof of Theorem 3.1 proves that $Z_{k,n}$ is a $(k-1)$-connected compact orientable manifold with $b_k(Z_{k,n}) = 1$. Let us see that it is non-formal by checking that it has a non-vanishing triple Massey product.

Let us see that the map $i^* : H^*(C_k) \cong H^*(W_{k,n}) \longrightarrow H^*(Z_{k,n})$ is injective. We have a commutative diagram

$$\begin{array}{ccc} H^*(W_{k,n}) & \xrightarrow{i^*} & H^*(Z_{k,n}) \\ \downarrow\cong & & \downarrow \\ H^*(W_k) & \xrightarrow{i^*} & H^*(Z_k). \end{array}$$

Since the bottom row is injective by the proof of Theorem 3.1, the top one is also. Thus there is a Massey product $\langle i^*\tilde{a}, i^*\tilde{a}, i^*\tilde{b}\rangle = i^*\tilde{c}$.

Now assume $n \neq 5k$, and let us prove that $i^*\tilde{a} \cup H^{2k}(Z_{k,n}) + H^{2k-1}(Z_{k,n}) \cup i^*\tilde{b} = 0 \subset H^{3k}(Z_{k,n})$. As in the proof of Theorem 3.1, we have an exact sequence

$$\cdots \longrightarrow H_{n+1-i}(C_k) \xrightarrow{j^*} H^i(C_k) \xrightarrow{i^*} H^i(Z_{k,n}) \xrightarrow{\partial^*} H_{n-i}(C_k) \longrightarrow \cdots$$

Then $H^{2k}(Z_{k,n}) = i^*H^{2k}(C_k)$ since $H_{n-2k}(C_k) = 0$. Then $i^*\tilde{a} \cup H^{2k}(Z_{k,n}) = 0$. Analogously $H^{2k-1}(Z_{k,n}) \cup i^*\tilde{b} = 0$, unless $n = 5k-1$. If $n = 5k-1$ (in particular, $k \geq 2$) then $H^{2k-1}(Z_{k,n}) = (\partial^*)^{-1}H_{3k}(C_k)$, generated by an element $\hat{c}$. Now $i^*\tilde{b} \cup \hat{c} = 0 \in H^{3k}(Z_{k,n})$, by Poincaré duality since $i^*\tilde{b} \cup \hat{c} \cup \hat{c} = 0$ (as $\hat{c}$ has odd degree). So $H^{2k-1}(Z_{k,n}) \cup i^*\tilde{b} = 0$.
It follows that $i^*\tilde{a} \cup H^{2k}(Z_{k,n}) + H^{2k-1}(Z_{k,n}) \cup i^*\tilde{b} = 0$ and so the above Massey product is non-zero in

$$\frac{H^{3k}(Z_{k,n})}{i^*\tilde{a} \cup H^{2k}(Z_{k,n}) + H^{2k-1}(Z_{k,n}) \cup i^*\tilde{b}} = H^{3k}(Z_{k,n}),$$

which completes the proof. □

4. Nilpotency of the constructed examples

In section 3 we provide with some examples of non-simply connected manifolds, the manifolds $Z_{1,n}$, $n \geq 5$. As we are studying a rational homotopy property, it is a natural question whether or not they are nilpotent spaces. Here we collect a rather non-conclusive collection of remarks on this question.

The fundamental group of $Z_{1,n}$ is $\pi_1(Z_{1,n}) = \langle a \rangle \cong \mathbb{Z}$, abelian. To check nilpotency of $Z_{1,n}$, need to describe the action of a on the higher homotopy groups $\pi_i(Z_{1,n})$. For instance, suppose $n \geq 6$ (so we already know that $Z_{1,n}$ is non-formal by Theorem 3.2). Then $\pi_2(Z_{1,n}) \cong \pi_2(C_1)$. Since $[a,[a,b]] = 0 \in \pi_2(C_1)$, the action of a on this homotopy group is nilpotent. In general, the nilpotency of the action of a on higher homotopy groups $\pi_i(Z_{1,n})$, $i > 2$ reduces to two issues:

