ON MINIMAL LIGHTLIKE SURFACES IN MINKOWSKI SPACE *

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We discuss a way to define *minimal* lightlike surfaces in Minkowski space.

1. Introduction

The minimal two-dimensional surfaces in the Euclidean space E^n have various characteristic properties which can be viewed as definitions of minimality. The classical definition says that a surface F^2 in E^n, $n \geq 3$, is minimal if its mean curvature vanishes, $H \equiv 0$. Minimal surfaces are critical points of the area functional, the position-vector of a minimal surface in E^n is harmonic in terms of isothermic coordinates, etc [1]. One of the most useful properties of minimal surfaces is their deformability. Namely, a surface in E^n admits a continuous family of conformal G-transformations different from translations and homotheties if and only if it is minimal [2]. Here a G-transformation is defined as a regular mapping $F^2 \longrightarrow \tilde{F}^2$ which preserves the Gauss image, i.e. the planes tangent to F^2 and $\tilde{F}^2$ at corresponding points are parallel.

The aim of this article is to introduce a natural analogue of minimal surfaces for the class of lightlike surfaces in the Minkowski space M^n. Recall that there are three types of two-dimensional planes in M^n: a two-dimensional plane Π^2 in M^n is called spacelike, timelike or lightlike if it contains zero, two or one null directions respectively [3],[5]. A two-dimensional surface F^2 in M^n is referred to as spacelike (timelike, lightlike), if all planes tangent to F^2 are spacelike (timelike, lightlike respectively). Every space- or time-

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like surface in M^n has a non-degenerate semi-Riemannian metric induced from ds^2_M, so we can consider all the fundamental intrinsic and extrinsic geometric notions. In particular, one can define such notions as the mean curvature and the minimal (maximal) space- and timelike surfaces. As for the lightlike surfaces, whose induced metrics are degenerate by definition, it is not clear what kind of lightlike surfaces may be treated as minimal and what is the mean curvature. For our best knowledge, a unique attempt to define the mean curvature for lightlike submanifolds was accomplished in [4] for the lightlike hypersurfaces in semi-Euclidean spaces, see also [5].

Our idea is to use the mentioned deformability property of minimal surfaces in E^n. It turns out to be true that the space- and timelike surfaces in M^n have the same property [6]: *a space- or timelike surface F^2 in M^n admits a continuous family of nontrivial conformal G-transformations if and only if the mean curvature of F^2 vanishes, $H \equiv 0$* . Moreover, *for every spacelike (timelike) surface F^2 with vanishing mean curvature in M^n its conformal G-transformations are well represented by pairs of conjugate harmonic functions (two functions of one argument, respectively) in terms of isothermic coordinates on F^2.* It is useful to recall that a spacelike (timelike) surface F^2 in M^n is parameterized by isothermic coordinates iff its position-vector $\rho(u^1, u^2)$ satisfies

$$\langle \partial_{u^1}\rho, \partial_{u^1}\rho \rangle = \epsilon \langle \partial_{u^2}\rho, \partial_{u^2}\rho \rangle, \qquad \langle \partial_{u^1}\rho, \partial_{u^2}\rho \rangle = 0,$$

and then the vanishing of the mean curvature reads

$$\partial_{u^1u^1}\rho + \epsilon \partial_{u^2u^2}\rho = 0,$$

here $\langle , \rangle$ stands for the inner product in M^n, $\epsilon = 1$ and $\epsilon = -1$ correspond to the spacelike and timelike cases respectively [7], [8], [9].

As for the lightlike case, it turns out that from the local point of view there are two classes of lightlike surfaces which admit non-trivial conformal G-deformations. One class consists of the null-ruled surfaces. A null-ruled surface in M^n is formed by null straight lines of M^n, it may be parameterized in such a way that its position-vector $\rho(u^1, u^2)$ satisfies

$$\langle \partial_{u^1}\rho, \partial_{u^1}\rho \rangle = 0, \qquad \langle \partial_{u^1}\rho, \partial_{u^2}\rho \rangle = 0, \qquad \partial_{u^1u^1}\rho = 0,$$

i.e. the coordinate curves $u^2 = const$ are null straight lines and u^1 is an *affine* parameter on them. This class of lightlike surfaces is formally relied to the space- and timelike surfaces with vanishing mean curvature: we have to set $\epsilon = 0$ instead of $\epsilon = \pm 1$.

