THE GRAPHICAL REPRESENTATION OF NEIGHBOURHOODS IN CERTAIN TOPOLOGIES *

EBERHARD MALKOWSKY †

Department of Mathematics, University of Gießen, Arndtstraße 2, D–35392 Gießen, Germany

AND

Department of Mathematics, Faculty of Science and Mathemtics, University of Niš, Višegradska 33, 18000 Niš, Serbia

E-mail: eberhard.malkowsky@math.uni-giessen.de, ema@BankerInter.net

We apply our own software package and its extensions ([16, 14, 15, 18]) for visualisations and animations in mathematics to the graphical representation of neighbourhoods in certain relative and weak topologies that arise in the theories of dual spaces of sequence spaces and of matrix transformations between FK spaces. The paper contains a short introduction to some of the main concepts of our software package, and its portability, and also an outline of the mathematical theory that motivates our graphical representations.

1. Introduction

In this paper, we apply our software package to the graphical representation of neighbourhoods in certain topologies in two– and three–dimensional space. Most of the topologies considered arise in the theory of sequence spaces in summability, in particular in FK and BK spaces, their α–, β–, γ– and continuous duals. FK spaces are of special interest in the characterizations of matrix transformations between sequence spaces. We give a short introduction to some of the main concepts of our software package, and its portability, and also an outline of the mathematical theory that motivates our graphical representations.

* *MSC 2000:* 53A05, 40H05.

Keywords: computer graphics, visualisation, paranormed spaces, FK spaces, dual spaces, matrix transformations.

† Work supported by the Ministry of Science under contract No *1646* and of the DAAD project *Multimedia Technologies for Mathematics and Computer Science Education*.

Visualisations and animations are of vital importance in modern mathematical education. They strongly support the understanding of mathematical concepts. We think that the application of most conventional software packages is neither a satisfactory approach for illustrating theoretical concepts nor can it be used as their substitute. The emphasis in the academic mathematical education should be put on teaching the underlying theories.

Thus we developed our own software package ([16, 14, 15, 18]) in Borland PASCAL and DELPHI to create our graphics for visualisations and animations, mainly of the results from classical differential geometry. Since the source files are available to the users, it can and has been extended to applications in physics, chemistry, crystallography ([3, 4]), and the engineering sciences. It also has various applications in research.

Our graphics can be exported to several formats such as BMP, PS, PLT, SCR (screen files under DOS), or GCLC, the Geometry Constructions Language Converter, developed at Belgrade University ([1, 6]). This is done directly in the PASCAL or DELPHI programmes themselves.

These formats can be converted to a number of other formats by means of any graphics converter software, for instance in Corel Draw to a CDR or GIF file, an EPS file to be included in a TEX or LATEX file, or a PNG or PDF file to be included in a TEX or LATEX file which is directly converted into a PDF file by means of PDFLATEX.

We also use the software packages Animagic GIF 32 to create an animation in animated GIF format from a number of GIF files of our graphics, and include the animation as an animated GIF image in an HTML file.

Originally our graphics software package was a collection of programmes written in Borland PASCAL. Recently Vesna Veličković translated the PASCAL code to DELPHI; she also created a user interface for the *interactive use of our software*. The users become independent of programming.

Currently we are working on an electronic textbook for differential geometry including visualisations and animations, and interactive graphics. The preliminary table of contents, list of figures, the index and a few sample pages are available at

`http://www.pmf.ni.ac.yu/visualization`

We emphasize that all the graphics in this paper were created with our software package and then processed in the way described above; we did not use any other software package.

2. Relative, Sup and Weak Topologies

First, we recall a few basic results and definitions.

There are many ways to introduce topologies on a set. A standard way to introduce a topology on a subset of a topological space is to use the relative topology. Sup topologies and their special cases, weak and product topologies, can be used to introduce topologies on sets in a more general case.

Let S be a subset of a topological space $(X, \mathcal{T})$. Then the relative topology $\mathcal{T}_S$ of X on S is given by $\mathcal{T}_S = \{O \cap S : O \in \mathcal{T}\}$ (Figures 1 and 2).

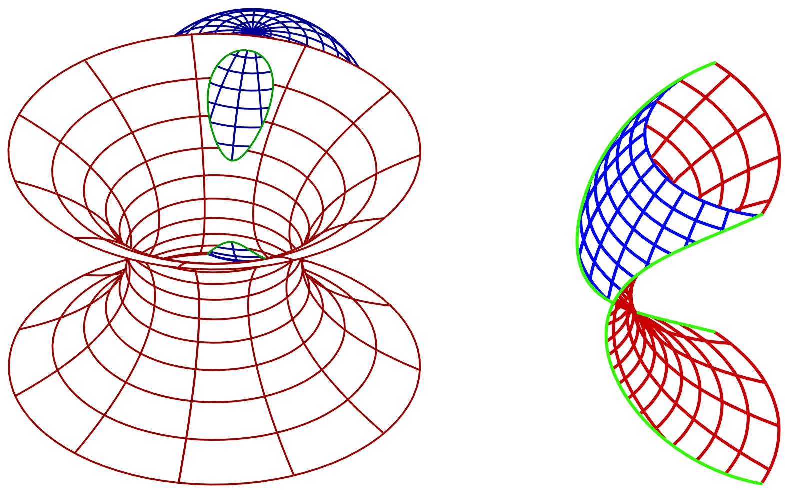

Figure 1. The intersection of a catenoid and a sphere

Let $(X, \mathcal{T})$ be a topological space. A subbase for $\mathcal{T}$ is a collection $\Sigma \subset \mathcal{T}$ such that, for every $x \in X$ and every neighbourhood N of x, there exists a finite subset $\{S_1, \ldots, S_n\} \subset \Sigma$ with $x \in \bigcap_{k=1}^{n} S_k \subset N$. If $X \neq \emptyset$ and Σ is a collection of sets with $\bigcup \Sigma = X$, then there is a unique topology $\mathcal{T}_\Sigma$ which has Σ as a subbase; $\mathcal{T}_\Sigma$ is the weakest topology with $\Sigma \subset \mathcal{T}_\Sigma$, and is called the topology generated by Σ. It consists of $\emptyset$, X and all unions of finite intersections of members of Σ.

If a set X is given a nonempty collection Φ of topologies and $\Sigma = \{\bigcup \mathcal{T} : \mathcal{T} \in \Phi\}$, then the topology $\bigvee \Phi = \mathcal{T}_\Sigma$ is called the sup–topology of Φ; it is stronger than each $\mathcal{T} \in \Phi$. If X has a countable collection $\{d_n : n \in \mathbb{N}\}$

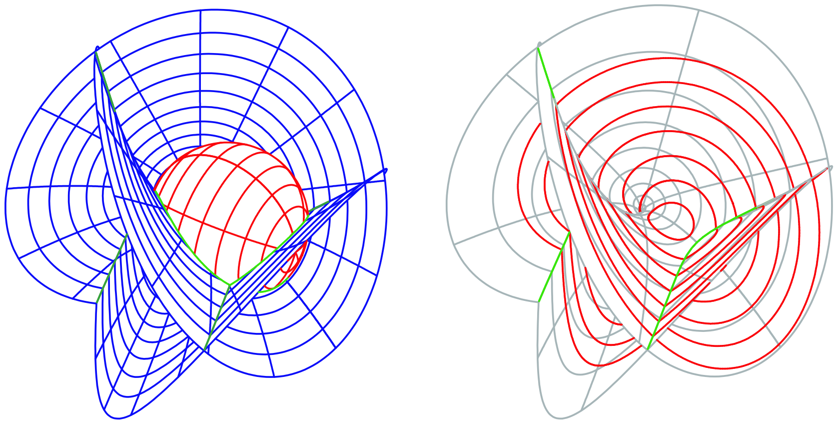

Figure 2. Neighbourhoods of a point in the relative topology of the Euclidean metric on Enneper's minimal surface

of semimetrics, then the sup–topology, denoted by $\bigvee d_n$ is semimetrizable, and given by the semimetric

$$d = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_n}{1 + d_n}; \tag{2.1}$$

if the collection is finite, then $d = \sum d_n$ may be used instead.