- The nilpotency of C_1. It is not clear whether the action of a on the higher homotopy groups $\pi_i(C_1)$, $i > 2$, is nilpotent, although this seems to be the case. Let us do the case $i = 3$.
  The Quillen model [12] of C_1 is $(\mathbb{L}(a,b,c), \partial)$, where $\mathbb{L} = \mathbb{L}(a,b,c)$ is the free Lie algebra generated by elements a, b, c of degrees $0, 1, 2$ and with $\partial a = 0$, $\partial b = 0$, $\partial c = [a,[a,b]]$. Then $\pi_i(C_1) \cong H^{i-1}(\mathbb{L}(a,b,c), \partial)$. Consider the map $p : \mathbb{L} \longrightarrow \mathbb{L}$, $p(x) = [a,x]$ of degree 0. This is a derivation. Moreover, a basis for the elements of degree 2 in $\mathbb{L}$ is $\{p^j(c), p^j([b,b]); j \geq 0\}$. Let $z = \sum \lambda_j p^j(c) + \mu_j p^j([b,b])$ be a closed element, defining a homology class $\bar{z} \in H^2(\mathbb{L}, \partial) = \pi_3(C_1)$. Hence $0 = \partial z = \sum \lambda_j p^{j+2}(c)$, so $\lambda_j = 0$ for all $j \geq 0$. So we can write $\bar{z} = \overline{\sum \mu_j p^j [b,b]}$. The map p descends to homology and $p^2(\bar{b}) = \overline{p^2(b)} = \overline{\partial c} = 0$. As p is a derivation, $p^3\left(\overline{[b,b]}\right) = 0$. Hence $p : \pi_3(C) \longrightarrow \pi_3(C)$ is nilpotent.
- Knowing the nilpotency of C_1, prove the nilpotency of $Z_{1,n}$. For $i \leq n-4$, $\pi_i(Z_{1,n}) \cong \pi_i(C_1)$. So nilpotency of the action of a on $\pi_i(Z_{1,n})$ would follow from the corresponding statement for C_1. Now suppose $i = n-3$. Then as $Z_{1,n}$ is of the homotopy type of $W_{1,n} - C_1 \subset \mathbb{R}^{n+1}$, we have that $\pi_{n-3}(Z_{1,n})$ is generated by $\pi_{n-3}(C_1)$ and the S^{n-3}-fiber f of the projection $Z_{1,n} \longrightarrow C_1$ over a smooth point of C_1. It is not clear that the action of a on f is nilpotent. Even more difficult is the case $i > n-3$.

Remark 4.1 In [7] it is claimed that the examples constructed there of non-formal manifolds with $b_1 = 1$ are not nilpotent (see Section 5 in [7]). In Lemma 9 in [7] it is proved that in our circumstances the action of any non-zero element $a \in \pi_1(M)$ on $\pi_2(M)$ is not trivial. This action is given as $h_a : \pi_2(M) \longrightarrow \pi_2(M)$, $h_a(A) = [a,A] + A = p(A) + A$. Therefore $h^k(A) \neq A$ for any $A \neq 0$, $k \geq 1$. However this does not mean that the action is not nilpotent, since nilpotency means that $p^N = (h - Id)^N = 0$, for some large N. It may happen that the examples of [7] are nilpotent, though the authors do not know the answer.

5. Non-formal examples with $b_k = 2$

Let $k \geq 1$. In this section, we are going to give a construction of a $(k-1)$-connected compact orientable non-formal manifold with $b_k = 2$ of any dimension $n \geq 4k$, by modifying slightly the construction in Section 3.

Consider now a wedge of two spheres $S^k \vee S^k \subset \mathbb{R}^{k+1}$. Let $a, b \in \pi_k(S^k \vee S^k)$ be the image of the generators of $\pi_k(S^k)$ by the inclusions of $S^k \hookrightarrow S^k \vee S^k$ as the first and second factors, respectively. The iterated Whitehead product $\gamma = [a, [a, b]] \in \pi_{3k-2}(S^k \vee S^k)$ yields a map $\gamma : S^{3k-2} \longrightarrow S^k \vee S^k$ and a $(3k-1)$-dimensional CW-complex