The other particular class consists of lightlike surfaces which are referred to as *strongly lightlike*. By definition, a strongly lightlike surface is not null-ruled and it may be parameterized in such a way that its position-vector $\rho(u^1, u^2)$ satisfies three conditions:

$$\langle \partial_{u^1}\rho, \partial_{u^1}\rho \rangle = 0, \qquad \langle \partial_{u^1}\rho, \partial_{u^2}\rho \rangle = 0,$$

i.e. the coordinate curves $u^2 = const$ in F^2 are null curves, and besides

$$\partial_{u^1u^2}\rho = P\partial_{u^1}\rho + Q\partial_{u^2}\rho.$$

This class of surfaces was described earlier by K. Ilienko [10], [11], [12] in terms of twistors and spinors but without any geometric interpretations.

Since in the lightlike case only the null-ruled and strongly lightlike surfaces admit continuous families of non-trivial conformal G-transformations, just like the minimal surfaces in E^n and the minimal (maximal) space- and timelike surfaces in M^n do, then the null-ruled and strongly lightlike surfaces may be also treated as minimal. It has to be remarked that every lightlike surface in M^3 is null-ruled; in M^4 the strongly lightlike surfaces are generic, whereas the null-ruled surfaces form a particular class of lightlike surfaces; in M^n, $n > 4$, generically a lightlike surface is neither strongly lightlike nor null-ruled.

The structures of null-ruled surfaces are quite simple, they were intensively studied from different points of view. On the other hand, the strongly lightlike surfaces were neglected, so our principal aim is to study this new class of surfaces. After proving the existence of continuous families of non-trivial conformal G-transformations for null-ruled and strongly lightlike surfaces, we will describe in more details the conformal G-transformations of strongly lightlike surfaces. Next, we will consider Laplace transformations between strongly lightlike surfaces and prove that Laplace transformations commute with conformal G-transformations. Finally, we will present some examples of lightlike surfaces in M^n.

2. General conformal G-transformations

Let x^0, ..., x^{n-1} be Cartesian coordinates in the n-dimensional Minkowski space M^n, the metric of M^n reads

$$ds_M^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - ... - (dx^{n-1})^2.$$

Consider a regular two-dimensional surface F^2 in M^n represented by a position-vector $x = \rho(u^1, u^2)$. The regularity of F^2 means that at every point $P \in F^2$ the vectors $\partial_{u^1}\rho$ and $\partial_{u^2}\rho$ are non-collinear and span the tangent plane $T_P F^2$ to F^2. The metric induced on F^2 from ds^2_M reads $ds^2 = g_{ij} du^i du^j$, where $g_{ij} = \langle \partial_{u^i}\rho, \partial_{u^j}\rho \rangle$ are corresponding inner products in M^n.

Suppose the given surface F^2 is transformed to another surface $\tilde{F}^2$ with position-vector $\tilde{\rho}(u^1, u^2)$. The transformation $\psi : F^2 \longrightarrow \tilde{F}^2$ is said to be a G-*transformation* if the tangent planes to F^2 and $\tilde{F}^2$ at corresponding points are parallel, i.e. the following equalities hold:

$$\partial_{u^1}\tilde{\rho} = A\,\partial_{u^1}\rho + B\,\partial_{u^2}\rho, \tag{1}$$

$$\partial_{u^2}\tilde{\rho} = C\,\partial_{u^1}\rho + D\,\partial_{u^2}\rho, \tag{2}$$

where A, B, C, D are some functions of variables u^1, u^2.

Clearly, translations and homotheties in M^n generate G-transformations of surfaces, they are referred to as trivial. Translations are described by $A = D \equiv 1$, $B = C \equiv 0$, whereas homotheties correspond to $A = D \equiv \mathit{const}$, $B = C \equiv 0$.

On the other hand, non-trivial G-transformations may exist too, so it is an interesting problem to study such kind of mappings. The existence of non-trivial G-transformations is relied to some restrictions for A, B, C, D, which arise from the compatibility of (1)-(2), $\partial_{u^1u^2}\tilde{\rho} = \partial_{u^2u^1}\tilde{\rho}$. These restrictions are following:

$$\begin{aligned}(\partial_{u^2}A - \partial_{u^1}C)\,\partial_{u^1}\rho + (\partial_{u^2}B - \partial_{u^1}D)\,\partial_{u^2}\rho \\ - C\,\partial_{u^1u^1}\rho \,+\, (A - D)\,\partial_{u^1u^2}\rho \,+\, B\,\partial_{u^2u^2}\rho \,=\, 0.\end{aligned} \tag{3}$$

