CURVATURE STRUCTURE AND GENERAL RELATIVITY *

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This paper presents some mathematical comparisons between those aspects of metric, connection, curvature and sectional curvature which are used in the geometrical description of Einstein's general relativity theory. It is argued that, generically, these four curvature "descriptors" are essentially equivalent.

1. Introduction

Newtonian gravitational theory essentially assumes the ability to distinguish between "true" forces and those "accelerative" forces which arise for an observer who "accelerates with respect to absolute space". Leaving aside the philosophical (and other) problems involved here, it is assumed from this that a preferred family of reference frames (observers), called *inertial*, exist in which accelerative forces are absent. In such frames, space is regarded as the set $\mathbb{R}^3$ in an intuitively obvious way and, less obviously, as "Euclidean". There are many different Euclidean "structures" one can put on $\mathbb{R}^3$. In fact, given a particular such structure (by, for example, specifying the subsets of $\mathbb{R}^3$ which are to be designated lines and planes according to Hilbert's axioms [1] or an axiomatic scheme based on a metric space [2]) and a bijective map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and using f to "relocate" the structure on $\mathbb{R}^3$ in an obvious way, another (in general different) Euclidean structure may be imposed. One may, in this sense, take the Euclidean geometry of an inertial frame to be that for which the term "straight" line becomes consistent with the use of the same term in Galileo's law of inertia (Newton's first law) for the motion of a particle in an inertial frame upon which no net

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force acts. In this sense one may think of an imposed Euclidean background which controls the motion of such free particles (that is, the physics) and, in this sense, may be regarded as a "geometrisation" of physics. However, the fact that the Euclidean background is itself unaffected by the physics is an unnerving failure of reciprocity.

If the net force on a particle in an inertial frame is not zero, the advent of the calculus enabled Newton, through his second law, to describe the particle's motion by means of differential equations. With Newton's postulate of the inverse square law, his gravitational theory has proved extremely successful. However, Newton's second law, as usually stated, refers only to an inertial reference frame. If one wishes to write it in a way which is applicable in any frame one must introduce the accelerative forces into the equation. In this sense, Newton's theory can be made "generally covariant" in a rather trivial way but only at the expense of introducing extra terms into the theory and which represent the non-inertial nature of the frames involved (that is, which represent the role played by absolute space).

The further development of the calculus enabled Riemann to develop his fundamental work on geometry and his introduction of the metric tensor. This metric was used in the elegant developments of analytical (Newtonian) mechanics and which resulted in a naturally "generally covariant" formulation of Newton's theory through variational principles. However, the absolute Euclidean background is still there, albeit in a less obvious way.

The most important use of Riemannian geometry in physics was Einstein's formulation of his general relativity theory. In this theory Einstein represents the universe (or, at least, that part of it under consideration) as a 4-dimensional manifold upon which a metric of Riemann's type and of Lorentz signature is assumed to exist. This metric represents the gravitational field and Einstein's field equations for its determination are *tensor equations* which involve, apart from any physical sources of the gravitational field, *only* this metric together with its first and second derivatives. In this sense the physics and the geometry "determine" each other and a more philosophically acceptable situation is achieved than is the case in Newtonian theory. Because of their tensor nature Einstein's equations are then formally the same in any coordinate system, that is, they are generally covariant, and contain only the "dynamical" variable which is to be determined (the metric). Thus general covariance is achieved without introducing any "absolute" elements (such as Newton's absolute space) into the theory. In this sense, Einstein achieves a complete geometrisation of physics.

The aim of this paper is to describe the geometrical aspects of general relativity theory and, in particular, the geometry of the metric, connection and curvature in Einstein's theory. It will concentrate on certain interrelations between these quantities and will also include a few brief remarks on the holonomy theory of space-times.