$$C'_k = (S^k \vee S^k) \cup_\gamma D^{3k-1}.$$

Clearly C'_k is $(k-1)$-connected and the only non-zero homology groups are

$$\begin{cases} H_k(C'_k) & = \langle a, b \rangle, \\ H_{3k-1}(C'_k) & = \langle c \rangle, \end{cases}$$

where a, b denote the images of the elements a, b under the map $\pi_*(S^k \vee S^k) \longrightarrow \pi_*(C'_k) \longrightarrow H_*(C'_k)$, and c is the element represented by the attached $(3k-1)$-cell to form C'_k from the wedge of the two spheres. Let $\tilde{a}, \tilde{b} \in H^k(C'_k)$ and $\tilde{c} \in H^{3k-1}(C'_k)$ be the cohomology classes dual to a, b, c, respectively. Then it is easy to see, as before, that the triple Massey product

$$\langle \tilde{a}, \tilde{a}, \tilde{b} \rangle = \tilde{c} \neq 0 \in \frac{H^{3k-1}(C'_k)}{\tilde{a} \cup H^{2k-1}(C'_k) + H^{2k-1}(C'_k) \cup \tilde{b}} = H^{3k-1}(C'_k).$$

Therefore by Lemma 2.1, C'_k is non-formal.

Now we take a PL embedding $C'_k \subset \mathbb{R}^{3k-1+(k+1)} = \mathbb{R}^{4k} \subset \mathbb{R}^{n+1}$, let $W'_{k,n}$ be a closed regular neighborhood of C'_k in $\mathbb{R}^{n+1}$. Let $Z'_{k,n} = \partial W'_{k,n}$ be its boundary. We can arrange easily that $Z'_{k,n}$ is a smooth manifold of dimension n.

Theorem 5.1 *$Z'_{k,n}$ is a $(k-1)$-connected compact orientable non-formal n-dimensional manifold with $b_k(Z'_{k,n}) = 2$, for any $n \geq 4k$.*

Proof. Note that the codimension of C'_k in $W'_{k,n}$ is $n + 1 - (3k-1) \geq k+2$. Therefore $\pi_i(Z'_{k,n}) \cong \pi_i(W'_{k,n} - C'_k) \cong \pi_i(W'_{k,n}) \cong \pi_i(C'_k) = 0$, for $1 \leq i \leq k-1$, so $Z'_{k,n}$ is $(k-1)$-connected. Also $\pi_k(Z'_{k,n}) \cong \pi_k(C'_k)$, so $b_k(Z'_{k,n}) = b_k(C'_k) = 2$ (this is the reason for the necessity of the condition $n \geq 4k$).

Now let us see that $Z'_{k,n}$ is non-formal. For this, we need to compute its cohomology. There is a long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^i(W'_{k,n}, Z'_{k,n}) &\xrightarrow{j^*} H^i(W'_{k,n}) \xrightarrow{i^*} H^i(Z'_{k,n}) \\ &\xrightarrow{\partial^*} H^{i+1}(W'_{k,n}, Z'_{k,n}) \longrightarrow \cdots, \end{aligned}$$

where $i : Z'_{k,n} \longrightarrow W'_{k,n}$ and $j : W'_{k,n} \longrightarrow (W'_{k,n}, Z'_{k,n})$ are the inclusions. Use that $H^*(C'_k) \cong H^*(W'_{k,n})$ and $H^*(W'_{k,n}, Z'_{k,n}) \cong H_{n+1-*}(W'_{k,n}) \cong H_{n+1-*}(C'_k)$. Hence we rewrite the above sequence as

$$\cdots \longrightarrow H_{n+1-i}(C'_k) \xrightarrow{j^*} H^i(C'_k) \xrightarrow{i^*} H^i(Z'_{k,n}) \xrightarrow{\partial^*} H_{n-i}(C'_k) \longrightarrow \cdots$$