Generically we have a system of algebraic and differential equations for A, B, C, D, whose compatibility depends on linear relations between $\partial_{u^1}\rho$, $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$, $\partial_{u^1u^2}\rho$ and $\partial_{u^2u^2}\rho$. Besides, since we consider *regular* G-transformations only, the functions A, B, C, D have to satisfy an additional constraint:

$$A\,D - B\,C \neq 0. \tag{4}$$

Now assume that the G-transformation $\psi : F^2 \longrightarrow \tilde{F}^2$ is *conformal*. It means that the metric $d\tilde{s}^2 = \tilde{g}_{ij} du^i du^j$ of $\tilde{F}^2$ is proportional to the metric of the initial surface F^2, $\dfrac{\tilde{g}_{11}}{g_{11}} = \dfrac{\tilde{g}_{12}}{g_{12}} = \dfrac{\tilde{g}_{22}}{g_{22}}$. It follows from (1)-(2) that ψ is conformal if and only if A, B, C and D satisfy the following conditions:

$$\frac{A^2 g_{11} + 2AB g_{12} + B^2 g_{22}}{g_{11}} = \frac{AC g_{11} + (AD + CB) g_{12} + BD g_{22}}{g_{12}}$$

$$= \frac{C^2 g_{11} + 2CDg_{12} + D^2 g_{22}}{g_{22}}. \tag{5}$$

Thus, the surface $F^2 \subset M^n$ being given, the existence of conformal G-transformations for F^2 depends on the solvability of (3)-(5) with respect to A, B, C, D. Generically we have trivial solutions only that correspond to translations and homotheties. However in some particular cases there are either finite sets or continuous families of non-trivial conformal G-transformations. We will study *continuous* families of conformal G-transformations which are referred to as conformal G-deformations. From the analytic point of view, the question is when there exists a continuous family of non-trivial solutions $A(u^1, u^2; \varepsilon)$, $B(u^1, u^2; \varepsilon)$, $C(u^1, u^2; \varepsilon)$, $D(u^1, u^2; \varepsilon)$ of (3)-(5) , here ε is a family parameter. It is natural to suppose that the desired family starts from a trivial solution $A(u^1, u^2; 0) = 1$, $B(u^1, u^2; 0) = 0$, $C(u^1, u^2; 0) = 0$, $D(u^1, u^2; 0) = 1$. For space- and timelike surfaces the discussion is quite simple and similar to the Euclidean case, if we apply isothermic coordinates. Our aim is to analyze the case of lightlike surfaces.

3. Conformal G-transformations of lightlike surfaces

Now assume that the regular surface F^2 is lightlike. It means that the differential form ds^2 induced in F^2 from ds^2_M is degenerate, $g_{11}g_{22} - g^2_{12} = 0$. In this case it is conventional to call ds^2 a *degenerate metric* [3], [4], [5]. In this case at every point $P \in F^2$ there exist a unique well defined null direction in the tangent plane $T_P F^2$. As consequence, F^2 is foliated into a one-parameter family of null curves. Without loss of generality we can specify the coordinates u^1, u^2 in such a way that the coordinate lines $u^2 = const$ are just the mentioned null curves in F^2. Then the metric of F^2 reads $ds^2 = g_{22}(du^2)^2$, i.e.

$$g_{11} = \langle \partial_{u^1}\rho, \partial_{u^1}\rho \rangle = 0, \tag{6}$$

$$g_{12} = \langle \partial_{u^1}\rho, \partial_{u^2}\rho \rangle = 0. \tag{7}$$

Remark that if we chose another local coordinates $u^1 = u^1(\hat{u}^1, \hat{u}^2)$, $u^2 = u^2(\hat{u}^2)$, then the coordinate lines $\hat{u}^2 = const$ in F^2 will still null and the metric will be written as $ds^2 = \hat{g}_{22}(d\hat{u}^2)^2$.

The planes tangent to F^2 and $\tilde{F}^2$ at points corresponding under the G-transformation $\psi : F^2 \longrightarrow \tilde{F}^2$ are parallel. Therefore F^2 and $\tilde{F}^2$ are of the same causal type, so $\tilde{F}^2$ is lightlike. Moreover, if we specify u^1, u^2 so that

(6)-(7) hold, then the conditions (5) read:

$$B = 0. \tag{8}$$

So, the G-transformation $\psi : F^2 \longrightarrow \tilde{F}^2$ is conformal iff it maps null directions tangent to F^2 into null directions tangent to $\tilde{F}^2$. Such mappings of lightlike submanifolds are sometimes referred to as radical preserving.

Because of (8), the equation (3) is rewritten as follows:

$$(\partial_{u^2} A - \partial_{u^1} C)\, \partial_{u^1} \rho \,-\, \partial_{u^1} D\, \partial_{u^2} \rho \,-\, C\, \partial_{u^1 u^1} \rho \,+\, (A - D)\, \partial_{u^1 u^2} \rho \,=\, 0. \tag{9}$$

Besides, the regularity condition (4) holds iff A and D don't vanish. We see that if the vectors $\partial_{u^1}\rho$, $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$ and $\partial_{u^1u^2}\rho$ are independent, then (9) has trivial solutions only, $C = 0$, $A = D = const$, they correspond to trivial conformal G-transformations.