Let X be a set, $(Y, \mathcal{T})$ be a topological space and $f : X \to Y$ be a map. Then $w(X, f) = \bigvee \{f^{-1}(O) : O \in \mathcal{T}\}$ is a topology for X, called the weak topology by f. The map $f : (X, w(X, f)) \to (Y, \mathcal{T})$ is continuous and $w(X, f)$ is the weakest topology on X for which this is true. If $\Sigma(Y)$ is a subbase for $\mathcal{T}$, then $\Sigma = \{f^{-1}(G) : G \in \Sigma(Y)\}$ is a subbase for $w(X, f)$. If the topology of Y is metrizable and given by the metric d, we may use the concept of the weak topology by f to define a semimetric δ on X by

$$\delta = d \circ f, \tag{2.2}$$

which is a metric whenever f is one–to–one. A neighbourhood $U_\delta(x_0, r)$ of a point x_0 with respect to the weak topology by f is thus given by

$$U_\delta(x_0, r) = \{x \in X : \delta(x, x_0) < r\} = \{x \in X : d(f(x), g(f_0)) < r\}.$$

More generally, let X be a set, Ψ be a collection of topological spaces, and for each $Y \in \Psi$, we assume given one or more functions $f : X \to Y$. Let the collection of all these functions be denoted by Φ. Then the topology $\bigvee \{w(X, f) : f \in \Phi\}$ is called the weak topology by Φ, and denoted by

$w(X, \Phi)$. Each $f \in \Phi$ is continuous on $(X, w(X, \Phi))$ and $w(X, \Phi)$ is the weakest topology on X such that this is true. If $\Sigma(Y)$ is a subbase for the topology of Y for each $Y \in \Psi$, then

$$\Sigma = \{f^{-1}(G) : f \in \Phi,\ f : X \to Y,\ G \in \Sigma(Y)\}$$

generates $w(X, \Phi)$. The weak topology by a sequence (f_n) of maps from a set X to a collection of semimetric spaces is semimetrizable.

The product topology for a product of topological spaces simply is the weak topology by the family of all projections from the product to the factors.

Example 2.1 Let $B = \mathbb{N}_0$ and $A_n = (\mathbb{C}, |\cdot|)$ for all $n \in \mathbb{N}_0$ where $|\cdot|$ is the absolute value on the set $\mathbb{C}$ of complex numbers. Then the product $\omega = \mathbb{C}^{I\!N_0}$ is the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Its product topology is given by the semimetric

$$d(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|} \text{ for all } x, y \in \omega. \tag{2.3}$$

If we define the sum and the multiplication by a scalar in a natural way by

$$x + y = (x_k + y_k)_{k=0}^{\infty} \text{ and } \lambda x = (\lambda x_k)_{k=0}^{\infty}\ (x, y \in \omega; \lambda \in \mathbb{C}),$$

then (ω, d) is a Fréchet space, that is a complete linear metric space, and convergence in (ω, d) and coordinatewise convergence are equivalent; this means $x^{(n)} \to x$ $(n \to \infty)$ if and only if $x_k^{(n)} \to x_k$ $(n \to \infty)$ for every k (Theorem 4.1.1, p. 54, [20]).

3. Some Metrizable Linear Topological Spaces

Here we consider certain sets of sequences with a metrizable linear topology and their dual spaces. Their common property is that they are continuously embedded in the Fréchet space (ω, d) of Example 2.1.

We recall that a paranorm is a real function g defined on a linear space, and satisfying the following conditions for all vectors x and y

- P.1 $g(0) = 0$
- P.2 $g(x) \geq 0$
- P.3 $g(-x) = g(x)$
- P.4 $g(x + y) \leq g(x) + g(y)$ (triangle inequality)
- P.5 If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ and (x_n) is a sequence of vectors with $g(x_n - x) \to 0$, then $g(\lambda_n x_n - \lambda x) \to 0$ (continuity of multiplication)

A paranorm g for which $g(x) = 0$ implies $x = 0$ is called total. The semimetric d of a linear semimetric space comes from a paranorm g, that is $d(x, y) = g(x - y)$, and the metric of a linear metric space comes from a total paranorm.

We write ℓ_∞, c, c_0 and ϕ, and bs, cs and ℓ_1 for the sets of all bounded, convergent, null and finite sequences, and for the sets of all bounded, convergent, and absolutely convergent series, and $\ell_p = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $0 < p < \infty$. As usual, e and $e^{(n)}$ ($n \in \mathbb{N}_0$) are the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. Given a sequence $x = (x_k)_{k=0}^{\infty} \in \omega$ and $m \in \mathbb{N}_0$, we write $x^{[m]} = \sum_{n=0}^{m} x_k e^{(k)}$ for the m–section of x.

An FK space X is a Fréchet subspace of ω which has continuous coordinates $P_n : X \to \mathbb{C}$ ($n = 0, 1, \dots$) where $P_n(x) = x_n$. An FK space $X \supset \phi$ is said to have AK, if $x = \lim_{m\to\infty} x^{[m]}$ for every sequence $x = (x_k)_{k=0}^{\infty} \in X$. A BK space is a normed FK space.

The following remark is for the benefit of the interested reader who may not be too familiar with the concept of FK and BK spaces.

Remark 3.1 (a) The letters F, B and K in FK and BK space stand for Fréchet, Banach and Koordinate, the German word for coordinate; AK stands for Abschnittskonvergenz, German for sectional convergence.
(b) The concept of an FK space is fairly general. An example of a Fréchet sequence space which is not an FK space can be found in [20], Problem 11.3.3, p. 205 and Example 7.5.6, p. 113.
(c) The importance of FK and BK spaces in the theory of matrix transformations comes from the fact that matrix mappings between FK spaces are continuous ([21], Corollary 11.3.5, p. 204 or [21], Theorem 4.2.8, p. 57).
(d) The FK topology of an FK space is unique; more precisely, if X and Y are BK spaces with $X \subset Y$, then the topology of X is stronger than that of Y, and they are equal if and only if X is a closed subspace of Y. This means there is at most one way to make a subspace of ω into an FK space ([21], Corollary 4.2.4, p. 56).
(e) Every Fréchet space with a Schauder basis is congruent to an FK space ([21], Corollary 11.4.1, p. 208).