The map $i^* : H^*(C'_k) \cong H^*(W'_{k,n}) \longrightarrow H^*(Z'_{k,n})$ is injective. In the case $n = 4k$, $H^*(W'_{k,4k}) \longrightarrow H^*(Z'_{k,4k})$ is injective, because $H_{4k+1-i}(C'_k) = 0$ for $i = k, 3k-1$. Now for $n \geq 4k$, we have a commutative diagram

$$\begin{array}{ccc} H^*(W_{k,n}) & \xrightarrow{i^*} & H^*(Z_{k,n}) \\ \downarrow\cong & & \downarrow \\ H^*(W_{k,4k}) & \xrightarrow{i^*} & H^*(Z_{k,4k}), \end{array}$$

where the bottom row is injective, hence the top one is also. This proves the injectivity of i^*. Thus $H^k(Z'_{k,n}) = \langle i^*\tilde{a}, i^*\tilde{b}\rangle$ and there is a well-defined Massey product $\langle i^*\tilde{a}, i^*\tilde{a}, i^*\tilde{b}\rangle = i^*\tilde{c}$.

Finally let us see that $i^*\tilde{a} \cup H^{2k-1}(Z'_{k,n}) + H^{2k-1}(Z'_{k,n}) \cup i^*\tilde{b} = 0 \subset H^{3k-1}(Z'_{k,n})$. First suppose $n \neq 5k-2$. Then $H^{2k-1}(Z'_{k,n}) = i^*H^{2k-1}(C'_k)$, since $H_{n-2k+1}(C'_k) = 0$. So $i^*\tilde{a} \cup H^{2k-1}(Z'_{k,n}) = i^*\tilde{a} \cup i^*H^{2k-1}(C'_k) = 0$. Analogously, $H^{2k-1}(Z'_{k,n}) \cup i^*\tilde{b} = 0$.

The remaining case is $n = 5k-2$. Then $k > 1$ since $n \geq 4k$. So $H^{2k-1}(Z'_{k,n}) = (\partial^*)^{-1}H_{3k-1}(C'_k)$. This is generated by an element $\hat{c}$. Now $i^*a \cup \hat{c} = 0 \in H^{3k-1}(Z'_{k,n})$, using Poincaré duality and $i^*a \cup \hat{c} \cup \hat{c} = 0$ (the degree of $\hat{c}$ is odd). So $i^*\tilde{a} \cup H^{2k-1}(Z'_{k,n}) = 0$. Analogously, $H^{2k-1}(Z'_{k,n}) \cup \tilde{b} = 0$.

This implies that the Massey product $\langle i^*\tilde{a}, i^*\tilde{a}, i^*\tilde{b}\rangle = i^*\tilde{c}$ is non-zero in

$$\frac{H^{3k-1}(Z'_{k,n})}{i^*\tilde{a} \cup H^{2k-1}(Z'_{k,n}) + H^{2k-1}(Z'_{k,n}) \cup i^*\tilde{b}} = H^{3k-1}(Z'_{k,n}).$$

The proof is complete. $\square$

6. Proof of Theorem 1.1

We start with some elementary lemmata.

Lemma 6.1 *If M has a non-trivial Massey product and N is any smooth manifold, then $M\#N$ has a non-trivial Massey product.*

Proof. Let $\langle a_1, a_2, a_3\rangle$ be a non-zero Massey product on M, $a_i \in H^{p_i}(M)$, $1 \le i \le 3$. Since $p_i > 0$, it is easy to arrange that $a_i = [\alpha_i]$, $1 \le i \le 3$, $\alpha_1 \wedge \alpha_2 = d\xi$ and $\alpha_2 \wedge \alpha_3 = d\eta$ where α_i, $1 \le i \le 3$, ξ and η are forms vanishing on a given disc in M (see [1]). Using this disk for performing the connected sum, we see that we can define the forms α_i, $1 \le i \le 3$, ξ and η in $M\#N$ by extending by zero. Let $a_i' = [\alpha_i] \in H^{p_i}(M\#N)$ be the cohomology classes thus defined. It follows easily that

$$\langle a_1', a_2', a_3'\rangle = \langle a_1, a_2, a_3\rangle \in \frac{H^{p_1+p_2+p_3-1}(M)}{a_1 \cup H^{p_2+p_3-1}(M) + H^{p_1+p_2-1}(M) \cup a_3} \subset$$
$$\subset \frac{H^{p_1+p_2+p_3-1}(M\#N)}{a_1' \cup H^{p_2+p_3-1}(M\#N) + H^{p_1+p_2-1}(M\#N) \cup a_3'}$$

is non-zero. □

Lemma 6.2 *Let M be an n-dimensional manifold with a non-trivial Massey product $\langle a_1, a_2, a_3\rangle$, $a_i \in H^{p_i}(M)$, $1 \le i \le 3$, with $p_1 + p_2 + p_3 < n$. Then the $(n+1)$-dimensional manifold $N = (M \times S^1)\#_{S^1} S^{n+1}$ has a non-trivial Massey product. Moreover, if M is $(k-1)$-connected then so is N and $b_k(N) = b_k(M)$.*