Proposition 3.1 *If a lightlike surface F^2 in M^n, with position-vector $x = \rho(u^1, u^2)$ and metric $ds^2 = g_{22}(du^2)^2$, admits a non-trivial conformal G-transformation, then the vectors $\partial_{u^1}\rho$, $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$ and $\partial_{u^1u^2}\rho$ are linearly dependent.*

The described necessary condition for the existence of non-trivial conformal G-transformations is invariant with respect to mentioned scaling changes of coordinates, $u^1 = u^1(\hat{u}^1, \hat{u}^2)$, $u^2 = u^2(\hat{u}^2)$, which preserve the nullity of one family of coordinate lines. How it is restrictive depends on the dimension n.

Lemma 3.1 *Let F^2 be a lightlike surface in the n-dimensional Minkowski space M^n. Assume that F^2 is represented by a position-vector $\rho(u^1, u^2)$ in such a way that coordinate lines $u^2 = const$ are null curves, i.e. the metric of F^2 reads $ds^2 = g_{22}(du^2)^2$.*

1) If $n = 3$, then $\partial_{u^1u^1}\rho$ and $\partial_{u^1u^2}\rho$ linearly depend on $\partial_{u^1}\rho$, $\partial_{u^2}\rho$.

2) If $n = 4$, then $\partial_{u^1}\rho$, $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$ and $\partial_{u^1u^2}\rho$ are linearly dependent.

Proof. The coordinates on F^2 are specified in such a way that (6)-(7) hold. If we differentiate these equalities, we obtain

$$\langle \partial_{u^1}\rho, \partial_{u^1u^1}\rho \rangle \,=\, 0, \tag{10}$$

$$\langle \partial_{u^1}\rho, \partial_{u^1u^2}\rho \rangle \,=\, 0, \tag{11}$$

$$\langle \partial_{u^2}\rho, \partial_{u^1u^1}\rho \rangle \,=\, 0, \tag{12}$$

$$\langle \partial_{u^2}\rho, \partial_{u^1u^2}\rho \rangle \,+\, \langle \partial_{u^1}\rho, \partial_{u^2u^2}\rho \rangle \,=\, 0, \tag{13}$$

It follows from (7), (10)-(11) that $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$ and $\partial_{u^1u^2}\rho$ are orthogonal to the null vector $\partial_{u^1}\rho$. On the other hand, the subspace of vectors orthogonal to a given null vector in M^n is $(n-1)$-dimensional and contains this null vector itself. Hence if $n = 3$, the subspace orthogonal to $\partial_{u^1}\rho$ is two-dimensional and it is spanned by $\partial_{u^1}\rho$ and $\partial_{u^2}\rho$, so $\partial_{u^1u^1}\rho$ and $\partial_{u^1u^2}\rho$ are linear combinations of $\partial_{u^1}\rho$ and $\partial_{u^2}\rho$.

If $n = 4$, the subspace orthogonal to $\partial_{u^1}\rho$ is three-dimensional and it contains $\partial_{u^1}\rho$, $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$ and $\partial_{u^1u^2}\rho$, so $\partial_{u^1}\rho$, $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$ and $\partial_{u^1u^2}\rho$ are linearly dependent, q.e.d.

If $n > 4$, the dimension of the subspace orthogonal to $\partial_{u^1}\rho$ is greater than or equal to 4. Therefore generically the vectors $\partial_{u^1}\rho$, $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$ and $\partial_{u^1u^2}\rho$ are linearly independent. However in some particular cases linear dependencies between the vectors in question may exist, it gives us a way to distinguish particular classes of lightlike surfaces in M^n, $n > 4$.

From the local point of view, three different classes of lightlike surfaces in M^n may be defined:

A) $\partial_{u^1u^1}\rho$ linearly depends on $\partial_{u^1}\rho$ and $\partial_{u^2}\rho$ at every point of F^2;

B) $\partial_{u^1}\rho$, $\partial_{u^2}\rho$ and $\partial_{u^1u^1}\rho$ are independent, whereas $\partial_{u^1u^2}\rho$ is its linear combination, at every point of F^2;

C) $\partial_{u^1}\rho$, $\partial_{u^2}\rho$, $\partial_{u^1u^1}\rho$ and $\partial_{u^1u^2}\rho$ are independent at every point of F^2.