2. General Relativity

Here, the universe is described by a 4-dimensional connected Hausdorff manifold M admitting a Lorentz metric g of signature $(-+++)$. The metric g represents the gravitational field, the existence of a "true" gravitational field, roughly speaking, being indicated by the non-vanishing of the curvature tensor associated with the Levi-Civita connection Γ of g. (More precisely, the vanishing of the curvature tensor on some non-empty open subset U of M will be interpreted as the vanishing of the gravitational field on U. In this paper, the *non-flat* condition, that no such open sets exist in M, will be assumed.) The paths of neutral small "test" particles are assumed to be described by timelike geodesics and those of electromagnetic radiation by null geodesics. A detailed elementary account may be found in [3].

Einstein's field equations for g are

$$R_{ab} - \frac{1}{2} R\, g_{ab} = \kappa\, T_{ab} \tag{1}$$

where, if $R^a{}_{bcd}$ are components of the type (1,3) curvature tensor associated with Γ, $R_{ab} \equiv R^c{}_{acb}$ are the Ricci tensor components, $R = R_{ab}\, g^{ab}$ the Ricci scalar, T_{ab} the components of the energy-momentum tensor and κ is a constant. The important *vacuum* field equations arise when $T_{ab} = 0$ and are equivalent to $R_{ab} = 0$.

A link between the geometry of M and the physics is provided by the equation of geodesic deviation and Jacobi vector fields [4]. The latter may be regarded as "instantaneous position vectors" between neighbouring (timelike) geodesics that is, between an observer and a neighbouring particle, and this deviation equation, with a Newtonian interpretation, can be used to show how the curvature tensor and its associated sectional curvature function, are related to the Newtonian force gradient (relative acceleration). The actual geodesic equation itself suggests a Newtonian interpretation of the connection coefficients (Christoffel symbols) in terms of "force".

These brief remarks (and several others - see e.g. [3]) reveal the geometrical description of the physics of the gravitational field in terms of the metric, the connection and the curvature tensor. The remainder of this paper will attempt to describe some relations between these geometrical objects. As is often the case, one gets drawn into the geometry and loses sight of the physics. The rest of this paper suffers from this in many respects!

3. The Curvature Map

Let Λ_p denote the vector space of type (2,0) skew-symmetric tensors at $p \in M$ and V_p the vector space of all type (1,1) tensors at p. Define the (linear) *curvature map* $f : \Lambda_p \longrightarrow V_p$ by

$$f : F^{ab} \longrightarrow R^a{}_{bcd}\, F^{cd} \qquad (F \in \Lambda_p) \tag{2}$$

and let $B_p(\subseteq V_p)$ denote the range space of f at p. Since each member of B_p is skew-self adjoint with respect to $g(p)$, B_p is a subspace of the Lorentz algebra $\mathcal{A}$ (the Lie algebra of the Lorentz group $\mathcal{L}$). Now if $P, Q \in B_p$ it is easily shown that the matrix commutator $[P, Q]$ is also skew-self adjoint with respect to $g(p)$ and so the closure $\overline{B_p}$ of B_p under the commutator operation is a subalgebra of $\mathcal{A}$. Members of B_p may, through the metric g, be naturally associated with members of Λ_p. A non-zero member of Λ_p has (matrix) rank 2 or 4 and if it is 2 it is called *simple* and otherwise *non-simple*. A simple member F may be written as $F = r \wedge s$ for r, s in the tangent space T_pM to M at p. The vectors r and s span a 2-dimensional subspace (2-space) of T_pM which is easily seen to be uniquely determined by F and called the *blade* of F. A non-simple member F of Λ_p can be shown to uniquely determine a pair of orthogonal 2-spaces at p, one spacelike and one timelike and which are called its *canonical pair of blades* [5]. Members of Λ_p are called *bivectors*.

The map f can be used to give a useful classification of the curvature tensor at p into five classes, **A, B, C, D** and **O** [6].

Class A. This is the most general class and it is allocated to the curvature tensor at p if it is not in any of the other classes below at p.

Class B. This is when B_p is 2-dimensional and spanned by two simple bivectors whose blades are spacelike and timelike, respectively, and orthogonal (with respect to $g(p)$).