Proof. Note that

$$N = (M \times S^1)\#_{S^1} S^{n+1} = ((M - D^n) \times S^1) \cup_{S^{n-1}\times S^1} (S^{n-1} \times D^2),$$

as there is only one way to embed S^1 in S^{n+1} since $n \ge 3$; otherwise M cannot have non-trivial Massey products. Any cohomology class $a \in H^*(M)$ of positive degree has a representative vanishing on the disc D^n. Therefore it defines in a natural way a cohomology class on N, giving a map $H^*(M) \longrightarrow H^*(N)$. A Mayer-Vietoris argument gives that the cohomology of N is $H^k(N) = H^k(M)$, $k = 0, 1$, $H^k(N) = H^k(M) \oplus H^{k-1}(M) \cdot [\eta]$, $2 \le k \le n-1$, $H^k(N) = H^{k-1}(M) \cdot [\eta]$, $k = n, n+1$ (where $[\eta]$ is the generator of $H^1(S^1)$). From this it follows easily the last sentence of the statement.

Now write $a_i = [\alpha_i]$, $1 \le i \le 3$, with $\alpha_1 \wedge \alpha_2 = d\xi$ and $\alpha_2 \wedge \alpha_3 = d\mu$. We arrange that α_i, $1 \le i \le 3$, ξ and μ are forms on M which vanish on the given disc $D^n \subset M$. This yields a Massey product $\langle a_1', a_2', a_3' \rangle$ on N, $a_i' = [\alpha_i] \in H^{p_i}(N)$. Since the map $H^*(M) \longrightarrow H^*(N)$ is injective for $* < n$, it follows that this Massey product is non-zero in N. The proof is complete. □

Proof of Theorem 1.1. First let us address the *only if* part of the theorem. Let M be a compact orientable $(k-1)$-connected n-manifold. If $n \leq 4k-2$ then M is formal by Theorem 2.2. If M has $b_k = 1$ and its dimension is $n \leq 4k$ then M is formal by Theorem 2.3. If M has $b_k = 0$ (this means that either M is k-connected, or else that $\pi_k(M)$ is torsion) then the minimal model of M is of the form $(\bigwedge V^{\geq(k+1)}, d)$. The argument of the proof of Theorem 2.2 proves the formality of M if $n \leq 4k+2$.

For the *if* part of the theorem, we have to give constructions of non-formal compact orientable $(k-1)$-connected n-manifolds for any $b_k = b \geq 0$ and $n \geq \max\{4k-1, 4k+3-2b\}$.

- Case $b_k = 0$ and $n \geq 4k+3$. We need examples of $(k-1)$-connected non-formal n-manifolds with $b_k = 0$. For instance, take the k-connected non-formal n-manifolds provided by [4], since $n \geq 4(k+1)-1$.
- Case $b_k = 1$ and $n \geq 4k+1$. The manifold $Z_{k,n}$ provided by Theorem 3.2 covers this case when $n \neq 5k$. For $k = 1$, in [7] are given examples of non-formal 5-dimensional manifolds with first Betti number $b_1 = 1$. For $n = 5k$ with $k \geq 2$, it is sufficient to consider the manifold $(Z_{k,5k-1} \times S^1)\#_{S^1} S^{5k}$, by Lemma 6.2.
- Case $b_k = 2$ and $n = 4k-1$. For the specific case $k = 2$, where we have $n = 7$ and $b_2 = 2$, Oprea [11] constructs an example of a compact non-formal manifold as the total space of a S^3-bundle over $S^2 \times S^2$ with Euler class 1 (actually Oprea gives a different construction, but it is easily seen to reduce to the above description). This was the first example of a non-formal simply connected compact manifold of dimension 7.