The case C is trivial because of Proposition 3.1. It is a generic situation for $n \geq 5$.

The case A describes null-ruled surfaces. To see this, suppose $\partial_{u^1u^1}\rho$ linearly depends on $\partial_{u^1}\rho$ and $\partial_{u^2}\rho$. Since $\partial_{u^1u^1}\rho$ and $\partial_{u^2}\rho$ are orthogonal, then $\partial_{u^1u^1}\rho$ is collinear to $\partial_{u^1}\rho$. One can chose an admissible scaling change of coordinates $u^1 = u^1(\hat{u}^1, \hat{u}^2)$, $u^2 = u^2(\hat{u}^2)$ on F^2 in such a way that $\partial_{\tilde{u}^1\tilde{u}^1}\rho = 0$, so $\rho = \xi(\tilde{u}^2)\tilde{u}^1 + \eta(\tilde{u}^2)$, where vector-function $\xi(\tilde{u}^2)$ and $\eta(\tilde{u}^2)$ satisfy $\langle\xi, \xi\rangle = 0$, $\langle\xi, \eta'\rangle = 0$. Thus the coordinate null curves $\tilde{u}^2 = const$ in F^2 are null straight lines of M^n, so the lightlike surface F^2 is ruled by null straight lines. It follows from Lemma 3.1, that every lightlike surface in M^3 is null-ruled. On the other hand, if $n > 3$, then the null-ruled surfaces form a very particular class of lightlike surfaces in M^n. A quite simple analysis shows that every null-ruled surface in M^n admits non-trivial conformal G-deformations and it still remains null-ruled under such transformations.

We consider in more details the case B. So suppose that at each point of F^2

the vectors $\partial_{u^1}\rho$, $\partial_{u^2}\rho$ and $\partial_{u^1u^1}\rho$ are linearly independent, whereas $\partial_{u^1u^2}\rho$ is its linear combination, $\partial_{u^1u^2}\rho = P\partial_{u^1}\rho + Q\partial_{u^2}\rho + R\partial_{u^1u^1}\rho$. In this case the lightlike surface F^2 is referred to as *strongly lightlike*. Clearly, F^2 is not null-ruled. Moreover, one can specify the choice of local coordinates u^1, u^2 in F^2, by applying a suitable scaling $u^1 = u^1(\tilde{u}^1, \tilde{u}^2)$, $u^2 = u^2(\tilde{u}^2)$, in such a way that

$$\partial_{u^1u^2}\rho = P\partial_{u^1}\rho \, + \, Q\partial_{u^2}\rho. \tag{14}$$

If (14) holds, then u^1, u^2 are called *Liouville coordinates* in F^2. They are determined uniquely up to scaling changes $u^1 \longrightarrow \tilde{u}^1(u^1)$, $u^2 \longrightarrow \tilde{u}^2(u^2)$.

It is easy to find the coefficients of (14), if we take into account (6)-(7), (10)-(12):

$$P = \frac{\langle \partial_{u^1u^1u^2}\rho, \partial_{u^1u^1}\rho \rangle}{\langle \partial_{u^1u^1}\rho, \partial_{u^1u^1}\rho \rangle}, \qquad Q = \frac{\langle \partial_{u^1u^2}\rho, \partial_{u^2}\rho \rangle}{\langle \partial_{u^2}\rho, \partial_{u^2}\rho \rangle}.$$

We remark that the lightlike surfaces described by K. Ilienko [10], [11], [12] in terms of spinor-twistor representations are just the strongly lightlike surfaces parameterized by Liouville coordinates.

Proposition 3.2 *A strongly lightlike surface F^2 in M^n admits non-trivial continuous conformal G-deformations. Each regular conformal G-transformation maps F^2 to another strongly lightlike surface $\tilde{F}^2$, and Liouville coordinates in F^2 are mapped to Liouville coordinates in $\tilde{F}^2$.*

Proof. In order to prove Proposition 3.2, we rewrite (9) by applying (14):

$$(\partial_{u^2}A - \partial_{u^1}C + P(A-D))\partial_{u^1}\rho + (Q(A-D) - \partial_{u^1}D)\partial_{u^2}\rho - C\partial_{u^1u^1}\rho = 0.$$

Since $\partial_{u^1}\rho$, $\partial_{u^2}\rho$ and $\partial_{u^1u^1}\rho$ are independent, then $C = 0$, and the non-vanishing functions A and D solve the following system of equations:

$$\partial_{u^2}A \;=\; P(D-A), \tag{15}$$

$$\partial_{u^1}D \;=\; Q(A-D). \tag{16}$$

It is easy to see that this system has many solutions different from $A = D = const$. For instance, one can find D from (15) and substitute it to (16), then we obtain a second-order linear pde of hyperbolic type for $A(u^1, u^2)$, which has a large variety of solutions. As consequence, the strongly lightlike surface F^2 admits many various non-trivial conformal G-transformations, which can generate non-trivial continuous conformal G-deformations of F^2.