Class C. This occurs if B_p is 2- or 3-dimensional and if there exists $0 \neq k \in T_pM$ such that each F in the range of f satisfies $F^a{}_b\, k^b = 0$.

Class D. This occurs when B_p is 1-dimensional. Then B_p is spanned by a single bivector which, because of the identity $R_{a[bcd]} = 0$, can easily be checked to be simple.

Class O. This occurs when the curvature tensor vanishes at p.

This classification is exhaustive and mutually exclusive and, in spite of the reference to the metric in class B, is independent of the metric in the sense that if two distinct Lorentz metrics give rise to the same type (1,3) curvature tensor, the curvature class is the same for each metric (c.f. section 5). Further, for class A, $dim B_p \geq 2$ and if $dim B_p \geq 4$, the class is necessarily A. Also of some related importance is the equation

$$R^a{}_{bcd}\, k^d = 0 \tag{3}$$

for $0 \neq k \in T_pM$. This equation has no solutions if the class at p is A or B, a unique independent solution if the class is C and exactly two independent solutions if the class is D. For a given space-time manifold M there exists a generic collection (an open dense subset in a certain Whitney topology) of Lorentz metrics on M such that, for each such metric, $dim B_p \geq 4$ for every $p \in M$ [7]. It follows that the collection of space-times which are of class A everywhere is generic in this sense.

4. Space-Time Holonomy

Since a space-time M is *connected*, it makes sense to talk about its *holonomy group* with respect to the Levi-Civita connection Γ of g. This holonomy group is denoted by Φ and is isomorphic to a subgroup of the Lorentz group $\mathcal{L}$. If we assume M is simply connected then Φ is a connected Lie group and is thus uniquely determined by its Lie algebra, which may be regarded as a subalgebra of the Lorentz algebra $\mathcal{A}$. The subalgebra structure of A is well known and a subalgebra classification given in [8] will be followed here. In this classification, 15 types arise, labelled $R_1, ..., R_{15}$, of which R_1 is the trivial case, R_{15} is the full algebra $\mathcal{A}$ and R_5 is impossible for a space-time [6, 9] - see also [10]. Space does not permit more than a few remarks regarding holonomy theory in general relativity. Suffice it to say that this holonomy classification scheme allows the holonomy algebras to be described in a convenient way for both the geometrical description and physical interpretation of space-times and it may be linked, through the associated infinitesimal holonomy group, to the curvature classification given in section 3 [6, 11]. Details of basic holonomy theory may be found in [12] and of the applications to Einstein's theory in [6]. Holonomy theory has a number of applications to the study of space-time symmetry [6] but suffers (as do many classification schemes in theoretical physics) from being too "general" in its general case (type R_{15}) and too "special" in the other cases.

The possible holonomy algebras (subalgebras of $\mathcal{A}$) are listed in table 1. In this table (l, n, x, y) is a null tetrad so that the only non-vanishing inner products between them are $l^a n_a = x^a x_a = y^a y_a = 1$. In the type R_{13}, (t, x, y, z) is a pseudorthonormal tetrad whose only non-vanishing inner products are $-t^a t_a = x^a x_a = y^a y_a = z^a z_a = 1$. In types R_5 and R_{12}, $0 \neq \rho \in \mathbb{R}$.

Table 1

	Basis	Dim		Basis	Dim
R_1	0	0	R_8	$l \wedge x, l \wedge y$	2
R_2	$l \wedge n$	1	R_9	$l \wedge x, l \wedge y, l \wedge n$	3
R_3	$l \wedge x$	1	R_{10}	$l \wedge x, n \wedge x, l \wedge n$	3
R_4	$x \wedge y$	1	R_{11}	$l \wedge x, l \wedge y, x \wedge y$	3
R_5	$l \wedge n + \rho x \wedge y$	1	R_{12}	$l \wedge x, l \wedge y, l \wedge n + \rho x \wedge y$	3
R_6	$l \wedge n, l \wedge x$	2	R_{13}	$x \wedge y, x \wedge z, y \wedge z$	3
R_7	$l \wedge n, x \wedge y$	2	R_{14}	$l \wedge x, l \wedge y, l \wedge n, x \wedge y$	4
			R_{15}	$(= L)$	6