Example 3.1 (a) The space (ω, d) with the metric in (2.3) is a locally convex FK space with AK; ϕ has no Fréchet topology ([21], 4.0.2, 4.0.5, p. 51).
(b) Let $p = (p_k)_{k=0}^{\infty}$ be a positive bounded sequence with $H = \sup_k p_k$. We

put $M = \max\{1, H\}$. Then the sets

$$\ell(p) = \left\{x \in \omega : \sum_{k=0}^{\infty} |x_k|^{p_k} < \infty\right\} \text{ and } c_0(p) = \left\{x \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\}$$

are FK spaces with AK with respect to their natural metrics $d_{(p)}$ and $d_{0,(p)}$ that come from the total paranorms

$$g_{(p)}(x) = \left(\sum_{k=0}^{\infty} |x_k|^{p_k}\right)^{1/M} \text{ and } g_{0,(p)}(x) = \left(\sup_k |x_k|^{p_k}\right)^{1/M}$$

([9], Theorem 1, [10], p. 318, and [12], Theorem 2). In

$$\begin{aligned} \ell_\infty(p) &= \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\} \qquad \text{and} \\ c(p) &= \{x \in \omega : x - \ell e \in c_0(p) \text{ for some } \ell \in \mathbb{C}\}, \end{aligned}$$

$g_{0,(p)}$ is a paranorm only in the trivial case $\inf_k p_k > 0$, when $\ell_\infty(p) = \ell_\infty$ and $c(p) = c$ ([19], Theorem 9). FK metrics for $\ell_\infty(p)$ and $c(p)$ using the concepts of co–echelon spaces and the inductive limit topology were given in [2].

(c) The spaces ℓ_p for $1 \le p < \infty$ are BK spaces with $\|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$, and p-normed FK spaces for $0 < p < 1$ with $\|x\| = \sum_{k=0}^{\infty} |x_k|^p$ in which case the corresponding topology is not localy comvex; c_0, c and ℓ_∞ are BK spaces with $\|x\|_\infty = \sup_k |x_k|$, ℓ_p and c_0 have AK, and c and c_0 are closed subspaces of ℓ_∞.

If X is a linear metric space, the set of all continuous linear functionals on X is denoted by X'; if X is a normed space, we write X^* for X' with the norm $\|f\| = \sup_{x\in B_X} |f(x)|$ $(f \in X')$ where B_X denotes the closed unit ball in X.

If x and y are sequences, and X and Y are subsets of ω, then we write $xy = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in \omega : ax \in X\}$ and

$$M(X, Y) = \bigcap_{x\in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$$

for the multiplier space of X and Y. We use the notations $x^\alpha = x^{-1} * \ell_1$, $x^\beta = x^{-1} * cs$ and $x^\gamma = x^{-1} * bs$, and $X^\alpha = M(X, \ell_1)$, $X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ for the α–, β– and γ–duals of X.

Obviously we have $X^\alpha \subset X^\beta \subset X^\gamma$. Also the following result holds.

Proposition 3.1 *(a) If $X \supset \phi$ is an FK space with AK then $X^\beta = X^\gamma$* ([21], Theorem 7.2.7 (iii), 106).
(b) Let X and Y be subsets of ω. If † denotes any of the symbols α, β or γ, then ([21], Theorem 7.2.2, p. 105 *and* [7], Lemma 2)

(i) $X \subset X^{\dagger\dagger}$, *(ii)* $X^{\dagger} = X^{\dagger\dagger\dagger}$, *(iii)* $X \subset Y$ *implies* $Y^{\dagger} \subset X^{\dagger}$.

If I *is an arbitrary index set and* $\mathcal{X} = \{X_\iota : \iota \in I\}$ *is a family of subsets of* X_ι *of* ω, *then*

(iv) $(\bigcup_{\iota \in I} X_\iota)^{\dagger} = \bigcap_{\iota \in I} X_\iota^{\dagger}$.

The following well–known result shows the close relation between the β– and continuous dual of an FK space.

Proposition 3.2 ([21], Theorem 7.2.9, p. 107) *Let* $X \supset \phi$ *be an* FK *space. Then* $X^\beta \subset X'$ *in the sense that each sequence* $a \in X^\beta$ *can be used to represent a function* $f_a \in X'$ *with* $f_a(x) = \sum_{k=0}^{\infty} a_k x_k$ *for all* $x \in X$, *and the map* $T : X^\beta \to X'$ *with* $T(a) = f_a$ *is linear and one–to–one. If* X *has* AK, *then* T *is an isomorphism.*

The boundedness of the sequence p is not needed in Part (a) of the next example.

Example 3.2 (a) We have $\ell(p)^\beta = \ell_\infty(p)$ for $0 < p_k \leq 1$ ([19], Theorem 7), and for $p_k > 1$ and $q_k = p_k/(p_k - 1)$,

$$\ell(p)^\beta = \mathcal{M}(p) = \bigcup_{N>1} \left\{ a \in \omega : \sum_{k=0}^{\infty} \left| \frac{a_k}{N} \right|^{q_k} < \infty \right\} \text{ ([11], Theorem 1);}$$

for all positive sequences ([11], Theorem 6, [7], Theorem 1 and [8], Theorem 2)

$$\begin{aligned} c_0(p)^\beta &= \mathcal{M}_0(p) = \bigcup_{N>1} \left\{ a \in \omega : \sum_{k=0}^{\infty} |a_k| N^{-1/p_k} < \infty \right\}, \\ c(p)^\beta &= c_0(p) \cap cs \qquad \text{and} \\ \ell_\infty(p)^\beta &= \mathcal{M}_\infty = \bigcap_{N>1} \left\{ a \in \omega : \sum_{k=0}^{\infty} |a_k| N^{1/p_k} < \infty \right\}. \end{aligned}$$

(b) If $1 < \inf_k p_k \leq p_k \leq \sup_k < \infty$ and $\ell(q)$ has its natural topology given by the total paranorm

$$g_{(p)}(a) = \left(\sum_{k=0}^{\infty} |a_k|^{q_k} \right)^{1/Q} \quad (a \in \ell(q)), \text{ where } Q = \sup_k q_k,$$

then $\ell(p)^*$ and $\ell(q)$ are linearly homeomorphic ([11], Theorem 4).

The classical special cases of the previous example are well known.

Example 3.3 We have $\ell_p^\beta = \ell_\infty$ for $0 < p \leq 1$, $\ell_p^\beta = \ell_q$ for $1 < p < \infty$ and $q = p/(p-1)$, $c_0^\beta = c^\beta = \ell_\infty^\beta = \ell_1$, $\omega^\beta = \phi$ and $\phi^\beta = \omega$. Furthermore, ℓ_p^*

$(0 < p < \infty)$ and c_0^* are norm isomorphic to their β–duals, and $f \in c^*$ if and only if

$$f(x) = \chi \lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} a_k x_k \text{ where } a \in \ell_1 \text{ and}$$
$$\chi = \chi(f) = f(e) - \sum_{k=0}^{\infty} f(e^{(k)}), \text{ and } \|f\| = |\lim_{k\to\infty} x_k| + \|a\|_1$$

([20], Examples 6.4.2, 6.4.3 and 6.4.4, p. 91). Finally ℓ_∞^* is not isomorphic to any sequence space ([20], Example 6.4.8, p. 93).