Let us analyze what kind of surfaces in M^n we can obtain, if we apply a regular conformal G-transformation to F^2. So assume that we have some non-trivial solution A, D of (15)-(16). In order to construct the corresponding non-trivial conformal G-transformation $\psi : F^2 \longrightarrow \tilde{F}^2$, we have to integrate the complete system of compatible equations (1)-(2) which now reads as follows:

$$\partial_{u^1}\tilde{\rho} = A\,\partial_{u^1}\rho, \tag{17}$$

$$\partial_{u^2}\tilde{\rho} = D\,\partial_{u^2}\rho. \tag{18}$$

The solution $\tilde{\rho}(u^1, u^2)$ of (17)-(18) is the position-vector of the transformed surface $\tilde{F}^2$. Let us discuss the properties of $\tilde{F}^2$. First of all, it is trivial to verify that $\tilde{F}^2$ is lightlike, the coordinate curves $u^2 = const$ in $\tilde{F}^2$ are null, the metric of $\tilde{F}^2$ is $d\tilde{s}^2 = \tilde{g}_{22}(du^2)^2$, where $\tilde{g}_{22} = g_{22}D^2$. Next, differentiate (17)-(18) and take into account (14) and (16):

$$\partial_{u^1u^1}\tilde{\rho} = \partial_{u^1}A\partial_{u^1}\rho + A\partial_{u^1u^1}\rho, \tag{19}$$

$$\partial_{u^1u^2}\tilde{\rho} = PD\partial_{u^1}\rho + QA\partial_{u^2}\rho. \tag{20}$$

It follows from (17)-(19), that $\partial_{u^1}\tilde{\rho}$, $\partial_{u^2}\tilde{\rho}$ and $\partial_{u^1u^1}\tilde{\rho}$ are independent and span the same three-dimensional subspace as $\partial_{u^1}\rho$, $\partial_{u^2}\rho$ and $\partial_{u^1u^1}\rho$ do.

Finally, find $\partial_{u^1}\rho$ and $\partial_{u^2}\rho$ from (17)-(18) and substitute into (20):

$$\partial_{u^1u^2}\tilde{\rho} \;=\; \tilde{P}\partial_{u^1}\tilde{\rho} + \tilde{Q}\partial_{u^2}\tilde{\rho}, \tag{21}$$

here $\tilde{P} = P\frac{D}{A}$ and $\tilde{Q} = Q\frac{A}{D}$. Therefore, $\tilde{F}^2$ is strongly lightlike and u^1, u^2 are Liouville coordinates in $\tilde{F}^2$.

4. Laplace transformations of strongly lightlike surfaces

Let the lightlike surface F^2 be strongly lightlike and parameterized by Liouville coordinates u^1, u^2 so that (14) holds. Consider the transformation $L : F^2 \longrightarrow \hat{F}^2$ represented by the following formula:

$$\hat{\rho} \;=\; \rho \;-\; \frac{1}{Q}\partial_{u^1}\rho. \tag{22}$$

The mapping L is completely determined by F^2, since the vector $(1/Q)\,\partial_{u^1}\rho$ is invariant under scaling changes of Liouville coordinates $u^1 \to \bar{u}^1(u^1)$, $u^2 \to \bar{u}^2(u^2)$. Following the traditional terminology [13], it is naturally to call the mapping L the Laplace transformation of F^2. Clearly, this transformation is defined if the coefficient Q from (14) does not vanish.

Differentiating (22) and taking into account (14) we obtain:

$$\partial_{u^1}\hat{\rho} = \left(1 + \frac{\partial_{u^1}Q}{Q^2}\right)\partial_{u^1}\rho \,-\, \frac{1}{Q}\,\partial_{u^1u^1}\rho, \tag{23}$$

$$\partial_{u^2}\hat{\rho} = \left(\frac{\partial_{u^2}Q}{Q^2} - \frac{P}{Q}\right)\partial_{u^1}\rho. \tag{24}$$

We see that the transformed surface $\hat{F}^2$ with position-vector $\hat{\rho}$ is regular (regularly parameterized by u^1, u^2) iff $\partial_{u^2}Q - PQ$ does not vanish. Moreover, it easily follows from (6) and (10), that $\hat{F}^2$ is lightlike, the coordinate curves $u^1 = const$ on $\hat{F}^2$ are null curves, and the metric of $\hat{F}^2$ is $d\hat{s}^2 = \hat{g}_{11}(du^1)^2$, where $\hat{g}_{11} = \langle\partial_{u^1u^1}\rho, \partial_{u^1u^1}\rho\rangle/Q^2$.