5. Curvature, Connection and Metric

Let M be a space-time manifold and let g, g' be space-time metrics on M whose Levi-Civita connections are identical (equivalently, their affinely parametrised geodesics agree). Then the holonomy algebra associated with their common connection may be regarded as a Lie algebra of matrices (under the commutator operation), every member of which is skew self adjoint with respect to g and g' (i.e. it is a subalgebra of each of the orthogonal algebras of g and g'). Each matrix F in this algebra then satisfies

$$g_{ca} F^c{}_b + g_{cb} F^c{}_a = 0 \qquad\qquad g'_{ac} F^c{}_b + g'_{cb} F^c{}_a = 0 \tag{4}$$

From (4) it can be checked [13, 14, 6] that if F is simple (respectively, non-simple) the blade of F (respectively, each of the pair of canonical blades of F) is an eigenspace of g' with respect to g. Thus the knowledge of the holonomy algebra enables a simple algebraic relationship between g and g' to be written down. The metric (zero covariant derivative) condition on g and g' then completes the solution to the problem. In fact, if the holonomy type of M is R_9, R_{12}, R_{14} or R_{15} (and this collection of space-times contains the collection of space-times which are of curvature class A everywhere and is, hence, generic - see section 3) g' and g are necessarily conformally related with a constant conformal factor [15].

Now suppose g and g' are assumed only to have the same type $(1,3)$ curvature tensor. Then each member of B_p (and, as is easily checked, of $\overline{B}_p$) satisfies (4). The algebraic result linking g and g', and given above, again allows a simple relationship between g and g' to be written down [13, 14, 6]. In the situation when the curvature class is A everywhere the metrics g and g' are necessarily conformally related and the Bianchi identity shows that this conformal factor is constant (and hence the Levi-Civita connection, from which the curvature arises, is unique) [14, 6]. However, the following class B space-time metrics (for example)

$$f(u,v)\,du\,dv + d\sigma^2 \qquad \text{and} \qquad \phi(u,v)f(u,v)\,du\,dv + d\sigma^2 \tag{5}$$

where $d\sigma^2$ is an arbitrary positive definite metric on some open subset of $\mathbb{R}^2$ and f an arbitrary nowhere zero function, have the same type $(1,3)$ curvature tensor [16] provided the nowhere-zero function ϕ satisfies $\phi\frac{\partial^2\phi}{\partial u\partial v} = \frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v}$. They will only have the same Levi-Civita connection if ϕ is a non-zero constant.

Another problem which may be discussed here is the following physical one. Suppose one wishes to determine the space-time metric by observing timelike geodesics (that is, paths of material particles). That is suppose that at $p \in M$ there is a timelike member $u' \in T_pM$ such that all timelike (unparametrised) geodesics through p in a direction spanned by $u \in T_pM$ for each u in some open neighbourhood of u' in T_pM are known. Then, if g and g' are Lorentz metrics on M whose respective Levi-Civita connections Γ and Γ' are consistent with such an observation, it is known that [17, 18]

$$\left(\delta^b_c T^a_{de} - \delta^a_c T^b_{de}\right) u^c u^d u^e = 0 \qquad (T^a_{bc} = \Gamma'^a_{bc} - \Gamma^a_{bc}) \tag{6}$$

for all such u. Thus $T^a_{bc} = \delta^a_b\psi_c + \delta^a_c\psi_b$ for some *closed* 1-form ψ (the closed condition arising from the fact that Γ and Γ' are metric connections). The condition that g' is compatible with Γ' is then

$$g'_{ab;c} = 2g'_{ab}\psi_c + g'_{ac}\psi_b + g'_{bc}\psi_a \tag{7}$$

where a semi-colon denotes a Γ covariant derivative. It also follows that the curvature tensors arising from Γ and Γ' are related by