Given any infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ of complex numbers and any sequence x, we write A_n for the sequence in the n–th row of A, $A_n(x) = \sum_{k=0}^{\infty} a_k x_k$ $(n = 0, 1, \dots)$ and $A(x) = (A_n(x))_{n=0}^{\infty}$. If X is a subset of ω then $X_A = \{x \in \omega : A(x) \in X\}$ denotes the matrix domain of A in X. Finally $(X, Y) = \{A : X_A \subset Y\}$ is the class of all matrices that map X into Y, that is $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all n, and $A(x) \in Y$ for all $x \in X$.

An infinite matrix $T = (t_{nk})_{n,k=0}^{\infty}$ is called a triangle, if $t_{nn} \neq 0$ for all n and $t_{nk} = 0$ for $k > n$.

The following result is well known

Proposition 3.3 ([21], Theorem 4.3.12, p. 63) *Let (X, d) be an FK space, T be a triangle and $Y = X_T$. Then (Y, d_T) is an FK space with*

$$d_T(y, y') = d(T(y), T(y')) \text{ for all } y, y' \in Y. \tag{3.1}$$

Remark 3.2 We observe that the metric d_T in (3.1) yields the weak topology $w(Y, L_T)$ by $L_T : X_T \to X$ on $Y = X_T$, where $L_T(y) = T(y)$ for all $y \in Y$.

Now we confine ourselves to the special case where $T = \Delta$ with $\Delta_{nn} = 1$, $\Delta_{n,n-1} = -1$ and $\Delta_{n,k} = 0$ (otherwise) for $n = 0, 1, \dots$; we use the convention that any term with a negative subscript is equal to zero.

Theorem 3.1 *Let $X \supset \phi$ be an FK space with AK, and the matrix $E = (e_{nk})_{n,k=0}^{\infty}$ be defined by $e_{nk} = 0$ for $0 \le k \le n-1$ and $e_{nk} = 0$ for $k \ge n$ $(n = 0, 1, \dots)$. Then $(X_\Delta)^\beta$ if and only if $a \in (X^\beta)_E$ and $W \in (X, c_0)$ where W is the matrix with*

$$w_{nk} = \begin{cases} \sum_{j=n}^{\infty} a_j & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad \text{for } n = 0, 1, \dots.$$

Furthermore, if $a \in (X_\Delta)^\beta$*, then*

$$\sum_{k=0}^{\infty} a_k y_k = \sum_{k=0}^{\infty} E_k(a)\Delta_k(y) \text{ for all } y \in X_\Delta. \tag{3.2}$$

Remark 3.3 The statement of Theorem 3.1 also holds for $X = \ell_\infty$.

Now we apply Theorem 3.1 and Remark 3.3. We write $bv(p) = (\ell(p))_\Delta$ and $c_0(p)(\Delta) = (c_0(p))_\Delta$.

Example 3.4 Let p be a bounded positive sequence.
(a) If $p_k \leq 1$ for all k, then $(bv(p))^\beta = cs$; if $1 > p_k$ and $q_k = p_k/(p_k - 1)$ for all k, then $a \in (bv(p))^\beta$ if and only if there is an integer $N > 1$ such that

$$\sum_{k=0}^{\infty} \left| \frac{1}{N} \sum_{j=k}^{\infty} a_j \right|^{q_k} < \infty \text{ and } \sup_n \sum_{k=0}^{n} \left| \frac{1}{N} \sum_{j=n}^{\infty} a_j \right|^{q_k} < \infty. \tag{3.3}$$

(b) We have $a \in (c_0(p)(\Delta))^\beta$ if and only if there is an integer $N > 1$ such that

$$\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} a_j \right| N^{-1/p_k} < \infty \text{ and } \sup_n \sum_{k=0}^{n} \left| \sum_{j=n}^{\infty} a_j \right| N^{-1/p_k} < \infty. \tag{3.4}$$

Proof. (a) First let $p_k \leq 1$ for all k. Then $bv(p) \subset bv = bv(e)$, and so $(bv(p))^\beta \supset bv^\beta = cs$ by Proposition 3.1 (b) (iii) and [21], Theorem 7.3.5 (iii), p. 110. Furthermore, it follows from Theorem 3.1, that if $a \in (bv(p))^\beta$, then R exists, and so $a \in cs$.
Now let $p_k > 1$ for all k. Then, by Theorem 3.1, $a \in (bv(p))^\beta$ if and only if $R = E(a) \in \ell(p)^\beta$ and $W \in (\ell(p), c_0)$. We obtain from Example 3.1 (a), that $R \in \ell(p)^\beta$ if and only if the first condition in (3.3) is satisfied. Also $W \in (\ell(p), c_0)$ if and only if ([8], Theorem 1 and [21], 8.3.6, p. 123) $\sup_n \sum_{k=0}^{\infty} |w_{nk}|^{q_k} N^{-q_k} = \sup_n \sum_{k=0}^{n} (|R_n|/N)^{q_k} < \infty$ for some $N > 1$, which is the second condition in (3.3), and

$$\lim_{n \to \infty} w_{nk} = 0 \text{ for all } k \tag{3.5}$$

which is redundant, since $R_n \to 0$ $(n \to \infty)$.

(b) Now $a \in (c_0(p)(\Delta))^\beta$ if and only if $R \in c_0(p)^\beta$ and $W \in (c_0(p), c_0)$. We obtain from Example 3.1 (a), that $R \in c_0(p)^\beta$ if and only if the first condition in (3.4) is satisfied. Furthermore, $W \in (c_0(p), c_0)$ by [7], Corollary 2 and [21], 8.3.6, p. 123 if and only if $\sup_n \sum_{k=0}^{\infty} |w_{nk}| N^{-1/p_k} < \infty$ for some

$N > 1$, which is the second condition in (3.3), and (3.5) holds which again is redundant. $\square$

Now we write $bv_p = (\ell_p)_\Delta$ for $p > 1$, $q = p/(p-1)$, and $c_0(\Delta) = (c_0)_\Delta$ and $\ell_\infty(\Delta) = (\ell_\infty)_\Delta$.

Example 3.5 (a) If $p > 1$, then $a \in bv_p^\beta$ if and only if $R \in \ell_q$ and $(nR_n)_{n=0}^\infty \in \ell_\infty$.
(b) We have $a \in (c_0(\Delta))^\beta$ if and only if $R \in \ell_1$ and $(nR_n)_{n=0}^\infty \in \ell_\infty$.
(c) We have $a \in (\ell_\infty(\Delta))^\beta$ if and only if $R \in \ell_1$ and $(nR_n)_{n=0}^\infty \in c_0$

Proof. Parts (a) and (b) are immediate consequences of (3.3) and (3.4).