Differentiate (24) with respect to u^2 and apply (14):

$$\partial_{u^2u^2}\hat{\rho} = \left(\partial_{u^2}\left(\frac{\partial_{u^2}Q}{Q^2} - \frac{P}{Q}\right) + P\left(\frac{\partial_{u^2}Q}{Q^2} - \frac{P}{Q}\right)\right)\partial_{u^1}\rho + \left(\frac{\partial_{u^2}Q}{Q} - P\right)\partial_{u^2}\rho.$$

Therefore $\partial_{u^1}\hat{\rho}$, $\partial_{u^2}\hat{\rho}$ and $\partial_{u^2u^2}\hat{\rho}$ are linearly independent, since $\partial_{u^2}Q - PQ \neq 0$ in view of regularity of $\hat{F}^2$.

Finally, differentiate (24) with respect to u^1 and replace $\partial_{u^1}\rho$ and $\partial_{u^1u^1}\rho$ by corresponding expressions in terms of $\partial_{u^1}\hat{\rho}$ and $\partial_{u^2}\hat{\rho}$. We obtain:

$$\partial_{u^1u^2}\hat{\rho} \,=\, \hat{P}\partial_{u^1}\hat{\rho} + \hat{Q}\partial_{u^2}\hat{\rho},$$

where

$$\hat{P} = P - \frac{\partial_{u^2}Q}{Q}, \quad \hat{Q} = \frac{Q^2\partial_{u^2}Q - \partial_{u^1}Q\partial_{u^2}Q - Q^3P + Q\partial_{u^1u^2}Q - Q^2\partial_{u^1}P}{Q(\partial_{u^2}Q - PQ)}.$$

Hence, the transformed surface $\hat{F}^2$ is strongly lightlike and parameterized by Liouville coordinates u^1, u^2. So the following statement holds.

Proposition 4.1 *Let $L : F^2 \longrightarrow \hat{F}^2$ be the Laplace transformation of a strongly lightlike surface F^2 parameterized by Liouville coordinates. If L is regular, then the transformed surfaces $\hat{F}^2$ is strongly lightlike and parameterized by Liouville coordinates.*

Thus one can apply the Laplace transformation in order to construct a strongly lightlike surface parameterized by Liouville coordinates from a given strongly lightlike surface parameterized by Liouville coordinates. It is easy to check that the Laplace transformation is involutive.

Remark that the null curves on F^2 are $u^2 = const$, whereas the null curves on $\hat{F}^2$ are $u^1 = const$. So the Laplace transformation is not conformal (radical preserving). Besides, the Laplace transformation is not a

G-transformation, since the planes tangent to F^2 and $\hat{F}^2$ at corresponding points are not parallel. On the other hand, we will demonstrate that conformal G-transformations and Laplace transformations commute, in the following sense.

Proposition 4.2 *Let F^2 be a strongly lightlike surface in M^n. Let $\Phi : F^2 \longrightarrow \tilde{F}^2$ be a conformal G-transformation, $L : F^2 \longrightarrow \hat{F}^2$ and $\tilde{L} : \tilde{F}^2 \longrightarrow \hat{\tilde{F}}^2$ the Laplace transformations. Suppose that the mapping L and $\tilde{L}$ are regular. Then the mapping $\tilde{L} \circ \Phi \circ L^{-1} : \hat{F} \longrightarrow \hat{\tilde{F}}^2$ is a conformal G-transformation.*

Proof. Without loss of generality we assume that the strongly lightlike surface F^2 is parameterized by Liouville coordinates u^1, u^2, and its position-vector $\rho(u^1, u^2)$ satisfies (14). Then the strongly lightlike surface $\tilde{F}^2$, that is obtained from F^2 by the given conformal G-transformation Φ, is parameterized by Liouville coordinates u^1, u^2 and its position-vector $\tilde{\rho}$ is relied to ρ by (17)-(18), where A and D satisfy (15)-(16). The Laplace transformation L is represented by the formula (22), and the Laplace transformation $\tilde{L}$ is represented by a similar formula:

$$\hat{\tilde{\rho}} = \tilde{\rho} - \frac{1}{\tilde{Q}}\, \partial_{u^1} \tilde{\rho}.$$