$$R'^a{}_{bcd} = R^a{}_{bcd} + \delta^a_d\,\psi_{bc} - \delta^a_c\,\psi_{bd} \qquad\qquad (\psi_{ab} = \psi_{a;b} - \psi_a\psi_b = \psi_{ba}) \quad (8)$$

and that the *projective tensor*, L, defined for Γ by

$$L^a{}_{bcd} = R^a{}_{bcd} - 1/3\,(\delta^a_c\,R_{bd} - \delta^a_d\,R_{bc}) \quad (9)$$

is the same whether calculated from Γ or Γ'. Now if, in the physical interpretation, g and g' are each *vacuum* metrics, the equality of the projective tensors L and L' and the vanishing of the Ricci tensors associated with g and g' give $R'^a{}_{bcd} = R^a{}_{bcd}$. Thus the theory of the previous paragraph applies. In fact, if one assumes that this common curvature tensor is not zero over any non-empty open subset of M, M decomposes into two regions, $M = M_1 \cup M_2$. Here, M_1 is an open submanifold of M on which $g' = \phi g, d\phi = 0$ and so, locally, g and g' are conformally related with a (positive) *constant* conformal factor. M_2 is a closed region whose interior, if non-empty, admits an open covering of subsets on each of which g and g' are related by $g'_{ab} = \alpha g_{ab} + \beta l_a l_b$ for appropriate *constants* α and β and l is a nowhere zero null vector field which is covariantly constant with respect to either Γ or Γ'. It can then be shown that Γ and Γ' are equal on M and an unambiguous concept of affine parameter automatically follows (and which is usually assumed on physical grounds) [19].

If one begins this problem again, this time dropping the vacuum condition on g and g' (in fact, imposing no restrictions on the Ricci tensors of g and g') but assumes, on physical grounds from observations of light beams, that the null cones of g and g' coincide everywhere on M, then $g' = \phi g$ for $\phi : M \longrightarrow \mathbb{R}$. But then (7) can be used to show that ϕ is constant on M and so Γ and Γ' are equal on M and one again has an unambiguous concept of affine parameter.

6. General Metric and Connection Compatibility

Suppose now that ∇ is a symmetric connection on a 4-dimensional non-flat (connected, Hausdorff) manifold M which is *not* assumed metric. Suppose also that h is a metric of Lorentz signature on M which satisfies

$$h_{ae}R^e{}_{bcd} + h_{be}R^e{}_{acd} = 0 \quad (10)$$

$$h_{ae}R^e{}_{bcd;f_1} + h_{be}R^e{}_{acd;f_1} = 0 \quad (11.1)$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$h_{ae}R^e{}_{bcd;f_1\ldots f_k} + h_{be}R^e{}_{acd;f_1\ldots f_k} \tag{11.k}$$

where $R^a{}_{bcd}$ are the curvature tensor components arising from ∇, a semicolon denotes a ∇-covariant derivative and k is some positive integer. Of course, if h were compatible with ∇ (so that $\nabla h = 0$), (10) and (11.k) would hold for each k. So the questions are: (i) does there exist a value of k such that (10) and (11.1) ... (11.k) guarantee a local metric (of arbitrary signature) in some neighbourhood of each point of M (or maybe of each point in some open dense subset of M) compatible with ∇ and (ii) if such a local metric exists, how is it related to h?

This question is closely related to the holonomy (or the local holonomy group) of M. For if the holonomy group Φ of M is known and is isomorphic to some (pseudo-) orthogonal group $O(p, q)$ then (quite generally) ∇ is compatible with a global metric on M of signature (p, q) [12, 20]. However, knowledge of Φ will not be assumed here.