(c) We have $a \in (\ell_\infty(\Delta))^\beta$ by Remark 3.3 if and only if $R \in \ell_\infty^\beta = \ell_1$, by Example 3.3, and $W \in (\ell_\infty, c_0)$. Now $W \in (\ell_\infty, c_0)$ by [21], Theorems 1.7.18 and 1.7.19, pp. 15–17 if and only if $\lim_{n\to\infty} \sum_{k=0}^\infty |w_{nk}| = \lim_{n\to\infty} n|R_n| = 0$ for all k, which is the second condition. $\square$

4. Some Neighbourhoods

We consider $\mathbb{R}^n$ for given $n \in \mathbb{N}$ as a subset of ω (Example 2.1) by identifying every point $X = (x^1, x^2, \cdots, x^n) \in \mathbb{R}^n$ with the real sequence $x = (x_k)_{k=1}^\infty \in \omega$ where $x_k = 0$ for all $k > n$, and introduce on $\mathbb{R}^n$ any of the metrics of Section 3.

We denote by $B_d(r, X_0) = \{X \in \mathbb{R}^n : d(X, X_0) < r\}$ the open ball in $(\mathbb{R}^n, d)$ of radius $r > 0$ with its centre in X_0, and consider the cases $n = 2$ and $n = 3$ for the graphical representation of neighbourhoods by the boundaries $\partial B_d(X_0)$ of $B_d(r, X_0)$.

4.1. *Neighbourhoods in* $\mathbb{R}^2$

The boundaries $\partial B_d(r, X_0)$ of $B_d(r, X_0)$ in $\mathbb{R}^2$ are given by the zeros of a real–valued function of two variables. Although our software provides an algorithm for this ([3, 4, 13]), it is more convenient and less time consuming if we can find a parametric representation for $\partial B_d(r, X_0)$. For instance, this can be achieved for the metrics d of Example 2.1 and $d_{(p)}$ of Example 3.1 (b).

Example 4.1 (a) We consider $\mathbb{R}^2$ with the metric d of Example 2.1 (left in Figure 3), that is

$$d(X, Y) = \frac{|x^1 - y^1|}{2(1 + |x^1 - y^1|)} + \frac{|x^2 - y^2|}{4(1 + |x^2 - y^2|)}. \tag{4.1}$$

Let $X_0 \in \mathbb{R}^2$ with the position vector $\vec{x}_0$, and $r < 1/4$. Then

$$\vec{x}(t) = \left\{ \frac{2r\,\mathrm{sgn}(\cos t)\cos^2 t}{1 - 2r\cos^2 t}, \frac{4r\,\mathrm{sgn}(\sin t)\sin^2 t}{1 - 4r\sin^2 t} \right\} \; + \; \vec{x}_0 \;\, (t \in (0, 2\pi))$$

is a parametric representation for $\partial B_d(r, X_0)$ which is not differentiable for $t = \pi/2, \pi, 3\pi/2$. The factors $1/2^k$ in the definition of the metric d for ω were introduced to ensure the convergence of the series; in fact they may be replaced by the terms of any positive convergent series. In the finite case, we may choose the factors to be equal to one, and consider the metric

$$\tilde{d}(X, Y) = \frac{|x^1 - y^1|}{1 + |x^1 - y^1|} + \frac{|x^2 - y^2|}{1 + |x^2 - y^2|} \tag{4.2}$$

and

$$\vec{x}(t) = \left\{ \frac{r\,\mathrm{sgn}(\cos t)\cos^2 t}{1 - r\cos^2 t}, \frac{r\,\mathrm{sgn}(\sin t)\sin^2 t}{1 - r\sin^2 t} \right\} \; + \; \vec{x}_0 \;\, (t \in (0, 2\pi)).$$

is a parametric representation for $\partial B_{d_{(p)}}(r, X_0)$ (right in Figure 3).

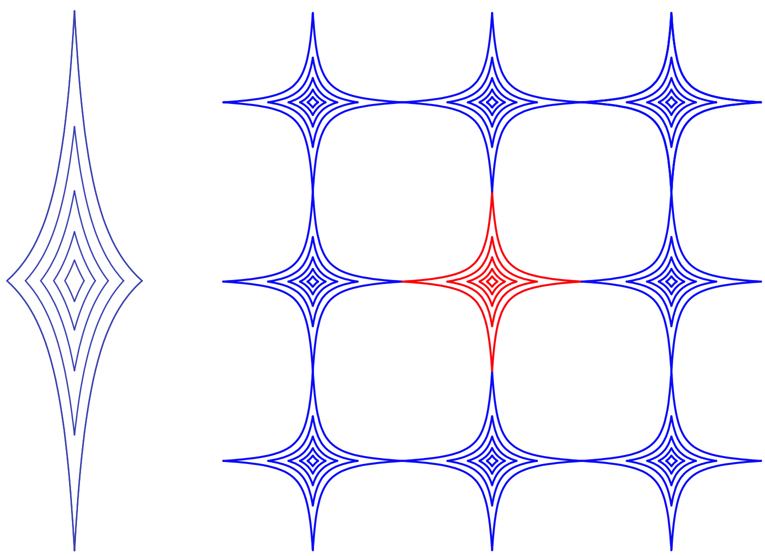

Figure 3. Boundaries of neigbourhoods in the metrics of (4.1) and (4.2) left: $\partial B_d(r, 0)$; right: $\partial B_{\tilde{d}}(r, X_0)$ for $r = n/30$ $(n = 0, 1, \ldots, 6)$.

(b) Now we consider the metric $d_{(p)}$ of Example 3.1 (b). Then

$$\vec{x}(t) = \{\phi_1(t), \phi_2(t)\} + \vec{x}_0 \;\, (t \in (0, 2\pi))$$

with

$$\phi_1(t) = r^{M/p_1}|\cos t|^{2/p_1}\operatorname{sgn}(\cos t) \tag{4.3}$$

and

$$\phi_2(t) = r^{M/p_2}|\sin t|^{2/p_2}\operatorname{sgn}(\sin t) \tag{4.4}$$

is a parametric representation for $\partial B_{d_{(p)}}(r, X_0)$ (Figure 4).

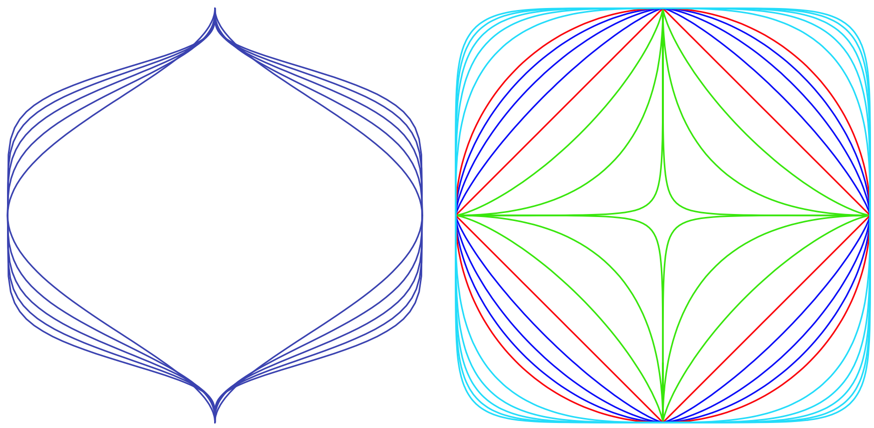

Figure 4. left: $\partial B_{d_{(p)}}(1, X_0)$ for $p_1 = 1/(n+1)$ and $p_2 = n+1$ $(n = 1, 2 \ldots, 5)$ right: $\partial B_{d_{(p)}}(1, X_0)$ for $p = 1/4 + (n+1)/4$ $(n = 0, 1, \ldots, 7)$ and $p = n - 5$ $(n = 8, 9, \ldots, 12)$