  This construction is generalized by Cavalcanti in Example 1 in [2] for any $k \geq 1$. The total space of a S^{2k-1}-bundle over $S^k \times S^k$ with Euler class 1, is a non-formal $(k-1)$-connected compact orientable $(4k-1)$-dimensional manifold with $b_k = 2$.
- Case $b_k = 2$ and $n \geq 4k$. The manifold $Z'_{k,n}$ provided by Theorem 5.1 covers this case. Otherwise, apply Lemma 6.2 repeatedly to the previous example.
- Case $b_k > 2$ and $n \geq 4k-1$. Let Z be a non-formal $(k-1)$-

connected orientable compact n-manifold with $b_k = 2$. Consider $Z\#(b_k - 2)\,(S^{k+1} \times S^{n-k-1})$, which is non-formal by Lemma 6.1.

□

Remark 6.1 Cavalcanti [2] gives also examples of non-formal $(k-1)$-connected compact orientable n-dimensional manifolds with $b_k = 1$, for $n \geq 4k+1$. The total space of a $S^{2k+2i-1}$-bundle over $S^k \times S^{k+2i}$ with Euler class 1 is non-formal, of dimension $4k+4i-1$ with $b_k = 1$, for $i > 0$. This covers the case $b_k = 1$, $n = 4k+4i-1 \geq 4k+3$ in the list above (in Example 1 in [2] it is shown an improvement to cover also the case $b_k = 1$, $n = 4k+4i$, with $i > 0$). Note that this method does not gives examples for the minimum possible value $n = 4k+1$.

Remark 6.2 The examples $Z_{k,n}$ are $(k-1)$-connected with $b_k = 1$. Hence by the proof of Theorem 2.2, $Z_{k,n}$ is $(2k-1)$-formal. It is not $2k$-formal by Lemma 2.2. Note that $n = 4k+1$ is the smallest dimension in which this can happen by Theorem 2.1.

The examples $Z'_{k,n}$ are $(k-1)$-connected. Hence by the proof of Theorem 2.3, $Z'_{k,n}$ is $(2k-2)$-formal. It is not $(2k-1)$-formal by Lemma 2.2. Again Theorem 2.1 says that $n = 4k-1$ is the smallest dimension in which this can happen.

7. Non-formal manifolds with small Betti numbers b_k and b_{k+1}

A natural question that arises from the proof of Theorem 3.1 is whether there are examples of compact non-formal k-connected n-manifolds, $n \geq 4k+1$ with $b_k = 1$ and b_{k+1} as small as possible. Our examples with $n > 4k+1$ satisfy that $b_{k+1} = 1$. But the examples with $n = 4k+1$ satisfy that $b_{k+1} = 2$.

Lemma 7.1 *If M is a $(k-1)$-connected compact orientable n-manifold with $b_k = 1$, $b_{k+1} = 0$ and $n \leq 4k+2$, $k \geq 1$, then M is formal.*

Proof. Work as in the proof of Theorem 2.3 to conclude that M is $(2k-1)$-formal. For $k > 2$, we have $V^{k+1} = 0$, since $b_{k+1} = 0$. So there is no product to be killed in $V^k \cdot V^{k+1}$ and hence $N^{2k} = 0$ (but maybe $C^{2k} \neq 0$) proving that M is $2k$-formal. If $k = 2$, then $V^2 = C^2 = \langle \xi \rangle$. If moreover $V^3 = 0$ then we conclude the 4-formality as above. Otherwise $V^3 = N^3 = \langle \eta \rangle$ with $d\eta = \xi^2$. So there is nothing closed in $V^2 \cdot V^3$, and we have $N^4 = 0$.

Therefore, M is 4-formal. Finally, if $k = 1$, then $V^1 = \langle \xi \rangle$ and $V^2 = 0$, hence M is 2-formal. Now use Theorem 2.1 to get the formality of M.

We end up with the following

Question: Let M be a $(k-1)$-connected compact orientable n-manifold with $b_k = 1$, $b_{k+1} = 1$ and $n \leq 4k+2$, $k \geq 1$. Is necessarily M formal?

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