The problem may be solved in the following manner using an adapted version of the curvature classification scheme outline in section 3 [21]. Let the curvature tensor of ∇ satisfy (10) and let f be the associated curvature map (2) and where, in the definition of the curvature types, any metric statement is assumed to be with respect to h. Let A, B, C, D and O represent the subsets of points of M at which the curvature has that class (this double use of symbols should cause no confusion). [One technical point needs to be resolved here regarding the subset B. First note that, from (10), if $R^{ab}{}_{cd} \equiv h^{be}R^a{}_{ecd}$ then $R^{ab}{}_{cd} = -R^{ab}{}_{dc} - R^{ba}{}_{cd}$ and then define the associated linear map $\tilde{f} : \Gamma_p \longrightarrow \Gamma_p$ by $\tilde{f} : F^{ab} \longrightarrow R^{ab}{}_{cd}F^{cd}$. By the class B definition, $\tilde{f}$ has only real eigenvalues and they are $0, \alpha$ and β, with $\alpha, \beta \neq 0$. Let $B = B_1 \cup B_2$ where B_1 (respectively, B_2) is the subset of points of B at which $\alpha \neq \beta$ (respectively, $\alpha = \beta$). This definition is made to ensure the local smoothness of α, β over $B_1 \cup intB_2$, where *int* denotes the interior operator on M.] Thus $M = A \cup B_1 \cup B_2 \cup C \cup D \cup O$. Now since the subset A may be shown to be open, and since the non-flat assumption gives $intO = \emptyset$, one may make a *disjoint* decomposition of M in the form

$$M = A \cup intB_1 \cup intB_2 \cup intC \cup intD \cup Z \tag{12}$$

where Z is closed and can be shown to satisfy $intZ = \emptyset$. Thus interest centers on the open dense subset $M \setminus Z$ of M.

Note that (10) and (11.1) - (11.k) are equivalent to (10) and

$$h_{ae;f_1}R^e{}_{bcd} + h_{be;f_1}R^e{}_{acd} = 0 \tag{13.1}$$

$$\vdots$$

$$h_{ae;f_1\ldots f_k}R^e{}_{bcd} + h_{be;f_1\ldots f_k}R^e{}_{acd} = 0 \tag{13.k}$$

An extension of the results in the second paragraph of section 5, together with the alternative formulation (10), (13.1) - (13.k) of (10), (11.1) - (11.k) may now be used to show that, in A (respectively, $intB_1$ or $intB_2$) if (10) and (11.1) hold then ∇ is compatible with a local Lorentz metric in some neighbourhood of any point of A (respectively, $intB_1$ or $intB_2$). It can also be shown that if (10), (11.1) and (11.2) hold on $intC$ then ∇ is compatible with a local Lorentz metric in some neighbourhood of any point in a certain open dense subset of $intC$ and that if (10), (11.1), (11.2) and (11.3) hold in $intD$ then ∇ is compatible with a local Lorentz metric in some neighbourhood of any point of a certain open dense subset of $intD$. A relationship between these local Lorentz metrics, the metric h and the natural geometrical features arising from the subsets A, B, C or D, as appropriate, can be easily found. Further details are available in [21].

7. Sectional Curvature

Let M be a space-time with metric g and let F be a *non-null* 2-dimensional subspace (a *2-space*) of T_pM. There exists a 2-dimensional non-null submanifold N of M generated locally by the geodesics of M through p with initial direction in F. The *sectional curvature* $\sigma_p(F)$ of F is defined to be the Gauss curvature of N at p. If r and s are independent members of F then

$$\sigma_p(F) = \frac{R_{abcd}F^{ab}F^{cd}}{(g_{ac}g_{bd} - g_{ad}g_{bc})F^{ab}F^{cd}} \qquad (F^{ab} = r^a s^b - s^a r^b) \tag{14}$$

and which is easily checked to be independent of the choice of r and s. It is noted that the denominator in (14) is non-zero if and only if F is non-null and, given that σ_p is not a constant function at p, no *continuous* extension of σ_p (regarded as a real valued function from an open submanifold of the Grassmann manifold of 2-spaces at p) to any null 2-space is possible [22] - see [6].