(c) Now we represent neighbourhoods in the metric $d_{(p)\circ\Delta}$ of $bv(p)$ and their dual neighbourhoods in the metric $d^*_{(p)\circ\Delta}$ of $(bv(p))^\beta$ (Example 3.3).
(d) Finally, we represent neighbourhoods in the metric

$$d = \frac{d_{(p_1)}}{1 + d_{(p_2)}} + \frac{d_{(p_1)}}{1 + d_{(p_2)}} \text{ ((2.1); Figure 6).}$$

Now we represent neighbourhoods in some weak topologies. Again, it is useful to obtain, if possible, parametric representations for the boundaries of the neighbourhood. Let d be a metric for $\mathbb{R}^2$ that comes from a paranorm g, $Y_0 \in \mathbb{R}^2$ and $\partial B_d(r, Y_0) \subset \mathbb{R}^2$ be given by a parametric representation $\vec{y}(t) = \{\phi_1(t), \phi_2(t)\}$ $(t \in I)$ where $I \subset \mathbb{R}$ is an interval. Then we have

$$Y \in \partial B_d(r, Y_0) \text{ if and only if } d\left((\phi_1(t), \phi_2(t)), Y_0\right) = r \text{ for some } t \in I.$$

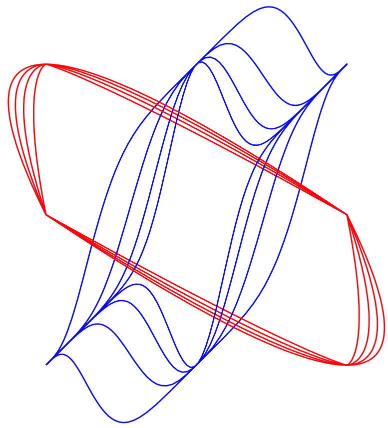

Figure 5. $\partial B_{d_{(p)}\circ\Delta}(1, X_0)$ and $\partial B^*_{d_{(p)}\circ\Delta}(1, X_0)$ for $p_1 = 1+4/(n+1)$ and $p_2 = 1/(4(n+1))$ $(n = 0, 1, 2, 3)$

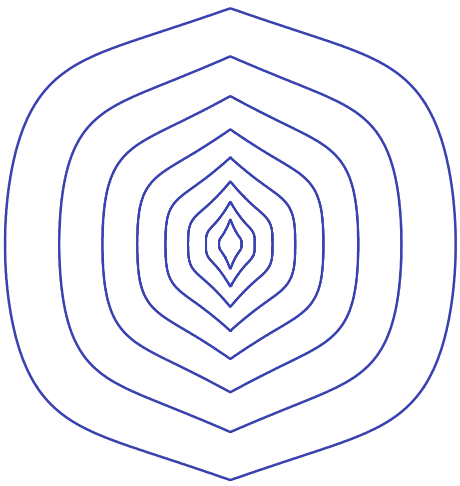

Figure 6. $\partial B_{d_{(p)}}(r, X_0)$ for $r = n/10$ $(n = 1, 2, \ldots, 9)$, $p_1 = (1, 2)$ and $p_2 = (5, 4)$ and the metric d of Example 4.1 (d)

We assume that $S \subset \mathbb{R}^2$ is a domain, and $f : S \to \mathbb{R}^2$ is bijective with inverse $h : \mathbb{R}^2 \to S$. Then the boundary $\partial B_\delta(r, X_0)$ of the weak neighbourhood $B_\delta(r, X_0)$ of $X_0 \in S$ in the metric δ of (2.1) of the weak topology

$w(S, f)$ has a parametric representation

$$\vec{x}(t) = h(\Phi(t) + f(X_0)) \ (t \in T) \text{ where } \Phi = (\phi_1, \phi_2),$$

since $X \in \partial B_\delta(r, X_0)$ if and only if

$$\begin{aligned}\delta(X, X_0) &= d\,(f(h(\Phi(t) + f(X_0)), f(X_0)) \\ &= d\,(\Phi(t) + f(X_0), f(X_0)) = g(\Phi(t)) = r.\end{aligned}$$

Example 4.2 (a) *Weak neighbourhoods in the unit circle C_1*
We use the function $f : C_1 = \{X = (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \to \mathbb{R}^2$ with

$$f(x, y) = \left(\frac{x}{1 - \sqrt{x^2 + y^2}}, \frac{y}{1 - \sqrt{x^2 + y^2}} \right) \tag{4.5}$$

to introduce the weak topology $w(C_1, f)$ on C_1. The inverse function $h : \mathbb{R}^2 \to C_1$ of f is given by

$$h(x, y) = \left(\frac{x}{1 + \sqrt{x^2 + y^2}}, \frac{y}{1 + \sqrt{x^2 + y^2}} \right).$$

We consider the metric $d_{(p)}$ of Example 3.1 (b) on $\mathbb{R}^2$, and write $\Phi = (\phi_1, \phi_2)$ where ϕ_1 and ϕ_2 are the functions in (4.3) and (4.4), and $\Psi = \Phi + f(X_0) = (\psi_1, \psi_2)$ for $X_0 \in C_1$. Then the boundary $\partial B_{\delta_{(p)}}(r, X_0) = \partial B_{d_{(p)} \circ f}(r, X_0)$ is given by a parametric representation $\vec{x}(t) = \{h_1(t), h_2(t)\}$ $(t \in (0, 2\pi))$ with

$$h_k(t) = h_k(\Psi(t)) = \frac{\psi_k(t)}{1 + \sqrt{(\psi_1(t))^2 + (\psi_2(t))^2}} \quad (k = 1, 2).$$

(b) Weak neighbourhoods in the square $[\,-1, 1]^2$
We use the functions $f, \tilde{f} : [\,-1, 1]^2 \to \mathbb{R}^2$ with

$$f(x, y) = \left(\tan\left(\frac{x\pi}{2}\right), \tan\left(\frac{y\pi}{2}\right) \right) \text{ and } \tilde{f}(x, y) = \left(\frac{x}{1 - |x|}, \frac{y}{1 - |y|} \right)$$

to introduce the metrics $\delta_{(p)} = d_{(p)} \circ f$ and $\tilde{\delta} = d_{(p)} \circ \tilde{f}$ in the square $[\,-1, 1]^2$.