Similarly to (23)-(24), one can write:

$$\partial_{u^1} \hat{\tilde{\rho}} = \left(1 + \frac{\partial_{u^1} \tilde{Q}}{\tilde{Q}^2}\right) \partial_{u^1} \tilde{\rho} - \frac{1}{\tilde{Q}}\, \partial_{u^1 u^1} \tilde{\rho}, \qquad \partial_{u^2} \hat{\tilde{\rho}} = \left(\frac{\partial_{u^2} \tilde{Q}}{\tilde{Q}^2} - \frac{\tilde{P}}{\tilde{Q}}\right) \partial_{u^1} \tilde{\rho}\ .$$

Recall that $\tilde{P} = P\frac{D}{A}$, $\tilde{Q} = Q\frac{A}{D}$. Taking into account (15)-(18), we obtain:

$$\partial_{u^1} \hat{\tilde{\rho}} = D\left(1 + \frac{\partial_{u^1} Q}{Q^2}\right) \partial_{u^1} \rho \ - \ \frac{D}{Q}\, \partial_{u^1 u^1} \rho,$$
$$\partial_{u^2} \hat{\tilde{\rho}} = \frac{(\partial_{u^2} Q - PQ) D - Q \partial_{u^2} D}{Q^2}\, \partial_{u^1} \rho\ .$$

Easy calculations which use (23)-(24) lead to the following final expressions:

$$\partial_{u^1} \hat{\tilde{\rho}} \ = \ D \partial_{u^1} \hat{\rho}, \tag{25}$$

$$\partial_{u^2} \hat{\tilde{\rho}} \ = \ \left(D - \partial_{u^2} D \frac{Q}{\partial_{u^2} Q - PQ}\right) \partial_{u^2} \hat{\rho}. \tag{26}$$

Therefore the planes tangent to $\hat{F}^2$ and $\hat{\tilde{F}}^2$ at corresponding points are parallel, i.e. the mapping $\tilde{L} \circ \Phi \circ L^{-1}$ is a G-transformation of $\hat{F}^2$ to $\hat{\tilde{F}}^2$.

Besides, it follows from (26) that the null curves $u^1 = const$ in $\hat{F}^2$ are mapped to the null curves $u^1 = const$ in $\hat{\hat{F}}^2$, so the mapping $\tilde{L} \circ \Phi \circ L^{-1}$ is conformal. $\square$

5. Examples

Example 1. Consider a surface of rotation F^2 in M^4 represented by a position-vector $\rho(u^1, u^2) = (u^1, f(u^1)\cos u^2, f(u^1)\sin u^2, h(u^1))$, where $f(u^1)$ and $h(u^1)$ satisfy $(f')^2 + (h')^2 = 1$, $f \neq 0$. It is easy to verify that F^2 is strongly lightlike provided $f''h' - h''f' \neq 0$.

Example 2. Consider a surface F^2 in M^5 represented by a position-vector $\rho(u^1, u^2) = (u^1,\ a\cos u^1,\ a\sin u^1,\ b\cos u^2,\ b\sin u^2)$, where constants a and b satisfy $a^2 + b^2 = 1$. It is easy to see that F^2 is strongly lightlike.

Example 3. Consider a Cartan surface N^2 in Euclidean space E^{n-1}, $n > 4$: by definition, N^2 carries a well-defined net of conjugate curves[13]. Choose a corresponding local parameterizations of N^2, $r(u^1, u^2) = (f^1(u^1, u^2), ..., f^{n-1}(u^1, u^2))$; the conjugacy of coordinate curves on N^2 means that $\partial_{u^1u^2} r$ is a linear combination of $\partial_{u^1} r$ and $\partial_{u^2} r$. Assume that the conjugate coordinates u^1, u^2 in N^2 are semi-geodesic, i.e. the metric of N^2 is $d\sigma^2 = (du^1)^2 + G(du^2)^2$. The surface N^2 being fixed as the base, consider the surface F^2 in M^n with position-vector $\rho(u^1, u^2) = (u^1,\ f^1(u^1, u^2),\ ..., f^{n-1}(u^1, u^2))$. It is easy to verify that F^2 is lightlike. If the base surface N^2 is not ruled by straight lines, then F^2 is strongly lightlike.

Example 4. Consider the two-dimensional surface of rotation F^2 in M^5 with position-vector $\rho = (u^1,\ f(u^1)\cos u^2,\ f(u^1)\sin u^2,\ h(u^1)\cos u^2,\ h(u^1)\sin u^2)$, where $f(u^1)$ and $h(u^1)$ are some functions which satisfy $(f')^2 + (h')^2 = 1$. The surface F^2 is lightlike. If $((f'')^2 + (h'')^2)(fh' - hf')$ does not vanish, then F^2 is neither strongly lightlike nor null-ruled.

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