Now suppose that g and g' are Lorentz metrics on M with identical sectional curvature functions, the latter being nowhere a constant function. The equality of these functions, and hence of their domains and the remark at the end of the last paragraph show that g and g' determine the same family of null 2-spaces at each point of M. From this it can be shown that $g' = \psi g$ for some function $\psi : M \longrightarrow \mathbb{R}$. Use of the Bianchi identities arising from the Levi-Civita connections of g and g' then reveal [23, 24, 6] that M may be disjointly decomposed as $M = U \cup V \cup W$, where U is open and, if $U \neq \emptyset$, then $g' = g$ on U, where V is open and, if $V \neq \emptyset$, g and g' are conformally related conformally flat "plane wave" metrics locally on V and W is closed with $intW = \emptyset$. If, in addition, g is a vacuum metric on M which satisfies the non-flat condition on M then $g' = g$ on M. Thus, generically [7] for space-times and always for non-flat vacuum space-times [23], the sectional curvature is "equivalent" to the metric and may be considered as an alternative to g as the gravitational field variable.

8. Conclusions

In this paper an attempt has been made to give an elementary description of general relativity theory, especially with regard to its geometrical features represented by the metric, the associated Levi-Civita connection and the corresponding type (1,3) curvature tensor and sectional curvature function. These quantities are the descriptors of the physics involved and it is interesting, mathematically, to enquire as to the relationships between them. It turns out that, generically (in a well-defined topological sense), any one of them uniquely determines each of the others except for an (expected) constant conformal factor in the metric, that is, a change of the unit of measurement. In solving this problem two technical devices, holonomy theory and the curvature map, were shown to be useful and were briefly reviewed. The work of section 6, though not directly related to the main problem discussed here, is an interesting link between curvature and metric structures and displays another use of the curvature map.

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References

1. D. Hilbert, *The Foundations of Geometry*, Open Court, Chicago, 1902.
2. L.M. Blumenthal, *A Modern View of Geometry*, Freeman, 1961.
3. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, McGraw-Hill, 2nd edition, 1975.
4. B. O'Neil, *Semi-Riemannian Geometry*, Academic Press, 1983.
5. R.K. Sachs, *Proc. Roy. Soc. (London)* **A264**, (1961), 309.
6. G.S. Hall, *Symmetries and Curvature Structure in General Relativity* World Scientific, Singapore, 2004.
7. A.D. Rendall, *J. Math. Phys.* **29**, (1988), 1569.
8. J.F. Schell, *J. Math. Phys.* **2**, (1961), 202.
9. G.S. Hall, *Gen. Rel. Grav.* **27**, (1995), 567.
10. C.D. Collinson and P.N. Smith, *Gen. Rel. Grav.* **19**, (1987), 95.
11. G.S. Hall and D.P. Lonie. *Class. Quant. Grav.* **17**, (2000), 1369.
12. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol 1, Interscience, New York, 1963.
13. G.S. Hall and C.B.G. McIntosh, *Int. J. Theor. Phys.* **22**, (1983), 469.
14. G.S. Hall, *Gen. Rel. Grav.* **15**, (1983), 581.
15. G.S. Hall, *Gen. Rel. Grav.* **20**, (1988), 399.
16. G.S. Hall and W. Kay, *J. Math. Phys.* **29**, (1988), 420.
17. L.P. Eisenhart, *Riemannian Geometry*, Princeton, 1966.
18. T.Y. Thomas, *Differential Invariants of Generalised Spaces* Cambridge, 1934.
19. G.S. Hall, In preparation.
20. B. Schmidt, *Comm. Math. Phys.* **29**, (1973), 55.
21. G.S. Hall and D.P. Lonie. *J. Phys. A: Gen.* **39**, (2006), 2995.
22. G.S. Hall, (1978 unpublished).
23. G.S. Hall, *Gen. Rel. Grav.* **16**, (1984), 79.
24. B. Ruh, *Math. Z.* **189**, (1985), 371.