4.2. *Neighbourhoods in $\mathbb{R}^3$*

Here we consider the case when the boundaries $\partial B_d(r, X_0)$ of the neighbourhoods in $\mathbb{R}^3$ are given by a parametric representation, as in the case of the metric $d_{(p)}$ of Example 3.1 (b). Then the principles of Subsection

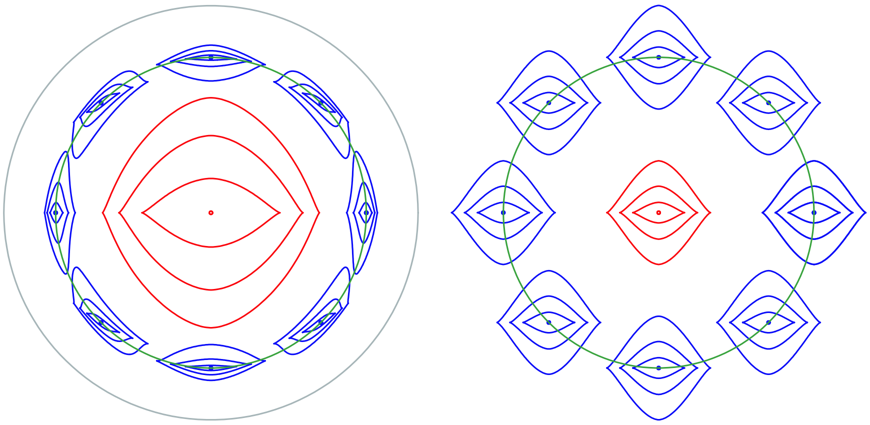

Figure 7. Weak neighbourhoods $\partial B_{\delta_{(p)}}(r, X_0)$ in the unit circle by the function f of (4.5) for $p = (7/4, 3/4)$ and corresponding neighbourhoods $\partial B_{d_{(p)}}(r, X_0)$ in $\mathbb{R}^2$

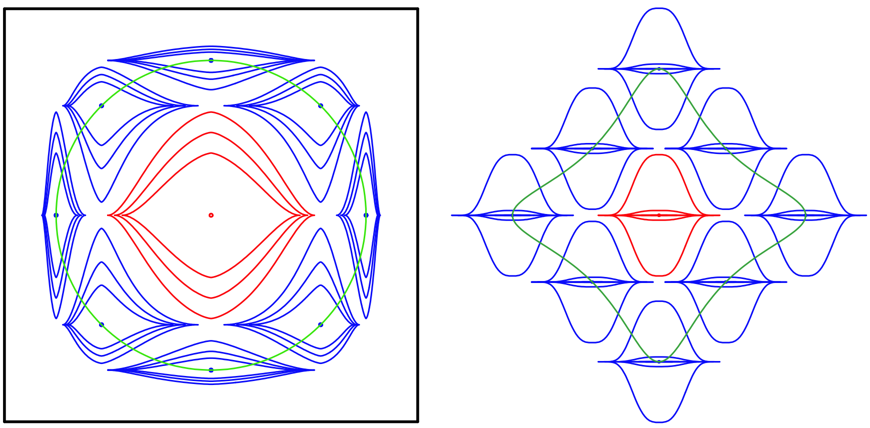

Figure 8. Weak neigbourhoods $\partial B_{\delta_{(p)}}(r, X_0)$ in $[-1, 1]^2$ and corresponding neighbourhoods $\partial B_{d_{(p)}}(r, X_0)$ in $\mathbb{R}^2$ for $p = (3, 1/8)$

4.1 can easily be extended and applied to the graphical representation of neighbourhoods in $\mathbb{R}^3$.

We consider the metric $d_{(p)}$ of Example 3.1 (b). Then

$$\vec{x}((u^1, u^2)) = \{\phi_1((u^1, u^2)),\ \phi_2((u^1, u^2)),\ \phi_3((u^1, u^2))\} + \vec{x}_0$$

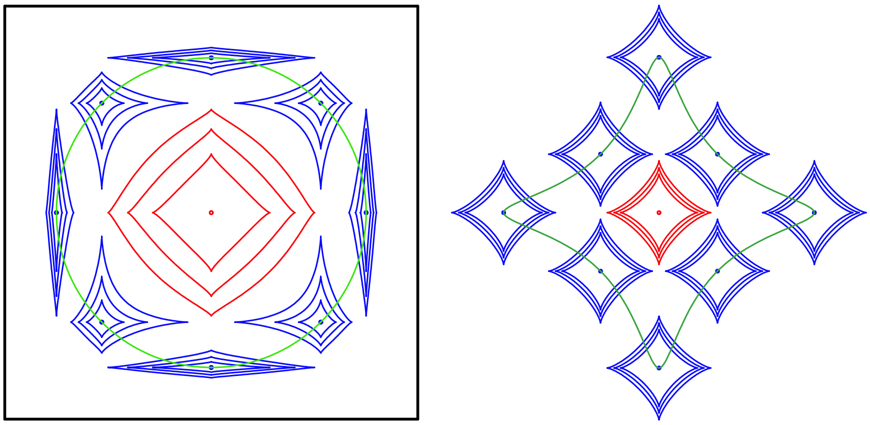

Figure 9. Weak neigbourhoods $\partial B_{\tilde{\delta}_{(p)}}(r, X_0)$ in $[-1,1]^2$ and corresponding neighbourhoods $\partial B_{d_{(p)}}(r, X_0)$ in $\mathbb{R}^2$ for $p = (3/4, 3/4)$

$((u^1, u^2) \in (-\pi/2, \pi/2) \times (0, 2\pi))$ with

$$\begin{aligned}
\phi_1((u^1, u^2)) &= r^{M/p_1}\,\mathrm{sgn}(\cos u^2)(\cos u^1 |\cos u^2|)^{2/p_1},\\
\phi_2((u^1, u^2)) &= r^{M/p_2}\,\mathrm{sgn}(\sin u^2)(\cos u^1 |\sin u^2|)^{2/p_2}, \text{and}\\
\phi_3((u^1, u^2)) &= r^{M/p_3}\,\mathrm{sgn}(\sin u^1)|\sin u^1|^{2/p_3}
\end{aligned}$$

is a parametric representation for $\partial B_{d_{(p)}}(r, X_0)$ which is not differentiable for $u^2 = \pi/2, \pi, 3\pi/2$ (Figure 10).

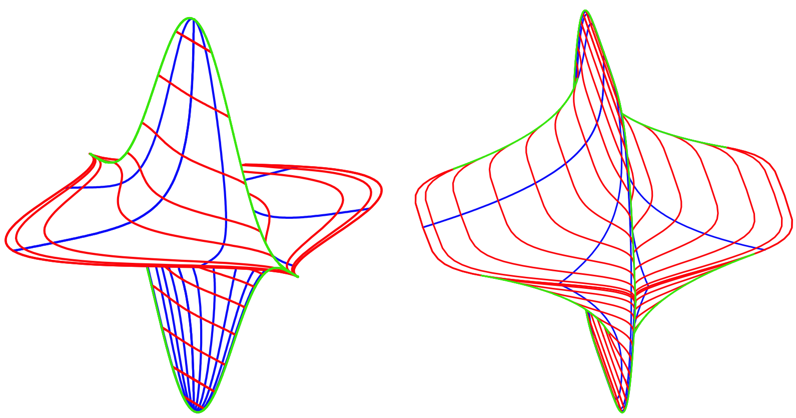

Figure 10. $\partial B_{d_{(p)}}(1, 0)$ for left: $p = (1/2, 2, 1/4)$ and right: $p = (1/4, 5, 1)$

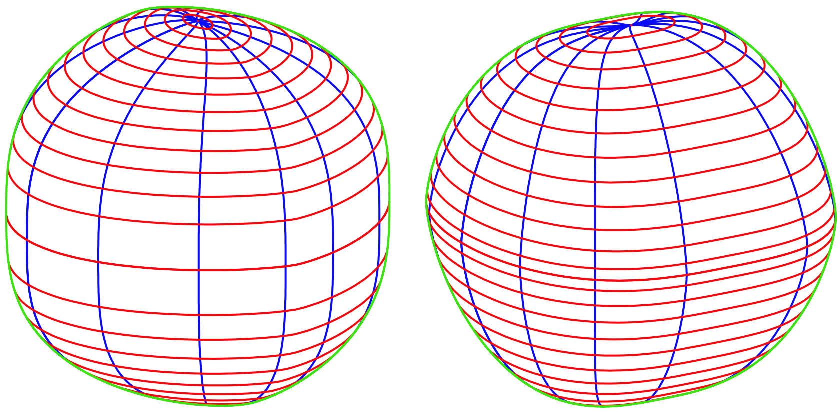

Figure 11. left: $\partial B_{d_{(p)}}(1,0)$ for $p = (3/2, 2, 3)$; right: its dual

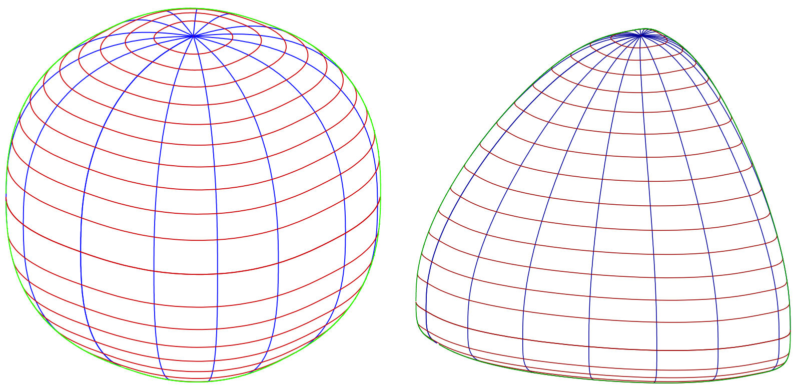

Figure 12. left: $\partial B_{d_p}(1,0)$ in $\mathbb{R}^3$; right: $\partial B_{d_p \circ f}(1,0)$ in $(0,\infty)^3$ with $f = (\log, \log, \log)$, for $p = 5/2$

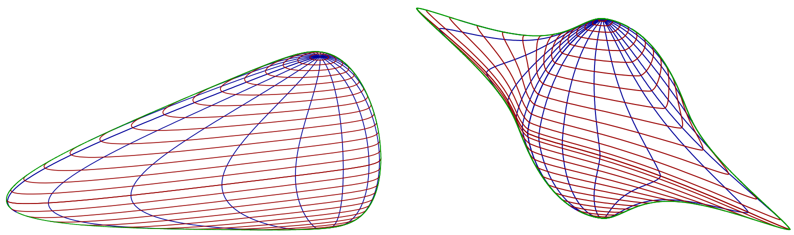

Figure 13. left: $\partial B_{d_p \circ f}(1,0)$ in $(0,\infty)^2 \times \mathbb{R}$ with $f = (\log, \log, id)$; right: $\tilde{f}(\partial B_{d_2}(1,0))$ with $\tilde{f} = (\sinh, \tan, id)$, for $p = 2$

References

1. M. Djorić and P. Janičić, *Constructions, instructions, teaching mathematics and its applications*, Oxford University Press **23(2)** (2004), 69–88.
2. K.G. Grosse–Erdmann, *The structure of the sequence spaces of Maddox*, Can. J. Math. **44(2)**, (1992) 298–307.
3. M. Failing, *Entwicklung numerischer Algorithmen zur computergrafischen Darstellung spezieller Probleme der Differentialgeometrie und Kristallograpie*, Ph.D. thesis, Gießen, Shaker Verlag Aachen, 1996.
4. M. Failing and E. Malkowsky, *Ein effizientes Nullstellenverfahren zur computergraphischen Darstellung spezieller Kurven und Flächen*, Mitt. Math. Sem. Giessen **229** (1996), 11–28.
5. A.M. Jarrah and E. Malkowsky, *Linear operators between some matrix domains*, Rend. Circ. Mat. Palermo **68** (2002), 641–655.
6. P. Janičić and I. Trajković, *WinGCLC–a workbench for formally describing figures*, In: Proceedings of the Spring Conference on Computer Graphics (SCCG 2003), April 24–26, ACM Press, New York, USA, 2003.
7. C.G. Lascarides, *A study of certain sequence spaces*, Pacific J. Math. **38(2)** (1971), 487–501.
8. C.G. Lascarides and I. J. Maddox, *Matrix transformations between some classes of sequences*, Proc. Camb. Phil. Soc. **68** (1970), 99–104.
9. I.J. Maddox, *Paranormed sequence spaces generated by infinite matrices*, Proc. Camb. Phil. Soc. **64** (1968), 335–340.
10. I.J. Maddox, *Some properties of paranormed sequence spaces*, J. London Math. Soc. **1** (1969), 316–322.
11. I.J. Maddox, *Continuous and Köthe–Toeplitz duals of certain sequence spaces*, Proc. Camb. Phil. Soc. **65** (1969), 431–435.
12. I.J. Maddox and J. W. Roles, *Absolute convexity in certain topological linear spaces*, Proc. Camb. Phil. Soc. **66** (1969), 541–545.
13. E. Malkowsky, *An open software in OOP for computer graphics and some applications in differential geometry*, In: Proceedings of the 20th South African Symposium on Numerical Mathematics, , 51–80, 1994.
14. E. Malkowsky, *A software for the visualisation of differential geometry*, Visual

Mathematics **4(1)**, (2002) (Electronic publication)
`http://www.mi.sanu.ac.yu/vismath/malkovsky/index.htm`

15. E. Malkowsky, *Visualisation and animation in mathematics and physics*, In: Proceedings of the Institute of Mathematics of NAS of Ukraine **50(3)**, 1415–1422, 2004.
`http://www.imath.kiev.ua-snmp2003/Proceedings/Proceedings2003.html`
16. E. Malkowsky and W. Nickel W, *Computergrafik und Differentialgeometrie*, Vieweg–Verlag, Braunschweig, 1993.
17. E. Malkowsky and V. Rakočević, *On matrix domains of triangles*, under communication
18. E. Malkowsky and V. Veličković, *Analytic transformations between surfaces with animations*, In: Proceedings of the Institute of Mathematics of NAS of Ukraine **50(3)**, 1496–1501, 2004.
`http://www.imath.kiev.ua-snmp2003/Proceedings/Proceedings2003.html`
19. S. Simons, *The sequence spaces $\ell(p_\nu)$ and $m(p_\nu)$*, Proc. London Math. Soc. **15**, 422–436, 1965.
20. A. Wilansky, *Functional Analysis, Blaisdell Publishing Co.*, New York, Toronto, London, 1964.
21. A. Wilansky, *Summability through Functional Analysis*, North–Holland Mathematics Studies **No. 85**, North–Holland, Amsterdam, New York, Oxford, 1984.